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Informal Report

STOICHAISTIC COOLING IN GENERAL AND ITS APPLICATION TO

$\bar{p}$ STORAGE IN PARTICULAR

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N O T I C E

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Stochastic cooling in general and its
application to p storage in particular

I present here some basic formulae for stochastic cooling. The formulae are not exact but suffice to get order of magnitude results.

1. Simplest Calculation

Consider a ring of some large circumference S containing N_T particles traveling near the velocity of light. Now suppose that we have some scheme that detects the mean lateral position ($\bar{y}$) of an element of the beam at a "pick up station" of length ℓ . The number of particles in that element we call N .

$$N = \frac{\ell}{S} N_T \quad (1)$$

If the signals corresponding to such displacements ($\bar{y}$) are amplified and used to correct these displacements at a "corrector station" at a point further round the ring, then the mean squared displacement σ^2 will be slowly reduced. First we assume:

- 1) negligible noise in the amplifier;
- 2) adequate band width, ie band width F

$$F \geq 2\pi\beta c/\ell \quad \text{cycles/sec}$$

where βc is the velocity of the particles;

- 3) particle transit time between pickup and corrector station constant to a time small compared with $\ell/\beta c$.
- 4) transit time for a complete rotation around the machine must have random fluctuations large compared with $\ell/\beta c$

Given these conditions we can now calculate what might be called the ideal stochastic cooling rate. Let y_i be the displacement of the i th particle. The mean square displacement will be

$$\sigma^2 = (\sum_{N_T} y_i^2) / N_T \quad (3)$$

The mean positions detected

$$\bar{y} = (\sum_N y_i) / N \quad (4)$$

will themselves have a distribution and will have a mean square value

$$\langle \bar{y}^2 \rangle = \sigma^2 / N \quad (5)$$

If the signals corresponding to these mean positions are amplified by a factor (a) and used to correct the trajectories, then the y position of each particle after correction will be $y_i - a \bar{y}$ and the resultant mean square displacement σ' will be given by:

$$\begin{aligned} (\sigma')^2 &= \sum_N (y_i - a \bar{y})^2 / N \\ &= \left(\sum_N y_i^2 \right) - 2a \bar{y} \sum_N y_i + N a^2 \bar{y}^2 / N \end{aligned}$$

$$(\sigma')^2 = \sigma^2 - 2a\bar{y}^2 + a^2\bar{y}^2$$

On average $\langle (\sigma')^2 \rangle = \sigma^2 - (2a - a^2) \langle \bar{y}^2 \rangle$

$$= \sigma^2 \left(1 - \frac{2a - a^2}{N}\right)$$

ie on average the mean square deviation is reduced by the fraction

$$f = \frac{2a - a^2}{N} \quad (6)$$

Clearly maximum cooling is obtained with $a = 1$ and

$$f_{\max} = \frac{1}{N} \quad (7)$$

If, as at the ISR, $N_T = 4 \cdot 10^{14}$, $l = 3$ cm, $F = 2$ GHz, then $N = 10^{10}$ and $f = 10^{-10}$. The time τ to reduce σ^2 to about $1/e$ is then $10^{10} \times 2 \text{ usec} = 6$ hours.

The CERN expected time τ is 24 hours but it turns out that at CERN our assumption (3) is not satisfied, i.e. there is significant transit time spread between pick-up and corrector stations.

2. Consider transit time spread between pick-up and corrector

If on average a fraction ϵ of the particles in the length l at the pick-up station remain in a length l at the correction station then equation (6) must be modified (see Appendix I):

$$f = \frac{2a\epsilon - a^2}{N} \quad (8)$$

Maximum reduction is obtained for $a=\epsilon$ and for this value

$$f_{\max} = \frac{\epsilon^2}{N} \quad (9)$$

At CERN $\epsilon = 1/2$ and thus the cooling time τ is increased to ≈ 24 hours. This agrees with the CERN calculated number as reported in Months foreign travel report.

At CERN the transit time fluctuations for a complete cycle are very large compared with $l/\beta c$ and thus no correction is needed for this. Only in a machine near transition or a machine with very small momentum spread would this become a problem. If then the fluctuations in transit time are Δt and we define $\delta = \frac{l}{\beta c \Delta t}$ then we find that the maximum cooling fraction is given by

$$f_{\max} = \frac{\epsilon^2(1-2\delta)}{N}$$

Since δ is in general negligible we will not consider this effect further.

3. Consideration of Noise

At CERN the noise is less than 10% of the pick-up signal and has a negligible effect. The situation is very different, however, in the case of a $\bar{p}$ storage ring. In a $\bar{p}$ storage ring the total number of particles N_T is unlikely to be more than 10^{12} finally and not more than 10^7 would be made per AGS pulse. If these were in a storage ring like the ISR the number N in a length of 3 cm would be only 200. Under such circumstances the cooling time is only 400 μsec . Noise in the pick-up system however makes this unattainable.

Let b be the noise quoted in terms of the apparent RMS y position of one particle in cm's. At CERN $b \approx 10^4$ cm, ie unless we have more than 10^4 particles one could not find where they were to better than 1 cm!

