DUAL TREES AND RESUMMATION THEOREMS

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ABSTRACT

Various resummation theorems known for graphical expansions in statistical physics and field theory are exhibited as special cases of a single symmetrical, easily remembered theorem on a generalized structure we call a dual tree.

I. Introduction

Graphical summation is now a common technique in physics. Graphs are constructed according to some simple recursive rule, and to each graph is associated a value determined algebraically according to the topology of the graph. The physical quantity of interest is expressed as the (infinite) sum of the values of all graphs. In a resummation theorem, the sum is proved identical to a sum derived in some other way, typically from a smaller set of graphs with modified rules for assigning values.

Many resummation theorems have been proved for various graphical systems. These theorems have a family resemblance, although the detailed statement and proof varies from system to system according to the kind of graph considered and the way its value is determined. In this paper we shall bring a whole family of such theorems into a single theorem, stated and proved once for all. To this end we define a particular kind of graph, which we call a dual tree, composed of two elements designated arbitrarily as squares and circles. By suitable interpretation of these elements, the system of dual trees can be transformed into any of a remarkably wide variety of known graphical systems.

In Section II we shall define the dual tree system and state its resummation theorem. We have two proofs, both derived from arguments already in the literature. The proofs are given in Appendices A and B. In the following four sections we shall derive a number of well-known results from this theorem, thereby exhibiting the flexibility of the dual tree concept. We conclude in Section VII.

Dual Trees

II. Central Theorem

By a "dual tree" we mean an assemblage of n_S squares and n_C circles ($n_S + n_C \geq 1$) joined by n_L lines under the following restrictions:

1. Each line joins a circle to a square—never a square to a square or a circle to a circle.
2. The whole graph is simply connected—it cannot be decomposed without severing a line, but the removal of any single line would render it decomposable.

The use of squares and circles has no geometrical significance; it is an aid to visual representation.

The n_S squares are distinguishable and to that end are labeled with integers from 1 to n_S . Similarly the n_C circles.

Two dual trees (from now on we shall drop the word "dual") are identical if they have the same topology, with the labeling taken into account. Figure 1 shows the twelve distinct trees having $n_S = 3$, $n_C = 2$.

We assume that for each non-negative integer n , two symmetric functions on n arguments are given, which we call S_n and C_n . The arguments of all these functions vary over the same domain X . For $n = 0$ there are no arguments, so that S_0 and C_0 are constants.

Let $S(\xi)$, $C(\xi)$ represent arbitrary functions on an argument $\xi \in X$. We define the functionals

$$\mathcal{D}(S) \equiv \sum_{n=0}^{\infty} n!^{-1} \int_{\xi_1 \dots \xi_n} C_n(\xi_1 \dots \xi_n) \prod_{i=1}^n S(\xi_i) \quad (1)$$

$$\mathcal{T}(C) \equiv \sum_{n=0}^{\infty} n!^{-1} \int_{\xi_1 \dots \xi_n} S_n(\xi_1 \dots \xi_n) \prod_1^n C(\xi_1) \quad (2)$$

where $\int_{\xi_1 \dots \xi_n}$ represents integration or summation with respect to a measure defined on X . The term in $n=0$ has the value C_0 in (1), or S_0 in (2).

To each tree T we assign a value $w(T)$ as follows. The n_L lines are given an arbitrary order. To the i^{th} line we associate a variable $\xi_i \in X$. To each square (attached to lines $i_1 \dots i_m$) we associate the quantity $S_m(\xi_{i_1} \dots \xi_{i_m})$. To each circle (attached to lines $i_1 \dots i_q$) we associate $C_q(\xi_{i_1} \dots \xi_{i_q})$. The product of all these S - and C -functions we call $\Pi(\xi_1 \dots \xi_{n_L})$. We obtain $w(T)$ by summing Π over all the ξ 's and dividing by $n_S! \cdot n_C!$. Thus the first six trees shown in Figure 1 have each the value

$$w = \frac{1}{3! \cdot 2!} \int_{\xi_1 \xi_2 \xi_3 \xi_4} S_1(\xi_1) S_2(\xi_2, \xi_3) S_1(\xi_4) C_2(\xi_1, \xi_2) C_2(\xi_3, \xi_4)$$

and the last six have each the value

$$w = \frac{1}{3! \cdot 2!} \int_{\xi_1 \xi_2 \xi_3 \xi_4} S_1(\xi_1) S_2(\xi_2, \xi_3) S_1(\xi_4) C_3(\xi_1, \xi_3, \xi_4) C_1(\xi_2).$$

We now define

$$\tau \equiv \sum_T w(T) \quad (3)$$

where the sum goes over all distinct trees. ⁽¹⁾ Regarding τ as a functional on $S_1, C_1, S_2, C_2, \dots$, we define

$$\overline{S}(\xi) \equiv \frac{\delta \tau}{\delta C_1(\xi)}, \quad \overline{C}(\xi) \equiv \frac{\delta \tau}{\delta S_1(\xi)} \quad (4)$$

Our central result is the following theorem.

Theorem 1: given the above definitions, the following relations hold

$$\mathcal{B}(\bar{S}) = \sum_T n_C w(T) , \quad \mathcal{F}(\bar{C}) = \sum_T n_S w(T) , \quad (5)$$

$$\bar{S}(\xi) = \left. \frac{\delta \mathcal{F}}{\delta C(\xi)} \right|_{C=\bar{C}} , \quad \bar{C}(\xi) = \left. \frac{\delta \mathcal{B}}{\delta S(\xi)} \right|_{S=\bar{S}} , \quad (6)$$

$$\tau = \mathcal{B}(\bar{S}) + \mathcal{F}(\bar{C}) - \int_{\xi} \bar{S}(\xi) \bar{C}(\xi) . \quad (7)$$

We shall defer the proof (two versions) to appendices.

Corollary: let the above definitions be modified as follows. Let $S_1, S_2, S_3, \dots$ all be functions of one argument. For each tree T , let $\Pi(\xi_1 \dots \xi_{n_S})$ be defined by attaching ξ_i to the i^{th} square, associating $S_m(\xi_i)$ to the i^{th} square where m is the number of attached lines, and associating $C_q(\xi_{i_1} \dots \xi_{i_q})$ to each circle (attached to squares $i_1 \dots i_q$). Obtain w from Π as before. Replace (2) by

$$\mathcal{F}(C) \equiv \sum_{n=0}^{\infty} n!^{-1} \int_{\xi} S_n(\xi) [C(\xi)]^n . \quad (8)$$

Retain (1), (3), and (4). Then (5), (6), (7) are still true.

This follows immediately from Theorem 1. For the values of $w(T)$ and $\mathcal{F}(C)$ so obtained are the same as would have followed from the old definitions if the S_n had the form

$$S_n(\xi_1 \dots \xi_n) = \delta(\xi_1 \dots \xi_n) S_n(\xi_1) \quad (9)$$

where $\delta(\xi_1 \dots \xi_n)$ is a suitable product of Kronecker or Dirac δ 's, depending on whether X is discrete or continuous.

