UR-524

SU(4), SU(8) MASS FORMULAS AND WEAK INTERACTIONS

MASTER

BY

SUSUMU OKUBO

THE UNIVERSITY OF ROCHESTER
DEPARTMENT OF PHYSICS AND ASTRONOMY
ROCHESTER, NEW YORK

DISCLAIMER

This report was prepared as an account of work sponsored by an agency of the United States Government. Neither the United States Government nor any agency Thereof, nor any of their employees, makes any warranty, express or implied, or assumes any legal liability or responsibility for the accuracy, completeness, or usefulness of any information, apparatus, product, or process disclosed, or represents that its use would not infringe privately owned rights. Reference herein to any specific commercial product, process, or service by trade name, trademark, manufacturer, or otherwise does not necessarily constitute or imply its endorsement, recommendation, or favoring by the United States Government or any agency thereof. The views and opinions of authors expressed herein do not necessarily state or reflect those of the United States Government or any agency thereof.

DISCLAIMER

Portions of this document may be illegible in electronic image products. Images are produced from the best available original document.

SU(4), -SU(8)-MASS-FORMULAS AND WEAK INTERACTIONS*

by

Susumu Okubo

Department of Physics and Astronomy,

University of Rochester,

Rochester, N. Y. 14627

Abstract

Mass formulas for SU(4) and SU(8) groups have been obtained, and are especially very simple for degenerate representations. Also, tensor representations of baryon multiplets are discussed together with some applications to weak decay processes.

* Work supported in part by the U. S. Atomic Energy Commission.
Individual copies will be supplied as long as supply lasts

DISTRIBUTION OF THIS DOCUMENT UNLIMITED

1. Infinitesimal Generators of U(n) Group

Recent spectacular discoveries¹ of ψ and ψ' renewed a great interest in the study of the SU(4) symmetry proposed by many authors² some years ago, since the SU(4) group is essential in the construction of a consistent unified gauge theory.^{3,4}

In this note, we will study mass formulas of the SU(4) as well as SU(8) groups. We shall also discuss some applications of the SU(4) for weak decay processes. As we shall see shortly, SU(4) mass formula is much more complicated than the ordinary SU(3) formula, and we shall consider mostly special classes of the SU(4) irreducible representations which are of physical relevance. With this in view, we shall, in this section, recapitulate some basic properties of the irreducible representation theory of the n-dimensional unitary group U(n). We shall consider here the U(n) group rather than the SU(n) group, since it is far easier to deal with for our purpose. Besides, the transition to its SU(n) sub-group is trivial. Some possible and physically different consequences between two groups will be discussed in Section 4.

Let A_{ν}^{μ} ($\mu, \nu = 1, 2, \dots, n$) be infinitesimal generators of the U(n) group satisfying the Lie commutation relation

$$[A_{\nu}^{\mu}, A_{\beta}^{\alpha}] = \delta_{\beta}^{\mu} A_{\nu}^{\alpha} - \delta_{\nu}^{\alpha} A_{\beta}^{\mu}. \quad (1.1)$$

If we are interested in the SU(n) sub-group, then we have to consider traceless generators B_{ν}^{μ} given by

$$B_{\nu}^{\mu} = A_{\nu}^{\mu} - \frac{1}{n} \delta_{\nu}^{\mu} \sum_{\lambda=1}^n A_{\lambda}^{\lambda} \quad (1.2)$$

which satisfy the same commutation relation as A_{ν}^{μ} .

Hereafter, we shall restrict ourselves to a given irreducible representation of the $U(n)$ specified by n integers $f_1, f_2, \dots, f_n$ satisfying

$$f_1 \geq f_2 \geq f_3 \geq \dots \geq f_n \quad (1.3)$$

The dimension N of the representation is calculable from the Weyl's formula⁵

$$N = \frac{\prod_{\mu < \nu}^n (\ell_{\mu} - \ell_{\nu})}{1! \cdot 2! \cdot \dots \cdot (n-1)!} \quad (1.4)$$

where we have set

$$\ell_{\lambda} = f_{\lambda} + n - \lambda, \quad \lambda = 1, 2, \dots, n. \quad (1.5)$$

If n^2 operators T_{ν}^{μ} ($\mu, \nu = 1, 2, \dots, n$) in this space satisfy commutation relation

$$[A_{\nu}^{\mu}, T_{\beta}^{\alpha}] = \delta_{\beta}^{\mu} T_{\nu}^{\alpha} - \delta_{\nu}^{\alpha} T_{\beta}^{\mu}, \quad (1.6)$$

then we call T_{ν}^{μ} to be a vector operator. Especially, A_{ν}^{μ} itself is a vector operator. For any two vector operators S_{ν}^{λ} and T_{ν}^{μ} , we can define a product vector operator R_{ν}^{μ} by

$$R_{\nu}^{\mu} = \sum_{\lambda=1}^n S_{\nu}^{\lambda} T_{\lambda}^{\mu}. \quad (1.7)$$

For simplicity, we shall hereafter write this relation simply as

$$R = ST, \quad (1.7')$$

dropping all Greek indices. The product thus defined obeys the associative law $(ST)U = S(TU)$ for three vector operators. Moreover, the unit vector operator I is given by

$$I_{\nu}^{\mu} = \delta_{\nu}^{\mu} E, \quad (1.8)$$

where E is an identity operator in our irreducible space. Then we can define j -th power A^j recursively by

$$A^{j+1} = A^j A, \quad A^0 = I, \quad j=0, 1, 2, \dots \quad (1.9)$$

Next, for any vector operator T_{ν}^{μ} , we can associate a scalar $\langle T \rangle$ by

$$\sum_{\lambda=1}^n T_{\lambda}^{\lambda} = \langle T \rangle E, \quad (1.10)$$

since $\sum_{\lambda=1}^n T_{\lambda}^{\lambda}$ commutes with all A_{ν}^{μ} so that it must be a constant multiple of a unit operator E due to the Schur's lemma. Especially, if we set

$$M_j^{(n)} = \langle A^j \rangle, \quad j=0, 1, 2, \dots, \quad (1.11)$$

then these are eigenvalues of generalized Casimir operators of the $U(n)$ group. Its explicit values have been computed by Louck and Biedenharn⁶ to be

$$M_j^{(n)} = \sum_{\lambda=1}^n (l_{\lambda})^j \prod_{\nu=1}^n \frac{1+l_{\nu}-l_{\lambda}}{l_{\nu}-l_{\lambda}}, \quad (1.12)$$

where the product on v omits the singular point $v = \lambda$.

Now as we have proved elsewhere⁷ (hereafter referred to as I), the $U(n)$ generators A_v^u satisfy a simple n -th order polynomial equation

$$A(\ell_1) A(\ell_2) \cdots A(\ell_{n-1}) A(\ell_n) = 0 \quad (1.13)$$

where $A(\ell_j)$ is defined by

$$A(\ell_j) = A - \ell_j I \quad (1.14)$$

and the product is the vector product in the sense of (1.7'). However, we can find a stronger identity, if we have $f_\mu = f_\nu$ for some $\mu \neq \nu$. Let us suppose that we have $f_{j+1} = f_j$ for some integer j . Then we call $A(\ell_j)$ to be redundant factor, and f_j to be the corresponding redundant signature. In such a case, we can omit⁷ all redundant factors in Eq. (1.13). As an illustration, let us consider a special case $n=8$ with

$$f_1 = f_2 = f_3 > f_4 = f_5 > f_6 > f_7 = f_8 \quad (1.15)$$

Then, all factors $A(\ell_1)$, $A(\ell_2)$, $A(\ell_4)$ and $A(\ell_7)$ are redundant, and we have a stronger identity

$$A(\ell_3) A(\ell_5) A(\ell_6) A(\ell_8) = 0 \quad (1.16)$$

Of course, the validity of (1.16) automatically implies that of (1.13) with $n=8$. As we have shown in I, the Eq. (1.16) is the minimum polynomial equation satisfied by A_v^u for this case.

