Dr-882

1-74
LA-5631-MS
Informal Report

UC-38

Reporting Date: May 1974

Issued: June 1974

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Using a Transfer Capacitor and a
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UNITED STATES
ATOMIC ENERGY COMMISSION
CONTRACT W-7405-ENG. 36

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Printed in the United States of America. Available from
National Technical Information Service
U.S. Department of Commerce
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Price: Printed Copy \$4.00 Microfiche \$1.45

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METHODS OF ENERGY TRANSFER FROM A MAGNETIC ENERGY STORAGE SYSTEM USING A TRANSFER CAPACITOR AND A SUPERCONDUCTING SWITCH

by

D. M. Weldon and J. D. G. Lindsay

ABSTRACT

Three circuits for magnetic energy storage using a transfer capacitor and a superconducting switch are analyzed, and a set of equations are derived which can be used for parameter studies of these circuits.

I. INTRODUCTION

Large magnetic energy storage systems making use of superconducting storage coils and superconducting switches have been considered for use in a future generation of θ -pinch plasma experiments.^{1,2} The circuit usually considered is shown in Fig. 1. The operational sequence for this storage system calls for building up current in coil L_p and switch R_{SW} while they are both in the superconducting state. To release the stored energy, the switch is driven normal by means of some auxiliary circuitry. The sudden appearance of resistance in the switch causes current to decay in the primary circuit. As the primary current decays energy is coupled inductively into the secondary L_s , which in turn drives experiment L_c . This circuit suffers from the disadvantage that at best only 25% of the stored energy

is delivered to L_c , while 50% of the stored energy is dissipated in resistors R_p and R_{SW} .

Efficiency of energy transfer can be greatly increased by inserting a capacitor across the circuit in any of the ways shown in Fig. 2. In order to understand the operation of these circuits, consider circuit (a) with $L_B = L_E$ and a persistent

Fig. 1. Schematic representation of a magnetic energy storage system. Cryogenic regions are enclosed in dashed lines.

Fig. 2. Magnetic energy storage circuits using transfer capacitors.

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current flowing through superconducting elements L_B and R . When switch R is driven normal a voltage builds up across R , and as shown in Fig. 3, this causes current both to build up in L_E and to flow into capacitor C . After reaching a peak voltage, capacitor C begins to discharge and current flows from the capacitor increasing the current in L_E and further decreasing the current in L_B . In the limit that R is infinite, the current in L_B eventually becomes zero and the current in L_E equals the current originally in L_B . Crowbar switch CB is then closed and current flows through load coil L_E and switch CB until damped out by stray resistance. In the transformer coupled systems current waveforms

Fig. 3. Current in the circuit elements as a function of time for a direct coupled magnetic energy storage system using a transfer capacitor. Circuit parameters are $R = 16\Omega$, $I_B = 25$ kA, $L_B = L_E = 2$ mH, $C = 400$ μ F.

are similar although there are some additional effects if the coupling between secondary and primary is less than perfect ($k < 1$). The capacitor size is considerable in all cases, since at peak voltage it must store about 1/2 of the energy originally in the storage coil.

Circuit analysis has been carried out by Caruthers³ for case (a) of Fig. 2 in a system switched by mechanical switches and ignitrons. Resistance of the mechanical switch replacing R was assumed to be infinite and the opening instantaneous. In this report we will analyze this circuit using a superconducting switch which has a finite resistance when open although it may still be thought of as opening instantaneously. We will also extend results to the transformer coupled circuits of Fig. 2, including the situation in which the coupling is less than perfect.

IX. PARAMETERS FIXED BY THE PLASMA EXPERIMENT

As far as the magnetic energy storage system is concerned, the plasma experiment may be thought of as only a large multi-turn toroidal solenoid with a 28.6 m major radius and a 16 cm minor radius. Other details such as the shock heating apparatus, vacuum system, stabilizing windings, etc., can be ignored. Risettime of the magnetic field in the experiment is designated by τ and has a value of 1 msec. Peak field in the experiment coil is 60 kG. The energy storage coil is a second torus set next to the experiment and having a slightly different major radius so it does not intersect the experiment. Both the experiment and storage coil are modularized for convenience of construction and maintenance, and to limit the voltage seen across the coil leads. Since the experiment and storage coil, as well as the transformer secondary if there is one, have almost the same major radius, the module length will be considered the same for all three coils. Calculations will be made for one of the modules, and because of the toroidal shape of the final configuration the long solenoid formulas for magnetic field and inductance can be safely used. Inductance of a long solenoid in henries is given by

$$L = 4\pi \times 10^{-9} \frac{n^2 r^2}{l}, \quad (1)$$

where n is the total number of turns in a coil module, r is the coil radius in cm, and ℓ is the module length in cm. Using the radius of the compression coil this can be written as

$$L_E = \frac{n^2}{\ell} \frac{E^2}{2}, \quad (2)$$

with $E = 1.01 \times 10^{-5}$ H-cm/turn². Energy in joules stored in the magnetic field of a solenoid is given by

$$W = 10^{-7} \frac{r^2 \ell B^2}{8}, \quad (3)$$

where B is the magnetic field in gauss. Using the radius and magnetic field of the experiment, this can be written as

$$W_E = \eta \ell, \quad (4)$$

where $\eta = 1.15 \times 10^4$ J/cm.

III. DIRECT COUPLING OF STORAGE COIL AND EXPERIMENT

A. Currents in the Circuit Elements

The network equations for circuit (a) of Fig. 2 can be written as:

$$L_B \frac{di_B}{dt} + i_R R = 0,$$

$$L_B \frac{di_B}{dt} + L_E \frac{di_E}{dt} = 0,$$

$$C L_B \frac{d^2 i_B}{dt^2} + i_B - i_R - i_E = 0.$$

Appropriate initial values at $t = 0$ are: $i_B = I_B$,

$$i_E = 0, \quad \frac{di_B}{dt} = \frac{di_E}{dt} = \frac{di_C}{dt} = 0.$$

Using these initial values the network equations have the solution:

$$i_E = I_B \frac{L_B}{L_B + L_E} \left(1 - e^{-\gamma t} \cos \omega t - \frac{\gamma}{\omega} e^{-\gamma t} \sin \omega t \right), \quad (5)$$

$$i_B = I_B \frac{L_E}{L_B + L_E} \left(\frac{L_B}{L_E} + e^{-\gamma t} \cos \omega t + \frac{\gamma}{\omega} e^{-\gamma t} \sin \omega t \right), \quad (6)$$

$$i_C = I_B \left(e^{-\gamma t} \cos \omega t - \frac{\gamma}{\omega} e^{-\gamma t} \sin \omega t \right), \quad (7)$$

$$i_R = 2 I_B \frac{\gamma}{\omega} e^{-\gamma t} \sin \omega t, \quad (8)$$

where

$$\gamma = \frac{1}{2RC}, \quad (9)$$

$$\omega = \sqrt{\frac{1}{L_T C} - \gamma^2}, \quad (10)$$

and

$$L_T = \frac{L_E L_B}{L_E + L_B}.$$

B. Times of Peak Voltage and Crowbar Application

The crowbar time is one of the known input parameters (τ), and the crowbar is applied when i_E is a maximum. Substitution of $t = \tau$ in the expression for $\frac{di_E}{dt} = 0$ gives

$$\omega = \frac{\gamma}{\tau}. \quad (11)$$

For any practical energy storage system the damping which occurs before the crowbar is applied must be small, and this can be ensured by demanding $\frac{\omega}{\gamma} \gg 1$. We define the quantity β by

$$\beta = \frac{\omega}{\gamma}. \quad (12)$$

Parameter β is another of the known input parameters and we assume $\beta \gg 10$. In some of the expressions to be derived, β^{-1} will be used as an expansion parameter and results will be carried out to order β^{-2} .

