VIBRATIONS OF A ROW OF CIRCULAR
CYLINDERS IN A LIQUID

by

S. S. Chen

Components Technology Division

U.S.C.-AUA-USERDA

Base Technology

April 1975

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PREFACE

The work reported herein was performed as part of the base technology activity under the Flow Induced Vibration Programs (189a No. 02659 and 02683) sponsored by ERDA/RRD. The overall objective of the activity is to develop new and/or improved analytical methods and guidelines for designing LMFBR components to avoid detrimental flow induced vibration.

Heat exchanger tubes and reactor fuel pins are long, slender, beam-like components typically arranged in bundles and immersed in a flowing liquid. As such, they are susceptible to flow induced vibration. The excitation mechanism may be associated with vortex-shedding, fluidelastic interaction, or random pressure fluctuations in the turbulent flow. Designing to avoid large amplitude motion, that is, to avoid a resonance condition or instability condition, and the prediction of component response, require knowledge of the dynamic behavior of the components. However, cylinders in a closely spaced bundle do not respond as single cylinders immersed in a liquid, rather, interaction with the liquid causes coupled motion of groups of cylinders. The fundamental natural frequency of the coupled system will be lower than that of a single cylinder immersed in a liquid.

Understanding and modeling fluid/structure interaction in cylinder bundles is a basic requirement in the development of analytical methods and guidelines for designing heat exchanger and reactor fuel assemblies that are free from component vibration problems. As a step toward satisfying this requirement, in this report a method of analysis based on potential flow theory is developed for modeling the response of a row of cylinders vibrating in a liquid. Added mass coefficients are given as a function of the number of cylinders in a row for various values of gap to radius ratios. A graphical method of determining natural frequencies

from a curve of the real part of the propagation constant as a function of dimensionless frequency is illustrated for a specified cylinder row. To demonstrate the method of analysis and coupling effects natural frequencies, normal modes, and forced vibration response are presented for a numerical example.

The analysis method presented will be extended to account for fluid/structure coupling in bundles and will be used in the development of sets of design curves corresponding to various values of displaced-fluid-mass to cylinder-mass ratio, gap to radius ratio, and number of cylinders in row. These curves will be included in a future design guide and will permit determination of the added mass coefficients and natural frequencies of coupled cylinder row and bundle systems.

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NOMENCLATURE

a_{jn}, b_{jn}	Coefficients of series solutions in Eqs. (1) and (2)
c_j	Damping coefficient of cylinder j
d_j	The distance from the center of cylinder j to the origin
E_j	Young's modulus of cylinder j
g_1	Magnitude of force
g_{jn}	Generalized force
G	Gap
G_j	Exciting force
H_j	The component of the hydrodynamic force in the x direction
I_j	Moment of inertia
k	Number of cylinders
m_j	Mass per unit length
q_{jn}	Generalized coordinate
(r_j, θ_j)	Cylindrical coordinates
R_{ij}	Distance between the centers of cylinders i and j
t	Time
u_j	The displacement of cylinder j in the x_j direction
U	Fluid velocity in the radial direction
v_j	The displacement of cylinder j in the y_j direction
(x_i, y_i)	Cartesian coordinates
z	Axial coordinate
α_{in}, β_{in}	Added mass coefficients
γ_{jn}	Generalized coordinate
η_{jn}	Damping ratio of uncoupled mode in liquid
ξ_{jn}	Damping ratio of coupled mode
ρ	Fluid density
σ	Mass ratio of fluid to cylinder

ϕ	Fluid velocity potential (total)
ϕ_j	Fluid velocity potential associated with the motion of cylinder j
$\phi_n(z)$	Orthonormal modal function of cylinders in vacuo
ψ_{ij}	Defined in Fig. 2
ω_{jn}	Natural frequency of the nth mode of cylinder j
Ω	Circular frequency
Ω_{jn}	Natural frequency of coupled mode

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ABSTRACT

The effects of interaction with surrounding liquid on the dynamic behavior of a row of circular cylinders are studied analytically. First, the hydrodynamic forces associated with cylinder motions are obtained using the potential flow theory. Then, a method of solution is presented for free vibration of a row of cylinders. Finally, steady state responses to harmonic excitations are presented. The results of this study have important application in the modeling of crossflow-induced vibration of heat-exchanger tubes.