Next, following the terminology of I, let us name a given irreducible representation to be degenerate, if we have

$$f_1 = f_2 = \dots = f_j > f_{j+1} = f_{j+2} = \dots = f_n \quad (1.17)$$

for an integer j . Then, the above redundant factor rule implies the validity of

$$A(l_j) A(l_n) = 0 \quad (1.18)$$

or equivalently of

$$\sum_{\lambda=1}^n A_{\nu}^{\lambda} A_{\lambda}^{\mu} = (l_j + l_n) A_{\nu}^{\mu} - l_j l_n \delta_{\nu}^{\mu} E, \quad (1.18')$$

The most interesting special case of the degenerate representation is that of the completely symmetric representation with $f_2 = f_3 = \dots = f_n = 0$, i.e., $f_j = 0$ for $j \neq 1$, where we can prove^{7,8} an extra stronger relation

$$(A_{\nu}^{\mu} + \delta_{\nu}^{\mu}) A_{\rho}^{\alpha} = (A_{\rho}^{\mu} + \delta_{\rho}^{\mu}) A_{\nu}^{\alpha}. \quad (1.19)$$

We remark that such a simple relation does not necessarily apply for other degenerate representations. If we relax the condition to a weaker one $f_3 = f_4 = \dots = f_n = 0$, i.e., $f_j = 0$ for $j \neq 1, 2$, we find a slightly more complicated formula⁹

$$\begin{aligned} & (A_{\nu}^{\mu} + \delta_{\nu}^{\mu}) K_{\rho}^{\alpha} + (A_{\rho}^{\alpha} + \delta_{\rho}^{\alpha}) K_{\nu}^{\mu} \\ &= (A_{\rho}^{\mu} + \delta_{\rho}^{\mu}) K_{\nu}^{\alpha} + (A_{\nu}^{\alpha} + \delta_{\nu}^{\alpha}) K_{\rho}^{\mu} \end{aligned} \quad (1.20a)$$

where K_{ν}^{μ} is given by

$$K_{\nu}^{\mu} = (AA)_{\nu}^{\mu} - \frac{1}{2} (f_1 + f_2 + 2n - 4) A_{\nu}^{\mu} \quad (1.20b)$$

We note that (1.20) is consistent with (1.15') and (1.19) for a special case, $f_2=0$. Also, it becomes a trivial identity for the case $f_1=f_2$ in view of (1.18').

Last, we can prove⁷ that any vector operator T_ν^μ can be expressed as

$$T_\nu^\mu = \sum_{j=0}^{n-1} a_j (A^j)_\nu^\mu \quad (1.21)$$

in the irreducible representation space under consideration where a_j ($j=0,1,2,\dots,n-1$) are some unspecified constants. Because of (1.18) this formula can be further simplified to become

$$T_\nu^\mu = a \delta_\nu^\mu E + b A_\nu^\mu \quad (1.22)$$

for any degenerate representation.

2. SU(4) Mass Formula

As in the SU(3) theory, we assume that the SU(4) breaking interaction H_1 has the following tensor structure

$$H_1 = T_3^3 + \gamma T_4^4 \quad (2.1)$$

where γ is a real constant. Then, as we noted in (1.21), T_ν^μ can be expressed as

$$T_\nu^\mu = a \delta_\nu^\mu E + b A_\nu^\mu + c (AA)_\nu^\mu + d (AAA)_\nu^\mu \quad (2.2)$$

in any given irreducible representation so that the most general SU(4)

mass formula contains 5 unknown parameters. Now, physical states are labeled by quantum numbers specified by the canonical chain decomposition

$$U(4) \supset U(3) \supset U(2) \supset U(1) \quad (2.3)$$

Explicit expressions for matrix elements of $(AA)_4^4$ and $(AAA)_4^4$ in this canonical decomposition can be easily obtained.¹⁰ However, the difficulty is to evaluate matrix elements of operators $(AA)_3^3$ and $(AAA)_3^3$ in the closed form. Of course, any matrix element of generator A_V^u and hence of $(AA)_4^4$ and $(AAA)_4^4$ can be calculated in principle from the general formula given by Baird and Biedenharn.¹¹ However, this procedure is in reality very complicated to be of much practical use, as has recently been noted by Bose¹², who also derived Eq. (2.2). Hence, we shall restrict ourselves to special classes of representations which are of physical interest.

First, let us consider the case of degenerate representations:

(a) Degenerate Representations

As we noted in the previous section, this implies that we have either $f_1 > f_2 = f_3 = f_4$, or $f_1 = f_2 > f_3 = f_4$, or $f_1 = f_2 = f_3 > f_4$. Especially, this class contains 20-dimensional representations (3,0,0,0), and 4 dimensional one (1,1,1,0) for baryons as well as another 20 dimensional representation (1,1,-1,-1) for possible exotic bosons. For degenerate representations, we can express $(AA)_V^u$ and $(AAA)_V^u$ in terms of $\delta_V^u E$ and A_V^u because of Eq. (1.18'). Therefore, (2.2) can be simplified to become Eq. (1.22), i.e.,

$$T_V^u = a \delta_V^u E + b A_V^u.$$

As a result, we have the following mass formula

$$M = M_0 + b (N_3 + \gamma N_4) \quad (2.5a)$$

where we have set

$$N_j = A_j^j, \quad j = 1, 2, 3, 4. \quad (2.5b)$$

Note that in the quark model, N_j is simply the number operator of the j -th quark. Then, our mass formula (2.5) implies that it gives exactly the same answer as is given by a naive additive quark model where the mass difference in the multiplet results solely from masses of constituent quarks. In that case, the parameter y is related to the bare quark mass m_j for the j -th quark q_j by

$$y = \frac{m_4 - m_1}{m_3 - m_1}. \quad (2.6)$$

As usual, we identify the completely symmetric representation $(3,0,0,0)$ as the lowest $J^P = \frac{3}{2}^+$ baryon multiplet. Then, the mass formula (2.5) gives the familiar equal spacing mass rule. However, since their explicit mass relations have been given elsewhere¹³, we shall not reproduce them here. The same equal-spacing relation also applies for completely anti-symmetric case $(1,1,1,0)$, of which $Y_0^*(1405 \text{ MeV})$ is presumably a member. With respect to bosons, so far no exotic state has been found. However, Iwasaki¹⁴ attempts to classify $\psi'(3.700 \text{ MeV})$ in an exotic representation which may correspond to $(1,1,-1,-1)$. In that case, we can again apply our formula (2.5) with $b=0$ since the mass of the bosons must be invariant under $N_j \rightarrow -N_j$ because of the charge conjugation invariance.

