LEGAL NOTICE

This report was prepared as an account of Government sponsored work. Neither the United States, nor the Commission, nor any person acting on behalf of the Commission:

A. Makes any warranty or representation, express or implied, with respect to the accuracy, completeness, or usefulness of the information contained in this report, or that the use of any information, apparatus, method, or process disclosed in this report may not infringe privately owned rights; or

B. Assumes any liabilities with respect to the use of, or for damages resulting from the use of any information, apparatus, method, or process disclosed in this report.

As used in the above, "person acting on behalf of the Commission" includes any employee or contractor of the Commission to the extent that such employee or contractor prepares, handles or distributes, or provides access to, any information pursuant to his employment or contract with the Commission.

AECU-3227

Photostat Price $ 3.30
Microfilm Price $ 2.40

Available from the
Office of Technical Services
Department of Commerce
Washington 25, D. C.

267-54-51

RADIAL VIBRATIONS IN SHORT, HOLLOW CYLINDERS OF BARIUM TITANATE

C. V. Stephenson - 5143

ABSTRACT

The mathematics has been developed for the determination of the radial coupling coefficient for a hollow cylinder of electrostrictive material whose length is small compared to its outside diameter.

Case No. 590.00

December 10, 1954

700-1

DISCLAIMER

This report was prepared as an account of work sponsored by an agency of the United States Government. Neither the United States Government nor any agency Thereof, nor any of their employees, makes any warranty, express or implied, or assumes any legal liability or responsibility for the accuracy, completeness, or usefulness of any information, apparatus, product, or process disclosed, or represents that its use would not infringe privately owned rights. Reference herein to any specific commercial product, process, or service by trade name, trademark, manufacturer, or otherwise does not necessarily constitute or imply its endorsement, recommendation, or favoring by the United States Government or any agency thereof. The views and opinions of authors expressed herein do not necessarily state or reflect those of the United States Government or any agency thereof.

DISCLAIMER

Portions of this document may be illegible in electronic image products. Images are produced from the best available original document.

2

THIS PAGE

WAS INTENTIONALLY

LEFT BLANK

3

RADIAL VIBRATIONS IN SHORT, HOLLOW CYLINDERS OF BARIUM TITANATE

INTRODUCTION

Frequently in the use of piezoelectric and electrostrictive materials, one is concerned with a method of measuring the coupling coefficient of the element for some mode of vibration. Normally the shape of the material is that of long rods or thin plates or discs. The coupling coefficient of a crystal for shapes such as these can accurately be determined by measurement of resonant and anti-resonant frequencies of the first harmonic. The mathematics for the necessary calculations for these shapes has been previously published.

Occasionally, however, a use arises for a peculiar shape and, with it, a need for a method of determining the coupling coefficient of materials of this shape. This is particularly true since the advent of electrostrictive ceramics such as barium titanate. Recently the need has arisen for a method of determining the coupling coefficient of a hollow cylinder of electrostrictive material, whose length is small compared to its outside diameter. This report concerns itself with the mathematics which allows calculation of coupling coefficients for such shapes for the radial mode of vibration. Electrostrictive equations will be used rather than piezoelectric since anyone working with such a shape will probably be working with one of the electrostrictive ceramics. However, it can easily be shown that the results of the electrostrictive case will carry over to the piezoelectric case.

RADIAL VIBRATIONS

The configuration with which we will be concerned is shown in Figure 1. The thickness l_t is small compared with the outside radius a. There will be no restriction on the inside radius b. Radial vibrations in a solid disc which has been treated by Mason[1] will become a limiting case of the present treatment.

For radial vibrations, it is best to transform the usual electrostrictive equations into cylindrical coordinates[1]. They then take the following form:[2]

[1] Mason, W. P., Piezoelectric Crystals and Their Application to Ultrasonics, D. Van Nostrand Company, 1950.

[2] These equations differ from those appearing in the first edition of Mason's book in that a correction term to the impermeability constant has been dropped after verbal communications with Mason.

700-2

4

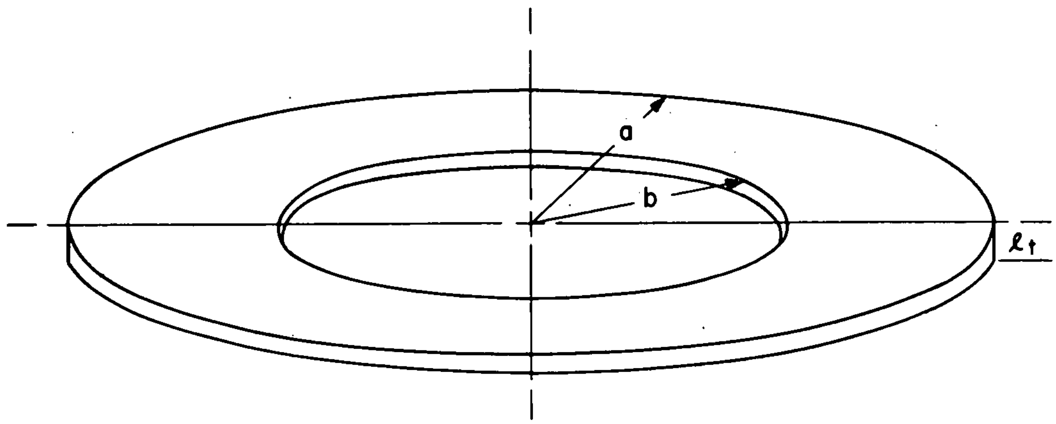

FIGURE 1 - HOLLOW CYLINDER WHOSE LENGTH IS SHORT COMPARED TO OUTSIDE DIAMETER

200-3

5

$$S_{rr} = s^D_{1111} T_{rr} + s^D_{1122} \left(T_{\theta\theta} + T_{zz}\right) + \left[Q_{1111}\delta_r^2 + Q_{1122}\left(\delta_\theta^2 + \delta_z^2\right)\right]$$

$$S_{\theta\theta} = s^D_{1122}\left(T_{rr} + T_{zz}\right) + s^D_{1111} T_{\theta\theta} + \left[Q_{1122}\left(\delta_r^2 + \delta_z^2\right) + Q_{1111}\delta_\theta^2\right]$$

$$S_{zz} = s^D_{1111} T_{zz} + s^D_{1122}\left(T_{rr} + T_{\theta\theta}\right) + \left[Q_{1111}\delta_z^2 + Q_{1122}\left(\delta_r^2 + \delta_\theta^2\right)\right]$$