I. INTRODUCTION

The dynamic response behavior of a group of circular cylinders, such as heat exchanger tubes and fuel rods, is influenced by the surrounding liquid. Interaction with liquid causes coupled motion of a group of cylinders. The dynamics of a single cylinder vibrating in a liquid are well understood. The comparable dynamic response for multiple cylinders, however, is more difficult to understand and model [1].*

Several studies have been made to investigate the coupling effects of multiple structures in a liquid. Laird and Warren [2] studied experimentally the variations of hydrodynamic forces with cylinder spacing during oscillations of a group of twenty-four cylinders in water. Dalton and Helfinstine [3] analyzed the problem of an accelerating flow past a group of rigid circular cylinders. Mazur [4] obtained a solution for the hydrodynamic forces of two cylinders moving in an ideal fluid and Chen [5] applied Mazur's results to study the coupled modes and responses of two cylinders in a liquid. Shimogo and Inui [6] considered the coupled vibration of two and four identical cylinders in water. Connors [7] and Blevins [8] studied the critical flow velocity for a tube-row subjected to a cross flow. Chen [9] and Chen and Rosenberg [10] investigated the coupled modes of two coaxial tubes and coaxial cylindrical shells separated by a liquid. A brief review of these published literature reveals that a general method of analysis for the vibration of a group of cylinders does not exist. The objective of this paper is to present a method of analysis which can be employed to study the response of a row of cylinders in a liquid.

*Numbers in brackets designate References at the end of paper.

II. ADDED MASS COEFFICIENTS

Consider the motion of a row of k circular cylinders in an ideal incompressible fluid (Fig. 1). The axes of the cylinders are perpendicular to the x - y plane and the centers of the cylinders are on the x -axis. Let R_j and $(d_j, 0)$ be the radius and the coordinates of the center of the cylinder j . The cylinders may vibrate in the x and y directions.

The fluid velocity potential associated with the motion of the cylinder j , assuming all other cylinders are stationary, can be written

$$\phi_j = \sum_{n=1}^{\infty} \frac{R_j^{n+1}}{r_j^n} a_{jn} \cos n \theta_j, \quad (1)$$

for the cylinder j vibrating in the x direction, and

$$\phi_j = \sum_{n=1}^{\infty} \frac{R_j^{n+1}}{r_j^n} b_{jn} \sin n \theta_j, \quad (2)$$

for the cylinder j vibrating in the y direction. r_j and θ_j are cylindrical coordinates referred to the cylinder j as shown in Fig. 1, and a_{jn} and b_{jn} are constants to be determined. The total field at a point in the fluid is made up of the partial fields generated by all cylinders; i.e.,

$$\phi = \sum_{j=1}^k \phi_j. \quad (3)$$

Note that ϕ_j given in equations (1) and (2) is written in terms of the local coordinates referred to the cylinder j . To satisfy the boundary conditions at a particular cylinder, the velocity potentials associated with all cylinders must be expressed in terms of the coordinates of that cylinder. The transformation is accomplished by use of the following relationships:

$$\frac{\cos n \theta_j}{r_j^n} = (-1)^n \sum_{m=0}^{\infty} \frac{(n+m-1)! r_i^m}{m! (n-1)! R_{ij}^{n+m}} \cos[m\theta_i - (m+n)\psi_{ij}], \quad (4)$$

and

Fig. 1. A row of circular cylinders vibrating in a liquid.

$$\frac{\sin n\theta_i}{r_j^n} = (-1)^{n+1} \sum_{m=0}^{\infty} \frac{(n+m-1)! r_i^m}{m! (n-1)! R_{ij}^{n+m}} \sin[m\theta_i - (m+n)\psi_{ij}] \quad (4)$$

(Contd.)

R_{ij} and ψ_{ij} are defined in Fig. 2. Equations (4) can be established from Fig. 2 and using Taylor series expansion of the function $[R_{ij} \exp(i\psi_{ij}) - r_i \exp(i\theta_i)]^{-n}$. Let the superscript i denote a variable written in terms of the local coordinates associated with the cylinder i ; e.g., the partial potential ϕ_j and total potential ϕ in terms of the coordinates of the cylinder i are ϕ_j^i and ϕ^i , respectively. Therefore,

$$\phi^i = \phi_i + \sum_{j=1}^{k*} \phi_j^i, \quad (5)$$

where $\sum^{k*}$ denotes the summation for j from 1 to k except $j = i$. Using equations (1), (2) and (4) gives

$$\phi_j^i = \sum_{n=1}^{\infty} \sum_{m=0}^{\infty} \frac{(-1)^{n+1} (n+m-1)! R_j^{n+1} r_i^m}{m! (n-1)! R_{ij}^{n+m}} a_{jm} \cos[m\theta_i - (m+n)\psi_{ij}] ,$$

and (6)

$$\phi_j^i = \sum_{n=1}^{\infty} \sum_{m=0}^{\infty} \frac{(-1)^{n+1} (n+m-1)! R_j^{n+1} r_i^m}{m! (n-1)! R_{ij}^{n+m}} b_{jn} \sin[m\theta_i - (m+n)\psi_{ij}] ,$$

for motions in the x and y directions, respectively.