For completely symmetric representation $(3,0,0,0)$, we could considerably simplify second-order mass formula for the second-order perturbation interaction of the form

$$H_2 = T_{33}^{33} + \alpha T_{44}^{44} + \beta T_{34}^{34}. \quad (2.7)$$

For this case, we can utilize (1.18) as well as (1.19) to reduce complicated tensors formed out of generators A_V^μ . Then the final result is simply to add terms proportional to $A_3^3 A_3^3$, $A_4^4 A_4^4$ and $A_3^3 A_4^4$ to our mass formula (2.5a). We need not consider a term $A_3^4 A_4^3$ since (1.19) leads to

$$A_3^4 A_4^3 = A_3^3 A_4^4 + A_3^3$$

The situation is very much analogous¹⁵ to the electromagnetic mass difference of the decouplet baryon state in the SU(3) symmetry.

(b) Representations with One Degeneracy

This is the case where only one pair among f_1 , f_2 , f_3 and f_4 coincide i.e., we have either $f_1=f_2>f_3>f_4$, or $f_1>f_2=f_3>f_4$ or $f_1>f_2>f_3=f_4$.

Especially, this class contains physically interesting cases of 20'-dimensional baryon representation (2,1,0,0) with $f_3=f_4=0$ and 15-dimensional boson representation (1,0,0,-1) with $f_2=f_3=0$. Then, our redundant factor rule implies that we can express now $(AAA)_V^\mu$ in a linear combination of $\delta_V^\mu E$, A_V^μ and $(AA)_V^\mu$. Therefore, the mass formula is rewritten as

$$M = M_0 + b (A_3^3 + \gamma A_4^4) + c [(AA)_3^3 + \gamma (AA)_4^4] \quad (2.8)$$

In terms of the baryon number B, hypercharge Y and the charm quantum number C, we can identify¹⁶

$$\begin{aligned} N_3 = A_3^3 &= B - C - Y, \quad N_4 = A_4^4 = C, \\ \sum_{\lambda=1}^4 A_\lambda^\lambda &= 3B \end{aligned} \quad (2.9)$$

Then, $(AA)_4^4$ is easily computed as in the SU(3) case^{17,10} to be

$$2(AA)_4^4 = M_2^{(4)} - M_1^{(4)} - M_2^{(3)} + (N_4)^2 + 4N_4 \quad (2.10)$$

where the j -th Casimir operator $M_j^{(n)}$ of the $U(n)$ group with $n=3,4$ can be evaluated by means of Eq. (1.12). Note that $M_2^{(3)}$ is the second-order Casimir operator of the $U(3)$ group contained in the canonical chain (2.3).

Similarly, $(AA)_3^3$ can be expressed as

$$2(AA)_3^3 = M_2^{(3)} - M_1^{(3)} - M_2^{(4)} + (M_3)^2 + 2N_3 + 2A_3^4 A_4^3 \quad (2.11)$$

or equivalently¹⁷ as

$$(AA)_3^3 = \frac{1}{4} Y^2 - I(I+1) - (2B - C + \frac{3}{2}) Y - (1+B) C + \frac{1}{2} (C^2 - B^2) + \frac{1}{2} M_2^{(3)} + A_3^4 A_4^3. \quad (2.11')$$

However, the difficult part is the evaluation of the matrix element $A_3^4 A_4^3$ contained in (2.11). As we remarked earlier, we could compute its matrix element from the formula of Baird and Biedenharn¹¹ or from that of Sen¹⁸. But the general expression proves to be too complicated to be of practical use, the main reason being that two irreducible representations of the $U(3)$ sub-group now mix together by this operator. Hence, it is far easier in practice to handle specific cases separately.

(i) Representation (2,1,0,0)

This is a 20-dimensional representation which contains the following $U(3)$ components:

$$(2, 1, 0) \oplus (2, 0, 0) \oplus (1, 1, 0) \oplus (1, 0, 0) \\ = \underline{8} \oplus \underline{6} \oplus \underline{\bar{3}} \oplus \underline{3}. \quad (2.12)$$

Note that except for the 8 dimensional case, all other $U(3)$ components are degenerate. Then, the final mass formula is calculated to be

$$\begin{aligned}
 \underline{8} : M(8) &= m_0 + \alpha Y + \beta [I(I+1) - \frac{1}{4} Y^2], \\
 \underline{6} : M(6) &= m_0 - (\gamma-1)\alpha + \frac{1}{2}(3\gamma+1)\beta + \alpha Y, \\
 \underline{3} : M(3) &= m_0 - 2(\gamma-1)\alpha - \beta + (\alpha - \frac{3}{2}\beta)Y, \\
 \underline{\bar{3}} : M(\bar{3}) &= m_0 - (\gamma-1)\alpha - \frac{1}{2}(\gamma-1)\beta + (\alpha - \beta)Y, \\
 \underline{6-\bar{3}} : M(\bar{3}-6) &= -\frac{\sqrt{3}}{2} \beta \sum_{j=1}^2 (|S_j\rangle \langle A_j| + |A_j\rangle \langle S_j|),
 \end{aligned} \tag{2.13}$$

where m_0 , α , and β are some unknown parameters, and $M(6-\bar{3})$ is the mass mixing operator between two states $|S_j\rangle$ and $|A_j\rangle$ ($j=1,2$) with $I=\frac{1}{2}$, $Y=1$ and $C=1$ belonging to $\underline{6}$ and $\underline{\bar{3}}$ representations, respectively. The index $j=1,2$, refers to two isotopic spin states $I_z = \pm \frac{1}{2}$. Also S and A designate symmetric and antisymmetric tensor states, respectively, as we will show in Section 4.

If we diagonalize the mixing problem, we can obtain physical masses in terms of three unknown parameters, m_0 , α , and β . When we eliminate these, we find various mass relations which have been given in the references 4 and 13. We should also mention that the mixing between S and A is very small¹³ and we can practically ignore its effect.

(ii) Representation (1,0,0,-1)

This contains the following U(3) components:

$$\begin{aligned}
 (1, 0, -1) \oplus (1, 0, 0) \oplus (0, 0, -1) \oplus (0, 0, 0) \\
 = \underline{8} \oplus \underline{3} \oplus \underline{\bar{3}} \oplus \underline{1}.
 \end{aligned} \tag{2.14}$$

