1

THE TRANSFER FUNCTION OF A
WATER BOILER REACTOR

BY:
R. E. SKINNER
D. L. HETRICK

ATOMICS INTERNATIONAL

A DIVISION OF NORTH AMERICAN AVIATION, INC.
P.O. BOX 309 CANOGA PARK, CALIFORNIA

CONTRACT: AT(11-1)-GEN-8
ISSUED: AUGUST 1, 1957

DISCLAIMER

This report was prepared as an account of work sponsored by an agency of the United States Government. Neither the United States Government nor any agency Thereof, nor any of their employees, makes any warranty, express or implied, or assumes any legal liability or responsibility for the accuracy, completeness, or usefulness of any information, apparatus, product, or process disclosed, or represents that its use would not infringe privately owned rights. Reference herein to any specific commercial product, process, or service by trade name, trademark, manufacturer, or otherwise does not necessarily constitute or imply its endorsement, recommendation, or favoring by the United States Government or any agency thereof. The views and opinions of authors expressed herein do not necessarily state or reflect those of the United States Government or any agency thereof.

DISCLAIMER

Portions of this document may be illegible in electronic image products. Images are produced from the best available original document.

DISTRIBUTION

This report has been distributed according to the category "Physics and Mathematics" as given in "Standard Distribution Lists for Unclassified Research and Development Reports" TID-4500 (13th Edition), January 15, 1957. A total of 620 copies was printed.

Copies are available from the Office of Technical Services, U.S. Department of Commerce, Washington 25, D.C.

ACKNOWLEDGMENT

The authors wish to express their thanks to B. M. Durand, who performed most of the necessary laborious calculations.

TABLE OF CONTENTS

	Page No.
Abstract	6
I. Introduction	7
II. The Nonlinear and Linear Space-Independent Kinetic Equations	7
III. The Transfer Function	9
A. General.	9
B. The Water Boiler Reactor Transfer Function	11
C. Asymptotic Expressions	13
D. Curves	18
IV. Other Representations of the Transfer Function	23
A. Inverse Polar Plot	23
B. Parallel Impedance Model	25
C. Nyquist Criterion	26
Appendix I	31
Appendix II	36
Appendix III	38
References	41

LIST OF FIGURES

1. Amplitude of Normalized Water Boiler Reactor Power Transfer Function, $G_P(i\omega; P_0)$, (6 Groups of Delayed Neutrons).	19
2. Phase Shift, $\delta_1(i\omega; P_0)$, for Water Boiler Reactor Power Transfer Function (6 Groups of Delayed Neutrons)	20
3. Tangent of Phase Shift for Water Boiler Reactor Power Transfer Function (6 Groups of Delayed Neutrons)	22
4. Temperature Transfer Function: Amplitude and Tangent of Phase Shift for a Water Boiler Reactor	23
5. Inverse of Normalized Water Boiler Reactor Power Transfer Function: $G_P(i\omega; P_0)^{-1}$, (6 Groups of Delayed Neutrons)	24

LIST OF FIGURES (Continued)

	Page No.
6. Parallel Impedance Model Block Diagram	25
7. $\{\beta_{eff}/[P_0 Z_p(i\omega; P_0)]\}^{-1}$ for Water Boiler Reactor Power Transfer Function	27
8. Block Diagram Representation for Application of Nyquist Criterion	28
9. Nyquist Plot for Water Boiler Reactor Power Transfer Function: $1 + G(i\omega)H(i\omega; P_0)$, (6 Groups of Delayed Neutrons) .	30

5

ABSTRACT

The power and temperature transfer functions for a water boiler reactor are obtained for the case of m groups of delayed neutrons including the effects of temperature and radiolytic gas generation. The nature of the water boiler reactor transfer function is investigated and important results discussed. A number of asymptotic and approximate expressions are given. Representations other than Bode plots are investigated. These include the inverse polar plot, complex power coefficient of reactivity, and modified Nyquist plots. The derivation and limitations of the space-independent reactor kinetics equations used are discussed briefly.

NOTE: This report is based in part on a paper presented at the Second Annual Meeting of the American Nuclear Society, Chicago, Illinois: Session VII, paper 3, June 6, 1956.

Work by R. E. Skinner was done in part while a National Science Foundation Predoctoral Fellow at the California Institute of Technology, 1955-56.

I. INTRODUCTION

The concept of a transfer function as applied to reactors has been treated by several investigators.¹⁻⁶ The transfer function provides a relatively simple relationship from which it is possible to gain an insight into the effects of delayed neutrons, temperature, radiolytic gas generation, fission product poisons, and other variables on the kinetic behavior of the reactor for the case of small deviations from equilibrium. Further, the transfer function provides a possible method of measuring the parameters which appear in the kinetic equations.

Such determinations from the transfer function of l (the effective prompt neutron generation time) and of β_{eff} (the effective total fraction of all neutrons which are delayed) have been made by measuring the high frequency portion by pile oscillator experiments.^{3, 7} At least one attempt has been made to determine the temperature effects by measuring the transfer function.⁵ Also, a knowledge of the transfer function of a reactor is a necessary and important item in the design of reactor control systems.

Here we consider the transfer function of a water boiler reactor, including the effects of delayed neutrons, temperature, and radiolytic gas.

II. THE NONLINEAR AND LINEAR SPACE-INDEPENDENT KINETIC EQUATIONS

The space-independent kinetic equations for a water boiler reactor, including the effects of delayed neutrons, temperature, and radiolytic gas, are taken to be †, ††

$$\frac{dP(t)}{dt} + \frac{\beta_{eff} - \rho(t)}{l} P(t) - \sum_{j=1}^m \lambda_j C_j(t) = 0. \quad \dots (1.1)$$

† The derivation and limitations of these equations are discussed in Appendix I. See Appendix II for the definition of the quantities appearing in these equations.