The velocity components of the cylinder i in the x and y directions are $\frac{\partial u_i}{\partial t}$ and $\frac{\partial v_i}{\partial t}$ respectively. The fluid velocity components in the r direction are U . In terms of the local coordinates of the cylinder i ,

$$U^i = \frac{\partial \phi^i}{\partial r_i} \quad (7)$$

At the interfaces of the cylinders and fluid, the following conditions must be satisfied:

Fig. 2. Relation of two cylindrical coordinates associated with the cylinders i and j .

$$U^1 \Big|_{r_1=R_1} = \frac{\partial u_1}{\partial t} \cos \theta_1, \quad (8)$$

$$i = 1, 2, 3, \dots k,$$

for motions in the x direction; and

$$U^1 \Big|_{r_1=R_1} = \frac{\partial v_1}{\partial t} \sin \theta_1, \quad (9)$$

$$i = 1, 2, 3, \dots k,$$

for motions in the y direction.

Substituting equations (7) into (8) and (9), and using equations

(5) and (6), a_{jn} and b_{jn} are determined,

$$a_{in} = \sum_{\ell=1}^k \alpha_{in\ell} \frac{\partial u_\ell}{\partial t}, \quad (10)$$

and

$$b_{in} = \sum_{\ell=1}^k \beta_{in\ell} \frac{\partial v_\ell}{\partial t},$$

where $\alpha_{in\ell}$ and $\beta_{in\ell}$ are solutions of the following equations:

$$\begin{aligned} (-n)\alpha_{in\ell} + \sum_{j=1}^k \sum_{m=1}^{\infty} \frac{(-1)^m (n+m-1)! R_1^{n-1} R_j^{m+1}}{(n-1)!(m-1)! R_{ij}^{n+m}} \alpha_{jml} \cos[(m+n)\psi_{ij}] &= \delta_{n1} \delta_{i\ell}, \\ (-n)\beta_{in\ell} + \sum_{j=1}^k \sum_{m=1}^{\infty} \frac{(-1)^{m+1} (n+m-1)! R_1^{n-1} R_j^{m+1}}{(n-1)!(m-1)! R_{ij}^{n+m}} \beta_{jml} \sin[(m+n)\psi_{ij}] &= \delta_{n1} \delta_{i\ell}, \end{aligned} \quad (11)$$

$$i, j, \ell = 1, 2, 3, \dots k,$$

$$n, m = 1, 2, 3, \dots \infty.$$

A finite number of undetermined coefficients were determined by inverting the matrix formed by truncating the infinite sets of equations (11).

The fluid forces acting on the cylinders can be calculated from fluid pressure p;

$$p = -\rho \frac{\partial \phi}{\partial t}, \quad (12)$$

where ρ is fluid density. The two components in the x and y directions of fluid force acting on the cylinder i are H_i and V_i ;

$$H_i = - \int_0^{2\pi} p^i \Big|_{r_i=R_i} \cdot R_i \cos \theta_i d\theta_i ,$$

and (13)

$$V_i = - \int_0^{2\pi} p^i \Big|_{r_i=R_i} \cdot R_i \sin \theta_i d\theta_i .$$

Using equations (5), (6), (10), and (12), and integrating equations (13) give

$$H_i = -\rho\pi \sum_{\ell=1}^k \left(\frac{R_i + R_\ell}{2} \right)^2 \alpha_{i\ell} \frac{\partial^2 u_\ell}{\partial t^2} ,$$

and (14)

$$V_i = -\rho\pi \sum_{\ell=1}^k \left(\frac{R_i + R_\ell}{2} \right)^2 \beta_{i\ell} \frac{\partial^2 v_\ell}{\partial t^2} ,$$

where

$$\alpha_{i\ell} = \frac{4R_i^2}{(R_i + R_\ell)^2} \left\{ -\alpha_{i\ell\ell} - \sum_{j=1}^{k*} \sum_{n=1}^{\infty} \left(\frac{R_j}{R_{ij}} \right)^{n+1} (-1)^n \alpha_{jn\ell} \cos[(n+1)\psi_{ij}] \right\} ,$$

and (15)

$$\beta_{i\ell} = \frac{4R_i^2}{(R_i + R_\ell)^2} \left\{ -\beta_{i\ell\ell} + \sum_{j=1}^{k*} \sum_{n=1}^{\infty} \left(\frac{R_j}{R_{ij}} \right)^{n+1} (-1)^n \beta_{jn\ell} \cos[(n+1)\psi_{ij}] \right\} .$$