Now, the operator $A_3^4 A_4^3$ mixes $\underline{8}$ and $\underline{1}$ multiplets. After some calculations, we find the following mass formulas

$$\begin{aligned}
 \underline{8} : M(8) &= m_0 + \beta [I(I+1) - \frac{1}{4} Y^2], \\
 \underline{1} : M(1) &= m_0 + \frac{1}{4} \beta (7-9\gamma), \\
 \underline{1-\bar{8}} : M(1-\bar{8}) &= \frac{1}{\sqrt{2}} \beta (|w_0\rangle \langle w_2| + |w_2\rangle \langle w_0|), \\
 \underline{3}, \underline{\bar{3}} : M(3) = M(\bar{3}) &= m_0 + \frac{1}{2} \beta (1-3\gamma) + 3\beta [I(I+1) - \frac{1}{4} Y^2],
 \end{aligned} \tag{2.15}$$

where we assumed the charge-conjugation invariance of the mass. Diagonalizing the mixing in the sector containing $\underline{1}$ and $\underline{\bar{6}}$, and eliminating unknown parameters m_0 and β , we find one mass relation

$$\begin{aligned} & \left[\phi - \frac{1}{3} (4K^* - \rho) \right] \left[\omega - \frac{1}{3} (4K^* - \rho) \right] \\ &= -\frac{2}{9} (K^* - \rho)^2, \end{aligned} \quad (2.16)$$

which has been originally found by Bjorken and Glashow² and by Gerstein and Whippman¹⁹. Note that this formula differs from the more familiar Schwinger mass formula¹⁹ by a factor of 4 on the right hand side. For 1^- nonet, this formula is badly satisfied. Moreover, (2.15) predicts very low masses for charmed mesons with masses lower than that of ρ . Hence, the pure 15-plet assignment for 1^- mesons is ruled out. Indeed, more popular assignment^{20,21} is to assume that the 1^- nonet as well as $\psi(3100 \text{ MeV})$ form a part of $\underline{15} \oplus \underline{1}$ dimensional multiplet of the $U(4)$ group. In this case, we have to consider an additional mixing between the 15-plet and the singlet. Since this problem has been discussed by many authors^{20,21}, I will not repeat it.

With respect to 0^- mesons, the mass formula (2.16) is well-satisfied for nonet π , K , $\bar{K}$, η and $\eta'(960 \text{ MeV})$. However, (2.15) again predicts then very low masses ($\approx 750 \text{ MeV}$) for charmed mesons. Besides, this assignment gives the value $y=2.43$ which is at variance with the value $y=20.7$ determined^{13,21} from $\underline{15} \oplus \underline{1}$ plets of 1^- mesons. For these reasons, we have to assign $\underline{15} \oplus \underline{1}$ structure rather than a pure 15-plet also for 0^- bosons.

3. SU(8) Mass Formula

As a generalization of the SU(6) quark model²², we may consider SU(8) group.²³

Let us designate the capital Latin indices A and B to represent pairs such as

$$A = (\mu, j), \quad B = (\nu, k), \quad (3.1)$$

where the Greek indices $\mu, \nu = 1, 2, 3, 4$ refer to the U(4) group, while $j, k = 1, 2$ refer to the U(2) spin sub-group of the U(8). In order to avoid possible confusion, we shall now use a notation X_B^A ($A, B=1, 2, \dots, 8$) to represent infinitesimal generators of the U(8) group.

As usual, we assume that the U(8)-breaking interaction is given by

$$H_1 = T_{(3,j)}^{(3,j)} + \gamma T_{(4,j)}^{(4,j)} + S_{(\mu,k),(\nu,j)}^{(\mu,j),(\nu,k)}, \quad (3.2)$$

where the tensor S_{CD}^{AB} ($A, B, C, D = 1, 2, \dots, 8$) represents spin-spin interaction between two quarks, and repeated indices over j and k and over μ and ν imply automatical summations on values 1, 2, and on values 1, 2, 3, 4, respectively.

We can write down the most general expressions for H_1 in terms of the U(8) generators X_B^A . However, the result is too complicated with little gain so that we shall concentrate on the most interesting case of the completely symmetric baryon representation with $f_1=3$, $f_j=0$ ($j=2, 3, \dots, 8$), corresponding to 120-dimensional degenerate representation. As has been noted by many authors²³, this has a nice decomposition.

$$\underline{120} = (\underline{20}', 2) \oplus (\underline{20}, 4)$$

with respect to the chain $U(8) \supset U(4) \otimes SU(2)$. For the completely

symmetric representation, we can express $(XX)_F^E$ in terms of δ_F^E and X_F^E because of (1.18'). Also, (1.19) is now rewritten as

$$X_c^B X_q^F - X_q^B X_c^F = \delta_q^B X_c^F - \delta_c^B X_q^F, \quad (3.3)$$

Defining $U(4)$ and $U(2)$ generators by⁹

$$\begin{aligned} A_\nu^\mu &= \sum_{j=1}^2 X_{(\nu,j)}^{(\mu,j)}, \quad \mu, \nu = 1, 2, 3, 4, \\ B_k^j &= \sum_{\mu=1}^4 X_{(\mu,k)}^{(\mu,j)}, \quad j, k = 1, 2, \end{aligned} \quad (3.4)$$

the general mass formula based upon (3.2) is rewritten as

$$M = m_0 + \alpha (A_3^3 + \gamma A_4^4) + \beta J(J+1), \quad (3.5)$$

if we use (3.3) and (1.18) together with (3.4). Here, J is the spin of the system given by

$$\begin{aligned} J(J+1) &= \frac{1}{2} \sum_{j,k=1}^2 B_k^j B_j^k - \frac{1}{4} B^2, \\ B &= \sum_{j=1}^2 B_j^j = \sum_{\mu=1}^4 A_\mu^\mu. \end{aligned} \quad (3.6)$$

Our formula (3.5) predicts exactly the same mass relations as the ordinary simple quark model. Especially, it leads to $m(\Lambda) = m(\Sigma)$ which is typical of the quark model. If we want to uplift this degeneracy, then we have only to introduce additional interaction of the form:

$$H_2' = S_{(3,j),(\nu,k)}^{(3,j),(\nu,k)} + T_{(4,j),(\nu,k)}^{(4,j),(\nu,k)} \quad (3.7)$$

Then, again utilizing the identity (3.3), the effect of H_2' is essentially equivalent to add new terms

$$\alpha'(AA)_3^3 + \beta'(N_4)^2 + \gamma'N_3 + \delta'N_4 + e'E \quad (3.8)$$

to the right hand side of (3.5). The evaluation of the matrix element of $(AA)_3^3$, has been already performed in the previous section. The final formula is very analogous to (but slightly different from) mass formulas proposed by some authors.²³

With respect to bosons, we may identify them to belong to a representation with a signature $f_1=1$, $f_2=f_3=\dots=f_7=0$ and $f_8=-1$. Then, 5 signatures f_2, f_3, f_4, f_5 and f_6 are redundant so that our redundant factor rule demands now the validity of

$$(XXX)_B^A = a \delta_B^A E + b X_B^A + c (XX)_B^A \quad (3.9)$$

However, the evaluation of these matrix elements is rather involved.

Besides we have an additional complication of the following nature.

The multiplet decomposes into²³

$$\underline{63} = (\underline{1}, 3) \oplus (\underline{15}, 3) \oplus (\underline{15}, 1)$$

under $U(8) \rightarrow U(4) \otimes SU(2)$. This implies that we have 15-plets of 1^- and 0^- mesons as well as a 1^- singlet. However, a pure 15-plet for the 0^- mesons leads to various undesirable results as we have already noted in the previous section. Therefore, we have to consider the $\underline{63} \oplus \underline{1}$ multiplet instead of a pure $\underline{63}$ -plet assignment. This gives an additional mixing problem, just as in the $U(4)$ case. The treatment of such a case

can be better handled by the tensor method which will be explained in the next section. However, we will not discuss the boson case further in this note.