$$S_{rz} = \left(s^D_{1111} - s^D_{1122}\right) T_{rz} + \left(Q_{1111} - Q_{1122}\right)\delta_r\delta_z$$

$$S_{r\theta} = \left(s^D_{1111} - s^D_{1122}\right) T_{r\theta} + \left(Q_{1111} - Q_{1122}\right)\delta_r\delta_\theta$$

$$S_{\theta z} = \left(s^D_{1111} - s^D_{1122}\right) T_{\theta z} + \left(Q_{1111} - Q_{1122}\right)\delta_\theta\delta_z$$

$$E_r = 4\pi\beta^T_{11}\delta_r - 2\left\{Q_{1111}\left[\delta_r T_{rr} + \delta_\theta T_{r\theta} + \delta_z T_{rz}\right] + Q_{1122}\left[\delta_r\left(T_{\theta\theta} + T_{zz}\right) - \left(\delta_\theta T_{r\theta} + \delta_z T_{rz}\right)\right]\right\}$$

$$E_\theta = 4\pi\beta^T_{11}\delta_\theta - 2\left\{Q_{1111}\left[\delta_\theta T_{\theta\theta} + \delta_r T_{r\theta} + \delta_z T_{\theta z}\right] + Q_{1122}\left[\delta_\theta\left(T_{rr} + T_{zz}\right) - \left(\delta_r T_{r\theta} + \delta_z T_{\theta z}\right)\right]\right\}$$

$$E_z = 4\pi\beta^T_{11}\delta_z - 2\left\{Q_{1111}\left[\delta_z T_{zz} + \delta_r T_{rz} + \delta_\theta T_{\theta z}\right] + Q_{1122}\left[\delta_z\left(T_{rr} + T_{\theta\theta}\right) - \left(\delta_r T_{rz} + \delta_\theta T_{\theta z}\right)\right]\right\}$$

In these equations E_r, E_θ and E_z are the component of the electric field in the r, θ and z directions, δ_r, δ_θ and δ_z are the components of the electric displacement divided by 4π, S_{ij} and T_{ij} are the ij^{th} components of the strain tensor and stress tensor respectively, s^D_{ijkl} are the elastic compliance constants measured

700 · 4

at constant electric displacement, β^T_{mn} the dielectric impermeability constants (inverse of dielectric constants) measured at constant stress, and Q_{ijno} are the electrostrictive constants.

In solving the equations of motion, it is also necessary to know the strains in terms of the mechanical displacements in the r, θ, and z directions. Denoting these displacements by u_r, u_θ and u_z the strains are:

$$S_{rr} = \frac{\partial u_r}{\partial r}$$

$$S_{\theta\theta} = \frac{1}{r}\frac{\partial u_\theta}{\partial \theta} + \frac{u_r}{r}$$

$$S_{zz} = \frac{\partial u_z}{\partial z}$$

$$S_{r\theta} = \frac{\partial u_\theta}{\partial r} - \frac{u_\theta}{r} + \frac{1}{r}\frac{\partial u_r}{\partial \theta}$$

$$S_{rz} = \frac{\partial u_r}{\partial z} + \frac{\partial u_z}{\partial r}$$

$$S_{\theta z} = \frac{1}{r}\frac{\partial u_z}{\partial \theta} + \frac{\partial u_\theta}{\partial z}$$

We will assume that the thickness is so small that the change of stress in the z direction is negligible. Since the stresses are zero at the surface, we can set

$$T_{zz} = T_{rz} = T_{\theta z} = 0.$$

Furthermore, since we shall consider only motion that is entirely radial, $T_{r\theta} = 0$ and also $u_\theta = u_z = 0$. We will consider the case in which the field is applied only in the z direction so that $\delta_r = \delta_\theta = 0$. The electrostrictive equations now become:

$$S_{rr} = s^D_{1111}\ T_{rr} + s^D_{1122}\ T_{\theta\theta} + Q_{1122}\delta_z^2$$

$$S_{\theta\theta} = s^D_{1122}T_{rr} + s^D_{1111}T_{\theta\theta} + Q_{1122}\delta_z^2$$

$$E_z = 4\pi\,\beta^T_{11}\delta_z - 2Q_{1122}\delta_z\left(T_{rr} + T_{\theta\theta}\right)$$

700-5

In the case of electrostrictive ceramics the electric displacement may be represented by $\delta_z = \delta_{zo} + \delta_z e^{j\omega t}$ where δ_{zo} is the remanent electric displacement caused by polarization and δ_z the alternating component. Solving the above equations simultaneously, the alternating component of the stress and displacement are given by the equations

$$T_{rr} = \left(\frac{Y_o^E}{1-\sigma^2}\right)\left(S_{rr} + \sigma S_{\theta\theta}\right) - \frac{2Q_{1122}\delta_{zo} Y_o^E E_z}{4\pi\beta_{11}^T (1-\sigma)}$$

$$T_{\theta\theta} = \left(\frac{Y_o^E}{1-\sigma^2}\right)\left(S_{\theta\theta} + \sigma S_{rr}\right) - \frac{2Q_{1122}\delta_{zo} Y_o^E E_z}{4\pi\beta_{11}^T (1-\sigma)}$$

$$\delta_z = \frac{E_z}{4\pi\beta_{11}^T} + \frac{2Q_{1122}\delta_{zo}}{4\pi\beta_{11}^T}\left(T_{rr} + T_{\theta\theta}\right)$$

where

$$-\, s_{1122}^E / s_{1111}^E = \sigma \quad \text{is the Poisson's ratio}$$

and

$$1/s_{1111}^E = Y_o^E \quad \text{is Young's modulus in which}$$

$$s_{1122}^E = \frac{s_{1122}^D}{1 - \dfrac{s_{1111}^D}{s_{1122}^D}\,\dfrac{Q_{1122}^2\delta_{zo}^2}{\pi\beta_{11}^T s_{1111}^E}}$$

and

$$s_{1111}^E = \frac{s_{1111}^D}{1 - \dfrac{Q_{1122}^2\delta_{zo}^2}{\pi\beta_{11}^T s_{1111}^E}}$$

700-6

8

The only remaining equation which is needed is the force equation which becomes for the described conditions

$$\rho \ddot{u}_r = \frac{\partial T_{rr}}{\partial r} + \frac{\left(T_{rr} - T_{\theta\theta}\right)}{r}$$

Since now $\quad S_{rr} = \frac{\partial u_r}{\partial r} \quad$ and $\quad S_{\theta\theta} = \frac{u_r}{r}$

the equation of motion becomes

$$\frac{Y_o^E}{1 - \sigma^2} \left[\frac{\partial^2 u_r}{\partial r^2} + \frac{1}{r} \frac{\partial u_r}{\partial r} - \frac{u_r}{r^2} \right] = \rho \frac{\partial^2 u_r}{\partial t^2} = - \omega^2 \rho u_r$$

for simple harmonic motion.