†† The equations, as given, assume that all the delayed neutrons are emitted with the same energy. A discussion is given in Appendix III of the case where the mean emission energy of the delayed neutrons in a group is not the same for all groups.

$$\frac{dC_j(t)}{dt} + \lambda_j C_j(t) = \frac{\beta_{eff} a_j}{l} P(t), (j = 1, 2, \dots, m), \quad \dots (1.2)$$

$$\frac{dT(t)}{dt} + \gamma T(t) = K [P(t) - P_0], \quad \dots (1.3)$$

$$\frac{dV(t)}{dt} + \sigma V(t) = G [P(t) - P_0], \quad \dots (1.4)$$

and

$$\rho(t) = \rho_1(t) + \alpha T(t) + \phi V(t). \quad \dots (1.5)$$

The system of nonlinear space-independent kinetic equations, Eqs. 1.1-1.5, can be linearized for the case

$$\frac{\delta P(t)}{P_0} \ll 1.$$

where $\delta P(t) = P(t) - P_0$ by neglecting the term $[\rho(t) \delta P(t)/l]$ in Eq. 1.1. Making this approximation, combining Eqs. 1.1 and 1.5, and rewriting all equations in terms of deviations from equilibrium, we obtain the linearized space-independent kinetic equations†

$$\frac{d\delta P(t)}{dt} + \frac{\delta P(t)}{\tau^*} - \frac{\alpha^*}{l^*} P_0 T(t) - \frac{\phi^*}{l^*} P_0 V(t) - \sum_{j=1}^m \lambda_j \delta C_j(t) = \frac{P_0}{l^*} \rho_1^*(t), \quad \dots (2.1)$$

$$\frac{d\delta C_j(t)}{dt} + \lambda_j \delta C_j(t) = \frac{a_j}{l^*} \delta P(t), (j=1, 2, \dots, m). \quad \dots (2.2)$$

† The derivation and limitations of these equations are discussed in Appendix I. See Appendix II for the definition of the quantities appearing in these equations.

$$\frac{dT(t)}{dt} + \gamma T(t) = K \delta P(t). \quad \dots (2.3)$$

and

$$\frac{dV(t)}{dt} + \sigma V(t) = G \delta P(t). \quad \dots (2.4)$$

where $\delta C_j(t) = C_j(t) - C_{j0}$; C_{j0} is the value of $C_j(t)$ when operating in steady state at the equilibrium total fission rate P_0 .

III. THE TRANSFER FUNCTION

A. GENERAL

The generalized definition of the transfer function $G(iw)$ in terms of Fourier transform is given by

$$G(iw) = \frac{\mathcal{F}[r(t); iw]}{\mathcal{F}[f(t); iw]}, \quad \dots (3)$$

where $r(t)$ is the output function corresponding to the input function $f(t)$; $\mathcal{F}[f(t); iw]$ is the Fourier transform of $f(t)$ and is given by

$$F(iw) = \mathcal{F}[f(t); iw] = \int_{-\infty}^{\infty} f(t) e^{-iwt} dt. \quad \dots (4.1)$$

and the inverse transform, $f(t) = \frac{1}{2\pi} \mathcal{F}[F(iw); -it]$. is given by

$$f(t) = \frac{1}{2\pi} \mathcal{F}[F(iw); -it] = \frac{1}{2\pi} \int_{-\infty}^{\infty} F(iw) e^{iwt} dw. \quad \dots (4.2)$$

Further it is seen that for all functions $f(t)$ and $r(t)$ whose Laplace transforms exist,⁸ the definition of the transfer function $G(i\omega)$, given by Eq. 3, reduces to the usual definition in terms of Laplace transforms, upon setting $s = i\omega$. Thus,

$$G(i\omega) = G(s) = \frac{\mathcal{L}[r(t); s]}{\mathcal{L}[f(t); s]}. \quad \dots (5)$$

where $\mathcal{L}[f(t); s]$ is the Laplace transform of $f(t)$.

In particular it is seen that if $f(t) = f_0 e^{i\omega t}$ (the case of forced oscillations), the output for a stable system may be written as $r(t) = A(\omega) e^{i\omega t}$, where $A(\omega)$ is a complex function of the input frequency. The transform of the input function is⁹

$$F(i\omega) = \mathcal{F}[f(t); i\omega] = f_0 \lim_{g \rightarrow \infty} \left\{ \frac{2 \sin [g(\omega - \omega)]}{(\omega - \omega)} \right\} = 2\pi f_0 \delta(\omega - \omega) \quad \dots (6.1)$$

and the transform of the output function is

$$R(i\omega) = \mathcal{F}[r(t); i\omega] = 2\pi A(\omega) \delta(\omega - \omega) \quad \dots (6.2)$$

where $\delta(\omega - \omega)$ is the Dirac δ function. Therefore,

$$f_0 G(i\omega) \delta(\omega - \omega) = A(\omega) \delta(\omega - \omega). \quad \dots (7)$$

whence

$$G(i\omega) = \frac{A(\omega)}{f_0}. \quad \dots (8)$$

This shows that the transfer function for $\omega = \omega$, ω real, is given by the response function $A(\omega)$ to pure oscillatory inputs divided by the input amplitude f_0 .

Thus, it is seen that the important aspects of the generalized definition, Eq. 3, of the transfer function are: (1) it provides in explicit form, in terms of the transforms of the corresponding input and output functions, the analytic continuation of the transfer function as defined by Laplace transforms, and (2) it provides the mathematical foundation and justification for combining data from pile oscillator experiments and other types of experiments (e.g., nonperiodic inputs) when measuring the transfer function for the case $w = \omega$, ω real.

B. THE WATER BOILER REACTOR TRANSFER FUNCTION

For a physical system that is governed by a system of linear differential equations, one may define a transfer function for each variable or for certain combinations of variables. For present purposes, it is sufficient to define:

1. The power transfer function, $G_P(iw; P_0)$.

$$G_P(iw; P_0) = \frac{\mathcal{F}[\delta P_1(t); iw]}{\mathcal{F}[\rho_1(t); iw]}, \quad \dots (9.1)$$

2. The normalized power transfer function (reactivity in dollars), $G_P(iw; P_0)$.

$$G_P(iw; P_0) = \frac{\beta_{eff}}{P_0} G_P(iw; P_0). \quad \dots (9.2)$$

3. The temperature loop transfer function, $G_T(iw; P_0)$,

$$G_T(iw; P_0) = \frac{\mathcal{F}[T_1(t); iw]}{\mathcal{F}[\delta P_1(t); iw]}, \quad \dots (9.3)$$

where $\delta P_1(t)$ the power response and $T_1(t)$ is the variation in the space average temperature as a function of time for the reactivity input, $\rho_1(t)$, at the equilibrium fission rate, P_0 .