4. Tensor Representations

In many calculations involving the $U(4)$ group, the tensor notation is quite often very convenient. First, let us consider the representation $(3,0,0,0)$, corresponding to the completely symmetric tensor $S_{\mu\nu\lambda}$,

$$S_{\mu\nu\lambda} = S_{\nu\mu\lambda} = S_{\lambda\nu\mu} = S_{\mu\lambda\nu} \quad (4.1)$$

Then, properly orthonormalized state is simply given by

$$|n_1, n_2, n_3, n_4\rangle = \left[\frac{3!}{n_1! n_2! n_3! n_4!} \right]^{\frac{1}{2}} S_{\mu\nu\lambda} \quad (4.2)$$

where n_j is the number of the integer j ($=1,2,3,4$) contained in the indices μ, ν, λ . For example, we have $n_1=2, n_2=1, n_3=n_4=0$ for S_{112} .

A slightly more complicated case is the representation $(2,1,0,0)$. This is specified by a tensor $\psi_{\mu\nu\lambda}$ satisfying conditions.

$$\psi_{\mu\nu\lambda} + \psi_{\nu\lambda\mu} + \psi_{\lambda\mu\nu} = 0 \quad (4.3)$$

$$\psi_{\mu\nu\lambda} = \psi_{\nu\mu\lambda} \quad (4.4)$$

Properly orthonormalized states under the chain decomposition (2.3) are given below:

(a) 8: (C=0)

i) $I=1, Y=0$,

$$\Sigma^+ = \psi_{113}, \quad \Sigma^0 = \sqrt{2} \psi_{123}, \quad \Sigma^- = \psi_{223}$$

ii) $I=0, Y=0$,

$$\Lambda = \sqrt{\frac{2}{3}} (\psi_{321} - \psi_{312})$$

(4.5a)

iii) $I=\frac{1}{2}, Y=1$; $p = \psi_{112}, \quad n = \psi_{221}$,

iv) $I=\frac{1}{2}, Y=-1$; $\Xi^0 = \psi_{331}, \quad \Xi^- = \psi_{332}$,

(b) 6: (C=1)

i) $I=1, Y=0$,

$$B^{++} = \psi_{114}, \quad B^+ = \sqrt{2} \psi_{124}, \quad B^0 = \psi_{224}$$

ii) $I=\frac{1}{2}, Y=-1$; $B^+ = S_1 = \sqrt{2} \psi_{134}, \quad B^0 = S_2 = \sqrt{2} \psi_{234}$, (4.5b)

iii) $I=0, Y=-2$; $B^0 = \psi_{334}$

(c) 3: (C=1)

i) $I=\frac{1}{2}, Y=-1$,

$$B^+ = A_1 = \sqrt{\frac{2}{3}} (\psi_{413} - \psi_{431}), \quad B^0 = A_2 = \sqrt{\frac{2}{3}} (\psi_{423} - \psi_{432})$$
 (4.5c)

ii) $I=0, Y=0$,

$$B^+ = A^+ = \sqrt{\frac{2}{3}} (\psi_{421} - \psi_{412})$$
,

(d) 3: (C=2)

i) $I=\frac{1}{2}, Y=-1$,

$$B^{++} = \psi_{441}, \quad B^+ = \psi_{442},$$

(4.5d)

ii) $I=0, Y=-2$,

$$B^+ = \psi_{443}$$

For $\underline{6}$ and $\bar{3}$: S_j and A_j ($j=1,2$) are the same objects we encountered in section 2. Actually, we have to diagonalize the mass matrix as a linear combination of S_j and A_j because of the mixing. However, since the mixing is very small,¹³ we can neglect the complication in practice.

We have normalized our states as

$$\sum_{j=1}^{20} N_j^* N_j = \frac{2}{3} \sum_{\lambda, \mu, \nu=1}^4 \psi_{\mu\nu\lambda}^* \psi_{\mu\nu\lambda}, \quad (4.6)$$

where N_j ($j=1,2,\dots,20$) are twenty states listed in (4.5). This can be easily shown if we utilize Eqs. (4.3) and (4.4) which especially lead to a relation such as

$$\psi_{jjkk} = -2 \psi_{kkjj} = -2 \psi_{jkjk}, \quad j \neq k.$$

We can easily reproduce the mass formula of the third section by the tensor method¹⁷ if we write the mass operator as

$$\begin{aligned} M = & \frac{2}{3} m_0' \sum_{\lambda, \mu, \nu=1}^4 \psi_{\mu\nu\lambda}^* \psi_{\mu\nu\lambda} \\ & + \alpha' \sum_{\lambda, \nu=1}^4 (\psi_{3\nu\lambda}^* \psi_{3\nu\lambda} + \gamma \psi_{4\nu\lambda}^* \psi_{4\nu\lambda}) \\ & + \beta' \sum_{\lambda, \nu=1}^4 (\psi_{3\nu\lambda}^* \psi_{3\nu\lambda} + \gamma \psi_{4\nu\lambda}^* \psi_{4\nu\lambda}) \end{aligned} \quad (4.7)$$

with some unknown parameters m_0' , α' and β' .

We remark that the $(2,1,0,0)$ representation can be also represented by another tensor $\phi_{\mu\nu\lambda}$ satisfying

$$\phi_{\mu\nu\lambda} + \phi_{\nu\lambda\mu} + \phi_{\lambda\mu\nu} = 0, \quad (4.8)$$

$$\phi_{\mu\nu\lambda} = -\phi_{\nu\mu\lambda} \quad (4.9)$$

instead of $\psi_{\mu\nu\lambda}$ satisfying (4.3) and (4.4). However, they are related to each other by

$$\begin{aligned}\phi_{\mu\nu\lambda} &= \frac{1}{\sqrt{3}} (\psi_{\mu\lambda\nu} - \psi_{\nu\lambda\mu}), \\ \psi_{\mu\nu\lambda} &= \frac{1}{\sqrt{3}} (\phi_{\mu\lambda\nu} + \phi_{\nu\lambda\mu}).\end{aligned}\tag{4.10}$$

In terms of $\phi_{\mu\nu\lambda}$, we can rewrite (4.5). For example, we find

$$\Lambda = \sqrt{2} \phi_{213}, \quad \Sigma_0 = \sqrt{\frac{2}{3}} (\phi_{132} + \phi_{231}).$$

Next, as is well-known⁵, all $U(4)$ representations with signature $(f_1 + e, f_2 + e, f_3 + e, f_4 + e)$ represent the same single $SU(4)$ representation for all integral values of e . Especially choosing $e = -1$, we see that $(2, 1, 0, 0)$ is equivalent to $(1, 0, -1, -1)$ as far as the $SU(4)$ sub-group is concerned. This implies that we could have used a traceless tensor $T_{\lambda}^{\mu\nu}$ satisfying

$$\begin{aligned}T_{\lambda}^{\mu\nu} &= -T_{\lambda}^{\nu\mu}, \\ \sum_{\lambda=1}^4 T_{\lambda}^{\lambda\nu} &= 0\end{aligned}\tag{4.11}$$

instead of $\psi_{\mu\nu\lambda}$ or $\phi_{\mu\nu\lambda}$ to describe the $J^P = \frac{1}{2}^+$ baryon multiplet. Indeed, we can achieve this fact by setting

$$T_{\lambda}^{\mu\nu} = \frac{1}{2} \sum_{\alpha, \beta=1}^4 \epsilon^{\mu\nu\alpha\beta} \phi_{\alpha\beta\lambda},\tag{4.12}$$