This is a Bessel's equation of the first order which has the solution

$$u_r = \alpha J_1\left(\frac{\omega r}{v}\right) + \beta K_1 \left(\frac{\omega r}{v}\right) \; ; \; v^2 = \frac{Y_o^E}{\left(1 - \sigma^2\right)\rho}$$

where $J_1 (\frac{\omega r}{v})$ and $K_1(\frac{\omega r}{v})$ are Bessel functions of the first and second kind. The boundary conditions are that the stress $T_{rr} = 0$ when $r = a$ and when $r = b$, a and b being the outside and inside radii.

$$T_{rr} = \frac{Y_o^E}{1 - \sigma^2} \left\{ \alpha \left[\frac{\omega}{v} J_o\left(\frac{\omega r}{v}\right) - \frac{(1 - \sigma)}{r} J_1 \left(\frac{\omega r}{v}\right)\right] + \beta \left[\frac{\omega}{v} K_o\left(\frac{\omega r}{v}\right) - \frac{(1 - \sigma)}{r} K_1\left(\frac{\omega r}{v}\right)\right]\right\}$$

$$- \frac{Q_{1122} \delta_{zo} Y_o^E E_z}{2\pi \beta_{11}^T (1 - \sigma)}$$

700-7

Inserting the boundary conditions and solving for α and β we get

$$\alpha = \frac{Q_{1122}\,\delta_{zo}\,E_z\,(1+\sigma)}{2\pi\beta_{11}^{T}}$$

$$\times \left\{ \frac{\left[\frac{\omega}{v} K_o\left(\frac{\omega a}{v}\right) - \frac{(1-\sigma)}{a} K_1\left(\frac{\omega a}{v}\right)\right] - \left[\frac{\omega}{v} K_o\left(\frac{\omega b}{v}\right) - \frac{(1-\sigma)}{b} K_1\left(\frac{\omega b}{v}\right)\right]}{\left[\frac{\omega}{v} K_o\left(\frac{\omega a}{v}\right) - \frac{(1-\sigma)}{a} K_1\left(\frac{\omega a}{v}\right)\right]\left[\frac{\omega}{v} J_o\left(\frac{\omega b}{v}\right) - \frac{(1-\sigma)}{b} J_1\left(\frac{\omega b}{v}\right)\right] - \left[\frac{\omega}{v} K_o\left(\frac{\omega b}{v}\right) - \frac{(1-\sigma)}{b} K_1\left(\frac{\omega b}{v}\right)\right]\left[\frac{\omega}{v} J_o\left(\frac{\omega a}{v}\right) - \frac{(1-\sigma)}{a} J_1\left(\frac{\omega a}{v}\right)\right]} \right\}$$

$$\beta = \frac{Q_{1122}\,\delta_{zo}\,E_z\,(1+\sigma)}{2\pi\beta_{11}^{T}}$$

$$\times \left\{ \frac{\left[\frac{\omega}{v} J_o\left(\frac{\omega b}{v}\right) - \frac{(1-\sigma)}{b} J_1\left(\frac{\omega b}{v}\right)\right] - \left[\frac{\omega}{v} J_o\left(\frac{\omega a}{v}\right) - \frac{(1-\sigma)}{a} J_1\left(\frac{\omega a}{v}\right)\right]}{\left[\frac{\omega}{v} K_o\left(\frac{\omega a}{v}\right) - \frac{(1-\sigma)}{a} K_1\left(\frac{\omega a}{v}\right)\right]\left[\frac{\omega}{v} J_o\left(\frac{\omega b}{v}\right) - \frac{(1-\sigma)}{b} J_1\left(\frac{\omega b}{v}\right)\right] - \left[\frac{\omega}{v} K_o\left(\frac{\omega b}{v}\right) - \frac{(1-\sigma)}{b} K_1\left(\frac{\omega b}{v}\right)\right]\left[\frac{\omega}{v} J_o\left(\frac{\omega a}{v}\right) - \frac{(1-\sigma)}{a} J_1\left(\frac{\omega a}{v}\right)\right]} \right\}$$

This gives

$$T_{rr} = \left(\frac{Y_o^E}{1-\sigma^2}\right)\left\{ \alpha\left[\frac{\omega}{v} J_o\left(\frac{\omega r}{v}\right) - \frac{(1-\sigma)}{r} J_1\left(\frac{\omega r}{v}\right)\right] + \beta\left[\frac{\omega}{v} K_o\left(\frac{\omega r}{v}\right) - \frac{(1-\sigma)}{r} K_1\left(\frac{\omega r}{v}\right)\right]\right\}$$

$$- \frac{Q_{1122}\,\delta_{zo}\,Y_o^E\,E_z}{2\pi\beta_{11}^{T}\,(1-\sigma)}$$

$$T_{\theta\theta} = \left(\frac{Y_o^E}{1-\sigma^2}\right)\left\{ \alpha\left[\frac{\sigma\omega}{v} J_o\left(\frac{\omega r}{v}\right) + \frac{(1-\sigma)}{r} J_1\left(\frac{\omega r}{v}\right)\right] + \beta\left[\frac{\sigma\omega}{v} K_o\left(\frac{\omega r}{v}\right) + \frac{(1-\sigma)}{r} K_1\left(\frac{\omega r}{v}\right)\right]\right\}$$

$$- \frac{Q_{1122}\,\delta_{zo}\,Y_o^E\,E_z}{2\pi\beta_{11}^{T}\,(1-\sigma)}$$

700-8

Substituting these values we find that

$$\delta_z = \frac{E_z}{4\pi\,\beta_{11}^T} - \frac{2Q_{1122}^2\,\delta_{zo}^2\,Y_o^E\,E_z}{\pi\beta_{11}^T\,(1-\sigma)} + \frac{Q_{1122}\,\delta_{zo}\,Y_o^E}{2\pi\beta_{11}^T(1-\sigma)}\left[\alpha\,\frac{\omega}{v}\,J_o\!\left(\frac{\omega r}{v}\right) + \beta\,\frac{\omega}{v}\,K_o\!\left(\frac{\omega r}{v}\right)\right]$$

The next step is to obtain an expression for the electrical admittance. The admittance is equal to the current into the element divided by the voltage across it. But for simple harmonic motion the current is $i = \frac{dQ}{dt} = j\omega Q$ where Q is the surface charge. This gives for the admittance $\frac{1}{z} = \frac{i}{E_z l_t} = \frac{i\omega Q}{E_z l_t}$. We need now to find an expression for the surface charge Q.