$\alpha_{i\ell}$ and $\beta_{i\ell}$ are added mass coefficients. It can easily be shown that

$$\alpha_{i\ell} = \alpha_{\ell i} ,$$

and (16)

$$\beta_{i\ell} = \beta_{\ell i} .$$

For a group of k cylinders, there are only $\frac{k}{2}(k+1)$ independent components for $\alpha_{i\ell}$ or $\beta_{i\ell}$ because of the symmetry. The diagonal ones α_{ii} and β_{ii} are self added mass coefficients, which are proportional to the hydrodynamic

force acting on the cylinder i due to its own acceleration; while the off diagonal ones are mutual-added mass coefficients, which are proportional to the hydrodynamic force acting on the cylinder i due to the acceleration of the cylinder l . α_{il} and β_{il} may be referred to as the added mass coefficient tensor. For a group of k cylinders it is possible to find k principal axes such that $\alpha_{il} = 0$, or $\beta_{il} = 0$ for $i \neq l$.

α_{il} and β_{il} are in terms of series solutions. Only a limited number of terms are needed in calculations. The added mass coefficients for two cylinders with different radii ($R_2 = 2R_1$) and five identical cylinders ($R_i = R$; $i = 1, 2, 3, 4, 5$), equally spaced, are shown in Fig. 3 as functions of the number of terms n taken in calculations. In both cases, α_{il} and β_{il} are referred to the x-y coordinates and the gap G between two cylinders is equal to $0.1 R_1$. Calculations also have been made for many other cylinder-rows. It is found that the rate of convergence increases when cylinders are far apart and are of the same radius. In general, $n = 10$ will give results with sufficient accuracy.

The vibration of a tube-row consisting of identical cylinders, such as heat exchanger tubes, is of importance. It is interesting to know the effect of the neighboring cylinders. Figures 4 illustrate the self-added mass coefficient α_{11} and mutual-added mass coefficient α_{1k} as functions of the number of cylinders in the tube-row. For $k = 1$, it corresponds to a single cylinder in an infinite fluid, α_{11} is equal to 1. As other cylinders 2, 3, 4, etc., are placed on the x-axis, α_{11} increases. The two adjacent cylinders 2 and 3 have a significant effect on α_{11} , while the effects of other cylinders are negligible. Similar effect is observed for mutual-added mass coefficients; α_{1k} is small except α_{12} and α_{13} . From Figs. 4, it is clear that, in in-plane motion of a row of cylinders, the coupling of a cylinder with other cylinders can be neglected

Fig. 3. Variations of added mass coefficients with the number of terms n used in calculations.

Fig. 4. Added mass coefficients α_{11} and α_{1k} as functions of the number of cylinders k .

except the two adjacent ones. The added mass coefficients β_{11} and β_{1k} have also been calculated as functions of the number of cylinders, which are shown in Fig. 5. In this case, the coupling between any two cylinders is much larger.

Added mass coefficients depend on R_i , R_{ij} , and k . For a tube-row consisting of identical cylinders at equal distance, added mass coefficients depend on G/R and k only. Figure 6 shows α_{il} and β_{il} for three cylinders as functions of the gap-radius ratio. For large G/R , α_{ii} and β_{ii} approach one, and α_{il} and β_{il} for $i \neq l$ approach zero. As G/R decreases, both α_{il} and β_{il} increase. It is noted that the mutual added mass coefficients α_{il} are negative and β_{il} are positive. In all cases the absolute values of α_{il} are smaller than those of β_{il} ; that is, the hydrodynamic effect on cylinder motions is more important in the out-of-plane vibration of a row of cylinders.

The hydrodynamic forces presented in this paper are based on a two dimensional theory. In the case of a group of cylinders vibrating in a liquid, the fluid field is not two dimensional. However, the three dimensional effect is small for large wave lengths [9]. Consequently, the two dimensional hydrodynamic forces will be employed to study the vibration of cylinder rows.

Fig. 5. Added mass coefficients β_{11} and β_{1k} as functions of the number of cylinders k .