III. The Number of Cayley Trees

As a simple application of Theorem 1, we derive a theorem due to Lee and Yang⁽²⁾ on the number of doubly labeled Cayley trees of order ℓ . Such a tree is just a simply connected graph composed of ℓ labeled points joined by $\ell - 1$ labeled lines. Letting A_ℓ denote the number of ways to do this⁽³⁾, Lee and Yang showed that

$$-\sum_{\ell=1}^{\infty} A_\ell (-x)^\ell / (\ell - 1)!^2 \ell = D + \frac{1}{2} D^2 \quad (10)$$

where

$$D e^D = x. \quad (11)$$

To reproduce this result from Theorem 1, we transform Cayley trees into dual trees by inserting a square into each line, and replacing each point by a circle, as shown in Figure 2. In this way we obtain only those dual trees in which all squares have order 2. We insure that all other dual trees have no value, by setting $S_n = 0$ for $n \neq 2$. We let the domain X consist of one point, so that all the S_n , C_n , $\bar{S}$, $\bar{C}$ are numbers instead of functions. We let $S_2 = -1$, and $C_n = x$, independently of n . Then we have $n_C = \ell$, $n_S = \ell - 1$, and hence

$$w(T) = \frac{1}{\ell!} \frac{1}{(\ell - 1)!} (-)^{\ell-1} x^\ell \quad (12)$$

for any dual tree T that corresponds to a Cayley tree of ℓ points. It follows that τ is just the left side of Eq. (10).

Putting our definitions of C_n , S_n into (1) and (2), we have

$$D(S) = \sum_{n=0}^{\infty} n!^{-1} \times S^n = x e^S, \quad (13)$$

$$T(C) = 2!^{-1}(-1) C^2 = -\frac{1}{2} C^2 \quad (14)$$

and so (6) becomes

$$\bar{S} = -\bar{C}, \quad \bar{C} = x e^{\bar{S}} \quad (15)$$

from which we immediately obtain (11) on setting $D \equiv \bar{C}$.

Now if we use (15) to write (13) as

$$D(\bar{S}) = \bar{C} \quad (16)$$

and substitute, with (14) and (15), into (7), we obtain

$$\begin{aligned} \tau &= \bar{C} + \left(-\frac{1}{2} \bar{C}^2\right) - (-\bar{C}) \bar{C} \\ &= \bar{C} + \frac{1}{2} \bar{C}^2 \end{aligned} \quad (17)$$

which is equivalent to (10).

IV. The Generating Functional of Irreducible Parts

For our second application, we have in mind any system in which a sequence of Green's functions is generated by some functional $\mathcal{U}$ on an external field J .

Suppose that we have a sequence

$$G_2(x_1, x_2), \quad G_3(x_1, x_2, x_3), \quad \dots$$

where the x 's may be position vectors in 3-space or in space-time, or may also contain intrinsic information (spin, etc.). We introduce a complementary sequence $\Gamma_2(x_1, x_2)$, $\Gamma_3(x_1, x_2, x_3)$, etc. The first has a special definition:

$$\Gamma_2 = -G_2^{-1} \quad (18)$$

where the meaning of (18) is that

$$\int_{x_2} \Gamma_2(x_1, x_2) G_2(x_2, x_3) = -\delta(x_1, x_3) \quad (19)$$

The higher Γ 's are uniquely defined as the irreducible vertex parts in a graphical expansion for the higher G 's. That is, each G_n ($n \geq 3$) can be obtained from G_2 and $\Gamma_3, \Gamma_4, \dots$ by the following prescription: Take all possible unlabeled Cayley trees with n labeled endpoints, and with no vertices of order 2. Associate the variables $x_1 \dots x_n$ with the endpoints in the specified order. In each tree, associate extra variables x_i ($i > n$) with each insertion of a line into an internal vertex. To each vertex of order m , assign a factor $\Gamma_m(x_{i_1}, x_{i_2}, \dots, x_{i_m})$. To each line, assign a factor $G_2(x_i, x_j)$.

Integrate the product of all these factors over the x_i ($i > n$) and sum the result over all the permitted Cayley trees with n endpoints. The result is $G_n(x_1, x_2, \dots, x_n)$.

Thus,

$$G_3(x_1, x_2, x_3) = \int_{x_4 x_5 x_6} \Gamma_3(x_4, x_5, x_6) G_2(x_1, x_4) G_2(x_2, x_5) G_2(x_3, x_6), \quad (20)$$

$$G_4(x_1, x_2, x_3, x_4) = \int_{x_5 \dots x_{10}} \Gamma_3(x_5, x_6, x_7) \Gamma_3(x_8, x_9, x_{10}) G_2(x_7, x_{10}) \\ G_2(x_1, x_5) G_2(x_2, x_6) G_2(x_3, x_8) G_2(x_4, x_9) \\ + \text{cyclic permutations on } x_1, x_2, x_3$$

$$+ \int_{x_5 \dots x_8} \Gamma_4(x_5, x_6, x_7, x_8) G_2(x_1, x_5) G_2(x_2, x_6) G_2(x_3, x_7) G_2(x_4, x_8) \quad (21)$$

and so on. (See Figure 3.)

It is easily seen that if m is the order of a vertex,

$$\sum (m - 2) = n - 2 \quad (22)$$

and therefore only a finite number of trees contribute to G_n , involving only $\Gamma_3 \dots \Gamma_n$, and only one tree involves Γ_n . Therefore $\Gamma_3, \Gamma_4, \dots$ are uniquely determined by (20), (21), etc.

Now let $\mathcal{H}$ and $\mathcal{K}$ be the generating functionals

$$\mathcal{H}(J) = \sum_2^{\infty} n!^{-1} \int_{x_1 \dots x_n} G_n(x_1 \dots x_n) \prod_1^n J(x_i), \quad (23)$$

$$K(A) = \sum_2^{\infty} n!^{-1} \int_{x_1 \dots x_n} \Gamma_n(x_1 \dots x_n) \prod_1^n A(x_i) . \quad (24)$$

It is a well-known theorem⁽⁴⁾, but one whose proof is usually indicated rather than given in detail, that $\mathfrak{H}$ and K are related by a Legendre transformation.

That is, if we let

$$A(x) = \frac{\delta \mathfrak{H}(J)}{\delta J(x)} , \quad (25)$$

then

$$\mathfrak{H}(J) = K(A) + \int_x J(x) A(x) . \quad (26)$$

We shall now obtain this result from Theorem 1.

By combining (23) with (20), (21), etc., we obtain $\mathfrak{H}(J)$ as a sum over all Cayley trees. In this sum, each m -vertex ($m > 2$) contributes a factor Γ_m , each line a G_2 , and each endpoint a J . Any 2-vertex contributes a factor 0. Only the endpoints are labeled, and we are to divide by $n!$ where n is the number of endpoints. The term of (23) with $n=2$ is included naturally if we count the Cayley tree with one line and no vertex.

We convert each Cayley tree into a dual tree as follows. First we insert a square into each line as in the previous section. The variables x_i are now associated with the newly formed lines.

The labeling of the endpoints renders all n_S squares inequivalent. We may therefore label all the squares and compensate by dividing the value of each tree by $n_S!$, since there are now $n_S!$ as many labeled trees as before.

The labeling of the squares now renders the endpoints inequivalent. Therefore we may drop the labeling of endpoints and compensate by omitting the factor $n!^{-1}$.