Since the value of δ_z at the surface is equal to the surface charge density we can find Q by performing the integration $Q = \int_o^{2\pi} d\theta \int_b^a \delta_z\, r dr$

Evaluating this integral and making the substitution

$$\frac{1}{4\pi\,\beta_{11}^T}\left[1 - \frac{2Q_{1122}^2\,\delta_{zo}^2\,Y_o^E}{\pi\beta_{11}^T\,(1-\sigma)}\right] = \frac{1}{4\pi\beta_{11}^{RC}}$$

where β_{11}^{RC} is the radially clamped impermeability constant, we have

$$Q = \frac{E_z\left(a^2 - b^2\right)}{4\beta_{11}^{RC}} + \frac{Q_{1122}\,\delta_{zo}\,Y_o^E}{\beta_{11}^T(1-\sigma)}\left\{\alpha\left[aJ_1\!\left(\frac{\omega a}{v}\right) - bJ_1\!\left(\frac{\omega b}{v}\right)\right] + \beta\left[aK_1\!\left(\frac{\omega a}{v}\right) - bK_1\!\left(\frac{\omega b}{v}\right)\right]\right\}$$

The radial coupling coefficient can be expressed as $k^2 = \frac{2Q_{1122}^2\,\delta_{zo}\,Y_o^E}{\pi\beta_{11}^T(1-\sigma)}$

Using this and the two expressions for the constants α and β we arrive at the formidable expression

$$\frac{1}{z} = \frac{j\omega(a^2 - b^2)}{4\beta_{11}^{RC}\, l_t}\left\{ 1 + \frac{k^2}{1 - k^2}\,\frac{(1+\sigma)}{a^2 - b^2} \right.$$

$$\left. \times \frac{\left[\begin{array}{l}\left[a J_1\left(\frac{\omega a}{v}\right) - b J_1\left(\frac{\omega b}{v}\right)\right]\left\{\left[\frac{\omega}{v} K_o\left(\frac{\omega a}{v}\right) - \frac{(1-\sigma)}{a} K_1\left(\frac{\omega a}{v}\right)\right] - \left[\frac{\omega}{v} K_o\left(\frac{\omega b}{v}\right) - \frac{(1-\sigma)}{b} K_1\left(\frac{\omega b}{v}\right)\right]\right\} \\ + \left[a K_1\left(\frac{\omega a}{v}\right) - b K_1\left(\frac{\omega b}{v}\right)\right]\left\{\left[\frac{\omega}{v} J_o\left(\frac{\omega b}{v}\right) - \frac{(1-\sigma)}{b} J_1\left(\frac{\omega b}{v}\right)\right] - \left[\frac{\omega}{v} J_o\left(\frac{\omega a}{v}\right) - \frac{(1-\sigma)}{a} J_1\left(\frac{\omega a}{v}\right)\right]\right\}\end{array}\right]}{\left[\frac{\omega}{v} K_o\left(\frac{\omega a}{v}\right) - \frac{(1-\sigma)}{a} K_1\left(\frac{\omega a}{v}\right)\right]\left[\frac{\omega}{v} J_o\left(\frac{\omega b}{v}\right) - \frac{(1-\sigma)}{b} J_1\left(\frac{\omega b}{v}\right)\right] - \left[\frac{\omega}{v} K_o\left(\frac{\omega b}{v}\right) - \frac{(1-\sigma)}{b} K_1\left(\frac{\omega b}{v}\right)\right]\left[\frac{\omega}{v} J_o\left(\frac{\omega a}{v}\right) - \frac{(1-\sigma)}{a} J_1\left(\frac{\omega a}{v}\right)\right]} \right\}$$

The resonant frequency occurs when $A = \infty$ or when

$$\frac{\frac{\omega a}{v} K_o\left(\frac{\omega a}{v}\right) - \left(1 - \sigma\right) K_1\left(\frac{\omega a}{v}\right)}{\frac{\omega a}{v} J_o\left(\frac{\omega a}{v}\right) - \left(1 - \sigma\right) J_1\left(\frac{\omega a}{v}\right)} = \frac{\frac{\omega b}{v} K_o\left(\frac{\omega b}{v}\right) - \left(1 - \sigma\right) K_1\left(\frac{\omega b}{v}\right)}{\frac{\omega b}{v} J_o\left(\frac{\omega b}{v}\right) - \left(1 - \sigma\right) J_1\left(\frac{\omega b}{v}\right)}$$

This means that the function $f\left(\frac{\omega r}{v}\right) = \dfrac{\frac{\omega r}{v} K_o\left(\frac{\omega r}{v}\right) - \left(1 - \sigma\right) K_1\left(\frac{\omega r}{v}\right)}{\frac{\omega r}{v} J_o\left(\frac{\omega r}{v}\right) - \left(1 - \sigma\right) J_1\left(\frac{\omega r}{v}\right)}$

must have two roots for any possible value of the function. One root corresponds to $A = \frac{\omega_r a}{v}$ where ω_r is the resonant frequency, and the other is $B = \frac{\omega_r b}{v}$.