Using the above definitions and taking the Fourier transform of the linear system, Eqs. 2.1-2.4, it is found that the normalized water boiler power transfer function is given by

$$G_P(i\omega; P_0) = \left[i\omega l^* + i\omega \sum_{j=1}^m \frac{a_j}{i\omega + \lambda_j} - \frac{\alpha^* K P_0}{i\omega + \gamma} - \frac{\phi^* G P_0}{i\omega + \sigma} \right]^{-1}, \quad \dots (10.1)$$

and the temperature loop transfer function by

$$G_T(i\omega; P_0) = \frac{K}{i\omega + \gamma} \quad \dots (10.2)$$

Letting $i\omega = u + i\omega$, Eqs. 10.1 and 10.2 can be written as

$$G_P(u + i\omega; P_0) = \{ [E_1(u + i\omega; P_0)]^2 + [E_2(u + i\omega; P_0)]^2 \}^{-1/2} \exp[i\delta_1(u + i\omega; P_0)] \quad \dots (11.1)$$

and

$$G_T(u + i\omega; P_0) = K [\omega^2 + (\gamma + u)^2]^{-1/2} \exp[i\delta_2(u + i\omega; P_0)] \quad \dots (11.2)$$

where

$$\delta_1(u + i\omega; P_0) = \tan^{-1} [-E_2/E_1]. \quad \dots (11.3)$$

$$\delta_2(u + i\omega; P_0) = \tan^{-1} [-\omega/(\gamma + u)]. \quad \dots (11.4)$$

$$E_1(u + i\omega; P_0) = \left[l^* u + \sum_{j=1}^m \frac{a_j [(\lambda_j + u) u + \omega^2]}{\omega^2 + (\lambda_j + u)^2} - \frac{(\gamma + u) \bar{K}}{\omega^2 + (\gamma + u)^2} - \frac{(\sigma + u) \bar{G}}{\omega^2 + (\sigma + u)^2} \right], \dots (11.5)$$

$$E_2(u + i\omega; P_0) = \omega \left[l^* + \sum_{j=1}^m \frac{a_j \lambda_j}{\omega^2 + (\lambda_j + u)^2} + \frac{\bar{K}}{\omega^2 + (\gamma + u)^2} + \frac{\bar{G}}{\omega^2 + (\sigma + u)^2} \right], \dots (11.6)$$

and

$$\bar{K} = K\alpha^* P_0, \bar{G} = G\phi^* P_0. \dots (11.7)$$

From Eqs. 11.1-11.7, it follows immediately that the water boiler transfer function depends only on the kinetic parameters $l^*, K\alpha^*, G\phi^*, \lambda_j, a_j$, and the equilibrium power P_0 . Further, by means of any measurements based on the space-independent kinetic equations, it is possible to measure the reactivity only in units of "dollars," that is to measure ρ^* . Consequently, it is possible to measure within the framework of the space-independent kinetic equations only the quantities $l^*, K\alpha^*, G\phi^*, \lambda_j, a_j$, and ρ^* . The parameter G may be measured by a static experiment, and the parameter K can be determined either by a static experiment or from the measurement of G_T . As a result, the parameters α^* and ϕ^* can be determined by the addition of two more measurements, one of which may be kinetic.

C. ASYMPTOTIC EXPRESSIONS

The following asymptotic expressions are easily obtained; for the case $u = 0$:

as $\omega \rightarrow 0$

$$|G_P(i\omega; P_0)| \sim \begin{cases} [(\bar{K}/\gamma) + (\bar{G}/\sigma)]^{-1}; & P_0 \neq 0 \quad \dots (12.1) \\ \left\{ \omega \left[l^* + \sum_{j=1}^m (a_j/\lambda_j) \right] \right\}^{-1}; & P_0 = 0 \quad \dots (12.2) \end{cases}$$

$$\tan \delta_1(i\omega; P_0) \sim \begin{cases} \frac{\omega \left\{ l^* + \sum_{j=1}^m (a_j/\lambda_j) + \left[(\bar{K}/\gamma^2) + (\bar{G}/\sigma^2) \right] \right\}}{[(\bar{K}/\gamma) + (\bar{G}/\sigma)]} \rightarrow 0; P_0 \neq 0 & \dots (12.3) \\ - \frac{\left[l^* + \sum_{j=1}^m (a_j/\lambda_j) \right]}{\omega \sum_{j=1}^m (a_j/\lambda_j^2)} \rightarrow \infty; & P_0 = 0 \quad \dots (12.4) \end{cases}$$

as $\omega \rightarrow \infty$

$$|G_P(i\omega; P_0)| \sim \frac{1}{\omega l^*} \quad \dots (12.5)$$

$$\tan \delta_1(i\omega; P_0) \sim (-\omega l^*). \quad \dots (12.6)$$

The quantity $[(K\alpha^*/\gamma) + (G\phi^*/\sigma)]$ appearing in Eqs. 12.1 and 12.3 is called the zero frequency power coefficient of reactivity.

Equations 12.1-12.4 are special cases of the general result that, if ω is much less than the smallest of $\lambda_1, \dots, \lambda_m, \gamma$, and σ . then

$$|G_P(i\omega; P_0)| \cong \left[\left\{ \omega^2 \sum_{j=1}^m (a_j/\lambda_j^2) - [(\bar{K}/\gamma) + (\bar{G}/\sigma)] \right\}^2 + \omega^2 \left\{ l^* + \sum_{j=1}^m (a_j/\lambda_j) + [(\bar{K}/\gamma^2) + (\bar{G}/\sigma^2)] \right\}^2 \right]^{-1/2} \quad \dots (13.1)$$

and

$$\tan \delta_1(i\omega; P_0) \cong \frac{-\omega \left\{ l^* + \sum_{j=1}^m (a_j/\lambda_j) + [(\bar{K}/\gamma^2) + (\bar{G}/\sigma^2)] \right\}}{\left\{ \omega^2 \sum_{j=1}^m (a_j/\lambda_j^2) - [(\bar{K}/\gamma) + (\bar{G}/\sigma)] \right\}} \quad \dots (13.2)$$

In a similar fashion Eqs. 12.5 and 12.6 are special cases of the general result that, if ω is much larger than the largest of $\lambda_1, \dots, \lambda_m, \gamma$, and σ , then

$$|G_P(i\omega; P_0)| \cong \left[\left\{ 1 - [(\bar{K}\gamma + \bar{G}\sigma)/\omega^2] \right\}^2 + \left\{ l^*\omega^2 + \sum_{j=1}^m (a_j\lambda_j) + (\bar{K} + \bar{G}) \right\}^2 \right]^{-1/2} \quad \dots (13.3)$$

$$\tan \delta_1(i\omega; P_0) \cong \frac{-\omega \left[l^*\omega^2 + \sum_{j=1}^m a_j\lambda_j + (\bar{K} + \bar{G}) \right]}{[\omega^2 - (\bar{K}\gamma + \bar{G}\sigma)]} \quad \dots (13.4)$$