We now replace each vertex and each endpoint by a circle. All the circles are inequivalent because the squares are labeled. Hence we may label the circles and divide by $n_C!$ where n_C is the total number of circles.

$\mathfrak{D}$ is now given as a sum over dual trees, and the contribution of each dual tree T can be identified with $w(T)$ as defined in Section II, provided that X is the domain of the x 's and

$$\left. \begin{aligned} S_2 &= G_2, & S_{n \neq 2} &= 0 \\ C_0 &= C_2 = 0, & C_1 &= J, & C_{n \geq 3} &= \Gamma_n \end{aligned} \right\} \quad (27)$$

Hence we may identify γ with $\mathfrak{D}$, and $\bar{S}$ with A . (Compare (25) with (4).)

Putting (27) into (1) and (2), we have

$$\begin{aligned} B(\bar{S}) &= \int_x J(x) A(x) + \sum_3^{\infty} n!^{-1} \int_{x_1 \dots x_n} \Gamma_n(x_1 \dots x_n) \prod_1^n A(x_i) \\ &= \int_x J(x) A(x) + K(A) - \frac{1}{2} \int_{x_1 x_2} \Gamma_2(x_1, x_2) A(x_1) A(x_2) \end{aligned} \quad (28)$$

and

$$\mathcal{F}(C) = \frac{1}{2} \int_{x_1 x_2} G_2(x_1, x_2) C(x_1) C(x_2), \quad (29)$$

whence

$$\bar{S}(x) = \int_{x'} G_2(x, x') \bar{C}(x') \quad (30)$$

by (6). Therefore

$$\begin{aligned} \mathcal{F}(\bar{C}) - \int_x \bar{S}(x) \bar{C}(x) &= -\frac{1}{2} \int_x \bar{S}(x) \bar{C}(x) = -\frac{1}{2} \int_{x_1 x_3} \bar{S}(x_1) \delta(x_1, x_3) \bar{C}(x_3) \\ &= \frac{1}{2} \int_{x_1 x_2} \Gamma_2(x_1, x_2) \bar{S}(x_1) \bar{S}(x_2) \end{aligned} \quad (31)$$

on account of (19) with (30).

If we add (28) to (31) and use (7), identifying $\mathcal{A}$ with τ on the left side and $\bar{S}$ with A on the right, we obtain the desired Eq. (26).

V. Mayer Cluster Expansion

It is well known⁽⁵⁾ that the pressure p and density ρ of a gas of short-range interacting particles obeying classical statistics are given in the thermodynamic limit by

$$p/kT = \sum_{\ell=1}^{\infty} b_{\ell} z^{\ell}, \quad \rho = \sum_{\ell=1}^{\infty} \ell b_{\ell} z^{\ell} \quad (32)$$

where z is the fugacity and b_{ℓ} is defined as follows. Take ℓ distinct (labeled) points and connect them by lines. Each pair of points may be connected by at most one line. Enough lines must be drawn to bind all ℓ points into a connected graph or cluster; it need not be simply connected. Now assign a position $\vec{r}_i$ to the i^{th} point. To each line (joining the i^{th} and j^{th} points) associate the quantity $f(\vec{r}_i - \vec{r}_j)$, where f is a certain temperature-dependent even function of short range. Multiply the f 's together, integrate with respect to all the $\vec{r}$'s over the volume V of the gas, sum over all clusters of ℓ points, and divide by $\ell!$. The result approaches b_{ℓ} as $V \rightarrow \infty$.

Mayer and coworkers⁽⁶⁾ observed that any cluster can be uniquely resolved into (one or more) irreducible clusters jointed together through common points in a simply connected manner. An irreducible cluster is one which cannot be severed by the removal of any single point and the lines attached to it. They showed that if $(k+1)^{-1} \beta_k$ is the contribution of single irreducible clusters to b_{k+1} then the elimination of z from (32) yields

$$p/kT = \rho \left[1 - \sum_{k=1}^{\infty} (k+1)^{-1} \beta_k \rho^k \right] \quad (33)$$

The same result can be obtained easily from the corollary to Theorem 1.

We first show how each cluster of l points yields a dual tree with $n_S = l$, unique except for the labeling of circles. Simply resolve the cluster into irreducible parts; for each irreducible part introduce a circle connected by lines to each of the constituent points; erase all the original lines between points, and replace the points by squares with the same labels. (See Figure 4.) There are now $n_C!$ distinct ways to label the circles—distinct because the squares are labeled already.

To a given tree may correspond more than one distinct cluster, because a circle with > 3 lines attached may represent any of a plurality of irreducible clusters. (See Figure 4, a and b.) All the clusters that reduce to the same tree may be grouped into a species.

To apply the Corollary, we take ξ_i as $\vec{r}_i$, the position of the i^{th} particle; X as the space occupied by the gas; $S_n(\vec{r}) = z$ for all $n, \vec{r}$; and $C_n(\vec{r}_1, \dots, \vec{r}_n) = \sum \prod f(\vec{r}_i - \vec{r}_j)$, where the sum goes over all irreducible clusters of n points, and the product is over those pairs i, j joined by a line of the selected irreducible cluster. It is easily seen that for any tree T , the contribution of the corresponding species to $z^{n_S} b_{n_S}$ is just $(n_C! / V) w(T)$. Since the species yields $n_C!$ distinct trees, the comparison of (3) and (5) with (32) yields

$$\tau = pV/kT, \quad \mathcal{F}(\bar{C}) = pV. \quad (34)$$

On the other hand, $\mathcal{F}$ is given explicitly through (8) as

$$\mathcal{F}(C) = \int_V d^3\vec{r} z \exp [C(\vec{r})] \quad (35)$$

and the use of (6) gives

$$\overline{S}(\vec{r}) = z \exp [\overline{C}(\vec{r})] . \quad (36)$$

in the limit $V \rightarrow \infty$, the range of f remaining finite, the functions $\overline{S}$ and $\overline{C}$ as determined by (4) or (6) will be $\vec{r}$ -independent except for a negligible region near the boundary of V . Therefore (35) and (36) yield

$$\mathcal{F}(\overline{C}) = V \overline{S} \quad (37)$$

and, from (34)

$$\overline{S} = \rho . \quad (38)$$

Finally, by comparing the definition of β_k with that of C_{k+1} and using (1), we obtain

$$\mathcal{O}(S) = V \sum_{k=1}^{\infty} (k+1)^{-1} \beta_k S^{k+1} \quad (39)$$

for any constant function S . Using (38) we have

$$\mathcal{O}(\overline{S}) = V \sum_{k=1}^{\infty} (k+1)^{-1} \beta_k \rho^{k+1} \quad (40)$$

and using (6) we get

$$\overline{C} = \sum_k \beta_k \rho^k . \quad (41)$$

The substitution of (34), (38), (40) and (41) into (7) yields (33) directly. Also from (36), (38) and (41) we may solve for z , obtaining

$$\ln z = \ln \rho - \sum_k \beta_k \rho^k . \quad (42)$$

The proof of Theorem 1 as applied to this example may be regarded as a relatively inefficient version of a derivation of (33) and (42) given by Uhlenbeck and Ford⁽⁷⁾. The advantage of the present treatment is that it places the result in a more general perspective.