Peak voltage across C (or R) occurs when $\frac{di_R}{dt} = 0$ and this expression gives the equation

$$\cot\left(\frac{\pi}{2} - \omega t_{VC}\right) = \beta.$$

From the expansion

$$\cot^{-1}\beta = \frac{1}{\beta} - \frac{1}{3\beta^3} + \frac{1}{5\beta^5} - \frac{1}{7\beta^7} + \dots, \quad (13)$$

we find that to order $\frac{1}{\beta^2}$

$$t_{VC} = \frac{\tau}{2} \left(1 - \frac{2}{\pi\beta}\right). \quad (14)$$

Peak energy in L_E is given by

$$W_E = \frac{1}{2} L_E I_E^2, \quad (15)$$

where I_E is the peak current in L_E .

Similarly the energy originally stored in L_B is

$$W_B = \frac{1}{2} L_B I_B^2. \quad (16)$$

Their ratio is

$$\frac{W_E}{W_B} = \frac{\frac{L_E}{L_B}}{\left(1 + \frac{L_E}{L_B}\right)^2} \left(1 + e^{-\frac{\pi}{\beta}}\right)^2.$$

Then for any β the maximum energy transfer will occur when

$$L_E = L_B, \quad (17)$$

and we will require that $I_E = I_B$ for the rest of the calculation.

C. Values of R, C, L_E , and I_E

Two more input parameters must now be chosen and those selected are I_B (the current in L_B at $t = 0$), and V_C (the peak voltage across the network). From

$$V_C = \frac{1}{C} \int_0^{t_{VC}} i_C dt,$$

and the integrals

$$\int e^{-\gamma t} \cos \omega t = \frac{e^{-\gamma t}}{\gamma^2 + \omega^2} (-\gamma \cos \omega t + \omega \sin \omega t)$$

$$\int e^{-\gamma t} \sin \omega t = -\frac{e^{-\gamma t}}{\gamma^2 + \omega^2} (\gamma \sin \omega t + \omega \cos \omega t),$$

the result obtained is

$$V_C = \frac{I_B}{C\omega} e^{-\left(\frac{\pi}{2\beta} - \frac{1}{\beta^2}\right)} \cos\left(\frac{1}{\beta}\right)$$

$$C = \frac{I_B \tau}{V_C \pi} \left[1 - \frac{\pi}{2\beta} + \left(\frac{2}{8} + \frac{1}{2}\right) \frac{1}{\beta^2}\right]. \quad (18)$$

Resistor R can be found by substituting Eq. (18) into Eq. (9), and the result is

$$R = \frac{V_C}{2I_B} \beta \left[1 + \frac{\pi}{2\beta} + \left(\frac{2}{8} - \frac{1}{2}\right) \frac{1}{\beta^2}\right]. \quad (19)$$

Inductance L_E is found from substituting Eqs. (11), (12), (17), and (18) into Eq. (10). The result is

$$L_E = L_B = \frac{2\tau V_C}{\pi I_B} \left[1 + \frac{\pi}{2\beta} + \left(\frac{2}{8} - \frac{3}{2}\right) \frac{1}{\beta^2}\right]. \quad (20)$$

Peak current I_E comes from substituting $\tau = \frac{\pi}{\omega}$ into Eq. (5) and the result is

$$I_E = I_B \left(1 - \frac{\pi}{2\beta} + \frac{2}{4\beta^2}\right). \quad (21)$$

D. Energy Dissipation and Efficiency

From Eqs. (16) and (20) the energy stored originally is

$$W_B = \frac{\tau V_C I_B}{\pi} \left[1 + \frac{\pi}{2\beta} + \left(\frac{2}{8} - \frac{3}{2}\right) \frac{1}{\beta^2}\right]. \quad (22)$$

Similarly Eqs. (15) and (20) give

$$W_E = \frac{\tau V_C I_B}{\pi} \left[1 - \frac{\pi}{2\beta} + \left(\frac{3\pi^2}{8} - \frac{3}{2}\right) \frac{1}{\beta^2}\right]. \quad (23)$$

The energy lost in resistor R up to crowbar time

(Q_R) is found by substituting Eq. (19) into

$$Q_R = R \int_0^\tau i_R^2 dt = I_B V_C \tau \left(\frac{1}{8} - \frac{\pi}{2\beta^2} \right). \quad (24)$$

At $t = \tau$ no energy is left in capacitor C since the voltage across C is zero ($\frac{di_E}{dt} = 0$).

However, there is a small amount left in L_B which we call Q_B .

$$Q_B = \frac{1}{2} L_B i_B^2 (t = \tau) = \frac{\pi}{4} \tau V_C I_B \frac{1}{\beta^2} \quad (25)$$

Notice that Q_B is an order of magnitude smaller than Q_R . As a consistency check the energy conservation equation,

$$W_B = W_E + Q_R + Q_B,$$

can be seen to be satisfied through order β^{-2} . Defining the efficiency by

$$\epsilon = \frac{W_E}{W_B}, \quad (26)$$

and substituting Eqs. (22) and (23) gives

$$\epsilon = 1 - \frac{\pi}{\beta} + \frac{3\pi^2}{4\beta^2}. \quad (27)$$

E. Module Length, Number of Turns, and Magnetic Field Strength

As has been mentioned, the module length for L_E and L_B is assumed to be the same and is found from Eqs. (4) and (23).

$$\ell = \frac{V_C I_B \tau}{\pi \eta} \left[1 - \frac{\pi}{2\beta} + \left(\frac{3\pi^2}{8} - \frac{3}{2} \right) \frac{1}{\beta^2} \right]. \quad (28)$$

Equations (2), (20), and (28) determine the number of turns on each module of the experiment coil.

$$n_E = \frac{\tau V_C}{\pi} \sqrt{\frac{2}{\eta \ell}} \left[1 + \left(\frac{\pi^2}{8} - \frac{3}{2} \right) \frac{1}{\beta^2} \right]. \quad (29)$$

One more input parameter must be chosen to fix the number of turns and peak magnetic field in the storage coil. The radius of the storage coil in cm (r_B) is the chosen parameter. Equation (3) applied to the storage coil yields the magnetic field in gauss.

$$B_B = \frac{\sqrt{8 \times 10^7 \eta}}{r_B} \left(1 + \frac{\pi}{2\beta} \right). \quad (30)$$

The number of turns on the storage coil is then calculated from

$$B_B = \frac{2\pi}{5} \frac{n_B I_B}{\ell}. \quad (31)$$

We obtain

$$n_B = \frac{10^4 \sqrt{5}}{2} \frac{V_C \tau}{r_B \sqrt{\eta}} \left[1 + \left(\frac{\pi^2}{8} - \frac{3}{2} \right) \frac{1}{\beta^2} \right]. \quad (32)$$

F. Method for Obtaining Approximate Results

Approximate expressions for all the equations just laboriously derived can be found simply by assuming $t_{vc} = \tau/2$ and setting $e^{-\gamma t} = 1$ when calculating the various circuit parameters from Eqs. (5) through (8). This is equivalent to taking only the zero order terms in most of the expansions. For $\beta \geq 20$, errors $\leq 8\%$ will be made in L_E , R , I_E , and C . The dissipation Q_R will be $\leq 8\%$ too large, but the efficiency will be correct through order β^{-1} .

G. Summary

Table I summarizes the quantities which have been calculated from the given set of input parameters. These input parameters are: the initial current in the storage coil in amperes (I_B), the radius of the storage coil in cm (r_B), the peak voltage across the network in volts (V_C), the transfer time to the experiment in seconds (τ), the ratio of the damping constant to angular frequency (β), the energy delivered to the experiment per unit length in J/cm (η), and an inductance constant for the experiment coil in H-cm/turn² (ℓ). This latter constant depends only on the coil radius.