Conversely, we can express $\phi_{\mu\nu\lambda}$ by

$$\phi_{\mu\nu\lambda} = \frac{1}{2} \sum_{\alpha, \beta=1}^4 \epsilon_{\mu\nu\alpha\beta} T_{\lambda}^{\alpha\beta},\tag{4.13}$$

where $\epsilon_{\mu\nu\alpha\beta}$ and $\epsilon^{\mu\nu\alpha\beta}$ are completely anti-symmetric tensors which are $SU(4)$ invariant but have signatures $(1, 1, 1, 1)$ and $(-1, -1, -1, -1)$ respectively under the $U(4)$ group. In terms of the quark model, $\phi_{\mu\nu\lambda}$ and $\psi_{\mu\nu\lambda}$ can be regarded as three quark system $q_{\mu} q_{\nu} q_{\lambda}$, while the tensor

$T_{\lambda}^{\mu\nu}$ corresponds to $\bar{q}_{\mu}\bar{q}_{\nu}q_{\lambda}$. Therefore, we conclude that the $J^P = \frac{1}{2}^+$ baryon multiplet could be represented by either of three quark systems with form $q_{\mu}q_{\nu}q_{\lambda}$ or $\bar{q}_{\mu}\bar{q}_{\nu}q_{\lambda}$. We note that the latter form has been suggested by most authors² in the early formulation of the SU(4) theory. As far as the SU(4) symmetry is concerned, we cannot tell the difference between two forms. However, other quantities such as electric charge and/or baryon quantum numbers depend upon the U(4) assignment rather than the SU(4), and they could lead to different consequences. In view of various successes of the $q_{\mu}q_{\nu}q_{\lambda}$ quark model, we take, however, the view that the baryon is really represented by the (2,1,0,0) rather than (1,0,-1,-1) representations of the U(4) group. We especially note that for the latter case, the $J^P = \frac{3}{2}^+$ multiplet must correspond to (2,-1,-1,-1) (rather than (3,0,0,0) multiplet) which corresponds to a complicated tensor $T_{\alpha\beta}^{\mu\nu\lambda} = \bar{q}_{\mu}\bar{q}_{\nu}\bar{q}_{\lambda}q_{\alpha}q_{\beta}$.

With respect to the $\underline{15} \oplus \underline{1}$ dimensional 0^- mesons, they can be represented by a tensor P_{ν}^{μ} which is not traceless. In the quark model, we can write it in a form $P_{\nu}^{\mu} = \bar{q}_{\mu}q_{\nu}$. A similar remark also applies to 1^- vector multiplet. Since the mass formulas based upon this identification have been investigated by various authors^{14,20,21}, we will not go into detail.

5. Applications to Weak Interactions

In the model of Glashow, Iliopoulos, and Miani³ (hereafter referred to as (G-I-M)), the charged hadronic weak currents are given by

$$j_{\lambda} = \cos \theta [\bar{\ell}_1 Q_{\lambda} \ell_2 + \bar{\ell}_4 Q_{\lambda} \ell_3] + \sin \theta [\bar{\ell}_1 Q_{\lambda} \ell_3 - \bar{\ell}_4 Q_{\lambda} \ell_2] \quad (5.1)$$

$$\begin{aligned} \bar{j}_\lambda &= \cos \theta [\bar{\ell}_2 Q_\lambda \ell_1 + \bar{\ell}_3 Q_\lambda \ell_4] \\ &\quad + \sin \theta [\bar{\ell}_3 Q_\lambda \ell_1 - \bar{\ell}_2 Q_\lambda \ell_4], \\ Q_\lambda &= \gamma_\lambda (1 + \gamma_5). \end{aligned} \quad (5.2)$$

in terms of four quarks q_μ , where θ is the Cabibbo angle. Then, the non-leptonic weak interaction responsible for $\Delta S \neq 0$ and/or $\Delta C \neq 0$ is given by

$$H_W = \frac{1}{\sqrt{2}} G_F j_\lambda \bar{j}_\lambda. \quad (5.3)$$

Also, the neutral current $j_\lambda^{(0)}$ and electromagnetic current $j_\lambda^{(em)}$ are expressed by

$$j_\lambda^{(0)} = \bar{\ell}_1 Q_\lambda \ell_1 + \bar{\ell}_4 Q_\lambda \ell_4 - \bar{\ell}_2 Q_\lambda \ell_2 - \bar{\ell}_3 Q_\lambda \ell_3, \quad (5.4)$$

$$j_\lambda^{(em)} = \frac{2}{3} (\bar{\ell}_1 \gamma_\lambda \ell_1 + \bar{\ell}_4 \gamma_\lambda \ell_4) - \frac{1}{3} (\bar{\ell}_2 \gamma_\lambda \ell_2 + \bar{\ell}_3 \gamma_\lambda \ell_3). \quad (5.5)$$

Let us now consider the following Weyl reflection:

$$W : \ell_1 \leftrightarrow -\ell_2, \quad \ell_3 \leftrightarrow \ell_4 \quad (5.6)$$

which is a special finite $U(4)$ transformation so that it defines an inner auto-morphism of the $U(4)$ group. Under this operation, we find

$$W : j_\lambda \leftrightarrow \bar{j}_\lambda, \quad j_\lambda^{(0)} \leftrightarrow -j_\lambda^{(0)} \quad (5.7)$$

However, $j_\lambda^{(em)}$ does not transform into itself. When we define

$$\begin{aligned} A^\dagger &= \sqrt{\frac{2}{3}} (\mathcal{V}_{421} - \mathcal{V}_{412}), \\ D^\dagger &= P_4^2, \quad D_1^\dagger = \frac{1}{\sqrt{2}} (P_4^1 - P_1^4), \end{aligned} \quad (5.8)$$

then under W , these transform as

$$\begin{aligned} W: \quad A^+ &\leftrightarrow \Lambda, & p &\leftrightarrow n, & \pi^+ &\leftrightarrow \pi^-, \\ D^+ &\leftrightarrow -\bar{K}^+, & D_1^0 &\leftrightarrow -K_1^0, & \pi^0 &\leftrightarrow -\pi^0. \end{aligned} \quad (5.9)$$

Note that A^+ is the particle with $I=0$ and $Y=0$ belonging to the representation $\bar{3}$, which Gaillard et al.⁴ designated as C_0^+ . The transformation $A^+ \leftrightarrow \Lambda$ is obvious from the explicit tensor representation (4.5).

Because of (5.7) and (5.9), we have relations

$$\begin{aligned} M(A^+ \rightarrow p \pi^0) &= -M(\Lambda \rightarrow n \pi^0), \\ M(A^+ \rightarrow n \bar{\ell} \nu) &= M(\Lambda \rightarrow p \ell \bar{\nu}) \end{aligned} \quad (5.10)$$

for the decay matrix elements of A^+ particles and

$$\begin{aligned} M(D^+ \rightarrow \pi^+ \pi^+ \pi^-) &= -M(\bar{K}^+ \rightarrow \pi^- \pi^- \pi^+), \\ M(D^+ \rightarrow \pi^+ \pi^0) &= M(\bar{K}^+ \rightarrow \pi^- \pi^0), \\ M(D_1^0 \rightarrow \pi^+ \pi^-) &= -M(K_1^0 \rightarrow \pi^- \pi^+), \\ M(D^+ \rightarrow \pi^0 \bar{\ell} \nu) &= M(\bar{K}^+ \rightarrow \pi^0 \ell \bar{\nu}) \end{aligned} \quad (5.11)$$

for the decays of $D^{+,0}$ in the exact $U(4)$ limit.