VI. Occupation-Number Expansion in Quantum Statistics

Lee and Yang⁽⁸⁾ found that the logarithm of the grand partition function for a system of interacting Bose particles could be expressed as the sum of connected diagrams ("primary 0-graphs") formed as follows:

Every line is directed; it begins at a vertex and ends at a vertex, possibly the same one. Every vertex has n lines entering and n lines leaving, where n is a positive integer pertaining to the vertex; a 1-vertex has $n = 1$, a 2-vertex has $n = 2$, etc. Each line is associated with a momentum variable k .

To evaluate a graph, we take a factor z for each line (z = fugacity) and a factor $Y_n(k_{i_1} \dots k_{i_n}; k_{j_1} \dots k_{j_n})$ for each n -vertex where $i_1 \dots i_n$ are the indices of the incoming lines and $j_1 \dots j_n$ are those of the outgoing lines, and Y_n is a function whose form need not concern us except that it is symmetric in the k_{i_j} 's, symmetric in the k_{j_j} 's, and vanishes unless momentum is conserved, that is unless

$$\sum_{v=1}^n k_{i_v} = \sum_{v=1}^n k_{j_v} . \quad (43)$$

We multiply all these factors together and divide by the symmetry number of the graph. Graphs differing only in the labeling of lines are not regarded as distinct.

Let us regard z as a bare propagator for these lines. In an obvious way, Lee and Yang introduced a clothed propagator $M(k)$, obtained by modifying a line of momentum k with all possible insertions (see their Eq. (IV.21)). They also defined a half-clothed propagator $m(k)$ by allowing only 1-vertex insertions; thus

$$m(k)^{-1} = z^{-1} - Y_1(k, k) . \quad (44)$$

Physically, the quantity $z^{-1}M(k) - 1$ is the mean occupation number of the particle state k , and $z^{-1}m(k) - 1$ is that occupation number in the absence of interaction, but still under the influence of Bose statistics.

After various manipulations, they found that the logarithm of the partition function could be written as

$$\mathcal{P} = \sum_k \ln [z^{-1}M(k)] - \sum_k m(k)^{-1} [M(k) - m(k)] + \mathcal{P}' \quad (45)$$

where $\mathcal{P}'$ is the sum over all irreducible graphs, excluding 1-vertices and replacing z with $M(k)$ for each line. (See their Eq. (IV.33).) By an irreducible graph is meant one that cannot be severed by removing two lines.

Our purpose in this section is to derive (45) from the Corollary to Theorem 1.

The treatment can be adapted to Fermi statistics by trivial modifications.

Other authors^{(9), (10)} have developed a field-theoretic graphical expansion for the logarithm of the partition function, more closely related to perturbation theory. From that expansion one obtains a formula (Eq. (47) of Ref. (9)) just like Eq. (45). However, each vertex then corresponds to a single application of the interaction Hamiltonian, whereas the vertices of Lee and Yang correspond to modified scattering matrices; and the variable k has not only the three momentum components but also a fourth energy-like component that varies discretely in steps proportional to the temperature.

The differences, however, between the formalism just described and that of Lee and Yang are completely inessential to what follows. Therefore the work of this

section is equally applicable to both, although we shall use the notation of Lee and Yang.

It is, on the other hand, essential here that k be conserved at each vertex as expressed by Eq. (43). This restriction distinguishes the subject matter of this section from that of Section IV, in which k -conservation was not assumed for the Fourier-transformed vertices.

Before describing the reduction of graphs to dual trees, we need some preliminary results concerning the graphs themselves.

Let us say that two vertices V_1 and V_2 are well-connected if the removal of any two lines from the graph leaves V_1 and V_2 still joined by a continuous path. Well-connectedness is clearly an equivalence relation.

Let us say that two lines i and j are related if the graph falls into two parts when i and j are both removed. By k -conservation, any two related lines must have the same k . But the converse also holds. For if i and j are not related, there must be a continuous path connecting the head of i to its tail, not traversing either i or j . Therefore it is possible to increment k_i without incrementing k_j , and still conserve k at each vertex by applying the same increment everywhere along this path.

Since i and j are related if and only if they are required to have the same k , relatedness is also an equivalence relation. We shall call the equivalence classes families of lines.

We now state a crucial topological lemma.

Lemma: the lines in a family have a unique cyclical ordering such that the head of each is well-connected to the tail of the one following.

The truth of this is intuitively evident, and it is used without proof in most treatments. For completeness we give a proof in Appendix C.

With the help of this lemma, we can express the logarithm of the partition function as a sum over dual trees. The transformation from graph to dual tree will have two parts: first we shall transform each graph G into a circuit graph J , and then we shall transform each circuit graph J into a dual tree T .

We start by labeling all the lines in G . This yields $n_\ell!/\sigma$ distinct labeled graphs, where n_ℓ is the number of lines and σ is the symmetry number. To compensate we divide by $n_\ell!$ instead of σ in evaluating each graph. Thus the value of a labeled graph is $n_\ell!^{-1} z^{n_\ell} \sum_{\{k\}} V_G(\{k\})$ where $V_G(\{k\})$ is the product of the functions Y_n for each n -vertex and $\{k\}$ represents the n_ℓ variables k .

Next, we form J by identifying all vertices in G that are well-connected, without destroying any lines. It follows from our lemma that the lines of each family now form a connected circuit, and for this reason we call J a circuit graph.

We now define the value of J to be the sum of the values of all labeled G 's that yield J under the foregoing operation. This sum may be evaluated from the structure of J in the following way.

If we let $\bar{P}$ be the set of vertices in G that condense into one vertex P in J , then the topology of $\bar{P}$ and of all the lines attached to it is completely contained in the graph I that is formed by deleting all vertices of G not in $\bar{P}$ and all lines of G not attached to $\bar{P}$, and connecting each line that leaves $\bar{P}$ with the related line that enters $\bar{P}$. I is an irreducible graph, since each family has only one line. (The reverse transformation, from I to G , may be regarded as the placing of insertions on the lines of I .)

It follows that G may be reconstructed from J if for each n -vertex P in J we are given the corresponding n -line irreducible graph I in G . Since $v(G)$ is just the product of $v(I)$ over all P in J , we can write the value of J as $n_\ell!^{-1} z^{n_\ell} \sum_{\{k\}} u_J(\{k\})$, where $u_J(\{k\})$ is the product of factors $C_n(k_1 \dots k_n)$ for each n -vertex P of J . It is now understood that each circuit (i.e., family) is assigned a variable k , and $k_1 \dots k_n$ are the variables assigned to the n circuits passing through P . The function $C_n(k_1 \dots k_n)$ is just the sum of $v_I(k_1 \dots k_n)$ over all irreducible graphs I .

The reduction of G to J is equivalent to that described by C. Bloch⁽¹¹⁾ in treating this problem. We now proceed to transform J into a dual tree T .

First, we label all the vertices (let their number be n_C) of J . This yields $n_C!$ different labeled circuit graphs, since each vertex is already specified uniquely as the head of some labeled line. We compensate by introducing an additional factor $n_C!^{-1}$ into the value. The vertices of J will be the circles of T .