TABLE I
CALCULATED PARAMETERS FOR A DIRECT COUPLED MAGNETIC ENERGY STORAGE SYSTEM

<u>Parameter</u>	<u>Equation</u>
Time of peak voltage (sec)	$t_{VC} = \frac{\tau}{2} \left(1 - \frac{2}{\pi\beta} \right)$
Capacitance (F)	$C = \frac{I_B \tau}{\pi V_C} \left[1 - \frac{\pi}{2\beta} + \left(\frac{\pi^2}{8} + \frac{1}{2} \right) \frac{1}{\beta^2} \right]$
Resistance (Ω)	$R = \frac{V_C \beta}{2 I_B} \left[1 + \frac{\pi}{2\beta} + \left(\frac{\pi^2}{8} - \frac{1}{2} \right) \frac{1}{\beta^2} \right]$
Inductance (H)	$L_B = L_E = \frac{2\tau V_C}{\pi I_B} \left[1 + \frac{\pi}{2\beta} + \left(\frac{\pi^2}{8} - \frac{3}{2} \right) \frac{1}{\beta^2} \right]$
Peak Current in L_E (A)	$I_E = I_B \left(1 - \frac{\pi}{2\beta} + \frac{\pi^2}{4\beta^2} \right)$
Energy transferred to L_E (J)	$W_E = \frac{\tau V_C I_B}{\pi} \left[1 - \frac{\pi}{2\beta} + \left(\frac{3\pi^2}{8} - \frac{3}{2} \right) \frac{1}{\beta^2} \right]$
Energy lost in R up to $t = \tau$ (J)	$Q_R = I_B V_C \tau \left(\frac{1}{\beta} - \frac{\pi}{2\beta^2} \right)$
Energy stored in L_B (J)	$W_B = \frac{\tau V_C I_B}{\pi} \left[1 + \frac{\pi}{2\beta} + \left(\frac{\pi^2}{8} - \frac{3}{2} \right) \frac{1}{\beta^2} \right]$
Energy left in L_B at $t = \tau$ (J)	$Q_B = \frac{\pi \tau V_C I_B}{4\beta^2}$
Efficiency	$\epsilon = 1 - \frac{\pi}{\beta} + \frac{3\pi^2}{4\beta^2}$
Module length (cm)	$\ell = \frac{V_C I_B \tau}{\pi \eta} \left[1 - \frac{\pi}{2\beta} + \left(\frac{3\pi^2}{8} - \frac{3}{2} \right) \frac{1}{\beta^2} \right]$
Number of turns on L_E	$n_E = \frac{\tau V_C}{\pi} \sqrt{\frac{2}{\eta \ell}} \left[1 + \left(\frac{\pi^2}{8} - \frac{3}{2} \right) \frac{1}{\beta^2} \right]$
Number of turns on L_B	$n_B = \frac{10^4 \sqrt{5}}{\pi^2} \frac{V_C \tau}{r_B \sqrt{\eta}} \left[1 + \left(\frac{\pi^2}{8} - \frac{3}{2} \right) \frac{1}{\beta^2} \right]$
Field of L_B (G)	$B_B = \frac{\sqrt{8 \times 10^7} \eta}{r_B} \left(1 + \frac{\pi}{2\beta} \right)$

IV. TRANSFORMER COUPLING OF STORAGE COIL AND EXPERIMENT WITH A TRANSFER CAPACITOR ACROSS THE TRANSFORMER SECONDARY

A. Currents in the Circuit Elements

1. Network Equations. The network equation for circuit (b) of Fig. 2 can be written as:

$$\begin{aligned} L_P \frac{di_P}{dt} + k \sqrt{L_P L_S} \frac{di_S}{dt} + i_P R &= 0, \\ L_S \frac{di_S}{dt} + L_E \frac{di_E}{dt} + k \sqrt{L_P L_S} \frac{di_P}{dt} &= 0, \\ CL_E \frac{d^2 i_E}{dt^2} + i_E - i_S &= 0. \end{aligned} \quad (33)$$

If $k = 1$, solution of these equations requires solving a second order differential equation, but if $k < 1$ the differential equation is third order. We take the general case $k \leq 1$ and rewrite Eqs. (33) as

$$Di_P = 0, \quad (34)$$

$$Di_E = \frac{kRI_P \sqrt{L_P L_S}}{CL_E} \quad (35)$$

$$Di_S = \frac{kRI_P}{C} \sqrt{L_P L_S} \left(\frac{1}{L_T} - \frac{1}{L_S} \right), \quad (36)$$

where

$$\begin{aligned} D &= L_P L_S (1-k^2) \frac{d^3}{dt^3} + RL_S \frac{d^2}{dt^2} \\ &+ \left[\frac{k^2 L_P}{C} + \frac{L_P L_S}{CL_T} (1-k^2) \right] \frac{d}{dt} + \frac{RL_S}{CL_T}, \end{aligned}$$

and

$$L_T = \frac{L_S L_E}{L_S + L_E}. \quad (37)$$

The quantity I_P is the current in L_P at $t = 0$. We define the parameters α and ρ by

$$\alpha = k^2 \frac{L_P}{L_S}, \quad (38)$$

and

$$\rho = \frac{R}{L_P (1-k^2)}. \quad (39)$$

The proper initial values at $t = 0$ for Eqs. (33) through (36) are $i_P = I_P$, $i_S = i_E = 0$,

$$\frac{di_P}{dt} = -I_P \rho, \quad \frac{di_S}{dt} = \sqrt{2} I_P \rho, \quad \frac{di_E}{dt} = 0, \quad \frac{d^2 i_P}{dt^2} = I_P \rho^2,$$

$$\frac{d^2 i_S}{dt^2} = -\sqrt{2} \rho^2 I_P, \quad \text{and} \quad \frac{d^2 i_E}{dt^2} = 0.$$

2. Solution of a Cubic Equation.⁴ The auxiliary equation connected with Eqs. (34) through (36) will be a cubic equation with one real root and two complex roots. If the auxiliary cubic equation is

$$z^3 + a_2 z^2 + a_1 z + a_0 = 0,$$

and we define

$$q = \frac{1}{3} a_1 - \frac{1}{9} a_2^2,$$

$$r = \frac{1}{6} (a_1 a_2 - 3a_0) - \frac{1}{27} a_3^2,$$

$$s_1 = \left[r + (q^3 + r^2)^{1/2} \right]^{1/3},$$

and

$$s_2 = \left[r - (q^3 + r^2)^{1/2} \right]^{1/3}, \quad (40)$$

Then the roots are given by

$$z_1 = (s_1 + s_2) - \frac{a_2}{3}, \quad (41)$$

$$z_2 = -\frac{1}{2} (s_1 + s_2) - \frac{a_2}{3} + i \frac{\sqrt{3}}{2} (s_1 - s_2), \quad (42)$$

and

$$z_3 = -\frac{1}{2} (s_1 + s_2) + \frac{a_2}{3} - i \frac{\sqrt{3}}{2} (s_1 - s_2). \quad (43)$$

Another useful relation will be

$$S_1 S_2 = -q. \quad (44)$$

3. Solution of the Network Equations. The general solution of Eqs. (34) to (36) is of the form

$$i = b_1 e^{-\gamma_0 t} + b_2 e^{-\gamma t} \sin \omega t + b_3 e^{-\gamma t} \cos \omega t + b_4 \quad (45)$$

where

$$\gamma_0 = S_1 + S_2 - \frac{a_2}{3}, \quad (46)$$

$$\gamma = \frac{1}{2} (S_1 + S_2) + \frac{a_2}{3}, \quad (47)$$

and

$$\omega = \frac{\sqrt{3}}{2} (S_1 - S_2). \quad (48)$$

Since the quantity a_2 is identified in Eq. (33) as

$$a_2 = \rho = \frac{R}{L_P(1-k^2)}, \quad (49)$$

we have

$$\gamma_0 = 2\gamma - \rho. \quad (50)$$

Forcing Eq. (45) to satisfy Eqs. (34) through (36) and the appropriate initial values gives expressions for the currents:

$$i_P = (I_P - H) e^{(2\gamma - \rho)t} + \frac{H(3\gamma - \rho) - 2\gamma I_P}{\omega} e^{-\gamma t} \sin \omega t + H e^{-\gamma t} \cos \omega t, \quad (51)$$

$$i_S = - \left[E + \sqrt{\alpha} I_P \left(1 - \frac{L_T}{L_S} \right) \right] e^{(2\gamma - \rho)t} + \frac{(3\gamma - \rho)E + 2\gamma \sqrt{\alpha} I_P - \sqrt{\alpha} I_P (2\gamma - \rho) \frac{L_T}{L_S}}{\omega} e^{-\gamma t} \sin \omega t + E e^{-\gamma t} \cos \omega t + \sqrt{\alpha} I_P \left(1 - \frac{L_T}{L_S} \right), \quad (52)$$

$$i_E = - \left(F + \frac{L_T}{L_E} \sqrt{\alpha} I_P \right) e^{(2\gamma - \rho)t} + \frac{(3\gamma - \rho)F + \frac{L_T}{L_E} \sqrt{\alpha} I_P (2\gamma - \rho)}{\omega} e^{-\gamma t} \sin \omega t + F e^{-\gamma t} \cos \omega t + \sqrt{\alpha} I_P \frac{L_T}{L_E}, \quad (53)$$

where

$$H = \frac{4\gamma(2\gamma - \rho) I_P}{(3\gamma - \rho)^2 + \omega^2},$$

$$E = \frac{(2\gamma - \rho) \left[(4\gamma - \rho) \frac{L_T}{L_S} - 4\gamma \right] \sqrt{\alpha} I_P}{(3\gamma - \rho)^2 + \omega^2},$$

and

$$F = - \frac{L_T}{L_E} I_P \sqrt{\alpha} \frac{(4\gamma - \rho)(2\gamma - \rho)}{(3\gamma - \rho)^2 + \omega^2}.$$