Fig. 6. Added mass coefficients as functions of gap-radius ratio for three equally spaced cylinders.

III. FREE VIBRATION

A. Frequency Equation

In-plane and out-of-plane vibrations can be studied independently.

The equations of motion for in-plane vibration are

$$E_j I_j \frac{\partial^4 u_j}{\partial z^4} + c_j \frac{\partial u_j}{\partial t} + m_j \frac{\partial^2 u_j}{\partial t^2} = G_j + H_j, \quad (17)$$

where m_j is mass per unit length, $E_j I_j$ is flexural rigidity, c_j is damping coefficient, H_j is hydrodynamic force, and G_j is external excitation force.

The hydrodynamic force is given in equations (14); substitution of equations (14) into (17) gives

$$E_j I_j \frac{\partial^4 u_j}{\partial z^4} + c_j \frac{\partial u_j}{\partial t} + m_j \frac{\partial^2 u_j}{\partial t^2} + \rho \pi \sum_l \left(\frac{R_j + R_l}{2} \right)^2 \alpha_{jl} \frac{\partial^2 u_l}{\partial t^2} = G_j. \quad (18)$$

In this analysis, it is assumed that all cylinders are of the same length l and have the same type of boundary conditions. Let

$$u_j = \sum_{n=1}^{\infty} q_{jn} \phi_n(z), \quad (19)$$

where $\phi_n(z)$ is an orthonormal modal function of cylinders in vacuo; i.e.,

$$\frac{1}{l} \int_0^l \phi_m(z) \phi_n(z) dz = \delta_{mn}. \quad (20)$$

Using equations (18), (19), and (20) gives

$$\begin{aligned} m_j \ddot{q}_{jn} + \sum_l r_{jl} \alpha_{jl} \ddot{q}_{ln} + 2m_j \zeta_{jn} \omega_{jn} \dot{q}_{jn} + m_j \omega_{jn}^2 q_{jn} &= g_{jn}, \\ j &= 1, 2, 3, \dots, k, \\ n &= 1, 2, 3, \dots, \infty, \end{aligned} \quad (21)$$

where ω_{jn} is the n th frequency in vacuo for the cylinder j , and

$$\begin{aligned} r_{jl} &= \rho \pi \left(\frac{R_j + R_l}{2} \right)^2, \\ \zeta_{jn} &= \frac{c_j}{2m_j \omega_{jn}}, \end{aligned} \quad (22)$$

and

$$g_{jn} = \frac{1}{L} \int_0^L G_j \phi_n dz \quad . \quad (22) \quad (Contd.)$$

Note that equations (21) can be applied to all values of n . For each n , there are k equations which are coupled. However, there is no coupling among the equations for different n . This is true for a group of cylinders having the same length and same type of boundary conditions. If the cylinders have different types of end conditions, a similar method of analysis can be used [5].

For a given n , equations (21) may be written

$$[M]\{\ddot{q}\} + [C]\{\dot{q}\} + [K]\{q\} = \{F\} \quad , \quad (23)$$

where $[M]$, $[C]$, and $[K]$ are symmetric matrices and

$$\begin{aligned} m_{jl} &= m_j \delta_{jl} + r_{jl} \alpha_{jl} \quad , \\ c_{jl} &= 2m_j r_{jl} \omega_{jn} \delta_{jl} \quad , \\ k_{jl} &= m_j \omega_{jn}^2 \delta_{jl} \quad , \end{aligned} \quad (24)$$

and

$$f_j = g_{jn} \quad .$$

For free vibration, neglect damping and forcing terms and let

$$\{q\} = \{\bar{q}\} \exp(i\Omega t) \quad . \quad (25)$$

Frequencies and normal modes of the system are computed from the undamped homogeneous equations

$$[K]\{\bar{q}\} = \Omega^2 [M]\{\bar{q}\} \quad . \quad (26)$$

Let $[P]$ be the weighted modal matrix formed from the columns of eigenvectors. It is easily shown that

$$[P^T][M][P] = [I] \quad ,$$

and

$$[P^T][K][P] = [\Lambda] \quad , \quad (27)$$

where $[I]$ is an identity matrix and $[\Lambda]$ is a diagonal matrix formed from the eigenvalues Ω^2 .

The same method of analysis can be applied to the out-of-plane vibration. This is easily done by replacing $\alpha_{j\ell}$ by $\beta_{j\ell}$ from equations (18) to (27), provided that all variables are referred to out-of-plane motions.