For small enough P_0 and angular frequencies larger than those for which Eqs. 12.1-12.4 are valid, but less than the smallest of the λ_j , it is found from Eqs. 13.1-13.2 that

$$|G_P(i\omega; P_0)| \sim \left\{ \omega \left[l^* + \sum_{j=1}^m (a_j/\lambda_j) \right] \right\}^{-1} \quad \dots (14.1)$$

and

$$\tan \delta_1(i\omega; P_0) \sim \left\{ -\omega \left[l^* + \sum_{j=1}^m (a_j/\lambda_j) \right] \right\} \left\{ \omega^2 \sum_{j=1}^m (a_j/\lambda_j^2) - [(\bar{K}/\gamma) + (\bar{G}/\sigma)] \right\}^{-1} \quad \dots (14.2)$$

Further, in the case of small P_0 , it is found, from Eqs. 13.3 and 13.4, that for ω larger than the largest of $\lambda_1, \dots, \lambda_m, \gamma$, and σ ,

$$|G_P(i\omega; P_0)| \sim \left\{ 1 + \left[l^* \omega^2 + \sum_{j=1}^m a_j \lambda_j \right]^2 \right\}^{-1/2} \quad \dots (14.3)$$

and

$$\tan \delta_1(i\omega; P_0) \sim - \left[l^* \omega^2 + \sum_{j=1}^m a_j \lambda_j \right] / \omega. \quad \dots (14.4)$$

If P_0 be small enough that $[(\bar{K}/\gamma) + (\bar{G}/\sigma)] < \omega^2$, Eq. 14.2 reduces to

$$\tan \delta_1(i\omega; P_0) \sim - \left[l^* + \sum_{j=1}^m (a_j/\lambda_j) \right] \left[\omega \sum_{j=1}^m (a_j/\lambda_j^2) \right]^{-1}. \quad \dots (14.5)$$

If $l^* \ll 1/\lambda$, Eq. 14.1 reduces to

$$|G_P(i\omega; P_0)| \sim \lambda/\omega. \quad \dots (14.6)$$

and Eq. 14.5 becomes

$$\tan \delta_1(i\omega; P_0) \sim - \left[\omega \lambda \sum_{j=1}^m (a_j / \lambda_j^2) \right]^1 \quad \dots (14.7)$$

where

$$1/\lambda = \sum_{j=1}^m (a_j / \lambda_j), \quad \dots (14.8)$$

with λ the "equivalent" one-group decay constant.

The major differences between the transfer function for m delayed neutron groups and the one delayed neutron group transfer function are displayed in Eq. 14.2 or 14.7 and Eqs. 14.3 and 14.4. These differences result from the fact that $(1/\lambda^2) \neq \sum_{j=1}^m (a_j / \lambda_j^2)$ and $\lambda \neq \sum_{j=1}^m a_j \lambda_j$. Numerically it is found, using the data given in Table I, that $(1/\lambda^2) = 145$ while $\sum_{j=1}^6 (a_j / \lambda_j^2) = 31.2$ and $\lambda = 0.0829$ while $\sum_{j=1}^6 a_j \lambda_j = 0.512$. Clearly, the constants $\Lambda_1, \Lambda_2, A_1$ and A_2 for a two delayed neutron group model are uniquely determined by the requirement that

$$\sum_{j=1}^2 (A_j / \Lambda_j) = \sum_{j=1}^m (a_j / \lambda_j), \quad \dots (14.91)$$

$$\sum_{j=1}^2 (A_j / \Lambda_j^2) = \sum_{j=1}^m (a_j / \lambda_j^2), \quad \dots (14.92)$$

$$\sum_{j=1}^2 A_j \Lambda_j = \sum_{j=1}^m a_j \lambda_j, \quad \dots (14.93)$$

and that

$$\sum_{j=1}^2 A_j = \sum_{j=1}^m a_j = 1. \quad \dots (14.94)$$

Correspondingly it is seen that a two delayed neutron group model will accurately represent the behavior described by the various asymptotic and approximate expressions given in the preceding discussion since the delayed neutron constants appear in the expressions only in combinations identical to those appearing on the right hand side of Eqs. 14.91-14.94.

D. CURVES

In Fig. 1 and 2 are plotted the Bode diagrams (amplitude and phase shift) of $G_p(i\omega; P_0)$ for $P_0 = 10^{-1}, 10^{-2}, 10^{-1}, 1, 10,$ and 50 kw. The values of the delayed neutron constants and of the kinetic parameters used in these and all other calculations for the case of six delayed neutron groups are given in Tables I and II respectively. It has been assumed here for purposes of calculation that the kinetic parameters are independent of power. The intercepts of the various asymptotes given by Eqs. 12.1-12.6, 14.1, and 14.2 are indicated.

TABLE I
PROPERTIES OF DELAYED NEUTRONS IN THERMAL FISSION OF U^{235} †

Group j	Half Life T_j Sec	Mean Lifetime τ_j Sec	Decay Constant λ_j Sec ⁻¹	Fraction α_j
1	54.31	78.36	0.01276	0.035
2	19.78	28.54	0.03504	0.230
3	5.50	7.935	0.126	0.180
4	2.11	3.04	0.3285	0.406
5	0.434	0.633	1.58	0.124
6	0.114	0.1645	6.08	0.025

†These values are as taken from Ref. 10. The values given in the post conference copy of Ref. 10 and the more recent values of Ref. 11 are slightly different.

Fig. 1. Amplitude of Normalized Water Boiler Reactor Power Transfer Function, $G_P(i\omega; P_0)$, (6 Groups of Delayed Neutrons)

Fig. 2. Phase Shift, $\delta_1(i\omega; P_0)$, for Water Boiler Reactor Power Transfer Function (6 Groups of Delayed Neutrons)

TABLE II
VALUES OF THE KINETIC PARAMETERS USED IN THE CALCULATIONS
REPORTED HERE †

$\beta_{eff} = 0.00855$ $l = 1.2 \times 10^{-4} \text{ sec.}$ $K = 0.01428 \text{ }^\circ\text{C/Kw-sec}$ $\gamma = 10^{-2} \text{ sec}^{-1}$	$\alpha = -2.6 \times 10^{-4} \text{ per } ^\circ\text{C}$ $G = 4.16 \text{ cm}^3/\text{Kw-sec at STP}$ $\sigma = 0.5 \text{ sec}^{-1} \text{ at STP}$ $\varphi = -7 \times 10^{-6} \text{ cm}^{-3} \text{ at STP}$
--	--

†Except for β_{eff} and K the values of the parameters are as given in Ref. 12. The value of β_{eff} is that measured for Lopo.⁷

The resonance in the transfer function that occurs for large enough P_0 (10 kw and 50 kw) in Fig. 1 indicates that for such P_0 the reactor is less than critically damped in the linear approximation. Hence, for such P_0 the transient solution contains a damped oscillatory term with an angular frequency at which the maximum of the resonance occurs. If both α and φ are negative, as they are for the water boiler, then the nonlinear transient solution will also contain a damped oscillatory term. No infinite resonance as discussed by Bethe¹³ is possible, since both α and φ are negative and there is no transport delay time in the model under consideration. It is interesting to note that there appears to be a correlation between the maximum of the resonance and the frequency at which $\delta_1 = 0$.