B. Ratio of L_E/L_S for Maximum Transfer Efficiency

The value of ω is found by substituting $t = \tau$

into $\frac{di_E}{dt} = 0$. The expression for $\frac{di_E}{dt} = 0$ is made up of $e^{(2\gamma - \rho)t}$, $e^{-\gamma t} \cos \omega t$, and $e^{-\gamma t} \sin \omega t$ terms since i_E contains these terms. Coefficients of the $e^{(2\gamma - \rho)t}$ and $e^{-\gamma t} \cos \omega t$ terms are identical, while the coefficient of the $e^{-\gamma t} \sin \omega t$ term is β^{-1} of the other two in order of magnitude. Parameter β is again defined by

$$\beta = \frac{\omega}{\gamma} \quad (54)$$

with $\beta \geq 10$. It will be shown later that we can write ρ as

$$\rho = \frac{4(1-k^2)}{k^2} \beta \omega (1 + \text{Order } \beta^{-2}). \quad (55)$$

With $\omega = \frac{\pi}{\tau}$ to zero order, the factor $e^{(2\gamma - \rho)\tau}$ is

roughly $e^{-\beta}$ and negligible compared to the $e^{-\gamma\tau} \sin \omega\tau$ and $e^{-\gamma\tau} \cos \omega\tau$ terms. In finding expressions for the various circuit parameters, an expansion in β^{-1} will be carried out to order β^{-1} . The two remaining terms in $\frac{di_E}{dt} = 0$ yield the relation

$$\cot \omega\tau = \frac{\rho-3\gamma}{\omega}, \quad (56)$$

which can be expanded to give

$$\omega = \frac{\pi}{\tau} \left(1 + \frac{4f}{\pi\beta} \right), \quad (57)$$

with

$$f = \frac{(1-k^2)}{k^2}. \quad (58)$$

With C , R , I_P , L_P , k , and L_S held constant, the value of L_E to be selected occurs when

$$W_E = \frac{1}{2} L_E i_E^2 \quad (t = \tau) \quad (59)$$

is maximized. Equivalently $\sqrt{L_E} i_E$ can be maximized. From Eqs. (53) and (56)

$$\sqrt{L_E} i_E = \sqrt{\alpha I_P} \frac{L_P}{\sqrt{L_E}} \left[1 - \frac{(2\gamma-\rho)}{(3\gamma-\rho)} e^{-\gamma\tau} \sin \omega\tau \right]$$

is a maximum when

$$L_S = L_E. \quad (60)$$

C. Expressions for the Currents when $L_S = L_E$

We assume from now on the $L_S = L_E$ and rewrite Eqs. (51) to (53) using this simplification.

$$\begin{aligned} i_P = I_P \left[\frac{(\gamma-\rho)^2 + \omega^2}{(3\gamma-\rho)^2 + \omega^2} e^{(2\gamma-\rho)t} + \frac{2\gamma}{\omega} \right. \\ \cdot \frac{(3\gamma-\rho)(\gamma-\rho) - \omega^2}{(3\gamma-\rho)^2 + \omega^2} e^{-\gamma t} \sin \omega t \\ \left. + \frac{4\gamma(2\gamma-\rho)}{(3\gamma-\rho)^2 + \omega^2} e^{-\gamma t} \cos \omega t \right], \quad (61) \end{aligned}$$

$$\begin{aligned} i_S = \frac{\sqrt{\alpha I_P}}{2} \left[1 - \frac{2\rho^2 + \gamma(\gamma-4\rho) + \omega^2}{(3\gamma-\rho)^2 + \omega^2} e^{(2\gamma-\rho)t} \right. \\ \left. - \frac{1}{\omega} \cdot \frac{\gamma(3\rho-2\gamma)(\rho-3\gamma) - (2\gamma+\rho)\omega^2}{(3\gamma-\rho)^2 + \omega^2} e^{-\gamma t} \sin \omega t \right. \\ \left. + \frac{(\rho-2\gamma)(\rho+4\gamma)}{(3\gamma-\rho)^2 + \omega^2} e^{-\gamma t} \cos \omega t \right], \quad (62) \end{aligned}$$

$$\begin{aligned} i_E = \frac{\sqrt{\alpha I_P}}{2} \left[1 - \frac{\gamma^2 + \omega^2}{(3\gamma-\rho)^2 + \omega^2} e^{(2\gamma-\rho)t} - \frac{1}{\omega} \right. \\ \cdot \frac{(\rho-2\gamma)(\rho\gamma-3\gamma^2 + \omega^2)}{(3\gamma-\rho)^2 + \omega^2} e^{-\gamma t} \sin \omega t \\ \left. - \frac{(\rho-4\gamma)(\rho-2\gamma)}{(3\gamma-\rho)^2 + \omega^2} e^{-\gamma t} \cos \omega t \right], \quad (63) \end{aligned}$$

$$\begin{aligned} i_C = \sqrt{\alpha I_P} \left[\frac{\rho(2\gamma-\rho)}{(3\gamma-\rho)^2 + \omega^2} e^{(2\gamma-\rho)t} - \frac{\rho}{\omega} \right. \\ \cdot \frac{\gamma(\rho-3\gamma) - \omega^2}{(3\gamma-\rho)^2 + \omega^2} e^{-\gamma t} \sin \omega t \\ \left. + \frac{\rho(\rho-2\gamma)}{(3\gamma-\rho)^2 + \omega^2} e^{-\gamma t} \cos \omega t \right]. \quad (64) \end{aligned}$$

Figure 4 shows current in the various circuit elements as a function of time for one choice of circuit parameters. A turns ratio of 10 between the secondary and primary has amplified the peak current in the secondary circuit by about a factor of 10 over the peak current in the primary circuit. Another noteworthy point is the large current spike in i_P just after the switch is opened. This current spike will produce a high voltage transient across R and also result in increased energy dissipation in R as shown in Fig. 5. Cause of the spike is the uncoupled inductance ($k = 0.95$) in the primary circuit, and the increased dissipation is essentially the energy stored in the uncoupled inductance. Both Figs. 4 and 5 were generated by numerical integration of Eqs. (33).

Fig. 4. Current in the circuit elements as a function of time for a transformer coupled system with a transfer capacitor across the load. Circuit parameters are $R = 160 \Omega$, $I_P = 25 \text{ kA}$, $L_P = 20.4 \text{ mH}$, $L_S = L_E = 204 \mu\text{H}$, $C = 4 \text{ mF}$, and $k = 0.95$. A current equal to $10 i_P$ has been plotted to expand the scale for i_P .

D. Calculation of t_{VC} , t_{VR} , R , and I_E

The time of maximum voltage across capacitor C is denoted by t_{VC} and is found by setting Eq. (64) for i_C equal to zero and solving for t_{VC} . The exponential term $e^{(2\gamma-\rho)t}$ is again neglected, and the result is

$$t_{VC} = \frac{\tau}{2} \left[1 + \left(\frac{4f}{\pi} - \frac{2}{\pi\beta} \right) \frac{1}{\beta} \right] \quad (65)$$

Fig. 5. Total energy loss in R as a function of time for the circuit described in Fig. 4.