Next, we place a new element, a square, at the "center" of each circuit of J , and connect it by "new" lines to each vertex of the circuit. The variable k associated with the circuit can now be regarded as belonging to the square. Let n_S be the number of squares—that is, the number of families in the original graph G . We label the squares, increasing the multiplicity of structures by $n_S!$ since each square is already distinguished by the labeled lines of its circuit. We compensate by introducing a factor $n_S!^{-1}$ into the value, which is now given by $n_S!^{-1} n_C!^{-1} n_\ell!^{-1} z^{n_\ell} \sum_{\{k\}} u_J(\{k\})$.

The final step is to erase the n_ℓ "old" lines, those forming the circuits of J . The remaining structure, T , is a dual tree since the "new" lines only join squares to

circles and the removal of any line of T (corresponding to the removal of two successive members of a family of G) splits T into two parts. However, the erasure has reduced the multiplicity of structures by a factor calculated as follows.

If we examine a square of n^{th} order in T , we find there are $(n-1)!$ ways to order the n neighboring circles cyclically, and hence $(n-1)!$ ways to draw in the n "old" lines of the circuit corresponding to this square. After all the "old" lines have been restored, there are $n_{\ell}!$ ways to label them, since each is already specified by the labeled square associated with its circuit and the labeled circle at its head. So the multiplicity has been reduced by $(n-1)!$ for each square of n^{th} order, as well as by an overall factor $n_{\ell}!$. To compensate, we omit the factor $n_{\ell}!^{-1}$ from the value and introduce a factor $(n-1)!$ for each square of n^{th} order in T , in addition to the n factors z inherited from the "old" lines of the corresponding circuit.

The value assigned to T is now identical with w_T , provided that we define

$$S_n(k_1 \dots k_n) = (n-1)! z^n \quad (46)$$

for $n = 1, 2, \dots$, and $S_0 = 0$. The cyclical ordering factor $(n-1)!$ is the distinguishing feature of this application and is the origin of the logarithm in (45). The whole transition from G to J to T is depicted in Figure 5.

The derivation of Eq. (45) is now straightforward. From (2) with (46) we have

$$\mathcal{T}(C) = \sum_{n=1}^{\infty} \sum_k z^n C(k)^n / n = \sum_k \ln [1 - z C(k)]^{-1} \quad (47)$$

and hence, from (6),

$$\bar{S}(k) = z / [1 - z \bar{C}(k)] \quad (48)$$

Now, $\bar{S}$ is just the M of Lee and Yang. This can be seen either from Eq. (A3) of Appendix A or from Eq. (4), noting that $M(k)$ is the functional derivative of the logarithm of the partition function with respect to a k -dependent point insertion in the propagator.

It follows, from (1) with our definition of C_n , that

$$\mathcal{D}(\bar{S}) = \sum_n \frac{1}{n!} \sum_I \sum_{k_1 \dots k_n} v_I(k_1 \dots k_n) \prod_{i=1}^n M(k_i) \quad (49)$$

where $\sum_I$ denotes the sum over all irreducible graphs I having n lines. But (49) is just the quantity $\mathcal{P}'$ appearing in (45), except that $\mathcal{P}'$ is restricted to graphs with no 1-vertex. However, the only irreducible graph that has a 1-vertex is the one consisting of one line, and its contribution to $\mathcal{D}(\bar{S})$ is

$$\sum_k M(k) Y_1(k) = \sum_k M(k) [z^{-1} - m(k)^{-1}] \quad (50)$$

on account of (44). Therefore

$$\mathcal{D}(\bar{S}) = \mathcal{P}' + \sum_k M(k) [z^{-1} - m(k)^{-1}] \quad (51)$$

If we write (47) and (48) in terms of $\bar{S}$ —that is, M —we obtain

$$\mathcal{F}(\bar{C}) = \sum_k \ln [z^{-1} M(k)] \quad (52)$$

and

$$\bar{C}(k) = z^{-1} - \bar{S}(k)^{-1} = z^{-1} - M(k)^{-1} \quad (53)$$

Substituting the last three equations into (7), we have

$$\begin{aligned} \tau &= \sum_k \ln [z^{-1} M(k)] + \mathcal{P}' + \sum_k M(k) [z^{-1} - m(k)^{-1}] \\ &\quad - \sum_k M(k) [z^{-1} - M(k)^{-1}] \\ &= \sum_k \ln [z^{-1} M(k)] + \mathcal{P}' + \sum_k [1 - m(k)^{-1}] M(k) \end{aligned} \quad (54)$$

which is equivalent to (45) since τ is the logarithm of the partition function.

In the discussion of Lee and Yang, a preliminary reduction was carried out by which the 1-vertices were eliminated. We have deliberately bypassed this reduction in order to show the power of the dual tree method. However, we can obtain Eq. (45) just as well by applying the Corollary to Theorem I after the reduction has been carried out.

The effect of the reduction is to replace Y_1 with 0 and z with $m(k)$ in evaluating the graphs. The sum of graphs found in this way must be augmented by a term we shall call $\mathcal{P}_1$ to obtain the logarithm of the partition function. $\mathcal{P}_1$ is the contribution of those original graphs containing only 1-vertices. Thus

$$\begin{aligned} \mathcal{P}_1 &= \sum_{l=1}^{\infty} \sum_k z^l Y_1(k)^l = \sum_k \ln [1 - z Y_1(k)]^{-1} \\ &= \sum_k \ln [z/m(k)] \end{aligned} \quad (55)$$

on account of (44).

In applying the Corollary to Theorem 1, we note that $\mathcal{B}(S)$ no longer receives the contribution of Eq. (50) since Y_1 has been replaced with 0. Therefore (51) becomes

$$\mathcal{B}(\bar{S}) = \mathcal{P}' \quad (56)$$

Likewise, we must replace z with $m(k)$ in (52) and (53). However, $\bar{S}$ is still the same as M . The expression for the logarithm of the partition function is now

$$\begin{aligned} \mathcal{P}_1 + \tau &= \sum_k \ln [z/m(k)] + \sum_k \ln [M(k)/m(k)] \\ &\quad + \mathcal{P}' - \sum_k M(k) [m(k)^{-1} - M(k)^{-1}] \\ &= \sum_k \ln [M(k)/z] + \mathcal{P}' + \sum_k [1 - m(k)^{-1} M(k)] \quad (57) \end{aligned}$$

which is the same as obtained before.

VII. Summary

It must be understood that we do not claim to have obtained results unknown before, or even to have found shorter proofs ab initio. The frequent references to earlier treatments should make it clear that the proofs given here are not original.

What is original is the explicit unification of diverse theorems into a single statement which is proved once and for all. The statement of Theorem 1 is aesthetically simple and easily memorized. Once it has been mastered and the flexibility of the dual tree concept is understood, one may derive a variety of results with great facility, each of which would otherwise require a separate argument of some complexity. The applications include not only those exhibited in the preceding four sections, but others known to the author, and doubtless unknown theorems as well, pertaining to graphical expansions yet to be studied.

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Appendix A

We shall first prove an auxiliary statement analogous to Eq. (5), namely

$$\int_{\xi} \bar{C}(\xi) \bar{S}(\xi) = \sum_T n_L w_T \quad (A1)$$

where n_L is the number of lines in T . By a similar argument, slightly more complicated, we shall prove Eq. (5). Once (A1) and (5) are established, Eq. (7) follows immediately because of the relation

$$n_S + n_C - n_L = 1 \quad (A2)$$

true of any dual tree.