In Fig. 3 is plotted a log plot of $|\tan \delta_1(i\omega; P_0)|$ as a function of the angular frequency ω . It is to be noted that the result for the intercept of the high frequency asymptote of the log log plot of $|\tan \delta_1(i\omega; P_0)|$ as a function of ω is completely general, while the result that $\delta_1(i\omega; P_0) = -45^\circ$ at $\omega = (1/l^*)$ in the high-frequency region is not valid in general for sufficiently large P_0 . The same remark holds for the low-frequency asymptotes of $|\tan \delta_1(i\omega; P_0)|$. In Fig. 4 are shown the log log plots of $|G_T(i\omega; P_0)|$ and $|\tan \delta_2(i\omega; P_0)|$ with the intercepts of the asymptotes indicated.

Fig. 3. Tangent of Phase Shift for Water Boiler Reactor Power Transfer Function (6 Groups of Delayed Neutrons)

Fig. 4. Temperature Transfer Function: Amplitude and Tangent of Phase Shift for a Water Boiler Reactor

IV. OTHER REPRESENTATIONS OF THE TRANSFER FUNCTION

A. INVERSE POLAR PLOT

In Fig. 5 is shown a polar plot of $G_P(i\omega; P_0)^{-1}$, the inverse of the normalized water boiler reactor power transfer function.† For the one delayed neutron group model it can be shown that at zero power, the inverse polar plot for small angular frequencies is a semi-circle passing through the origin with its diameter along the polar axis. At higher frequencies this semi-circle becomes distorted and, finally, for high enough frequencies, it becomes the straight vertical line seen in Fig. 5.

†In Fig. 5, 7, and 9 the solid line represents the function for the frequency range $0 \leq \omega < \infty$ and the dashed line represents the frequency range $-\infty < \omega \leq 0$.

Fig. 5. Inverse of Normalized Water Boiler Reactor Power Transfer Function: $G_P(i\omega; P_0)^{-1}$, (6 Groups of Delayed Neutrons)

B. PARALLEL IMPEDANCE MODEL

Another useful way to treat a reactor is as two complex impedances in parallel: one is the complex impedance presented to the reactivity input function $\rho_1^*(t)$ due to the effects of prompt and delayed neutrons alone, $Z_0(i\omega)$; the other is the impedance due to power reactivity feedback effects $Z_P(i\omega; P_0)$. For a water boiler, the power reactivity feedback effects are those due to temperature variations and radiolytic gas generation. This situation is illustrated in Fig. 6.

Fig. 6. Parallel Impedance Model Block Diagram

The overall complex impedance presented to the input function of reactivity $\rho_1^*(t)$ is then $Z(i\omega; P_0)$ where

$$Z(i\omega; P_0)^{-1} = Z_0(i\omega)^{-1} + Z_P(i\omega; P_0)^{-1}. \quad \dots (15)$$

Thus, $Z(i\omega; P_0)$ is simply the normalized reactor power transfer function $G_P(i\omega; P_0)$ (reactivity expressed in dollars), and $Z_0(i\omega) = G_P(i\omega; 0)$.

The advantage of this approach lies in the fact that it enables one to separate the effects of the reactivity power feedback on the transfer function. The function $[-P_0 Z_P(i\omega; P_0)]^{-1}$ is called the complex power coefficient of reactivity. It is to

be noted that Eqs. 13.1-13.4 are special cases of general asymptotic relationships which can be written, for a wide class of reactivity feedback loops,¹⁴ in terms of limiting forms of the complex power coefficient of reactivity.

For a water boiler reactor, Z_P is given by

$$Z_P(i\omega; P_0) = -P_0^{-1} \left[\frac{K\alpha}{i\omega + \gamma} + \frac{G\phi^*}{i\omega + \sigma} \right]^{-1}. \quad \dots (16)$$

The polar plot of the function $\{\beta_{eff}/[P_0 Z_P(i\omega; P_0)]\}$ for a water boiler reactor is shown in Fig. 7.

C. NYQUIST CRITERION

In order to apply the generalized Nyquist criterion¹⁵ in its usual form to the linear behavior of a water boiler reactor, it is convenient to formulate the derivation of the transfer function in a different manner. The reactor in the linear approximation can be regarded as a system which includes only the effects of prompt neutrons on the power output plus a series of parallel feedback loops. These feedback loops consist of one for each group of delayed neutrons, one for the effect of temperature on reactivity, and one for the effects of radiolytic gas bubbles on reactivity. The block diagram for this model is shown in Fig. 8. From this approach it is found that $G_P(i\omega; P_0)$ can be written in the form

$$G_P(i\omega; P_0) = G(i\omega)/[1 + G(i\omega)H(i\omega; P_0)] \quad \dots (17.1)$$

or

$$[1 + G(i\omega)H(i\omega; P_0)] = [G(i\omega)/G_P(i\omega; P_0)] \quad \dots (17.2)$$

where

$$H(i\omega; P_0) = \sum_{j=1}^m H_j(i\omega) + H_T(i\omega; P_0) + H_V(i\omega; P_0). \quad \dots (17.3)$$

(A)

Fig. 7. $\{\beta_{eff}/[P_0 Z_p(i\omega; P_0)]\}^{-1}$ for Water Boiler Reactor Power Transfer Function

Fig. 8. Block Diagram Representation for Application of Nyquist Criterion

$G(i\omega)H(i\omega; P_0)$ is then the open-loop transfer function. The expressions for $G(i\omega)$, $H_j(i\omega)$, $H_T(i\omega; P_0)$, and $H_V(i\omega; P_0)$ are given in Fig. 8. The generalized Nyquist stability criterion for the water boiler power transfer function reduces to the statement that the system is stable in the linear approximation if the polar plot of $[1 + G(i\omega)H(i\omega; P_0)]$ as ω goes from $-\infty$ to $+\infty$, does not encircle the point $[1 + G(i\omega)H(i\omega; P_0)] = 0$.†

Nyquist plots of the water boiler reactor power transfer function for $P_0 = 10^{-3}$, $P_0 = 1$, $P_0 = 10$, and $P_0 = 50$ kw are shown in Fig. 9. If the model used is valid, these plots show that the water boiler is stable in the linear approximation.† Even for the case where $P_0 = 0$, which is equivalent to the behavior with no power reactivity feedback terms, it can be shown that the Nyquist plot does not encircle the point zero. However, the Nyquist plot in this case passes through the point zero which indicates that $[1 + G(i\omega)H(i\omega; P_0)]$ vanishes for some real ω , namely $\omega = 0$. Thus, for this case the transient solution contains a constant, but is still stable in the linear approximation. It is a well-known fact that, in the actual nonlinear theory, a reactor is anything but stable for this case. This emphasizes the extremely important fact that even though a system is stable in the linear approximation, it can be disastrously unstable in the nonlinear theory.