Highest voltage across R occurs at $t = 0$ for coupling constant $k < 1$, but after dipping to almost zero it builds up to a secondary maximum as seen for example in Fig. 4 at about $t = 1 \text{ msec}$. Time of the secondary maximum (t_{VP}) is found by setting Eq. (61) for i_P to zero and solving for t_{VP} .

$$t_{VP} = \frac{\tau}{2} \left[1 + \left(\frac{64}{\pi} f^2 - \frac{2}{\pi} - \frac{4f}{\pi} \right) \frac{1}{\beta} \right] \quad (66)$$

Using Eq. (51), the voltage across R at $t = t_{VP}$ is $V_P = R i_P(t = t_{VP})$. Actually V_P is the given input parameter and this equation is used to find R .

$$R = \frac{V_P}{2I_P} \left(\beta + \frac{\pi}{2} \right) \quad (67)$$

Voltage across R at $t = 0$ is given by

$$V_P' = \frac{V_P}{2} \left(\beta + \frac{\pi}{2} \right) \quad (68)$$

Peak current in L_E comes from Eq. (63) evaluated at $t = \tau$ and is given by

$$I_E = \sqrt{\alpha} I_P \left(1 - \frac{\pi}{2\beta} \right) \quad (69)$$

E. Calculation of V_C , C , L_P , and L_E

Peak voltage across capacitor C is denoted by V_C and occurs at $t = t_{VC}$. In order to find V_C , we must relate V_C to V_0 (the voltage across R at $t = t_{VC}$); and V_0 in turn must be related to V_P (the voltage across R at $t = t_{VP}$). By using Eq. (61) at the two different times, it can be shown that $V_P = V_0$, through order β^{-1} at least. Then V_P is related to V_C by

$$V_P = \frac{R i_P(t = t_{VP})}{L_E \frac{di_E(t = t_{VC})}{dt}}$$

In this equation the currents and their derivatives come from Eqs. (61) and (65), while R/L_E is eliminated with Eqs. (38), (39), and (60). The final expression for V_C , good to order β^{-1} which vanishes, is

$$V_C = \frac{V_P}{\sqrt{\alpha}} \quad (70)$$

Capacitance C can be related to V_C by

$$V_C = \frac{1}{C} \int_0^{t_{VC}} i_C dt.$$

Performing the integral and using Eq. (70) yields

$$C = \frac{\alpha T_P \tau}{\pi V_P} \left[1 - \left(\frac{\pi}{2} + \frac{4f}{\pi} \right) \frac{1}{\beta} \right]. \quad (71)$$

In order to find L_P , we first use Eqs. (47), (48), and (49) to obtain

$$\frac{1}{9} \left[(3\gamma - \rho)^2 - 3\omega^2 \right] = S_1 S_2. \quad (72)$$

Eliminating $S_1 S_2$ between Eqs. (72) and (44) finally gives an equation in which L_P can be solved for in terms of quantities already calculated. The result is

$$L_P = \frac{2V_P \tau}{k^2 I_P \pi} \left[1 + \left(\frac{\pi}{2} - \frac{4f}{\pi} \right) \frac{1}{\beta} \right]. \quad (73)$$

Equations (38) and (60) then give L_S and L_E in terms of L_P . These expressions are given in Table II. With L_P known, Eq. (55), which was assumed to be true can now be checked by direct substitution and found to be consistent.

F. Energy Dissipation and Efficiency

Using Eq. (73) the energy originally stored in L_P is given by

$$\begin{aligned} W_P &= \frac{1}{2} L_P I_P^2 \\ &= \frac{I_P V_P \tau}{\pi k} \left[1 + \left(\frac{\pi}{2} - \frac{4f}{\pi} \right) \frac{1}{\beta} \right]. \end{aligned} \quad (74)$$

Energy delivered to the experiment is

$$\begin{aligned} W &= \frac{1}{2} L_E I_E^2 \\ &= \frac{I_P V_P \tau}{\pi} \left[1 - \left(\frac{\pi}{2} + \frac{4f}{\pi} \right) \frac{1}{\beta} \right]. \end{aligned} \quad (75)$$

Transfer efficiency is given by

$$\begin{aligned} \epsilon &= \frac{W_E}{W_P} \\ &= k^2 \left(1 - \frac{\pi}{\beta} \right). \end{aligned} \quad (76)$$

It is instructive to look at the energy loss in resistor R in more detail. The total loss (Q_{RT}) can be approximately divided into two parts, with the first part (Q_{UN}) due to the uncoupled inductance in the primary circuit, while the second part (Q_R) is due to the voltage induced in the primary circuit by changing current in L_S . The total loss is given by

$$Q_{RT} = (1 - k^2) \frac{V_P I_P \tau}{\pi k^2} \left[1 + \left(\frac{\pi}{2} - \frac{4f}{\pi} \right) \frac{1}{\beta} \right] + \frac{V_P I_P \tau}{\beta}, \quad (77)$$

where the first term on the right is identified with Q_{UN} and the second is identified with Q_R . To order β^{-1} it can be seen that

$$Q_{UN} = \frac{1}{2} (1 - k^2) L_P I_P^2.$$

There is some difficulty in carrying this division of energy loss to order β^{-2} because cross terms occur in i_P^2 between the $e^{(2\gamma - \rho)\tau}$ term and the oscillatory terms. These cross terms are of order β^{-2} and could reasonably be included in either Q_{UN} or Q_R . Including these terms in Q_R allows Q_R to be carried one order farther.

$$Q_R = \frac{V_P I_P \tau}{\beta} \left[1 - \frac{1}{\beta} \left(\frac{\pi}{2} + \frac{4f}{\pi} + \frac{32f^2}{\pi} \right) \right].$$

Since the β^{-2} terms in Q_{UN} originating from R have not been calculated, we cannot tell if Q_{RT} is more accurate including the β^{-2} term in Q_R or without it. Energy remaining in L_S at $t = \tau$ can be shown to be of order β^{-2} , and energy in L_P and in the mutual inductance belongs to still higher order.

G. Module Length, Number of Turns, and Magnetic Field Strength

From Eqs. (4) and (75) the module length is

$$L = \frac{I_P V_P \tau}{\pi \eta} \left[1 - \left(\frac{\pi}{2} + \frac{4f}{\pi} \right) \frac{1}{\beta} \right]. \quad (78)$$

TABLE II

CALCULATED PARAMETERS FOR A TRANSFORMER COUPLED MAGNETIC ENERGY STORAGE SYSTEM WITH A TRANSFER CAPACITOR ACROSS THE TRANSFORMER SECONDARY