(For completeness, we note that (A2) can be proved by induction on n_L . If $n_L = 0$ it is obviously true. If $n_L > 0$, remove an arbitrary line, and the tree will fall into two parts, having $n_L^{(1)}$ and $n_L^{(2)}$ lines respectively, where $n_L^{(1)} + n_L^{(2)} + 1 = n_L$. Since (A2) can be assumed true for each part, the number of squares and circles in the first part is $n_L^{(1)} + 1$, and in the second is $n_L^{(2)} + 1$, so that for the whole tree $n_S + n_C = (n_L^{(1)} + 1) + (n_L^{(2)} + 1) = n_L^{(1)} + n_L^{(2)} + 2 = n_L + 1$, q.e.d.)

This way of establishing Eq. (7) is the one used by C. Bloch⁽¹¹⁾ for the system treated in Section VI of this paper.

In order to prove Theorem 1, then, we need only establish (A1), (5), and (6). For this purpose we introduce the rooted tree, which is a dual tree in which one square or circle (the root) has an extra line projecting from it, the other end of which is unattached. The value of a rooted tree is computed in the same way as that of an ordinary

dual tree, except that the ξ -variable associated with the extra line is not summed over.

Hence the value is a function of one variable ξ .

Thus, if T is an unrooted dual tree containing a square with three lines attached (call them i_1, i_2, i_3) and T_S is the rooted tree formed from T by attaching an extra line to this square, then $w_{T_S}(\xi)$ is formed from w_T by replacing the factor $S_3(\xi_{i_1}, \xi_{i_2}, \xi_{i_3})$ with $S_4(\xi, \xi_{i_1}, \xi_{i_2}, \xi_{i_3})$.

Let T_S represent any rooted tree having a square for root, and T_C any one having a circle for root. Then we assert that

$$\bar{S}(\xi) = \sum_{T_S} w_{T_S}(\xi), \quad \bar{C}(\xi) = \sum_{T_C} w_{T_C}(\xi) \quad (A3)$$

where $\bar{S}$ and $\bar{C}$ are defined by (4).

To prove this, let T be any dual tree, containing n_C circles in all. Then

$\frac{\delta w_T}{\delta C_1(\xi)}$ is the sum of terms, one for each circle of order 1, obtained by replacing the corresponding factor $C_1(\xi_i)$ in w_T with $\delta(\xi_i, \xi)$. (We intend a Kronecker or Dirac δ -function according to the meaning of $\int_{\xi}$.) But the result of this replacement is precisely the function $\frac{1}{n_C} w_{T_S}(\xi)$ where T_S is the rooted tree obtained by deleting the circle in question, but leaving the line to which it was attached. The factor

$\frac{1}{n_C} = \frac{(n_C-1)!}{n_C!}$ arises from the fact that T_S has only $n_C - 1$ circles.

In forming T_S , it was necessary to relabel the remaining circles; let us specify that this was done so as to preserve their order. Thus, if the deleted circle bore the label j , then all circles with labels $< j$ in T have the same labels in T_S , while

all circles with labels $> j$ in T have those labels decreased by 1 in T_S , so that the labeling in T_S is consecutive. Then, if we add up all the terms of $\frac{w_T}{C_1(\xi)}$, and sum over all possible T , we shall form every possible T_S just n_C times, one for each value of j from 1 to n_C . (By n_C , now, we mean one more than the number of circles in T_S .) Thus

$$\sum_T \frac{w_T}{C_1(\xi)} = \sum_{T_S} n_C \frac{1}{n_C} w_{T_S}(\xi) \quad (A4)$$

which is equivalent to the first part of (A3). The second part is proved similarly.

We now prove (A1). Take any dual tree T , and any line in it. If this line is cut in two, we have two rooted trees T_S and T_C . Say that T contains n_S squares and n_C circles; T_S , n_S^S squares and n_C^S circles; and T_C , n_S^C squares and n_C^C circles. Then it is manifest that

$$w_T = \frac{n_S^S! n_C^S!}{n_S!} \cdot \frac{n_C^C! n_S^C!}{n_C!} \int_{\xi} w_{T_S}(\xi) w_{T_C}(\xi) \quad (A5)$$

However, T_S and T_C had to be relabeled so as to make each one consecutive. Say that this was done so as to preserve ordering within each set $n_S^S, n_C^S, n_C^C, n_S^C$. Then, if we let T range over all dual trees and cut each line of T in turn, we shall generate each pair (T_S, T_C) just $\frac{n_S^S!}{n_S^S! n_C^S!} \cdot \frac{n_C^C!}{n_C^C! n_S^C!}$ times, since that is the number of ways to divide the n_S square labels and the n_C circle labels each into subsets of the right size (once this is done, T is uniquely determined by T_S and T_C).

Hence the prefactors in (A5) cancel out and we have

$$\sum_T n_L w_T = \sum_{T_S} \sum_{T_C} \int_{\xi} w_{T_S}(\xi) w_{T_C}(\xi) \quad (A6)$$

which in view of (A3) is equivalent to (A1).

The proof of (5) is very similar. Let us define an S_m -structure as a dual tree in which a square of order m has been specially marked, and the lines attached to it have been numbered from 1 to m . Let the value of an S_m -structure be just that of the original tree. Since a dual tree T , containing n_{S_m} squares of order m , can be converted into an S_m -structure in $n_{S_m} m!$ ways, we have

$$\sum_T n_S w_T = \sum_{T_m} \sum_{S_m} n_{S_m} w_T = \sum_{m T} \sum_{S_m} n_{S_m} w_T = \sum_m F_m / m! \quad (A7)$$

where F_m is the sum of the values of all S_m -structures.

To find F_m , we note that each S_m -structure consists of m rooted trees T_C^i ($i = 1 \dots m$) whose extra lines are inserted into the marked square. The value of the S_m -structure is manifestly

$$\frac{\prod_{i=1}^m n_S^{(i)!}}{n_S!} \cdot \frac{\prod_{i=1}^m n_C^{(i)!}}{n_C!} \int_{\xi_1 \dots \xi_m} S_m(\xi_1 \dots \xi_m) \prod_{i=1}^m w_{T_C^i}(\xi_i) \quad (A8)$$

in an obvious notation. As in the previous arguments, the prefactors are canceled by the number of S_m -structures that reduce to the same m -tuple $(T_C^1 \dots T_C^m)$ upon consecutive relabeling. No complication arises from any topological identity among

the various T_C^i , since they are distinguished from one another once for all by the numbering of the m root-lines.

Aside from this relabeling factor, the sum over S_m -structures is equivalent to an independent summation over all the T_C^i , and we have

$$F_m = \int_{\xi_1 \dots \xi_m} S_m(\xi_1 \dots \xi_m) \prod_{i=1}^m \bar{C}(\xi_i) . \quad (A9)$$

Substituting into (A7), and comparing with Eq. (1), we have the first half of (5). The second half is proved in the same way.