B. Periodic Tube-Rows

When all cylinders are identical and equally spaced, considerable simplification can be made. In this case

$$\begin{aligned} R_1 &= R_2 = R_3 = \dots = R_k = R, \\ m_1 &= m_2 = m_3 = \dots = m_k = m, \\ \omega_{1n} &= \omega_{2n} = \omega_{3n} = \dots = \omega_{kn} = \omega_n, \\ \alpha_{11} &= \alpha_{22} = \alpha_{33} = \dots = \alpha_{kk} = \alpha_s, \end{aligned} \quad (28)$$

and

$$\alpha_{12} = \alpha_{23} = \alpha_{34} = \dots = \alpha_{(k-1)k} = \alpha_c.$$

As pointed out earlier, for in-plane vibration, the coupling effect can be neglected except for the two adjacent cylinders. Therefore, equations (26) can be written

$$\bar{q}_{(j-1)n} - 2 \cos \mu \bar{q}_{jn} + \bar{q}_{(j+1)n} = 0, \quad (29)$$

where

$$\cos \mu = \frac{\left(\frac{\omega_n}{\Omega}\right)^2 - 1 - \sigma \alpha_s}{2\sigma \alpha_c}, \quad (30)$$

and σ is the ratio of the displaced mass of fluid to the mass of the cylinder.

The solution of equation (29) is

$$\bar{q}_{jn} = \bar{q}_{(j-1)n} \exp(i\mu). \quad (31)$$

μ is the solution of equation (30). Since α_s and α_c depend on G/R , the

Fig. 7. The propagation constant as a function of the frequency ratio for periodic tube-row for $G/R = 0.1$ and $\sigma = 0.5$.

Fig. 8. Dependence of the propagation region on the mass ratio σ for (a) $G/R = 0.1$ and (b) $G/R = 1.0$.

From equations (33), the frequency equation is obtained:

$$\frac{\cos[(k+1)\pi] \sin[(k+1)\mu]}{\sin \mu} = 0 \quad (34)$$

The solutions of equation (34) satisfying equation (30) are

$$\mu = \frac{m\pi}{k+1}, \quad m = 1, 2, 3, \dots, k \quad (35)$$

In the propagation band, equation (35) permits k solutions. From equations (30) and (35), the natural frequencies of a group of periodically spaced cylinders can be obtained by a simple procedure as follows. For a row of k cylinders, divide the ordinate over the range 0 to π into $k+1$ equal parts. The projections on the abscissa (Ω/ω_n) of the points of intersection of the μ curve and horizontal lines corresponding to $m = 1, 2, \dots, k$ will give the frequencies. This procedure is the same as that of a periodically supported beam [11]. Substituting equation (35) into (33) and solving for $\{\bar{q}\}$ give

$$\begin{Bmatrix} \bar{q}_{1n} \\ \bar{q}_{2n} \\ \vdots \\ \bar{q}_{(k-1)n} \\ q_{kn} \end{Bmatrix} = \begin{Bmatrix} \sin\left(\frac{m\pi}{k+1}\right) \\ \sin\left(\frac{2m\pi}{k+1}\right) \\ \vdots \\ \sin\left[\frac{(k-1)m\pi}{k+1}\right] \\ \sin\left[\frac{km\pi}{k+1}\right] \end{Bmatrix}, \quad m = 1, 2, 3, \dots, k \quad (36)$$

For the purpose of illustration, Fig. 9 shows the discrete values of μ and the corresponding natural frequencies of in-plane vibration for a row of five cylinders with a rigid cylinder at each end. There are three frequencies for each n ; figure 9 can be applied to all n .

Fig. 9. Natural frequencies of a row of cylinders consisting of three identical cylinders and two rigid cylinders for $\sigma = 0.5$ and $G/R = 0.1$.

C. Numerical Examples

Figures 10 and 11 show the normal modes of two, three, four, and ten cylinders vibrating in a liquid. For a row of k cylinders, corresponding to each n there are k normal modes in in-plane or out-of-plane vibrations. In in-plane vibration, the fundamental frequency is associated with the mode in which every cylinder moves out-of-phase with respect to its two adjacent cylinders, while the k th frequency is associated with the mode in which all cylinders vibrate in-phase. On the other hand, in out-of-plane vibration, the fundamental frequency is corresponding to in-phase mode and the k th frequency is corresponding to the mode that every cylinder vibrates out-of-phase with respect to its two adjacent cylinders. It can be seen from Figs. 10 and 11 that between any two adjacent cylinders of the group, the total number of phase changes in the n th mode is $(k-n)$ for in-plane motion and $(n-1)$ for out-of-plane motion.