A plot of this function is given in Figure 2. In this plot, the Poisson ratio of barium titanate $\sigma = .30$ is assumed. The first U-shaped part of the curve corresponds to the first harmonic. It is noticed that for every value of $f(\frac{\omega r}{v})$ there are two values of $\frac{\omega r}{v}$ which satisfy this value of $f(\frac{\omega r}{v})$. It is also noticed that for any ratio of the crystal radii $a/b = A/B$ there is one and only one value of the function which will have the two roots A and B. This means that if this portion of the curve is plotted carefully, one can find the two roots A and B which correspond to each value of $f(\frac{\omega r}{v})$ and these values of A and B may be plotted against the

700-10

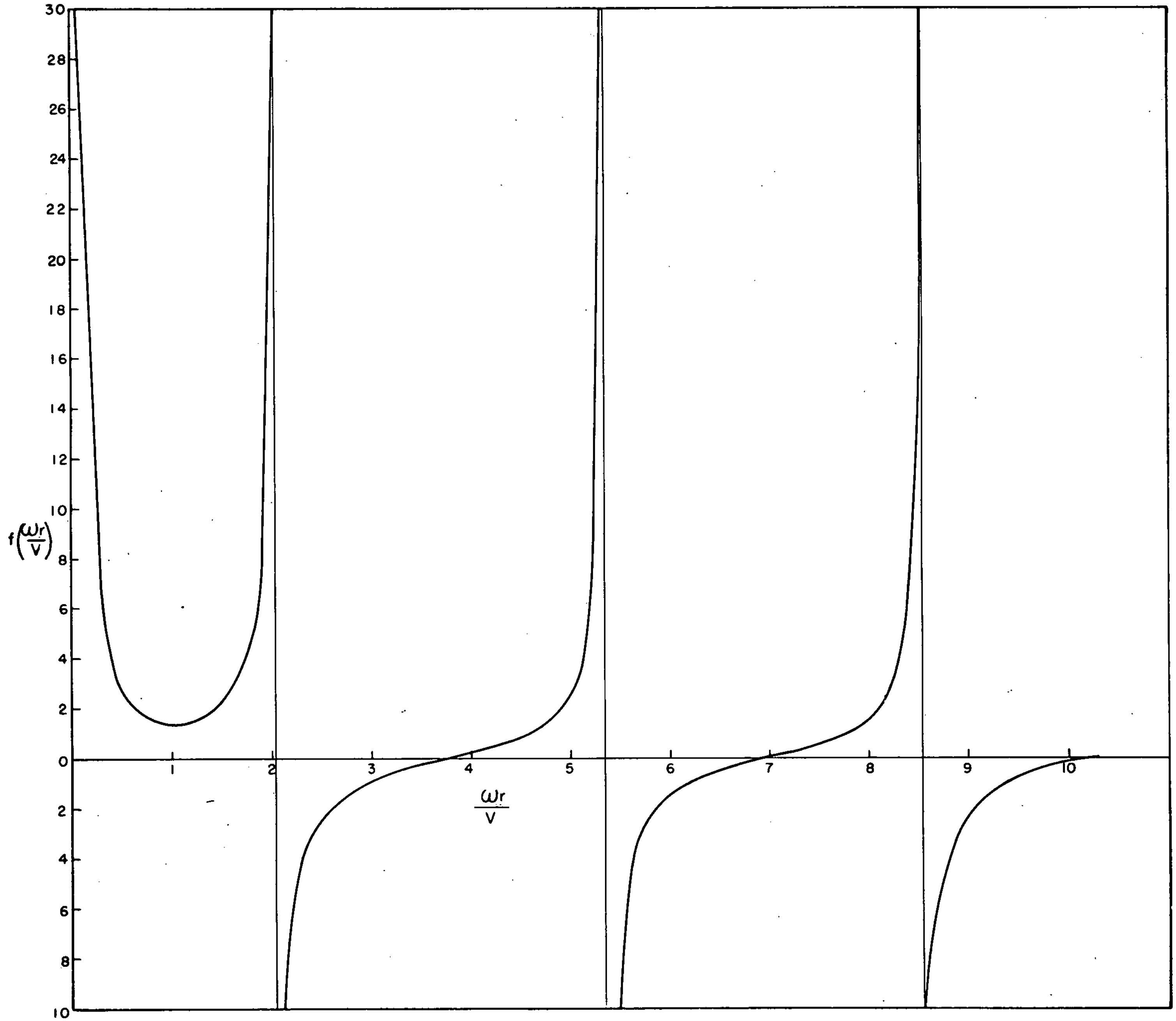

FIGURE 2 - PLOT OF FUNCTION $f\left(\frac{\omega r}{V}\right)$ WHICH OCCURS IN RESONANCE CONDITION

Jee-11

ratio of the radii a/b. Figures 3 and 4 show such plot. Thus given the values of a and b the resonant frequency is uniquely determined.

The other part of the curve in Figure 2 corresponds to higher harmonics. It is particularly interesting to note that whereas there is a first harmonic resonance for any ratio of a/b, for any higher harmonic there is only a specific ratio of a to b which will allow this harmonic to exist. Thus for any crystal of this shape, almost all of the higher harmonics are forbidden. This of course does not apply in the limiting case of b = o in which case all higher harmonics are permissible.

There remains the problem of determining what happens at anti-resonance. This occurs when the expression in brackets in the admittance equation reduces to zero. The frequency separation between resonance and anti-resonance can be obtained by developing the Bessel functions in Taylor series about the roots A and B. This gives

$$J_o\left(\frac{\omega a}{v}\right) = J_o(A) - A J_1(A)\frac{\Delta f}{f_r} + \cdots$$

$$K_o\left(\frac{\omega a}{v}\right) = K_o(A) - A K_1(A)\frac{\Delta f}{f_r} + \cdots$$

$$J_1\left(\frac{\omega a}{v}\right) = J_1(A) + A J_o(A)\frac{\Delta f}{f_r} - J_1(A)\frac{\Delta f}{f_r} + \cdots$$

$$K_1\left(\frac{\omega a}{v}\right) = K_1(A) + A K_o(A)\frac{\Delta f}{f_r} - K_1(A)\frac{\Delta f}{f_r} + \cdots$$

where f_r is resonant frequency. Similar expressions can be derived for $J_o\left(\frac{\omega b}{v}\right)$, $K_o\left(\frac{\omega b}{v}\right)$, $J_1\left(\frac{\omega b}{v}\right)$ and $K_1\left(\frac{\omega b}{v}\right)$ about the root B. Also we have $\frac{\omega a}{v} = A + A\frac{\Delta f}{f_r}$ and $\frac{\omega b}{v} = B + B\frac{\Delta f}{f_r}$. Inserting these values into the bracket expression, we get to a first approximation the second formidable expression