We can compute the decay rate of $A^+ \rightarrow p \pi^0$ as well as its asymmetry parameter α_A from the corresponding decay $\Lambda \rightarrow n \pi^0$ by means of (5.10).

However, we do not know how good or how bad the exact $U(4)$ will be.

Here, we take a view that when we express decay matrix elements such as $M(A^+ \rightarrow p \pi^0)$ in terms of Lorentz invariant amplitudes, then we can apply the consequences of the exact $U(4)$ symmetry for the amplitudes if they are dimensionless quantities. For example, writing

$$M(A^+ \rightarrow p \pi^0) = \left(\frac{M_A M_p}{2 E_A E_p E_\pi \sqrt{3}} \right)^{\frac{1}{2}} \bar{u}_p (S + \gamma_5 P) u_A,$$

we find that Lorentz-invariant amplitudes S and P are also dimensionless. Therefore, we can compute them from the known decay parameters²⁴ of $\Lambda \rightarrow n\pi^0$. Assuming $M_\Lambda \approx 2.90$ GeV as is determined¹³ from the quadratic baryon mass formula, then we estimate $\alpha_\Lambda \approx 0.5$ for the asymmetry parameter and

$$\Gamma(\Lambda^+ \rightarrow p \pi^0) \leq 1.2 \times 10^{11} \text{ sec}^{-1} \quad (5.12)$$

for the decay rate.

Also, the same method is applicable for $D^+ \rightarrow \pi^+ \pi^+ \pi^-$ decay and we find

$$\Gamma(D^+ \rightarrow \pi^+ \pi^+ \pi^-) \leq 3.5 \times 10^8 \text{ sec}^{-1} \quad (5.13)$$

where we assumed $M(D) \approx 1.8$ GeV for the reason which will become apparent soon. Note that the SU(4) mass formula predicts $M(D) \approx 2.1$ GeV. With respect to the decay $D^+ \rightarrow \pi^+ \pi^0$, the Lorentz-invariant amplitude has a dimension of the mass. Hence, we assume that in applying the exact U(4), it is proportional to the mass of the decaying parent meson. Under this assumption, (5.11) enables us to compute

$$\begin{aligned} \Gamma(D^+ \rightarrow \pi^+ \pi^0) &\leq 7.5 \times 10^7 \text{ sec}^{-1}, \\ \Gamma(D_1^0 \rightarrow \pi^+ \pi^-) &\leq 3.4 \times 10^{10} \text{ sec}^{-1} \end{aligned} \quad (5.14)$$

from the known decay rate of $K^+ \rightarrow \pi^+ \pi^0$ and $K_1^0 \rightarrow \pi^+ \pi^-$. All these rates (5.12), (5.13), and (5.14) are somewhat small, since they are really the minor decay modes obeying the $\Delta S = 0$ rule. The dominant decays in the GIM model are those⁴ satisfying $\Delta S = \pm 1$ rule, which are expected to be

larger by a factor of $\cot^2 \theta$ in comparison to the former.

Niu et al²⁵ discovered a particle X^+ decaying into $\pi^0 + \text{charged}$. If we identify the exact decay mode as $X^+ \rightarrow p\pi^0$ decay, then they find $M(X) = 2.95 \text{ GeV}$ with lifetime of $3.6 \times 10^{-14} \text{ sec}$. On the other hand, if the decay is $X^+ \rightarrow \pi^+\pi^0$, then we must have $M(X) = 1.8 \text{ GeV}$ with lifetime of $2.2 \times 10^{-14} \text{ sec}$. We note that the former mass is very near to the $SU(4)$ mass value of $m(A^+) = 2.9 \text{ GeV}$. Hence, we identify it with our A^+ particle. In that case, we can compute the decay branching ratio from (5.12) to be

$$R = \frac{\Gamma(A^+ \rightarrow p\pi^0)}{\Gamma(A^+ \rightarrow \text{all})} \approx 0.4 \times 10^{-2} \quad (5.15)$$

which is small but not unreasonably so. On the other hand, if X^+ is really D^+ , then we compute

$$R = \frac{\Gamma(D^+ \rightarrow \pi^+\pi^0)}{\Gamma(D^+ \rightarrow \text{all})} \approx 1.6 \times 10^{-6} \quad (5.16)$$

which is too small for a single event $D^+ \rightarrow \pi^+\pi^0$ to be fortuitously discovered. Hence, the identification of the particle of Niu et al, with the baryon A^+ is perhaps more realistic, unless it corresponds to other particles such as V_b^2 which is the 1^- counter part of $P_b^2 = D^+$.

As another application of our W symmetry, we find

$$\begin{aligned} \langle X | j_\lambda | p \rangle &= \langle \tilde{X} | \bar{j}_\lambda | \pi \rangle, \\ \langle X | j_\lambda | \pi \rangle &= \langle \tilde{X} | \bar{j}_\lambda | p \rangle \end{aligned} \quad (5.17)$$

for any state X , where $\tilde{X}$ is W -conjugate of X , i.e.,

$$\tilde{X} = W X \quad (5.18)$$

For example, if we identify $X = K^+ = P_1^3$, then we find $\tilde{X} = -P_2^4 = -\bar{D}^+$. In other words, if X is a strange particle, then $\tilde{X}$ is a charmed particle and vice versa. Therefore, roughly speaking, the validity of (5.17) indicates near-equality of charmed-particle production cross-section with strangeness-particle production cross-section in the neutrino reactions. This fact may be relevant to the recent experiment²⁶ on muon-pair production by neutrino. For the neutral neutrino reaction, we can write more precise relations

$$\begin{aligned}\sigma(\nu T \rightarrow \nu + Y + a\pi\gamma) &= \sigma(\nu T \rightarrow \nu + \tilde{Y} + a\pi\gamma), \\ \sigma(\bar{\nu} T \rightarrow \bar{\nu} + Y + a\pi\gamma) &= \sigma(\bar{\nu} T \rightarrow \bar{\nu} + \tilde{Y} + a\pi\gamma)\end{aligned}\quad (5.19)$$

for isoscalar target T in the exact $U(4)$ limit, where Y ($\tilde{Y} = WY$) denotes any particular single particle.