Figures 12 show the frequencies of a tube-row of five, equally spaced, identical steel tubes in water as functions of the gap. The tube outside radius is 1.27 cm (0.5 in.), wall thickness 0.159 cm (1/16 in.), length 1.27 m (50 in.) and is simply-supported at both ends. The fundamental frequencies of the constituent tube are 41.744 Hz in vacuo and 33.733 Hz in water, which is given by the dotted lines in Fig. 12. Figures 12 show that when the tubes are closely spaced, the frequencies are widely spread. As the gap increases, all frequencies approach that of a single tube in water. It is also noted that the lowest frequency is associated with the in-phase mode of out-of-plane motion.

Figure 12 shows the fundamental frequencies of in-plane and out-of-plane vibrations of a tube-row, consisting of the same tubes presented in Fig. 12, as functions of the number of tubes. As the number increases, the fundamental frequencies decrease. This is attributed to the increase in the self added mass and coupling effect.

IN-PLANE VIBRATION

OUT-OF-PLANE VIBRATION

Fig. 10. Normal modes of two, three, and four cylinders vibrating in a liquid.

Fig. 11. Normal modes of ten cylinders vibrating in a liquid.

Fig. 12. Natural frequencies of a tube row.

Fig. 13. Fundamental frequencies for in-plane and out-of-plane vibrations as functions of the number of tubes.

IV. FORCED VIBRATION

The response to an excitation can be calculated from equations (23) and the appropriate initial conditions. Assuming that the damping matrix $[C]$ is proportional to $[K]$, equations (23) can be reduced to a set of k uncoupled (modal) equations by letting

$$\{q\} = [P]\{\Gamma\} , \quad (37)$$

and premultiplying by the transpose $[P^T]$, the result being

$$[P^T M P] \{\ddot{\Gamma}\} + [P^T C P] \{\dot{\Gamma}\} + [P^T K P] \{\Gamma\} = [P^T]\{F\} . \quad (38)$$

In equations (38) the square matrices on the left side are diagonal matrices. Thus each equation reduces to that of a simple oscillator and has the form

$$\ddot{\gamma}_{jn} + 2\eta_{jn}\Omega_{jn}\dot{\gamma}_{jn} + \Omega_{jn}^2\gamma_{jn} = \sum_l P_{ljn}f_{ln} \quad (39)$$

$$j = 1, 2, 3, \dots k$$

$$n = 1, 2, 3, \dots \infty .$$

The solution of equation (39) is easily obtained. Finally the cylinder displacements, and other quantities of interest, can be calculated from equations (19) and (37).

In equation (39), Ω_{jn} is the natural frequency of a coupled mode and η_{jn} is the corresponding modal damping ratio. η_{jn} is related to the modal damping ratios of uncoupled modes. This can be shown by considering a special case that all cylinders are identical. Assuming that the frequency in vacuo of the constituent cylinders is ω_{jn} , the corresponding frequency in liquid is $\omega_{jn}/\sqrt{1+\sigma}$ (σ = mass ratio of fluid to cylinder) and the modal damping ratio is ξ_{jn} . It can be shown that

$$\xi_{jn} = \frac{\zeta_{jn}}{\sqrt{1+\sigma}} . \quad (40)$$

From equations (38) and (39), it is easily seen that

$$\eta_{jn} = \left(\frac{\Omega_{jn}}{\omega_{jn}} \right) \zeta_{jn} \quad (41)$$

Using equations (40) and (41) gives

$$\eta_{jn} = \left(\frac{\Omega_{jn}}{\omega_{jn}} \right) \sqrt{1 + \sigma} \xi_{jn} \quad (42)$$

Evidently, the modal damping ratio of a coupled mode η_{jn} may be smaller or larger than that of an uncoupled mode ξ_{jn} . Since the fundamental frequency of the coupled modes is always smaller than $\omega_{jn}/\sqrt{1 + \sigma}$, the corresponding damping ratio will be smaller than that of the corresponding uncoupled mode. Equation (42) also shows that the modal damping ratio increases as the frequency increases; that is, the higher modes will have a higher damping ratio.

The steady-state responses of a tube-row consisted of five, equally spaced, identical tubes have been considered. The tube properties are the same as those presented in free vibration study. Figures 14 and 15 show the responses of the tubes when tube 1 is subjected to a harmonic excitation $g_1 \sin \omega t$, uniformly distributed along the tube and $\zeta_{jn} = 0.05$. The magnification factor is defined as the ratio of the displacement amplitude at midspan to that of the deflection of tube 1 to a static load of the same magnitude. In Fig. 14, the gap-radius ratio is 0.1, the tubes are strongly coupled. The response of tube 5 is still large, although it is far away from tube 1. In Fig. 15, the gap-radius is 1.0. The response is small for those tubes not directly excited and the response of the tube 1 is similar to that of a corresponding isolated tube.