†The sense of the work "stable" as used here is that of the impossibility of divergent oscillations.

Fig. 9. Nyquist Plot for Water Boiler Reactor Power Transfer Function:
 $1 + G(i\omega)H(i\omega; P_0)$ (6 Groups of Delayed Neutrons)

APPENDIX I

DERIVATION AND LIMITATIONS OF THE SPACE-INDEPENDENT KINETIC EQUATIONS

There are limitations to the validity of the model represented by Eqs. 1.1-1.5, the seriousness of the limitations increasing with the magnitude of the deviations from equilibrium. Further, a number of complex phenomena are entirely neglected. It is the purpose of the following discussion to indicate how these kinetic equations are derived and the limitations imposed by the requisite approximations.

A. EQUATIONS FOR NEUTRON BEHAVIOR

Numerous derivations of Eqs. 1.1 and 1.2 for the behavior of prompt and delayed neutrons have appeared in the literature. A modification of the derivation given by Ussachoff¹⁶ has been carried out¹⁴ with the result that Eqs. 1.1 and 1.2 were obtained with the quantity $(\beta_{eff} a_j)$ replaced by $(\beta_{\epsilon_j} a_j)$, $\dagger \epsilon = \sum_{j=1}^m \epsilon_j a_j$, $l = [l_0/k_{eff}]$, and $C(t) = [\epsilon_j c_j(t)/(vl)]$ where l_0 is the prompt neutron lifetime and is given by the ratio of the integral importance of all neutrons to the rate of destruction of importance; ϵ_j is the relative importance of delayed neutrons belonging to the j^{th} delayed neutron group as compared to prompt neutrons and is given by the ratio of the rate of creation of importance (if all neutrons were born with the energy spectrum of the j^{th} delayed group) to the total rate of creation of importance; v is the average number of neutrons per fission and is dependent upon the neutron energy spectrum of the reactor; and $c_j(t)$ is the total number of delayed neutron precursor atoms for the j^{th} delayed neutron group at time t . The usual assumption that l is a constant is equivalent to the first assumption made by Bethe.^{13, \dagger\dagger} This assumption is also one that is made in the present derivation itself.

The other assumptions made in this derivation are that: (1) the delayed neutron precursor atoms stay where they are formed; and (2) that

^{\dagger}See Appendix III for a more detailed discussion of this point.

^{\dagger\dagger}The first assumption made by Bethe is that the excess multiplication, $k_{ex} = k_{eff} - 1$, can be neglected with respect to one.

(A)

$$\int dVol \int_{\hat{\Omega}} d\hat{\Omega} \int_0^\infty \left\{ \frac{1}{v} \frac{\partial}{\partial t} [F^+(\mathbf{r}, E, \hat{\Omega}, t) F(\mathbf{r}, E, \hat{\Omega}, t)] \right\} dE$$

$$>> \int dVol \int_{\hat{\Omega}} d\hat{\Omega} \int_0^\infty \left[\frac{F(\mathbf{r}, E, \hat{\Omega}, t)}{v} \frac{\partial F^+(\mathbf{r}, E, \hat{\Omega}, t)}{\partial t} \right] dE.$$

where $F(\mathbf{r}, E, \hat{\Omega}, t)$ is the neutron flux of energy E at point $\mathbf{r}$ at time t per unit energy per unit solid angle about $\hat{\Omega}$, and where $F^+(\mathbf{r}, E, \hat{\Omega}, t)$ is the adjoint flux or neutron importance function at point $\mathbf{r}$ at time t for neutrons of energy E with velocities in the direction $\hat{\Omega}$.

The second assumption is equivalent to the assumption that the change in the spatial distribution of the fission rate density takes place slowly in time or that it is localized. It is, however, not entirely clear as to what kinetic phenomena satisfy this assumption. What happens if this approximation is not valid has been discussed to some extent in a number of places, e.g., see Ref. 17. It is clear, however, that this approximation is certainly valid for a larger class of kinetic behavior than it would be if the space-independent kinetic equations were formulated in terms of the total importance of neutrons. In the latter case the second approximation is replaced by the assumption of separability of space and time. Further, from the discussion which follows, it is clear that the limitations imposed on the other equations are probably more restrictive than those for Eqs. 1.1 and 1.2.

B. EQUATION FOR RADIOLYTIC GAS

The equation governing the behavior of radiolytic gas, Eq. 1.4, is derived by assuming that all of the radiolytic gas that is formed appears immediately in the form of bubbles, i.e., that bubbles are formed instantaneously, and that the rate of removal of gas is directly proportional to the volume of gas present. Other possibilities have been considered by Wilkins.¹⁸ However, modifications of this equation cannot properly be made until the process of gas bubble formation

is better understood. In particular, recent experimental results indicate that the model should be extended to include an intermediate dissolved gas stage with some sort of average bubble formation time.¹⁹

C. EQUATION FOR TEMPERATURE

Equation 1.3, which is taken to describe the temperature behavior, can be derived by assuming that the heat generation at r at time t is directly proportional to the fission rate density $P(r, t)$ at r at time t , and that

$$\frac{d\theta(r, t)}{dt} = -f[\theta(r, t), v(r, t), P(r, t)] + K(r) P(r, t). \quad \dots (18)$$

where $\theta(r, t)$ is the temperature at r at time t , $v(r, t)$ is the radiolytic gas density at r at time t , $K(r)$ is directly proportional to the reciprocal heat capacity at r at time t , and $f(\theta, v, P)$ is directly proportional to the rate of removal of heat. The function f is then expanded in a Taylor series about the equilibrium temperature $\theta_0(r)$, radiolytic gas density $v_0(r)$, and fission rate density $P_0(r)$ at r for the equilibrium total fission rate P_0 . All terms except the one involving the first partial of f with respect to θ are neglected; and finally, integrating the resultant equation over the core, one obtains Eq. 1.3 where

