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THE ZEROS OF BESSEL FUNCTIONS AND $x J_{\nu+1}(x)/J_{\nu}(x) = \text{constant}$.

Joseph S. Rosen*

We outline a method for finding the zeros of the transcendental equation

$$f_0 - \frac{x J_{\nu+1}(x)}{J_{\nu}(x)} = 0, \quad (1)$$

in which f_0 is a constant and $J_{\nu}(x)$ is the Bessel function of order ν .

We first indicate some general theorems [1] on the roots of Eqn. (1).

Using the relation

$$x J'_{\nu}(x) = \nu J_{\nu}(x) - x J_{\nu+1}(x),$$

Eqn. (1) can be written in the form

$$ax J'_{\nu}(x) + b J_{\nu}(x) = 0 \quad (2)$$

where a and b are real numbers.

(1) If ν is real and α is a positive real zero of the function

(2), then $-\alpha$ is also a zero of this function.

(2) If ν is real and ≥ 0 , between two consecutive positive zeros of $J_{\nu}(x)$ there lies one and only one zero of the function in (2).

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(3) If a and b are real and $v > -1$, then the function (2) has only real zeros except that it has two purely imaginary zeros when $b/a + v < 0$ [when $f_0 < 0$ in Eqn. (1)]. If a and b are positive, and v real and ≥ 0 , the function (2) has no imaginary zeros.

(4) If $v > -1$ and if $ab' \neq a'b$, then the positive zeros of (2) are interlaced with those of $a'x J'_v(x) + b' J_v(x)$ (where a' and b' are constants), and no function of this type can have a repeated zero other than $x = 0$.

If we set $-u = x J_{v+1}(x)/J_v(x)$, then u satisfies the differential equation of Riccati

$$x \frac{du}{dx} + u^2 + 2vu + x^2 = 0, \quad (3)$$

and u can be represented by an infinite continued fraction or by a power series [2]. Thus,

$$\frac{x J_{v+1}(x)}{J_v(x)} = \frac{x^2}{2(v+1)} - \frac{x^2}{2(v+2)} - \frac{x^2}{2(v+3)} - , \text{ etc.} \quad (4)$$

and

$$\frac{x J_{v+1}(x)}{J_v(x)} = f_{v,1}x^2 + f_{v,2}x^4 + f_{v,3}x^6 + \dots \quad (5)$$

where the coefficients, as given by Kapteyn [2], are defined by

$$2(v+1) f_{v,1} = 1$$

$$2(v+n+1) f_{v,n+1} = f_{v,1} f_{v,n} + f_{v,2} f_{v,n-1} + \dots + f_{v,n} f_{v,1} \quad (6)$$

where $n = 1, 2, 3, \dots$

Accordingly, the zeros of Eqn. (1) may be obtained by solving the equation
(valid for $|x| < \text{the first root of } J_v(x)$)

$$f_0 = f_{v,1} x^2 + f_{v,2} x^4 + f_{v,3} x^6 + \dots \quad (7)$$

where f_0 is the given constant and where the coefficients in the right member are obtained from the recurrent relations of Eqn. (6).¹ The general expression for $f_{v,n}$ is difficult to formulate, but we give below the terms for some small values of n .

It is also possible to write the power series, valid for $|x| < |j_{v,1}|$,

$$\frac{x J_{v+1}(x)}{J_v(x)} = 2 \sum_{n=1}^{\infty} \sigma_{v,n} x^{2n} \quad (8)$$

where

$$\sigma_{v,n} = \sum_{s=1}^{\infty} \frac{1}{j_{v,s}^{2n}} \quad (9)$$

¹Other forms for determining these coefficients are given by Kapteyn in the references cited.

and where $j_{v,1}, j_{v,2}, \dots$ are the positive zeros of $x^{-v} J_v(x)$ arranged in ascending order of magnitude [3]. Comparing the coefficients of Eqs. (5) and (8), we see that

$$f_{v,n} = 2\sigma_{v,n}.$$

The recurrence relations of Eqn. (6) for calculating $f_{v,n}$, may therefore also be used for obtaining the smallest positive zero of $J_v(x)$ by Rayleigh's method [4], since

$$\lim_{n \rightarrow \infty} \sqrt{\frac{f_{v,n}}{f_{v,n+1}}} = j_{v,1}.$$

For small values of n , we can write for $f_{v,n}$ [4]:

$$f_{v,1} = \frac{1}{2(v+1)},$$

$$f_{v,2} = \frac{1}{2^3(v+1)^2(v+2)},$$

$$f_{v,3} = \frac{1}{2^4(v+1)^3(v+2)(v+3)},$$

$$f_{v,4} = \frac{5v+11}{2^7(v+1)^4(v+2)^2(v+3)(v+4)},$$

²Because of this property of the coefficients of Eqn. (7), all the terms in this equation after a given term may be approximated by an infinite geometric series.