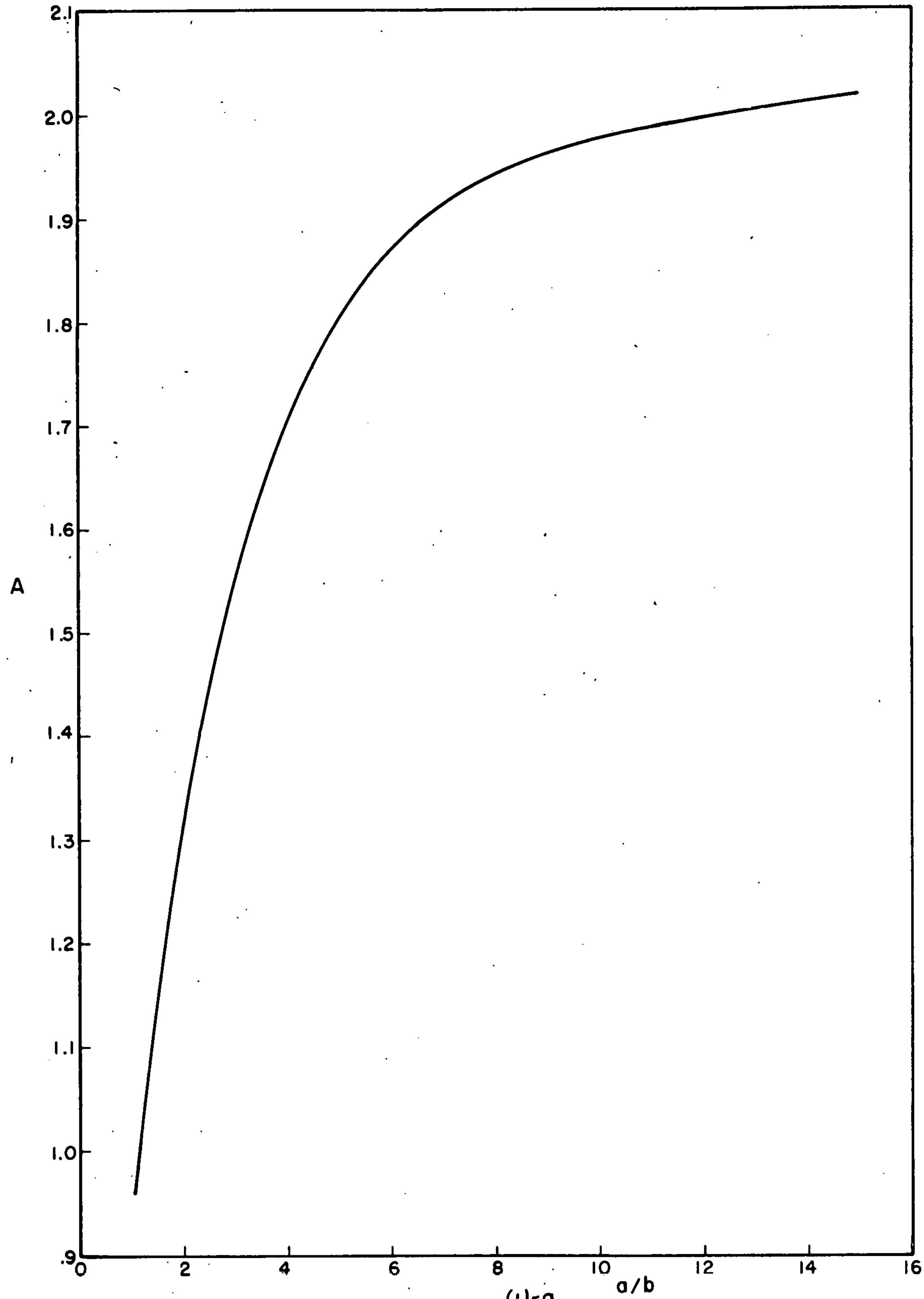

FIGURE 3 - PLOT OF $A = \frac{\omega_r a}{V}$ AS A FUNCTION OF THE RATIO a/b

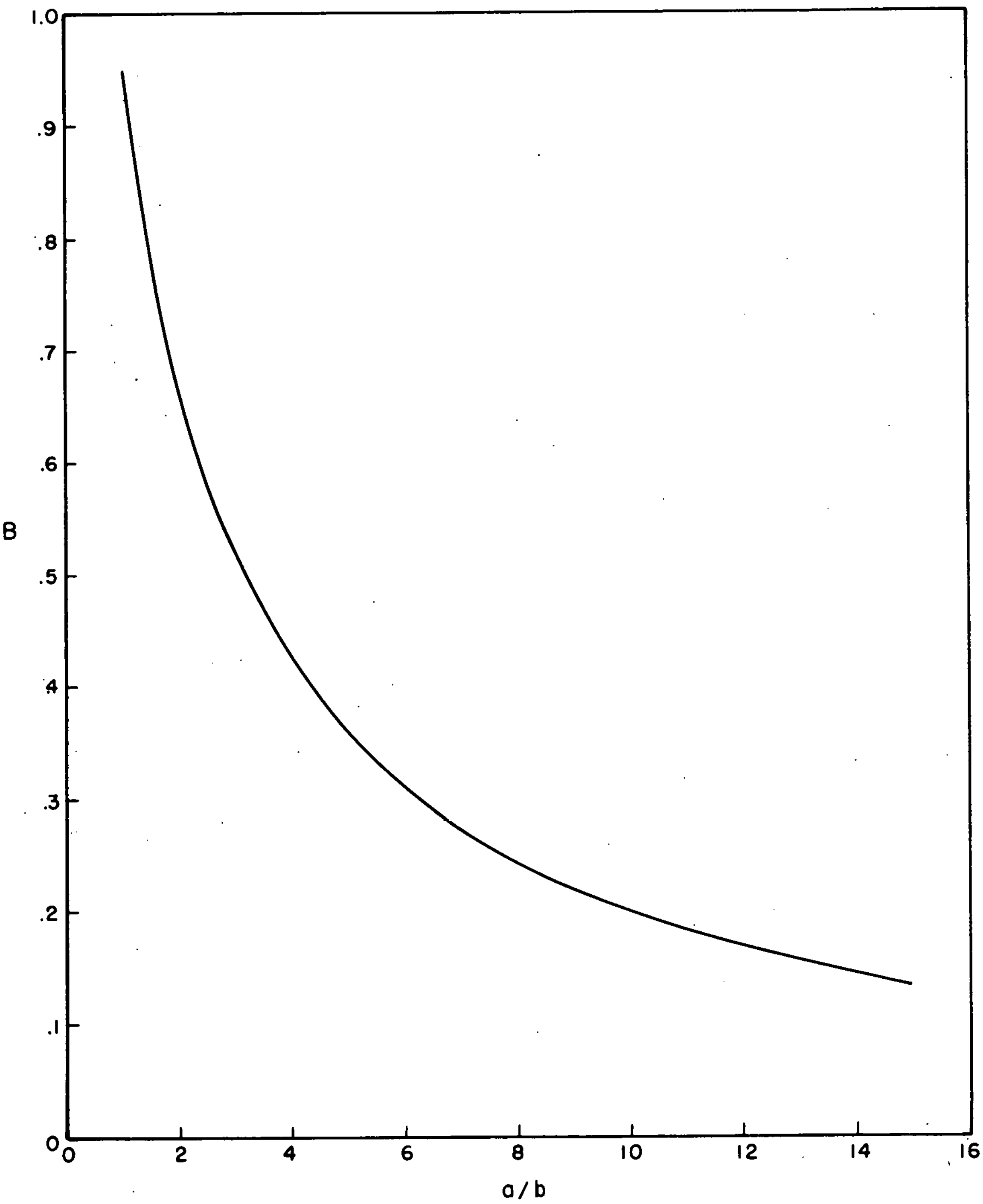

FIGURE 4 - PLOT OF $B = \frac{\omega_r b}{V}$ AS A FUNCTION OF THE RATIO a/b

$$
\frac{\Delta f}{f_r}\;
\frac{\left(\begin{aligned}
&2\Big[K_o(B)\,J_o(A) - K_o(A)\,J_o(B)\Big] + B\Big[K_o(A)\,J_1(B) - K_1(B)\,J_o(A)\Big] + A\Big[K_1(A)\,J_o(B) - K_o(B)\,J_1(A)\Big] \\
&+ (1-\sigma)\left(2\Big[K_o(A)\,J_o(B) - K_o(B)\,J_o(A)\Big] + \frac{A^2+B^2}{AB}\Big[K_1(B)\,J_1(A) - K_1(A)\,J_1(B)\Big]\right) \\
&+ (1-\sigma^2)\left(\frac{1}{B}\Big[K_1(B)\,J_o(A) - K_o(A)\,J_1(B)\Big] + \frac{1}{A}\Big[K_o(B)\,J_1(A) - K_1(A)\,J_o(B)\Big] + \frac{2}{AB}\Big[K_1(A)\,J_1(B) - K_1(B)\,J_1(A)\Big]\right)
\end{aligned}\right)}
{\left(\begin{aligned}
&A\Big[K_o(A)\,J_1(A) - K_1(A)\,J_o(A)\Big] + B\Big[K_o(B)\,J_1(B) - K_1(B)\,J_o(B)\Big] + B\Big[K_1(B)\,J_o(A) - K_o(A)\,J_1(B)\Big] \\
&+ A\Big[K_1(A)\,J_o(B) - K_o(B)\,J_1(A)\Big] + \frac{A^2-B^2}{AB}(1-\sigma)\Big[K_1(B)\,J_1(A) - K_1(A)\,J_1(B)\Big]
\end{aligned}\right)}
$$

$$
= \frac{k^2}{1-k^2}\;\frac{1+\sigma}{A^2-B^2}
$$

If the inner and outer radii and the Poisson's ratio is known this equation can be put into the form

$$
\frac{k^2}{1-k^2} = C\,\frac{\Delta f}{f_r} \;,
$$

where C is a constant. For barium titanate the value of this constant has been calculated for various ratios of a/b and is plotted in Figure 5.

EXPERIMENTAL

The results of the preceding section were checked by the following experiments. Three barium titanage elements were available which had had holes cut in their center. These elements were .125" thick and had an outside diameter of 1.047". The hole diameters were .126", .367" and .492". Unfortunately no resonance measurements were made on these elements before the holes were cut. However several elements of the same batch were available for measurements, and these elements had radial coupling coefficients of .26 $\pm$ .01. Using the results of the preceding section, the radial coupling coefficients of the three test samples were .25, .26, and .26.

The preceding section also predicts that, when compared with the resonant frequency of the solid disc, the resonant frequency of an element with a hole in the middle should decrease with increasing hole diameter if the outside diameter is kept constant. The ratio of the resonant frequency of a ring to the resonant frequency of the solid disc is plotted against the ratio a/b as the solid line in Figure 6. The ratios of the resonant frequencies of the three experimental elements to

