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THE ASYMPTOTIC TIME-DEPENDENT TRANSPORT EQUATION

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THE ASYMPTOTIC TIME-DEPENDENT TRANSPORT EQUATION

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ABSTRACT

A multiplying system is considered in which the neutron distribution has settled down to the normal mode but the population is varying with time as e^{at} . It is then shown that the customary diffusion equation

$$\frac{1}{v} \frac{\partial \phi}{\partial t} = D \nabla^2 \phi + (c-1) \sigma \phi$$

can be used far from boundaries if we let

$$D = \frac{(1-c) \sigma + \frac{a}{v}}{k^2}$$

where k^2 is obtainable from

$$\text{arth} \frac{k}{\sigma + \frac{a}{v}} = \frac{k}{c \sigma} .$$

An expansion is then obtained which is valid for $\left| 1 - \frac{c}{1 + \frac{a}{\sigma v}} \right| \ll 1$,

$$\frac{1}{D} \simeq 3 \sigma \left(\frac{1 + 4c}{5} + \frac{1}{5} \frac{a}{\sigma v} + \dots \right)$$

The coefficient of the $\frac{a}{\sigma v}$ term in this expansion is $1/5$ that to be expected from use of the time-dependent P_1 approximation which gives rise to the telegrapher's equation, indicating that the usual time-dependent diffusion equation is preferable to the telegrapher's equation for the treatment of the problem considered here.

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I. THE ASYMPTOTIC SOLUTION

A straightforward generalization of conventional methods¹ has been used to obtain a solution to the one-velocity time-dependent transport equation in a multiplying medium, which is valid far from boundaries and after the neutron distribution has settled down to the normal mode. We are then considering a solution which is asymptotic in time as well as in space and which varies with time as e^{at} . We are usually interested in the value of a for such a system. The motivation of this note was to obtain a valid expression for the diffusion coefficient, D , in the system. It turns out that D is a function of a , but is not as strongly dependent upon a as one would suspect from consideration of the "telegrapher's equation".^{2, 3}

Consideration is limited to the one-dimensional (plane) one-velocity transport equation with isotropic scattering. Extension of the result to anisotropic scattering is evident, though tedious. The transport equation may be written,

$$\frac{1}{v} \frac{\partial \psi}{\partial t} + \mu \frac{\partial \psi}{\partial x} + \sigma \psi(x, \mu, t) = \frac{c\sigma}{2} \int_{-1}^1 \psi(x, \mu', t) d\mu' \quad (1)$$

where $\psi(x, \mu, t)$ is the angular distribution of neutron path length, σ is the macroscopic total crosssection, μ the cosine of the angle between the neutron velocity and the x axis, and c the number of neutrons emergent per collision. We now postulate a solution which may hold asymptotically in space and time,

¹ E. G., Glasstone and Edlund, The Elements of Nuclear Reactor Theory, Van Nostrand (1952), pp 395-8.

² Weinberg and Noderer, Theory of Neutron Chain Reactions, Vol I., Chap. II, Oak Ridge, Report No. CF-51-5-98.

³ M. Clark, "The Time Dependent Diffusion Equation," UCRL-4174, Aug. 17, 1953.

$$\psi(x, \mu, t) = f(\mu) e^{at \pm kx} \quad (2)$$

and obtain an integral equation for $f(\mu)$,

$$\left(\frac{a}{v} \pm k\mu + \sigma \right) f(\mu) = \frac{c\sigma}{2} \int_{-1}^1 f(\mu') d\mu' \quad (3)$$

Differentiating with respect to μ , we obtain

$$\frac{df}{f} \pm \frac{k d\mu}{\frac{a}{v} \pm k\mu + \sigma} = 0$$

whose solution is

$$f(\mu) = \frac{C}{\frac{a}{v} + \sigma \pm k\mu} \quad (3.1)$$

where C must remain indeterminate because (3) is homogeneous. However, substitution of (3.1) into (3) will give a relation between k and a :

$$\frac{c\sigma}{k} \operatorname{arth} \frac{k}{\frac{a}{v} + \sigma} = 1 \quad (4)$$

If we let

$$\phi(x, t) = \int_{-1}^1 \psi(x, \mu, t) d\mu \quad (5.1)$$

be the total flux, we want it to satisfy the customary diffusion equation,

$$\frac{1}{v} \frac{\partial \phi}{\partial t} = D \frac{\partial^2 \phi}{\partial x^2} + (c-1) \sigma \phi \quad (5)$$