$$f_{v,5} = \frac{7v + 19}{2^8 (v+1)^5 (v+2)^2 (v+3) (v+4) (v+5)}$$

$$f_{v,6} = \frac{2(21v^3 + 181v^2 + 513v + 473)}{2^{11} (v+1)^6 (v+2)^3 (v+3)^2 (v+4) (v+5) (v+6)}$$

For $v = 0$, the numerical values for the first 13 terms of $f_{0,n}$ are:

n	$f_{0,n}$
1	.50000 00000
2	.06250 00000
3	.01041 66667
4	.00179 03646
5	.00030 92448
6	.00005 34623
7	.00000 92441
8	.00000 15984
9	.00000 02764
10	.00000 00478
11	.00000 00083
12	.00000 00014
13	.00000 00002

THE CALCULATION OF THE SMALLEST POSITIVE ZEROS OF THE BESSEL FUNCTION $J_\nu(x)$

By the use of high-speed computing machinery the coefficients $f_{\nu,n}$ in Eqn. (7) are readily calculated from

$$2(\nu+n+1) f_{\nu,n+1} = \sum_{i=1}^{i=n} f_{\nu,i} f_{\nu,n-i+1} ; \quad n = 1, 2, 3, \dots \quad (6')$$

the recurrence formula of Eqn. (6). The right hand member of this equation is easily programmed for automatic computation and requires few computing sequences. The solution of Eqn. (7) may then be obtained by the usual methods for solving transcendental equations, or by the method indicated below.

Furthermore, the successive ratios $f_{\nu,n}/f_{\nu,n+1}$ converge readily to the square of the smallest positive zero of $J_\nu(x)$ for values of $\nu > -1$.

WHITTAKER'S SOLUTION OF ALGEBRAIC AND TRANSCENDENTAL EQUATIONS

Whittaker [5] gives a method for finding the root of the equation

$$0 = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + a_4 x^4 + \dots \quad (10)$$

which is the smallest in absolute value. His result is somewhat unusual in that he derives a literal formula which gives the desired root on substitution of the coefficients of the equation. We will not, however, concern ourselves with Whittaker's formula, but consider instead the recurrence relations which pertain

$$-a_0 P_{n+1} = \sum_{i=1}^{i=n+1} a_i P_{n-i+1} \quad (11)$$

where $n = 0, 1, 2, 3, \dots$ and $P_0 = 1$. P_s denotes the sum of the homogeneous powers and products of the reciprocals of the roots taken s at a time, so that

$$\lim_{s \rightarrow \infty} P_{s-1}/P_s$$

tends to the limit which is the root of Eqn. (10) which has the smallest modulus.

It is evident that the expression for the recurrence relations in Eqn. (6') is a special case of that in Eqn. (11), and that the programming for the automatic computation of P_s , and the ratios of these quantities, is equally simple.

THE BERNOULLIAN NUMBERS

A useful relation [6] is given by

$$f_{\frac{1}{2}, r} = 2^{2r} B_r / (2r)! \quad (12)$$

where B_r is the r th Bernoullian number. Since the numbers $f_{\frac{1}{2}, r}$ are easily generated by automatic computation with $v = 1/2$ in Eqn. (6'), expansions involving the Bernoullian numbers appear to offer some advantages on digital computers. Furthermore, many Bernoullian expansions involve the factors shown in Eqn. (12), so that the use of the numbers $f_{\frac{1}{2}, r}$ is more natural than those of Bernoulli. For example, some of the generating functions of Bernoullian numbers are, in terms of the numbers $f_{\frac{1}{2}, r}$:

$$\frac{2x}{e^{2x} - 1} = 1 - x + f_{\frac{1}{2},1} x^2 - f_{\frac{1}{2},2} x^4 + \dots ,$$

$$x \coth x = 1 + f_{\frac{1}{2},1} x^2 - f_{\frac{1}{2},2} x^4 + f_{\frac{1}{2},3} x^6 - \dots ,$$

$$x \cot x = 1 - f_{\frac{1}{2},1} x^2 + f_{\frac{1}{2},2} x^4 - \dots .$$

An analogous relation is given by

$$f_{-\frac{1}{2},r} = (2^{2r} - 1) 2^{2r} B_r / (2r)! \quad (13)$$

where $f_{-\frac{1}{2},r}$ are the numbers in Eqn. (6') with $v = -\frac{1}{2}$. Expansions involving Bernoullian numbers often have associated with them the factors shown in Eqn. (13).

For example: '

$$\frac{2x}{1 + e^{-2x}} = x + f_{-\frac{1}{2},1} x^2 - f_{-\frac{1}{2},2} x^4 + \dots ,$$

$$\tan x = f_{-\frac{1}{2},1} x + f_{-\frac{1}{2},2} x^3 + f_{-\frac{1}{2},3} x^5 + \dots .$$

Other examples in which the $f_{\frac{1}{2},r}$ and $f_{-\frac{1}{2},r}$ numbers occur simply are the Euler-Maclaurin formula and expansions for the logarithm of the circular and hyperbolic functions [7] .

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