Parameter	Equation
Coupling constant	$k = \frac{r_p}{r_p + \Delta}$
Parameter f	$f = \frac{1 - k^2}{k^2}$
Time of peak voltage across C (sec)	$t_{VC} = \frac{\tau}{2} \left[1 + \left(\frac{4f}{\pi} - \frac{2}{\pi} \right) \frac{1}{\beta} \right]$
Time of highest voltage across R excluding initial transient (sec)	$t_{VR} = \frac{\tau}{2} \left[1 + \left(\frac{64}{\pi} f^2 - \frac{\pi}{2} - \frac{4f}{\pi} \right) \frac{1}{\beta} \right]$
Capacitance (F)	$C = \frac{\pi I_P \tau}{2V_P} \left[1 - \left(\frac{\pi}{2} + \frac{4f}{\pi} \right) \frac{1}{\beta} \right]$
Resistance (Ω)	$R = \frac{V_P}{2I_P} \beta \left[1 + \frac{\pi}{2\beta} \right]$
Inductance of L_P (H)	$L_P = \frac{2V_P \tau}{\pi k I_P} \left[1 + \left(\frac{\pi}{2} - \frac{4f}{\pi} \right) \frac{1}{\beta} \right]$
Inductance of L_E and L_S (H)	$L_E = L_S = \frac{2V_P \tau}{\pi I_P} \left[1 + \left(\frac{\pi}{2} - \frac{4f}{\pi} \right) \frac{1}{\beta} \right]$
Peak voltage across C (V)	$V_C = \frac{V_P}{\sqrt{2}}$
Peak current in L_E (A)	$I_E = \sqrt{2} I_P \left(1 - \frac{\pi}{2\beta} \right)$
Energy transferred to L_E (J)	$W_E = \frac{I_P V_P \tau}{\pi} \left[1 - \left(\frac{\pi}{2} + \frac{4f}{\pi} \right) \frac{1}{\beta} \right]$
Energy in uncoupled inductance (J)	$Q_{UN} = \frac{(1 - k^2) V_P I_P \tau}{\pi k^2} \left[1 + \left(\frac{\pi}{2} - \frac{4f}{\pi} \right) \frac{1}{\beta} \right]$
Energy lost in R excluding Q_{UN} (J)	$Q_R = \frac{V_P I_P \tau}{\beta}$
Energy stored in L_P (J)	$W_P = \frac{I_P V_P \tau}{\pi k^2} \left[1 + \left(\frac{\pi}{2} - \frac{4f}{\pi} \right) \frac{1}{\beta} \right]$
Efficiency	$\epsilon = k^2 \left(1 - \frac{\pi}{\beta} \right)$
Module length (cm)	$l = \frac{I_P V_P \tau}{\pi^2} \left[1 - \left(\frac{\pi}{2} + \frac{4f}{\pi} \right) \frac{1}{\beta} \right]$
Number of turns on L_E	$n_E = \frac{V_P \tau}{\pi} \sqrt{\frac{2}{\alpha \mu_0 \mu_r}} \left(1 - \frac{4f}{\pi} \right)$

TABLE II (continued)

Radius of secondary (cm)	$r_S = r_P + \Delta$
Number of turns on L_S	$n_S = \frac{\sqrt{5} \times 10^4}{\pi} \cdot \frac{V_P \tau}{(r_P + \Delta) \sqrt{\alpha \eta}} \left(1 - \frac{4f}{\pi \beta}\right)$
Initial field in L_P (G)	$B_P = \frac{\sqrt{8 \times 10^7 \eta}}{r_P^k} \left(1 + \frac{\pi}{2\beta}\right)$
Number of turns on L_P	$n_P = \frac{\sqrt{5} \times 10^4}{\pi} \cdot \frac{V_P \tau}{r_P^k \sqrt{\eta}} \left(1 - \frac{4f}{\pi \beta}\right)$
Peak voltage across L_P in transient (V)	$V_P' = \frac{\beta V_P}{2} \left(1 + \frac{\pi}{2\beta}\right)$

The number of turns on the experiment coil is found from Eqs. (2), (38), and (73) and is given by

$$n_E = \frac{V_P \tau}{\pi} \sqrt{\frac{2}{\alpha \eta}} \left(1 - \frac{4f}{\pi \beta}\right). \quad (79)$$

Again at this point another input parameter must be chosen, and the parameter chosen is r_P , the radius of the storage coil in cm. The secondary coil radius can now be found from

$$\frac{r_S}{r_P} = \frac{1}{k}.$$

Alternatively, if the difference between r_S and r_P is taken to be a constant (Δ), then

$$r_S = r_P + \Delta, \quad (80)$$

and the coupling constant can be calculated from

$$k = \frac{r_P}{r_P + \Delta}. \quad (81)$$

When r_S is known, the number of turns on the secondary (n_S) is determined because

$$\frac{n_S}{n_E} = \frac{r_E}{r_S}. \quad (82)$$

Substituting Eqs. (2), (79), and (80) into Eq. (82) gives

$$n_S = \frac{\sqrt{5} \times 10^4 V_P \tau}{\pi^2 (r_P + \Delta) \sqrt{\alpha \eta}} \left(1 - \frac{4f}{\pi \beta}\right). \quad (83)$$

Initial magnetic field in the primary is calculated from Eq. (3) and given by

$$B_P = \frac{\sqrt{8 \times 10^7 \eta}}{k r_P} \left(1 + \frac{\pi}{2\beta}\right). \quad (84)$$

The number of turns on the storage coil (n_P) can be found using Eq. (31), and the result is

$$n_P = \frac{\sqrt{5} \times 10^4 V_P \tau}{\pi^2 r_P^k \sqrt{\eta}} \left(1 - \frac{4f}{\pi \beta}\right). \quad (85)$$

H. Method for Obtaining Approximate Results

Setting $k = 1$ in Eqs. (33) reduces the order of the differential equations to be solved from third order to second order, and the problem is then similar to the direct coupled case. In solving the second order equations, initial values to be taken at $t = 0$ are: $i_E = i_P = 0$, $i_S = \sqrt{\alpha} i_P$, and

$\frac{di_E}{dt} = \frac{di_P}{dt} = \frac{di_S}{dt} = 0$. Resulting equations for the currents are:

$$i_E = \sqrt{\alpha} I_P \frac{L_S}{L_S + L_E} \left(1 - e^{-\gamma t} \cos \omega t - \frac{\gamma}{\omega} e^{-\gamma t} \sin \omega t \right),$$

$$i_S = \sqrt{\alpha} I_P \frac{L_E}{L_S + L_E} \left[\frac{L_S}{L_E} + e^{-\gamma t} \cos \omega t - \frac{\gamma}{\omega} \left(1 + \frac{2L_S}{L_E} \right) e^{-\gamma t} \sin \omega t \right],$$

$$i_C = \sqrt{\alpha} I_P \left(e^{-\gamma t} \cos \omega t - \frac{\gamma}{\omega} e^{-\gamma t} \sin \omega t \right),$$

and

$$i_P = 2 I_P \frac{\gamma}{\omega} e^{-\gamma t} \sin \omega t$$

where

$$\gamma = \frac{\alpha}{2RC}$$

and

$$\omega = \sqrt{\frac{L_E + L_S}{L_E L_S C} - \gamma^2}.$$

If the additional approximation $e^{-\gamma t} = 1$ is made, the expressions for the various circuit parameters become especially simple. Maximum energy transfer to L_E occurs again when $L_E = L_S$.

Heuristic arguments can then extend the results to the $k < 1$ case. Energy stored in the uncoupled primary inductance is

$$Q_{UN} = \frac{1}{2} (1 - k^2) L_P I_P^2,$$

and this energy is dumped in R as an initial transient. Energy coupled through the transformer is

$$W_{CO} = \frac{1}{2} k^2 L_P I_P^2$$

which suggests that the ratio of both Q_R and W_E to W_P must be reduced by a factor k^2 in going to the $k < 1$ case. Finally, the factor k^2 must be included in the definition of α so that for all expressions

$$\alpha = k^2 \frac{L_P}{L_S}.$$

The quantity $\sqrt{\alpha}$ must be used instead of $\sqrt{\frac{L_P}{L_S}}$

to compute primary voltage from secondary voltage, etc.

I. Summary

Table II summarizes the quantities which have been calculated for circuit (b) of Fig. 2 from the given set of input parameters. These input parameters are: the initial current in the primary in amperes (I_P), the radius of the primary in cm (r_P), the difference between the primary and secondary radius in cm (Δ), the maximum voltage across the primary excluding the initial transient (V_P), the transfer time to the experiment in seconds (τ), the ratio of the damping constant to angular frequency (β), a constant equal to the calculated coupling constant squared times the inductance ratio of the primary and secondary (α), the energy delivered to the experiment per unit length in J/cm ($\mathcal{E}$), and an inductance constant for the experiment coil in H-cm/turn² (ξ).