Substituting the solution (2) in (5.1) and (5), we see that (5) will be satisfied if

$$\frac{a}{v} = Dk^2 + (c-1) \sigma \quad (6.1)$$

This equation then gives us an expression for the correct diffusion coefficient to use in asymptotic diffusion theory. It will be most convenient, for the series expansions we are next to consider, to consider the reciprocal of the diffusion coefficient,

$$\frac{1}{D} = \frac{k^2}{(1-c)\sigma + \frac{a}{v}} \quad (6)$$

In order to obtain an expression for k^2 , let us make the substitution,

$$\frac{k}{\frac{a}{v} + \sigma} = \kappa \quad (7.1)$$

in which case (4) becomes

$$\text{arth } \kappa = \frac{\kappa}{\gamma} \quad (7)$$

$$\text{where } \gamma = \frac{c}{1 + \frac{a}{\sigma v}} \quad (7.2)$$

It is then possible to use the well-known expansions of the solution of (7) good for $|\gamma-1| < 1$. From Eq. (74) of Case, de Hoffmann and Placzek,⁴ we have

$$\kappa^2 = 3(1-\gamma) \left[1 - \frac{4}{5}(1-\gamma) + \frac{4}{175}(1-\gamma)^2 + \frac{4}{175}(1-\gamma)^3 + \dots \right] \quad (7.3)$$

Substituting (7.3) and (7.1) into (6), using (7.2), we obtain

$$\begin{aligned} \frac{1}{D} &= \frac{\kappa^2 \sigma^2 (1 + \frac{a}{\sigma v})^2}{\sigma [1 + \frac{a}{\sigma v} - c]} \\ &= \frac{\kappa^2 \sigma}{1-\gamma} (1 + \frac{a}{\sigma v}) \\ &= 3\sigma \left[\frac{1+4c}{5} + \frac{1}{5} \frac{a}{\sigma v} + \frac{4}{175} (1 + \frac{a}{\sigma v}) \left(1 - \frac{c}{1 + \frac{a}{\sigma v}} \right)^2 + \dots \right] \quad (8) \end{aligned}$$

The first two terms in the brackets are evidently sufficient for

$\left| 1 - \frac{c}{1 + \frac{a}{\sigma v}} \right| \ll 1$. For weakly anisotropic scattering, one can replace

σ in Eq. (8) by the transport cross section, σ_{tr} ,

$$\sigma_{tr} = \sigma(1 - c\bar{\mu}_0) \quad (8.1)$$

where $\bar{\mu}_0$ is the average value of the cosine of the scattering angle in the lab system.

⁴ Case, de Hoffmann and Placzek, Introduction to the Theory of Neutron Diffusion, Vol I, Los Alamos, 1953.

II. COMPARISON WITH TELEGRAPHER'S EQUATION

The time-dependent P_1 approximation to the transport equation is well-known.^{2,3} In our present notation the telegrapher-type equation obtained is

$$\frac{1}{3\sigma(1-c\bar{\mu}_0)} \frac{\partial^2 \phi}{\partial x^2} - \left[1 + \frac{1}{v\sigma(1-c\bar{\mu}_0)} \frac{\partial}{\partial t} \right] \left[\sigma(1-c) + \frac{1}{v} \frac{\partial}{\partial t} \right] \phi = 0 \quad (9)$$

If we again look for solutions of the form

$$\phi \sim \phi_0 e^{at},$$

we can make the replacement

$$1 + \frac{1}{v\sigma(1-c\bar{\mu}_0)} \frac{\partial}{\partial t} = 1 + \frac{a}{\sigma v(1-c\bar{\mu}_0)}$$

and Eq. (9) becomes the customary diffusion equation (5) if we let

$$D = \frac{1}{3\sigma(1-c\bar{\mu}_0) \left[1 + \frac{a}{\sigma v(1-c\bar{\mu}_0)} \right]}$$

Comparison of Eq. (10) with Eq. (8) for $c \simeq 1$ and $\left| \frac{a}{\sigma v} \right| \ll 1$ indicates that Eq. (10) overestimates the dependence of the diffusion coefficient on a by a factor of 5. Since $\left| \frac{a}{\sigma v} \right|$ is usually small with respect to one, it is then evident that the customary time-dependent diffusion equation (in which the dependence of D on a is ignored) gives a better approximation to the asymptotic time dependence than the neutronic form of the telegrapher's equation, and is therefore to be preferred for its accuracy as well as its simplicity in situations where the neutron population has settled down to its normal mode. However, the use of Eq. (8) is evidently sufficiently simple to make it preferable to the other recipes for the diffusion coefficient in cases where $\frac{a}{\sigma v}$ is not negligible.

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