It remains only to prove (6). We define a rooted S_m -structure as a rooted tree whose root is a square of order $m+1$ (m , without the extra line) and in which the m interior lines attached to the root have been numbered from 1 to m . Its value is that of the rooted tree. It is then manifest, from (A3), that

$$\bar{S}(\xi) = \sum_m F_m'(\xi) / m! \quad (A10)$$

where F_m' is the sum of the values of all rooted S_m -structures. But an argument identical to the one that led to (A9) gives

$$F_m'(\xi) = \int_{\xi_1 \dots \xi_m} S_{m+1}(\xi, \xi_1 \dots \xi_m) \prod_{i=1}^m \bar{C}(\xi_i) . \quad (A11)$$

Substituting (A11) into (A10), and replacing m by $n-1$, we have

$$\bar{S}(\xi) = \sum_{n=1}^{\infty} (n-1)!^{-1} \int_{\xi_1 \dots \xi_{n-1}} S_n(\xi, \xi_1 \dots \xi_{n-1}) \prod_{i=1}^{n-1} \bar{C}(\xi_i) . \quad (A12)$$

On the other hand, if we evaluate $\frac{\delta \mathcal{F}(C)}{\delta C(\xi)}$ directly from Eq. (2) and set

$C = \bar{C}$, we obtain the right side of (A12). This proves half of (6); the other half is proved in the same way. The proof of Theorem 1 is now complete.

Appendix B

Here we shall give an alternate proof of Theorem 1, not using rooted trees but emphasizing the concept of infinitesimal variation. With respect to Eqs. (5) and (6), this proof may be regarded as a rewording of the one given in Appendix A, but with respect to Eq. (7) it is essentially different. In Appendix A the proof of (7) depended on (5) and (A1) but not on (6). Here it will depend on (6) but not directly on (5) and not at all on (A1).

Our proofs rest on two preliminary assertions:

$$\frac{\delta \tau}{\delta C_1(\xi)} = \sum_m \int_{\xi_1 \dots \xi_m} S_{m+1}(\xi, \xi_1 \dots \xi_m) \frac{\delta \tau}{\delta S_m(\xi_1 \dots \xi_m)} \quad (B1)$$

and similarly with S, C interchanged; and

$$\frac{\delta \tau}{\delta S_m(\xi_1 \dots \xi_m)} = m!^{-1} \prod_{i=1}^m \bar{C}(\xi_i) \quad (B2)$$

and similarly with S, C interchanged.

It must be understood, since S_m is restricted to be symmetric in its arguments, that $\delta \tau / \delta S_m(\xi_1 \dots \xi_m)$ is also to be symmetric in the ξ 's and satisfy

$$\delta \tau = \int_{\xi_1 \dots \xi_m} \delta S_m(\xi_1 \dots \xi_m) \frac{\delta \tau}{\delta S_m(\xi_1 \dots \xi_m)} \quad (B3)$$

for arbitrary symmetric infinitesimal functions δS_m .

To prove (B1), suppose that $C_1(\xi)$ is incremented infinitesimally by $\delta C_1(\xi)$. Then the first-order increment in w_T is the sum of terms, each obtained by choosing a circle of order 1 and replacing C_1 with δC_1 for that circle. Each such term could also be obtained from a smaller tree T' , from which the circle in question and its attached line have been deleted, by replacing $S_m(\xi_1 \dots \xi_m)$ for the square at the other end of that line with $\frac{1}{n_C} \int_{\xi} S_{m+1}(\xi, \xi_1 \dots \xi_m) \delta C_1(\xi)$ where n_C is the number of circles in T .

Now if one sums over all terms coming from all choices of T , one obtains each T' , with each square singled out, just n_C times (see the proof of (A3) in the previous appendix) so that the factor n_C cancels out and one has

$$\delta \tau = \sum_m \int_{\xi, \xi_1 \dots \xi_m} \frac{\delta \tau}{\delta S(\xi_1 \dots \xi_m)} S_{m+1}(\xi, \xi_1 \dots \xi_m) \delta C_1(\xi) \quad (B4)$$

which yields (B1) upon elimination of the arbitrary increment δC_1 .

To prove (B2), we note that S_m can have only symmetric increments and that any symmetric function can be expressed as a linear combination of terms of the form $\prod_{i=1}^m f(\xi_i)$. Therefore we need only consider an increment

$$\delta S_m(\xi_1 \dots \xi_m) = \prod_{i=1}^m f(\xi_i) \quad (B5)$$

where f is infinitesimal.

The resulting increment in w_T has a term for each square of order m , consisting of a product of m factors. Each factor is (apart from factorials) the value of a tree

obtained by severing one line from the square, retaining everything on the other side of the line, and attaching the factor $f(\xi_i)$ to the severed end.

In summing over T and over the choice of square, the factorials cancel out and leave a factor $1/m!$, as in the proof of (5) in Appendix A. However, the sum over each one-line factor is of the form $\int_{\xi} f(\xi) P(\xi)$ where P is independent of m . It follows that

$$\frac{\delta \tau}{\delta S_m(\xi_1 \dots \xi_m)} = m!^{-1} \prod_1^m P(\xi_i) \quad (B6)$$

and by setting $m = 1$ and using (4), we have $P = \bar{C}$, so that (B2) follows.

With (B1) and (B2) established, we can easily prove Theorem 1. First, substitute (B2) into (B1) and replace m with $n - 1$. This yields (A12) from which half of (6) follows as in Appendix A. The other half is proved similarly.

Next, we introduce the short notation $S_m \frac{\delta}{\delta S_m}$ for $\int_{\xi_1 \dots \xi_m} S_m(\xi_1 \dots \xi_m) \frac{\delta}{\delta S_m(\xi_1 \dots \xi_m)}$. Multiplying (B2) by S_m , summing over the ξ 's and summing over m , we have, with the use of (2),

$$\sum_m S_m \frac{\delta \tau}{\delta S_m} = \tau(\bar{C}) \quad (B7)$$

On the other hand, if a tree T has n_S squares then w_T is homogeneous of degree n_S in $S_0, S_1, \dots$, so that by Euler's theorem

$$\sum_m S_m \frac{\delta w_T}{\delta S_m} = n_S w_T \quad (B8)$$

Summing over T , and comparing with (B7), we have half of Eq. (5); the other half is proved similarly.

Finally, let us define the quantity

$$\sigma \equiv \mathcal{F}(\bar{C}) + D(\bar{S}) - \int_{\xi} \bar{S}(\xi) \bar{C}(\xi) \quad (B9)$$

and consider its variation when the S_m are changed infinitesimally. This variation has two parts: one part due to the variation of $\bar{S}$ and $\bar{C}$, which depend on the S_m through Eqs. (3) and (4), and the other part due to the direct role of the S_m in the definition of $\mathcal{F}$, Eq. (2). But the first part is just

$$\int_{\xi} \frac{\delta \mathcal{F}}{\delta \bar{C}(\xi)} \delta \bar{C}(\xi) + \int_{\xi} \frac{\delta D}{\delta \bar{S}(\xi)} \delta \bar{S}(\xi) - \int_{\xi} [\bar{S}(\xi) \bar{C}(\xi) + \bar{C}(\xi) \bar{S}(\xi)] = 0 \quad (B10)$$

on account of (6). In other words, (B9) is completely stationary when regarded as a functional on $\bar{C}$ and $\bar{S}$, the S_m and C_m being fixed and Eq. (4) being disregarded in the variation but satisfied at the stationary point.