V. TRANSFORMER COUPLING OF STORAGE COIL AND EXPERIMENT WITH A TRANSFER CAPACITOR ACROSS THE TRANSFORMER PRIMARY

A. Circuit Parameters

The network equations for circuit (c) of Fig. 2 can be written as:

$$(L_E + L_S) \frac{di_S}{dt} + k \sqrt{L_P L_S} \frac{di_P}{dt} = 0,$$

$$L_P \frac{di_P}{dt} + R i_R + k \sqrt{L_P L_S} \frac{di_S}{dt} = 0,$$

and

$$CR \frac{di_R}{dt} - i_P + i_R = 0. \quad (86)$$

If we take $L_E = L_S$, the currents in the various circuit elements can be written as

$$i_P = I_P e^{-\gamma t} (\cos \omega t + \frac{\gamma}{\omega} \sin \omega t)$$

$$i_R = 2I_p \frac{\gamma}{x} e^{-\gamma t} \sin xt,$$

$$i_S = \frac{\sqrt{2}I_p}{2} \left(1 - e^{-\gamma t} \cos xt - \frac{\gamma}{x} e^{-\gamma t} \sin xt \right),$$

and

$$i_C = I_p \left(e^{-\gamma t} \cos xt - \frac{\gamma}{x} e^{-\gamma t} \sin xt \right), \quad (S7)$$

where

$$\gamma = \frac{1}{2RC},$$

$$x = \sqrt{\frac{1}{L_p C \left(1 - \frac{k^2}{2} \right)} - \gamma^2},$$

and

$$x = k^2 \frac{L_p}{L_S}.$$

Peak current occurs in L_E when $t = \frac{\pi}{x}$. Details of the remainder of the calculation are similar to those of the direct coupled circuit, and so only the results are given in Table III. The definitions of the input parameters are the same in Table III as in Table II. Expansions in β^{-1} are carried out to order β^{-2} .

B. Uncoupled Inductance

Energy loss from the uncoupled inductance requires some explanation because it takes a different form in circuit (c) than in circuit (b). Figure 6 shows the current in the various circuit elements as a function of time in circuit (c) for a particular choice of parameters. Although $k = 0.95$, the initial voltage transient across R found in circuit (b) does not occur nor does Fig. 7 show any transient energy loss. When the crowbar is applied at $t = \tau$, the uncoupled energy is stored in the transformer. The capacitor is completely discharged at that time. For $t > \tau$, there will be oscillations between C and the uncoupled inductance, with most of the uncoupled energy gradually lost in R and lead resistance. A second crowbar across capacitor C might be useful to damp the oscillations and prevent energy loss in R

Fig. 6. Current in the circuit elements as a function of time for a transformer coupled system with a transfer capacitor across the storage coil. Circuit parameters are $R = 16 \Omega$, $I_p = 25$ kA, $L_p = 2$ mH, $L_S = L_E = 20$ μ H, $C = 0.4$ mF, and $k = 0.95$. A current equal to $0.1 i_E$ has been plotted to fit it on the graph.

and the dewar leads. Referring to Fig. 6 again, it should be noted that when the crowbar is applied there are high currents flowing in both L_p and L_S . The direction of the currents is such that the magnetic fields tend to cancel and reduce the amount of stored energy.

The energy not coupled into L_E at $t = \tau$ is listed in Table II as Q_{UN} . It requires a more careful examination to see what part of this energy is lost in R and what part is coupled into the secondary. If the crowbar switch is assumed to have no resistance and is closed at $t = \tau$, the energy loss in R as $t \rightarrow \infty$ can be shown to be

TABLE III

CALCULATED PARAMETERS FOR A TRANSFORMER COUPLED MAGNETIC ENERGY STORAGE SYSTEM WITH A TRANSFER CAPACITOR ACROSS THE TRANSFORMER PRIMARY

Parameter	Equation
Coupling constant	$k = \frac{\tau_P}{\tau_P + \Delta}$
Time of peak voltage across L_P (sec)	$t_{VP} = \frac{\tau}{2} \left(1 - \frac{2}{\pi\beta} \right)$
Time of peak voltage across L_E (sec)	$t_{VE} = \frac{\tau}{2} \left(1 - \frac{2}{\pi\beta} \right)$
Capacitance (F)	$C = \frac{I_P \tau}{V_P \pi} \left[1 - \frac{\pi}{2\beta} + \left(\frac{\pi^2}{8} + \frac{1}{2} \right) \frac{1}{\beta^2} \right]$
Resistance (Ω)	$R = \frac{V_P \beta}{2I_P} \left[1 + \frac{\pi}{2\beta} + \left(\frac{\pi^2}{8} - \frac{1}{2} \right) \frac{1}{\beta^2} \right]$
Inductance of L_P (H)	$L_P = \frac{2\tau V_P}{(2 - k^2)\pi I_P} \left[1 + \frac{\pi}{2\beta} + \left(\frac{\pi^2}{8} - \frac{3}{2} \right) \frac{1}{\beta^2} \right]$
Inductance of L_S and L_E (H)	$L_S = L_E = \frac{2k^2 \tau V_P}{(2 - k^2)\pi \alpha I_P} \left[1 + \frac{\pi}{2\beta} + \left(\frac{\pi^2}{8} - \frac{3}{2} \right) \frac{1}{\beta^2} \right]$
Peak voltage across L_E (V)	$V_E = \frac{k^2 V_P}{(2 - k^2)\sqrt{\alpha}}$
Peak current in L_E (A)	$I_E = \sqrt{\alpha} I_P \left(1 - \frac{\pi}{2\beta} + \frac{\pi^2}{4\beta^2} \right)$
Energy transferred to L_E (J)	$W_E = \frac{k^2 \tau V_P I_P}{(2 - k^2)\pi} \left[1 - \frac{\pi}{2\beta} + \left(\frac{3\pi^2}{8} - \frac{3}{2} \right) \frac{1}{\beta^2} \right]$
Energy not coupled into L_E at $t = \tau$ (J)	$Q_{UN} = \frac{(1 - k^2) \tau V_P I_P}{(2 - k^2)\pi} \left[1 - \frac{3\pi}{2\beta} + \left(\frac{7\pi^2}{8} - \frac{3}{2} + \frac{\pi^2}{4(1 - k^2)} \right) \frac{1}{\beta^2} \right]$
Energy stored in L_P (J)	$W_P = \frac{\tau V_P I_P}{(2 - k^2)\pi} \left[1 + \frac{\pi}{2\beta} + \left(\frac{\pi^2}{8} - \frac{3}{2} \right) \frac{1}{\beta^2} \right]$
Efficiency	$\epsilon = k^2 \left(1 - \frac{\pi}{\beta} + \frac{3\pi^2}{4\beta^2} \right)$
Module length (cm)	$\ell = \frac{k^2 \tau V_P I_P}{(2 - k^2)\pi \eta} \left[1 - \frac{\pi}{2\beta} + \left(\frac{3\pi^2}{8} - \frac{3}{2} \right) \frac{1}{\beta^2} \right]$

TABLE III (continued)

Number of turns on L_E

$$n_E = \frac{k^2 V_P \tau}{(2-k)^2 \pi} \sqrt{\frac{2}{3\beta}} \left[1 + \left(\frac{\pi^2}{8} - \frac{3}{2} \right) \frac{1}{\beta^2} \right]$$

Radius of secondary (cm)

$$r_S = r_P + \Delta$$

Number of turns on L_S

$$n_S = \frac{\sqrt{5} \times 10^4}{\pi^2} \frac{k^2}{(2-k)^2} \frac{V_P \tau}{(r_P + \Delta) \sqrt{\beta}} \left[1 + \left(\frac{\pi^2}{8} - \frac{3}{2} \right) \frac{1}{\beta^2} \right]$$

Initial field in L_P (G)

$$B_P = \frac{\sqrt{8} \times 10^7}{r_P k} \left(1 + \frac{\pi}{2\beta} \right)$$

Number of turns on L_P

$$n_P = \frac{\sqrt{5} \times 10^4}{\pi^2} \frac{k V_P \tau}{(2-k)^2 \sqrt{\beta}} \left[1 + \left(\frac{\pi^2}{8} - \frac{3}{2} \right) \frac{1}{\beta^2} \right]$$

$$Q_R = \frac{(1-k^2)}{(2-k)^2} \frac{V_P I_P \tau}{\pi} \left[1 - \frac{3\pi}{2\beta} + \left(\frac{9\pi^2}{8} - \frac{3}{2} \right) \frac{1}{\beta^2} \right] \quad (88)$$

Energy finally trapped in the transformer secondary as $t \rightarrow \infty$ is

$$Q_S = \frac{k^2}{(2-k)^2} \frac{I_P V_P \tau}{\pi} \cdot \frac{\pi}{4\beta^2} \quad (89)$$

Thus for practical purposes all of Q_{UN} is lost in the primary, and this would continue to be the case if the crowbar had resistance, provided the crowbar resistance was small enough compared to R .