Fig. 14. Dynamic responses of five tubes to a harmonic excitation for $G/R = 0.1$.

Fig. 15. Dynamic responses of five tubes to a harmonic excitation for $G/R = 1.0$.

V. CONCLUSIONS

A rational method of analysis is presented for coupled motions of a row of cylinders in liquids. The method can be applied to a group of cylinders vibrating in-plane or out-of-plane, and the cylinders may have different properties. The method is useful in modeling the response of a group of cylinders to flow excitations.

Detailed studies have been made on the coupled cylinder/fluid motions. Several general conclusions can be drawn: (1) The lowest frequency of the coupled modes is associated with the in-phase mode of out-of-plane motion. (2) The fundamental frequency of a group of identical cylinders decreases with increasing cylinder number. (3) In a group of k identical cylinders, corresponding to each frequency of the constituent cylinder in vacuo, there are k frequencies of coupled modes for in-plane or out-of-plane vibrations. (4) The modal damping ratio of the fundamental coupled mode of a group of identical cylinders is lower than that of the corresponding mode of an isolated cylinder. (5) The coupled frequencies become more widely spread as the cylinder spacing becomes smaller.

It should be remarked that the added mass is affected by other parameters, such as vibration frequency, vibration amplitude, and fluid viscosity [12]. However, for small oscillations in practical system components, the potential flow theory will give results with sufficient accuracy. It should also be kept in mind that the force due to the added mass is not the only force acting on the cylinders. Other forces, such as drag force, lift force, and fluid elastic force are also important; these forces can not be obtained based on the potential flow theory.

REFERENCES

1. Chen, S. S., "Parallel-Flow-Induced Vibrations and Instabilities of Cylindrical Structures," *The Shock and Vibration Digest*, Vol. 6, No. 10, Oct. 1974, pp. 2-12.
2. Laird, A. D. K., and Warren, R. P., "Groups of Vertical Cylinders Oscillating in Water," *J. Eng. Mech. Div., Proc. ASCE*, Vol. 89 (EM 1), Feb. 1963, pp. 25-35.
3. Dalton, C., and Helfinstine, R. A., "Potential Flow Past a Group of Circular Cylinders," *J. of Basic Eng., Trans. ASME*, Vol. 93, Dec. 1971, pp. 636-642.
4. Mazur, V. Y., "Motion of Two Circular Cylinders in an Ideal Fluid," *Izvestiya Akademi Nauk SSSR, Mekhanika Zhiakostii Gaza*, Vol. 6, Dec. 1970, pp. 80-84.
5. Chen, S. S., "Dynamic Response of Two Parallel Circular Cylinders in a Liquid," To appear in *J. Pressure Vessels and Piping*, ASME.
6. Shimogo, T., and Inui, T., "Coupled Vibration of Many Elastic Circular Bars in Water," *Proc. 21st Japan National Congress for Applied Mechanics*, 1971, pp. 495-505.
7. Connors, H. J. Jr., "Fluidelastic Vibration of Tube Arrays Excited by Cross Flow," *Flow Induced Vibration in Heat Exchangers*, ASME, 1970, pp. 42-56.
8. Blevins, R. D., "Fluid Elastic Whirling of a Tube Row," Presented at the *Pressure Vessels and Piping Conference*, Miami Beach, Florida, June 24-28, 1974; ASME Paper No. 74-PVP-29.
9. Chen, S. S., "Dynamics of a Rod-Shell System Conveying Fluid," *Nucl. Eng. and Design*, Vol. 30, 1974, pp. 223-233.
10. Chen, S. S., and Rosenberg, G. S., "Dynamics of a Coupled Shell/Fluid System," Presented at the *Eighth Int. Cong. on Acoustics*, London, England, 1974.

REFERENCES (Continued)

11. Chen, S. S., "In-Plane Vibration of Continuous Curved Beams, "Nucl. Eng. and Design, Vol. 25 (3), Aug. 1973, pp. 413-431.
12. Wambsganss, M. W., Chen, S. S., and Jendrzeczyk, J. A., "Added Mass and Damping of a Vibrating Rod in Confined Viscous Fluids", ANL-CT-75-08, September 1974.