It follows that for all n we have

$$\begin{aligned} S_m \frac{\delta \sigma}{\delta S_m} &= S_m \left(\frac{\delta \sigma}{\delta S_m} \right)_{\bar{S}, \bar{C}} \\ &= S_m \left(\frac{\delta \mathcal{F}(\bar{C})}{\delta S_m} \right)_{\bar{S}, \bar{C}} \end{aligned} \quad (B11)$$

We sum over n , noting that $\mathcal{F}(\bar{C})$ for fixed $\bar{C}$ is a linear homogeneous functional on the S_m , and obtain

$$\begin{aligned} \sum_m S_m \frac{\delta \sigma}{\delta S_m} &= \mathcal{F}(\bar{C}) \\ &= \sum_m S_m \frac{\delta \tau}{\delta S_m} \end{aligned} \quad (B12)$$

on account of (B7).

Now let all the S_m be multiplied by the parameter u , varying from 0 to 1. Let the resulting values of Eqs. (3) and (B9) be called $\tau(u)$ and $\sigma(u)$. Then clearly $u \frac{d}{du} = \sum_m S_m \frac{\delta}{\delta S_m}$ and (B12) becomes

$$\frac{d\tau(u)}{du} = \frac{d\sigma(u)}{du} \quad (B13)$$

for all u .

If $u=0$ then all the S_m vanish, and the only tree that contributes to τ is the one with a circle and no square:

$$\tau(0) = C_0. \quad (B14)$$

Putting (B14) into (4), we find that $\bar{S}=0$ and hence

$$\sigma(0) = \mathcal{D}(0) = C_0 \quad (B15)$$

so that

$$\tau(0) = \sigma(0). \quad (B16)$$

Integrating (B13) from 0 to 1, and adding (B16), we have

$$\tau(1) = \sigma(1) \quad (\text{B17})$$

which is equivalent to Eq. (7) since putting $u = 1$ restores the original values of the S_m .

The proof of Theorem 1 is now complete. This method of deriving Eq. (7) is essentially the same as that given by Lee and Yang⁽⁸⁾ for the quantum statistical system treated in Section VI of this paper, and adapted by Luttinger and Ward⁽⁹⁾ for the field theoretic formulation. A proof along these lines was given by Uhlenbeck and Ford⁽⁷⁾ for the Mayer cluster expansion dealt with in Section V.

Appendix C

We shall prove the topological lemma stated in Section VI.

Let i be a line in the family F . If F contains no other line, the head of i is well-connected to its tail, and the lemma is satisfied. Otherwise, let j be another member of F . When i and j are removed, the graph falls into two parts A and B , one of which (say A) contains the head of i (call it h_i) and the tail of j (call it t_j).

Now h_i and t_j are connected through A and also outside of A . Hence if they are not well-connected, they can be separated by the removal of two lines just one of which (call it j') lies in A . Now if we remove only i and j' , h_i and t_j are connected neither through A (since j' is removed) nor outside of A (since i is removed). Therefore j' is also related to i , and when i and j' are removed the graph falls into two parts one of which (A') is a subset of A and contains h_i and t_j . Replacing j , A with j' , A' in the foregoing argument, we continue until we arrive at a line i' that belongs to F and whose tail is well-connected to the head of i .

We may now replace i with i' in the foregoing and develop a sequence $i, i', i'', \dots$ of members of F , each having its head well-connected with the tail of the next. Eventually the sequence must repeat, and therefore it contains a circuit H . It remains only to prove that H contains all the members of F , and that its ordering is unique.

Let p be a member of H , and q be a line not in H . The head of p is well-connected to the tail of p' , which follows p in H . Therefore this connection can be made so that it does not pass through p or q . Likewise the head of p' can be connected to the tail of p'' without passing through p or q . In this way we can

find a continuous path from the head of p to its tail, passing neither through p nor through q . But if p and q were related, this would be impossible, since it permits us to increment k_p and not k_q . Hence H contains all the members of F .

The ordering in H is unique apart from cyclic permutations. For if the head of p were well-connected to the tail of $p^{(n)}$, a circuit H' could be formed by deleting the lines $p', p'', \dots, p^{(n-1)}$ from H . Then, if $p^{(n)} \neq p'$, H' would not contain all the members of F , which has been proved impossible.

References and Footnotes

1. Equivalently, we could drop the labels on the squares and circles—thereby reducing the number of distinct trees—and calculate $w(T)$ by dividing not by $n_S! n_C!$ but by $\sigma(T)$, the symmetry number of the tree. For example, the twelve trees in Figure 1 would reduce to two, each having $\sigma = 2$, and their total contribution to τ would be the same as in our definition. That method is more convenient for evaluating τ , but the one we choose is equivalent and seems better adapted to proving theorems.

Our definitions allow trees consisting of a single square or circle and having the value S_0 or C_0 . However, the theorem loses no generality in practice if it is restricted to $S_0 = C_0 = 0$, which means that these two degenerate trees are omitted.

2. T. D. Lee and C. N. Yang, Phys. Rev. 117, 12 (1960), Appendix C.
3. It is known that $A_\ell = (\ell - 1)! \ell^{\ell-2}$, but this fact is not used either here or in Ref. 2.
4. G. Jona-Lasinio, Nuovo Cimento 34, 1790 (1964).
5. H. Ursell, Proc. Camb. Phil. Soc. 23, 685 (1927); J. E. Mayer, J. Chem. Phys. 5, 67 (1937).
6. J. E. Mayer and P. F. Ackermann, J. Chem. Phys. 5, 74 (1937), especially Eqs (9)ff, (17), and Note in Proof; J. E. Mayer and S. F. Harrison, J. Chem. Phys. 6, 87 (1938), especially Appendix.
7. G. E. Uhlenbeck and G. W. Ford, Studies in Statistical Mechanics (ed. DeBoer and Uhlenbeck) 1, 123 (1962), Sections III.2-3.
8. T. D. Lee and C. N. Yang, Phys. Rev. 117, 22 (1960).

9. J. M. Luttinger and J. Ward, Phys. Rev. 118, 1417 (1960).
10. C. Bloch, Studies in Statistical Mechanics (ed. DeBoer and Uhlenbeck) 3, 1 (1964),
also references cited therein.
11. See Ref. (10), Section 4.21. The connection with the Mayer cluster expansion is
indicated on p. 133.

Figure Captions

- Figure 1. All 12 inequivalent dual trees with $n_C = 2$, $n_S = 3$.
- Figure 2. Transformation of Cayley tree to dual tree.
- Figure 3. Graphical expansion of G_3 and G_4 in terms of irreducible vertices Γ_m and propagators G_2 .
- Figure 4. Transformation of Mayer cluster graphs to dual trees. Note that a) and b) give the same dual tree.
- Figure 5. Transformation of Lee and Yang's primary 0-graphs to dual trees.
- a) From labeled graph G to circuit graph J . The upper two vertices in G are well-connected and coalesce in J .
 - b) From circuit graph J to dual tree T (labeling not shown).
 - c) Another graph G' that yields the same J .
 - d) Another circuit graph J' that yields the same T .

Fig. 1

Fig. 2

FIG. 3 GRAPHICAL EXPANSION OF G_3 AND G_4 IN TERMS OF IRREDUCIBLE VERTICES Γ_m AND PROPOGATORS G_2

Fig. 4