Fig. 7. Total energy loss in R as a function of time for the circuit described in Fig. 6.

VI. COMPARISON OF THE THREE CIRCUITS

A. Economics

In the transformer coupled systems one extra cost item is, of course, the transformer secondary. The transformer gives the ability to step-up current, but it is not immediately obvious that this is much of an advantage. Without doubt the transformer system has imperfect coupling between the secondary and primary. In both transformer coupled circuits this represents energy which is wasted; in circuit (b) it is also energy which is inevitably dissipated at cryogenic temperatures. Another problem peculiar to the transformer systems is the large forces which can exist between the primary and secondary during energy transfer. These forces will exist if there is any misalignment between the primary and secondary, and will necessitate a stronger and more expensive dewar as well as extra bracing structure to support the secondary.

In a practical energy storage system, the individual modules would be slightly rearranged so that the switches and storage coils would be connected in series. Each storage coil would be separated from the next by one of the superconducting switches. With this arrangement, a minimum of only 2 high current leads entering the cryostat is needed to charge up the system. This arrangement might be realized with circuit (b), since it thermally isolates the primary and secondary circuit and couples them only

magnetically. However, both circuits (a) and (c) have high current leads entering the cryostat every module. These leads cause enhanced heat leak, although they could be designed for transient operation and have a small cross section.

As long as $k < 1$, circuit (b) has a large voltage transient appearing across the switch just as it is opened. One possible way to cut the transient down might be to install extra capacitors across R. Another way would be to reduce the module length, and hence the size of the voltage transient until the peak voltage was within safe limits. This would increase the number of modules beyond the optimum number, and the cost of the entire system would increase.

B. Safety

An adequate discussion of safety in these circuits would mean a calculation of energy deposition in the various circuit elements, not only for normal circuit operation, but for the more probable malfunctions as well. Unfortunately, this energy deposition depends upon many parameters which were ignorable until now, such as the resistance of the crowbar switch and the bus bar. Because the number and possible range of these parameters is large, we will derive only approximate results for the most likely parameter ranges and consider only two problems. These problems are the ultimate destination of the energy transferred to L_E during normal circuit operation and the destination of energy in L_P in the event of a crowbar prefire. A complete failure of the crowbar to fire would be another malfunction which could be investigated, but presumably a proper duplication of firing circuits could make this an unlikely occurrence.

1. Transformer Coupled Systems. The principal concern with a crowbar prefire in a transformer coupled system is the amount of energy deposited in the primary circuit because the primary is at low temperature. The given transformer system can be replaced by a circuit in which the secondary is removed and a resistance equal to $\frac{L_P}{L_S} R_{CB}$ is put in parallel with R across the primary circuit. Here R_{CB} is the resistance of the crowbar, and the coupling is taken as $k = 1$. When this substitution is made and if

$\frac{L_P}{L_S} R_{CB} \ll R$, the energy lost in R is of the order

$$Q_R = \frac{L_P R_{CB}}{L_S R} \left(\frac{1}{2} L_P I_P^2 \right). \quad (90)$$

Reasonable parameters to use in Eq. (90) are

$\frac{L_P}{L_S} = 100$, $R_{CB} = 0.01$, and $R = 100$. With these parameters no more than about 1% of the energy stored in L_P at the time of a crowbar prefire should be added to R after the prefire. Physically what happens is that the low impedance of the crowbar allows current to transfer very rapidly from the primary to the secondary circuit. From the type of argument given to establish Eq. (90), the same equation should also serve to estimate the energy fed back to L_P from L_E during normal circuit operation.

2. Direct Coupled System. One reason that the transformer coupled system turned out to be so safe during crowbar prefire was that resistor R was effectively connected in series with L_P for $t \gg t_{CB}$. This limited the amount of current which could flow in the primary circuit. In the direct coupled circuit, R is in parallel with R_{CB} and L_P . The concern now is not that excessive energy would be deposited in R, but that it would be deposited in the dewar leads or coil L_P if it should be driven normal. Both the crowbar failure and the final disposition of stored energy can be studied with the circuit of Fig. 8. For a crowbar failure at $t = 0$ or the decay of energy from the experiment coil during normal operation for $t > \tau$, the low resistance of the crowbar prevents any large voltage from building up across the capacitor. For this reason capacitor C and resistor R can be left out of the circuit in these two situations. In the crowbar case, L_X is the storage

Fig. 8. Simplified circuit used to consider safety problems in direct coupled magnetic energy storage systems.

coil inductance, R_X is the resistance of dewar leads and bus bar, R_Y is the resistance of the experiment coil and L_Y is the experiment coil inductance. For finding the final energy deposition in normal operation, the storage coil and experiment coil change places so that L_X is the experiment coil inductance, L_Y is the storage coil inductance, etc. In either case the initial conditions are $i_X = I_P$ and $i_Y = 0$. The inductances are taken as equal, so $L_X = L_Y = L$. Energy loss in R_X and R_Y is given by

$$Q_X = \left(\frac{1}{2} L I_P^2 \right) \frac{R_X}{(R_X - R_Y)^2 + 4R_{CB}^2} \left[\frac{(R_Y + R_{CB})(R_Y - R_X)^2 + R_{CB}^2(3R_Y - R_X + 2R_{CB})}{R_X R_Y + R_{CB}(R_X + R_Y)} + \frac{4R_{CB}^2}{R_X + R_Y + 2R_{CB}} \right] \quad (91)$$

and

$$Q_Y = \left(\frac{1}{2} L I_P^2 \right) \frac{R_{CB}^2 R_Y}{(R_X - R_Y)^2 + 4R_{CB}^2} \left[\frac{R_X + R_Y + 2R_{CB}}{R_X R_Y + R_{CB}(R_X + R_Y)} - \frac{4}{R_X + R_Y + 2R_{CB}} \right] \quad (92)$$

Qualitatively, the dangerous situation during a crowbar failure occurs when $R_X \gg R_{CB}$, since most of the stored energy will be dumped into the dewar leads (R_X). Q_X measures the energy dumped into the dewar leads. During normal decay of current in the storage coil (now L_Y), the dewar leads (now R_Y) would receive most of the energy only if $R_{CB} \gg R_Y \gg R_X$. This is an unlikely situation; however, a significant amount of energy could still be deposited in the leads, and the amount can be calculated from Eq. (92) for Q_Y .

A crowbar prefire is more probable at $t = \tau/2$ than at $t = 0$ because voltage across the crowbar is maximum at $t = \tau/2$ and zero at $t = 0$. The circuit of Fig. 8 can also be used to find the energy lost in the dewar leads during a $t = \tau/2$ prefire if $R_{CB} \ll L/R_X$ and $R_{CB} \ll L/R_Y$. Energy stored in the capacitor just before the prefire is 1/2 of the energy originally stored in L_X . With the assumed ordering of time constants, all of the energy stored in C is dissipated in R_{CB} before the current has time to change much in L_X and L_Y . The proper initial values for i_X and i_Y are now $i_X = I_P/2$. Energy deposited in the dewar leads is

$$Q_X = \left(\frac{1}{2} L I_P^2 \right) \frac{R_X}{4[(R_X - R_Y)^2 + 4R_{CB}^2]} \left[\frac{(R_X - R_Y)^2(R_Y + 2R_{CB}) + 8R_{CB}^3 + 2R_{CB}^2(3R_Y - R_X) - R_{CB}(R_X^2 - R_Y^2)}{R_X R_Y + R_{CB}(R_X + R_Y)} + \frac{4R_{CB}(R_X - R_Y)}{R_X + R_Y + 2R_{CB}} \right] \quad (93)$$

ACKNOWLEDGMENTS

We would like to thank H. L. Laquer for encouragement to reinvestigate the use of transfer capacitors in magnetic energy storage systems, and J. D. Rogers for many helpful discussions.

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