$$T(t) = V^{-1} \int_{core} T(r, t) dVol \quad \dots (19.1)$$

$$\gamma = [1/T(t)] \int_{core} \left\{ \left[\frac{\partial f[\theta_0(r), v_0(r), P_0(r)]}{\partial \theta_0(r)} \right] T(r, t) \right\} dVol \quad \dots (19.2)$$

and

$$K = [1/\delta P(t)] \int_{core} [K(r) \delta P(r, t)] dVol \quad \dots (19.3)$$

where

$$T(\mathbf{r}, t) = \theta(\mathbf{r}, t) - \theta_0(\mathbf{r}) \quad \dots (19.4)$$

$$\delta P(\mathbf{r}, t) = P(\mathbf{r}, t) - P_0(\mathbf{r}), \quad \dots (19.5)$$

$$\delta P(t) = \int_{core} \delta P(\mathbf{r}, t) dVol, \quad \dots (19.6)$$

$$P_0 = \int_{core} P_0(\mathbf{r}) dVol, \quad \dots (19.7)$$

and V is the volume of the core.

From the derivation, it is clear that Eq. 1.3 is strictly valid for small deviations from equilibrium or for times small compared with $(1/\gamma)$ after the initiation of a fast transient in which case $\theta(\mathbf{r}, t)$ is directly proportional to $P(\mathbf{r}, t)$. However, for certain heat-removal mechanisms (e.g., those based on Newton's law of cooling) the derivation is also valid for large deviations from equilibrium. In the case of solution-type reactors, the neglect of the first order term in the Taylor series expansion of f involving the radiolytic gas density $v(\mathbf{r}, t)$ may not be strictly valid.²⁰

This derivation is quite general and applies to all reactors. It also has the advantage that if one possesses an expression for $f(\theta, v, P)$ and for $K(\mathbf{r})$, it enables him to calculate the parameters γ and K .

D. REACTIVITY FEEDBACK EQUATION

The reactivity feedback equation, Eq. 1.5, also involves an approximation. It is assumed that all effects of temperature and radiolytic gas on the kinetic behavior of the fission rate will enter through the multiplication constant and can be represented by a function g which depends on the space average tempera-

ture $\theta(t)$, the total volume of radiolytic gas in the form of bubbles $v(t)$, and the total fission rate $P(t)$ at the time t . The expression for g is obtained from that given by Ussachoff¹⁶ for the reactivity. The term $[\alpha T(t) + \phi V(t)]$ appearing in Eq. 1.5 is then obtained by expanding g in a Taylor series about equilibrium with the result that

$$\alpha = \frac{\partial g(\theta_0, v_0, P_0)}{\partial \theta_0} \text{ and } \phi = \frac{\partial g(\theta_0, v_0, P_0)}{\partial v_0} . \quad \dots (20)$$

Clearly the validity of this approximation will in most cases be restricted to small deviations from equilibrium. However, as in the case of the temperature equation, the range of validity may in some cases be much larger.

From a pragmatic viewpoint, the range of validity of the space independent kinetic equations may be extended by regarding the parameters K , α , γ , G , ϕ , and σ as functions of the equilibrium power, with all other external experimental parameters constant, as measured from the experimental transfer function.

APPENDIX II NOMENCLATURE

- $P(t)$ = total fission rate (or power) at time t . †
- P_0 = the equilibrium or steady-state total fission rate (or power). †
- $C_j(t)$ = the effective "latent" fission rate of the reactor stored in the form of precursor atoms of the j^{th} group of delayed neutrons at the time t .
- $T(t)$ = the space average over the core of the temperature at the time t , measured with respect to the same space average when operating at the steady-state total fission rate P_0 .
- $V(t)$ = the total volume at STP of radiolytic gas in solution in the form of gas bubbles at the time t , measured with respect to the total volume of gas in solution in the form of gas bubbles when operating at the steady-state total fission rate (or power) P_0 .
- $\rho(t)$ = $(k_{eff} - 1)/k_{eff}$, the reactivity in "absolute" units at time t , where k_{eff} is the effective multiplication factor.
- $\rho_1(t)$ = the external input function of reactivity, in "absolute" units, e.g., by the motion of a control rod.
- β = the total fraction of all fission neutrons which are delayed.
- a_j = the fraction of all delayed neutrons which belong to the j^{th} delayed group, $\sum_{j=1}^m a_j = 1$.
- βa_j = the fraction of all fission neutrons which belong to the j^{th} group of delayed neutrons.
- ϵ = the relative effectiveness of delayed neutrons compared to prompt neutrons.
- β_{eff} = $\epsilon\beta$, the effective total fraction of all neutrons which are delayed.
- l = l_0/k_{eff} , the "effective" neutron generation time in the reactor (assumed to have negligible time dependence).
- l_0 = the neutron lifetime.
- K = the adiabatic rate of increase of the space average temperature per unit total fission rate (or power) for the reactor.

†The terms "power" and "total fission rate" or simply "fission rate" are used synonymously throughout this report.

- G = the adiabatic rate of increase of the total volume at STP of radiolytic gas in solution per unit total fission rate (or power) for the reactor.
- γ = the reciprocal of the characteristic mean time for loss of heat from the core, such that $\gamma T(t)$ is the rate of decrease of $T(t)$.
- σ = the reciprocal of the characteristic time for loss of radiolytic gas from the core, such that $\sigma V(t)$ gives the total rate of loss of gas from the "soup."
- α = the increase in reactivity of the reactor due to a unit increase in T , the temperature coefficient reactivity in "absolute" units. †
- ϕ = the increase in reactivity of the reactor due to a unit increase in radiolytic gas volume (STP), i. e., the void coefficient of reactivity in "absolute" units. †
- ρ^* = ρ/β_{eff} , the reactivity in dollars. ††
- l^* = l/β_{eff} . ††
- α^* = α/β_{eff} , the temperature coefficient of reactivity in units of dollars. ††
- ϕ^* = ϕ/β_{eff} , the void coefficient of reactivity in units of dollars. ††

†It is to be noted that in the case of all water boiler type reactors built to date α and ϕ are negative.

††The superscript * is used to denote the symbol without the * divided by β_{eff} .

APPENDIX III

EFFECT OF ENERGY DEPENDENCE OF DELAYED NEUTRONS

It has been assumed in writing Eqs. 1.1-1.5 that the delayed neutrons are all emitted with the same energy. Actually the delayed neutrons in each group are emitted with a spectrum of energies, and this spectrum is different for each group. In what follows we shall assume that the delayed neutrons in the j^{th} group may be characterized by a mean birth energy, E_j .