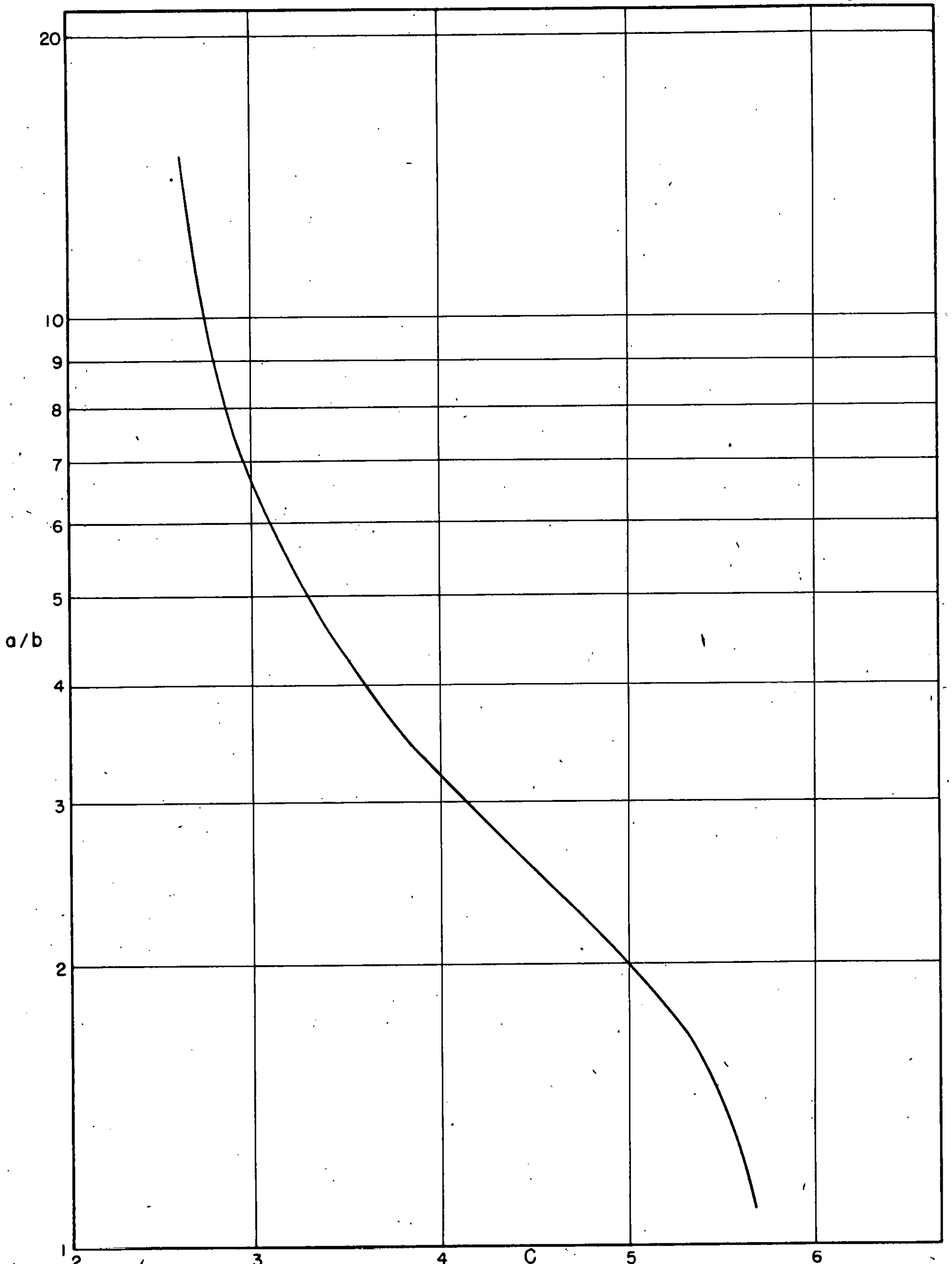

FIGURE 5 - PLOT OF THE CONSTANT C WHICH OCCURS IN THE EQUATION $\frac{k^2}{1-k^2} = C \frac{\Delta f}{f r}$ AS A FUNCTION OF THE RATIO a/b

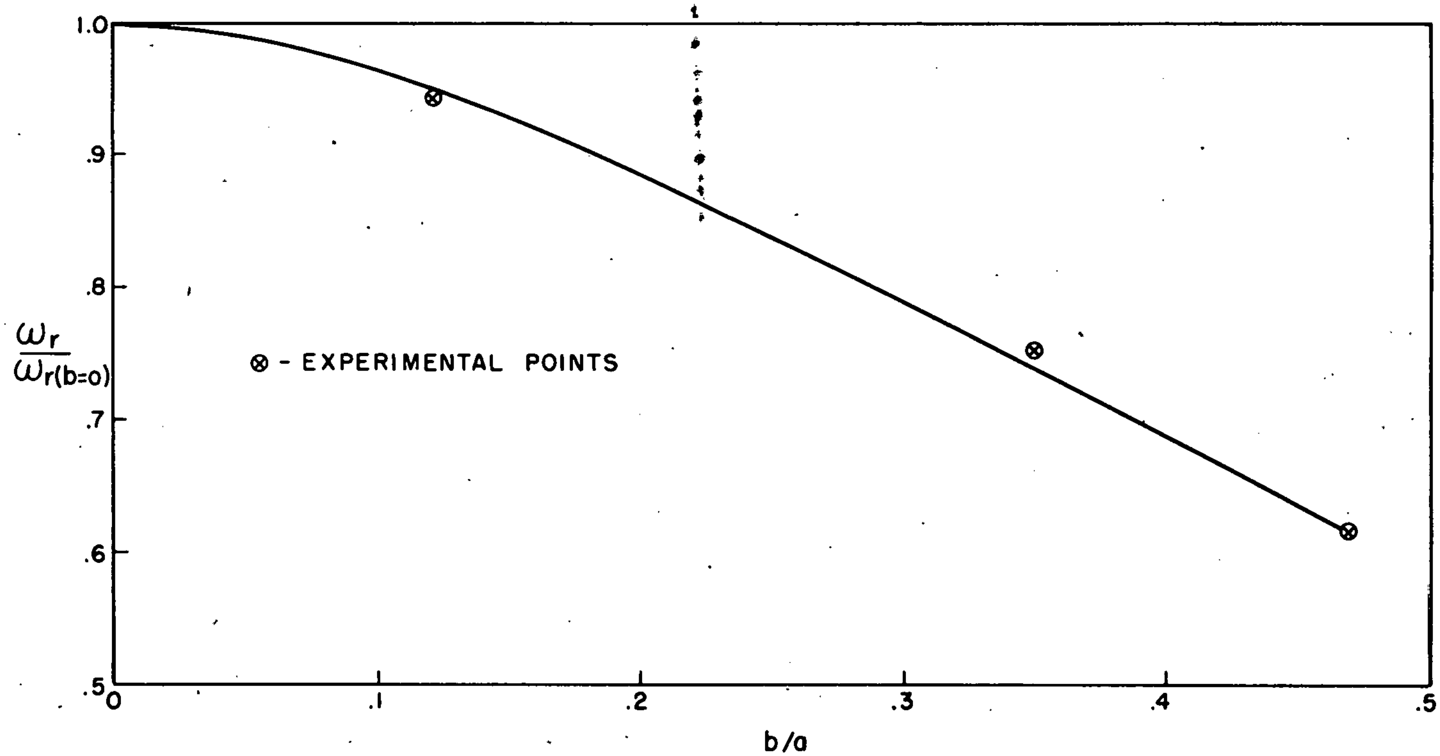

FIGURE 6 – PLOT OF THE RATIO OF THE RESONANT FREQUENCY OF A HOLLOW CYLINDER TO THE RESONANT FREQUENCY OF A SOLID DISC AS A FUNCTION OF THE RATIO b/a

the average of the resonant frequencies of other crystals of the same batch are plotted as experimental points. The agreement is within experimental error.

CONCLUSION

The mathematics has been developed for the determination of the radial coupling coefficient for a hollow cylinder of electrostrictive material whose length is small compared to its outside diameter. The relationship is $\frac{k^2}{1 - k^2} = C \frac{\Delta f}{f_r}$ where

k is the coupling coefficient f_r is the resonant frequency, Δf is the difference in frequency between resonant and antiresonant frequencies and C is a constant which is dependent on the ratio of outside to inside diameter.

C. V. STEPHENSON - 5143

Case No. 700 00

1-19

THIS PAGE
WAS INTENTIONALLY
LEFT BLANK