Given this definition of b we get instead of equation (8)

$$= \frac{2a \epsilon - a^2}{N} - \frac{a^2 b^2}{N^2 \sigma^2} \quad (10)$$

The maximum cooling is obtained for

$$a = \epsilon / (1 + b^2 / N \sigma^2) \quad (11)$$

If b is large then the reduction factor is given by:

$$f = \frac{\sigma^2 \epsilon^2}{b^2} \quad (12)$$

and this is not a function of N !

If we again use the CERN signal to noise ratio we can then derive this maximum cooling rate.

$$\frac{b}{\sigma} = \frac{\text{noise}}{\text{signal}} \sqrt{N} \approx \frac{1}{10} \cdot 10^5 = 10^4$$

$$\text{then } f = 10^{-8} \cdot \epsilon^2$$

$$\text{for } \epsilon \approx 1 \quad f \approx 10^{-8} \quad \tau = 2 \text{ } \mu\text{sec} \times 10^8 = 200 \text{ sec.}$$

Two ways present themselves for further improving this rate:

- (1) lower the noise.
- (2) increase the rate of corrections per second.

The former can possibly be achieved when it is noted that the band width is not specified and need not be nearly as large as that used at CERN. Theoretically this does not lower the noise ratio since the plate lengths have to be lengthened as the band width is lowered. Despite this, the freedom of choosing the band width could probably allow lower noise amplifiers.

The second solution suggests the use of a small ring. Although the number of particles per unit length will be larger this does not affect the rate. The increased number of corrections per sec, however, will be directly useful. For instance, if we capture $\bar{p}$'s at 3 GeV we can have a ring 1/10 the size of the ISR and get a cooling time of only 20 seconds. This consideration must, however, be weighed against the smaller total phase space capacity of the smaller ring and thus the smaller number of $\bar{p}$'s that can be captured prior to cooling.

We have shown that the cooling for a $\bar{p}$ ring is noise limited yet, when figures taken from CERN are used and the best gain selected, then cooling times as short as 20 sec are still possible.

I conclude then that Stochastic Cooling would allow the collection, cooling and stacking of $\bar{p}$'s from some quite large number of AGS pulses.

APPENDIX

Consideration of Transit Time Fluctuations between pick up and Corrector Stations.

We will consider the sample of particles (N_0) between the pick up electrodes, and the sample of particles (N_4) that lie between the corrector plates at a later time when the correction signal derived from the first sample is applied to the corrector plates. Ideally these two samples are identical and the time delay is obviously set to maximize the number of particles that are present in both samples. Let us now subdivide each of the two samples derived above (N_0 and N_4) into subsets:

Let N_1 be the number of particles found in both samples N_0 and N_4 .

Let N_2 be those in the sample N_0 that are not in N_1 : $N_2 = N_0 - N_1$

Let N_3 be those in the sample N_4 that are not in N_1 : $N_3 = N_4 - N_1$

We will denote sums over the samples N_k by $\sum_k$ and define $S_k = \sum_k y_i$.

The correction signal is now given by the amplification factor (a) times the mean position of the particles between the pick up electrodes; i.e. correction signal = $a S_0 / N_0 = a (S_1 + S_2) / N_0$. This signal is then used to correct the positions of all the particles between the corrector plates. Positions (y_i) of the particles i in the sample N_4 are then changed to $y_i - a(S_1 + S_2) / N_0$. The mean square deviation after correction $(\sigma')^2$ is then given by:

$$\begin{aligned}
 (\sigma')^2 &= \sum_4 \{y_i - a(S_1 + S_2)/N_0\}^2 / N_4 \\
 &= \frac{\sum_4 y_i^2}{N_4} - \frac{2a(S_1 + S_2)}{N_0 N_4} \sum_4 y_i + \frac{a^2 (S_1 + S_2)^2}{N_0^2}
 \end{aligned}$$

Now $\sum_4 y_i^2 / N_4 = \sigma^2$, $\sum_4 y_i = S_4 = S_1 + S_3$

$$(\sigma')^2 = \sigma^2 - \frac{2a(S_1 + S_2)(S_1 + S_3)}{N_0 N_4} + \frac{a^2(S_1 + S_2)^2}{N_0^2}$$

Now we must consider what the average or expectation values of the sums S are when averaged over some number of cycles. The mean square expectation value of any sum S_k is given by

$$\langle S_k^2 \rangle = \langle (\bar{y}_k N_k)^2 \rangle = \sigma^2 N_k$$

However the expectation value of the product of two uncorrelated samples, e.g. N_1 and N_2 , is zero

$$\langle S_1 \cdot S_2 \rangle = 0$$

Thus

$$\langle (\sigma')^2 \rangle = \sigma^2 - \frac{2a \sigma^2 N_1}{N_0 N_4} + \frac{a^2(\sigma^2 N_1 + \sigma^2 N_2)}{N_0^2}$$

We now note that $N_4 \approx N_0$, and define ϵ by $N_1 = \epsilon N_0$. In the ideal situation of no transit time fluctuations $\epsilon = 1$. As the fluctuations become significant so ϵ is reduced. At the CERN ISR $\epsilon \approx 0.5$.

So using this definition of ϵ we obtain.

$$\langle (\sigma')^2 \rangle = \sigma^2 \left\{ 1 - \frac{2a\epsilon + a^2}{N_0} \right\}$$

or if $\langle (\sigma')^2 \rangle = \sigma^2 \{ 1 - f \}$

$$f = \frac{2a\epsilon + a^2}{N}$$