A. EFFECT ON EQS. 1.1-1.5 AND DERIVATIVE EQUATIONS

First we consider the effect of including the difference between the mean birth energies amongst the delayed neutron groups upon the form of Eqs. 1.1-1.5 and the derivative equations. As pointed out in Appendix I, the derivation discussed there yields Eqs. 1.1-1.5 with the quantity $(\beta_{eff}a_j)$ replaced by $(\beta\epsilon_ja_j)$ and with $\beta_{eff} = \epsilon\beta$ where

$$\epsilon = \sum_{j=1}^m \epsilon_j a_j. \quad \dots (21)$$

Thus Eq. 1.2 becomes

$$\frac{dC_j(t)}{dt} + \lambda_j C_j(t) = \frac{\beta\epsilon_j a_j}{l} P(t). \quad \dots (22)$$

Let

$$e_j = (a_j \epsilon_j / \epsilon), \quad \dots (23.1)$$

then Eq. 22 can be written in the form

$$\frac{dC_j(t)}{dt} + \lambda_j C_j(t) = \frac{\beta_{eff} e_j}{l} P(t) \quad \dots (23.2)$$

where

$$\sum_{j=1}^m e_j = 1. \quad \dots (23.3)$$

Hence, the inclusion of the effect of the differences in the mean emission energy for the different delayed neutron groups is achieved simply by replacing a_j by e_j in Eqs. 1.1-1.5 and the subsequent equations.

B. PHYSICAL INTERPRETATION

The treatment has, so far, been formal. We now discuss the physical interpretation of the various quantities that have been introduced.

We define a relative effectiveness factor $\epsilon(E)$ for neutrons born with energy E where $\epsilon(E)$ is the relative importance of neutrons born with energy E compared to prompt neutrons. Then, for example, it follows from Appendix I that

$$\epsilon_j = \epsilon(E_j). \quad \dots (24)$$

In general (except for fast reactors) ϵ_j is always equal to or greater than one. This results from the fact that the E_j are always less than the mean birth energy of prompt neutrons, and consequently (except for fast reactors) delayed neutrons have less chance of leaking out of the reactor before being absorbed. Thus, for a thermal or epithermal reactor ϵ_j is approximately proportional to the ratio of the nonleakage probability for delayed neutrons belonging to the j^{th} delayed group to the nonleakage probability for prompt neutrons. In the case of a fast reactor, the mean energy of the neutron energy spectrum may be larger than E_j , in which case the interpretation of ϵ_j in terms of nonleakage probabilities breaks down.

It is possible to define a "mean" birth energy, $\bar{E}$, for all the delayed neutrons by the relationship:

$$\epsilon(\bar{E}) = \epsilon$$

where ϵ is given by Eq. 21.† Then ϵ is seen to be the relative effectiveness factor for delayed neutrons, all assumed to be born at the same energy $\bar{E}$. Correspondingly, the quantity (e_j/a_j) is the effectiveness of delayed neutrons belonging to the j^{th} group relative to neutrons born with energy $\bar{E}$.

†It is to be noted that in general $\bar{E} \neq \sum_j E_j a_j$. This implies that the use of the mean energy E of the spectrum for the j^{th} delayed group is not strictly valid. A more thorough treatment would extend the argument given here for the case of a discrete spectrum of energies E_j to include the continuous spectrum within a group.

REFERENCES

1. J. N. Grace, M. A. Schultz, and T. E. Fairey, "Inherent Reactor Stability," Proc. 1955 Conf. on Nucl. Eng. UCLA, (1955) B1-22.
2. J. M. Harrer, R. E. Boyar, and D. Krucoff, "Transfer Function of Argonne CP-2 Reactor," Nucleonics, 10, No. 8, 32-36 (1952).
3. Eric Hellstrand, "The Swedish Heavy Water Reactor - Two Kinetic Experiments," Atomics 6, No. 7, 187 (1955).
4. A. Lundby, "Effect of Photoneutrons on the Transfer Function of a Heavy-Water Reactor," J. Nucl. Energy 1: 53 (1954).
5. J. Pelser, "The Influence of the Temperature Coefficient on the Transfer Function of a Heavy Water Reactor," Proceedings of the International Conference on the Peaceful Uses of Atomic Energy, Geneva, 1955. (New York United Nations, 1956) P/934.
6. R. Siegel and H. Hurwitz, Jr., "The Effect of Temperature Coefficients on Reactor Stability and Reactor Transfer Function," KAPL 1138, March 1, 1955.
7. F. de Hoffman, "Criticality of the Water Boiler and Effective Number of Delayed Neutrons," AECD-3051, December 8, 1944.
8. W. Magnus and F. Oberhettinger, Formulas and Theorems for the Functions of Mathematical Physics, Trans. by J. Wermer, Chelsea Publishing Co., New York, (1954).
9. L. I. Schiff, Quantum Mechanics, McGraw-Hill Book Co., Inc., New York, (1948)
10. G. R. Keepin and T. F. Wimett, "Delayed Neutrons," Preprint of Geneva Conf. Paper No. 831 and Addendum 1 (1955).
11. G. R. Keepin, Private Communication, (1956).
12. M. E. Remley, J. W. Flora, D. L. Hetrick, and L. P. Inglis, "Program Review of the Water Boiler Reactor Kinetic Experiments," NAA-SR-1525, March 15, 1956.
13. H. A. Bethe, "Reactor Safety and Oscillator Tests," APDA-117, October 15, 1956.
14. R. E. Skinner, unpublished.
15. H. Chestnut and R. W. Mayer, Servomechanisms and Regulating System Design, John Wiley and Sons, Inc., New York, 39-42, (1951).
16. L. N. Ussachoff, "Equation for the Importance of Neutrons, Reactor Kinetics and the Theory of Perturbation." Proceedings of the International Conference on the Peaceful Uses of Atomic Energy, Geneva, 1955. (New York, United Nations, 1956) P/656.
17. A. M. Weinberg and H. C. Schweinler, "Theory of Oscillating Absorber in a Chain Reactor," Phys. Rev. 74, 851-63 (1948).

18. J. Ernest Wilkins, Jr., "A Note on the Kinetics of Homogeneous Piles," CP-3068 June 26, 1945.
19. D. L. Hetrick, J. W. Flora, E. L. Gardner, B. R. Moskowitz, D. R. Muller, M. E. Remley, and R. K. Stitt, "Preliminary Results on the Kinetic Behavior of Water Boiler Reactors," NAA-SR-1896, April 15, 1957.
20. F. P. Durham, "Radiolytic - Gas Bubbling Improves Convective Heat Transfer in SUPO," Nucleonics 13, No. 5, 42 (1955).