There is another interesting Weyl symmetry operation, W' , defined by

$$\begin{aligned}W': \quad \ell_4 &\rightarrow \ell_1 \rightarrow -\ell_4, \\ \ell_3 &\rightarrow \ell_2 \rightarrow -\ell_3.\end{aligned}\quad (5.20)$$

Under W' , all currents including the electromagnetic current J_μ^{em} are now invariant, i.e.,

$$\begin{aligned}W': \quad j_\lambda &\leftrightarrow j_\lambda, \quad \bar{j}_\lambda \leftrightarrow \bar{j}_\lambda, \quad j_\lambda^{(a)} \leftrightarrow j_\lambda^{(a)}, \\ j_\lambda^{(e,m)} &\leftrightarrow j_\lambda^{(e,m)}.\end{aligned}\quad (5.21)$$

Noting that under W' , we have

$$\begin{aligned}W': \quad K_2^0 &\leftrightarrow -K_2^0, \quad K_1^0 \leftrightarrow K_1^0, \\ D_1^0 &\leftrightarrow D_1^0,\end{aligned}\quad (5.22)$$

this implies that all decay matrix elements for $K_2^0 \rightarrow \gamma\gamma$ and $K_2^0 \rightarrow \mu\bar{\mu}$ as well as $K_1^0 \leftrightarrow K_2^0$ transition are forbidden in the exact $U(4)$ limit. This

fact has been previously noted by Gaillard and Lee²⁷ on the basis of a SU(2) sub-group of the U(4). As we noted, the neutral current $j_\mu^{(0)}$ is invariant under W' . Conversely, if we demand that $j_\mu^{(0)}$ is W' -invariant with CP = +1 parity, then it is easy to show that $j_\mu^{(0)}$ cannot contain any troublesome strangeness-changing component. Besides, it must be a U-spin scalar.²⁸ Therefore, the W' -invariance is perhaps one of the important ingredients of the usefulness of the GIM model, since it can effectively determine many essential features of all hadronic currents.

When we combine with W and W' symmetries, then we can generate another Weyl symmetry

$$W'' = W' W = -W W'; \quad \ell_1 \leftrightarrow \ell_3, \quad \ell_2 \leftrightarrow \ell_4 \quad (5.23)$$

which may be useful for some weak interaction processes.

Last, we briefly mention a new weak interaction scheme of Goto and Mathur²⁹ who assign electric charges 2/3, -1/3, -1/3 and -4/3 for four quarks, q_1, q_2, q_3 and q_4 , respectively. In this theory, the weak charged current is now given by

$$j_\lambda = \cos \theta [\bar{\ell}_1 Q_\lambda \ell_2 + \bar{\ell}_2 Q_\lambda \ell_4] \\ + \sin \theta [\bar{\ell}_1 Q_\lambda \ell_3 + \bar{\ell}_3 Q_\lambda \ell_4]$$

rather than (5.1). Unfortunately, all Weyl symmetry operations so far considered are no longer symmetry of the new theory, so that we cannot derive any analogous relations. However, the following operation

$$W''': \quad \ell_1 \leftrightarrow \ell_4, \quad \ell_2 \leftrightarrow \ell_2, \quad \ell_3 \leftrightarrow \ell_3$$

leaves

$$j_\lambda \leftrightarrow \bar{j}_\lambda, \quad j_\lambda^{(0)} \leftrightarrow -j_\lambda^{(0)}$$

which may be useful for some problems.

References

1. J. J. Aubert et al., Phys. Rev. Lett. 33, 1404 (1974); J.E. Augustine et al., ibid 33, 1406 (1974); C. Bacci et al., ibid 33, 1408 (1974); G. S. Abrams et al., ibid 33, 1453 (1974); W. C. Braunschweig et al., Phys. Lett. 53B, 393 (1974); 53B, 491 (1975); L. Criegee et al., ibid 53B, 489 (1975).
2. D. Amati, H. Bacry, J. Nuyts and J. Prentki, Phys. Lett. 11, 190 (1964); B. J. Bjorken and S. L. Glashow, ibid 11, 255 (1964); Y. Hara, Phys. Rev. 134, B701 (1964); Z. Maki and Y. Ohnuki, Prog. Theor. Phys. 32, 144 (1964); P. Tarjanne and V. L. Teplitz, Phys. Rev. Lett. 11, 447 (1963).
3. S. L. Glashow, J. Illiopoulous and L. Miani, Phys. Rev. D2, 1285 (1972).
4. M. K. Gaillard, B.W. Lee and J. L. Rosner, Fermi National Accelerator Laboratory, preprint (1974).
5. H. Weyl, "Classical Group" (Princeton University Press, Princeton, N. J. (1939)).
6. J. D. Louck and L. C. Biedenharn, J. Math. Phys. 11, 2368 (1970).
7. S. Okubo, University of Rochester preprint, UR-502 (1974), to be published in J. Math. Phys. (March 1975).
8. J. D. Louck, J. Math. Phys. 6, 1786 (1965); N. Mukunda, ibid 8, 1069 (1967).
9. S. Okubo, University of Rochester preprint, UR-474 (1974) (unpublished). Although this paper deals with a model nuclear Hamiltonian, most of the results of this paper are immediately applicable to SU(8) symmetry.
10. For some earlier papers on this subject, see:
P. Tarjanne, Phys. Rev. 136, B1532 (1964); M.L. Whippman, Nuovo Cimento 37, 824 (1965).

11. G. E. Baird and L. C. Biedenhorn, J. Math. Phys. 4, 1449 (1963).
12. S. K. Bose, Notre Dame University, preprint (1975).
13. S. Okubo, V. S. Mathur, and S. Borchardt, Phys. Rev. Lett. 34, 236 (1975).
14. Y. Iwasaki, Kyoto University preprint (1975).
15. S. Okubo, J. Phys. Soc. Japan, 19, 1509 (1964).
16. The assignment of all quantum numbers such as Q, Y and C is the same as in reference 3.
17. S. Okubo, Prog. Theor. Phys. 27, 949, 28, 24 (1962).
18. R. N. Sen, J. Math. Phys. 8, 896 (1967).
19. I. S. Gerstein and M. L. Whippman, Phys. Rev. 137, B1522 (1965).
See also, R. E. Marshak, S. Okubo and J. Wojtaszek, Phys. Rev. Lett. 15, 463 (1965).
20. Z. Maki, T. Maskawa and Z. Umemura, Prog. Theor. Phys. 47, 1682 (1972).
21. e. g. see
V. S. Mathur, S. Okubo and S. Borchardt, Univ. of Rochester preprint (1975); Y. Hara, Tokyo Univ. of Education, preprint (1975);
E. Takasugi and S. Oneda, Univ. of Maryland preprint (1975).
22. e.g., F. J. Dyson, "Symmetry Groups in Nuclear and Particle Physics" (Benjamin, N.Y. 1966).
23. e.g., J. W. Moffat, Phys. Rev. 140, B1681 (1965) and Toronto Univ. preprint (1974); Y. Hara, see ref. 21.
24. e.g., R. E. Marshak, Riazuddin and C. P. Ryan, "Theory of Weak Interaction in Particle Physics" (Wiley-Interscience, N.Y. 1969).
25. K. Niu, E. Mikumo and Y. Maeda, Prog. Theory. Phys. 46, 1644 (1971).
26. A. Benvenuti et al., Phys. Rev. Lett. 34, 419 (1975),
ibid 34, 597 (1975).

27. M. K. Gaillard and B. W. Lee, Phys. Rev. D10, 897 (1974).
28. V. S. Mathur, S. Okubo and J. E. Kim, Phys. Rev. D10, 3648 (1974);
and to be published in Phys. Rev.
29. T. Goto and V. S. Mathur, University of Rochester preprint
UR-519 (1975).

ERRATUM

P. 28 - Ref. 7: S. Okubo; J. Math. Phys. 16, 528 (1975)

P. 29 - Ref. 23:

S. Iwao; Ann. Phys. (N.Y.) 35, 1 (1965)

J. W. Moffat; Phys. Rev. 140B, 1681 (1965) and preprint, Univ. of Toronto (1975)

D. B. Lichtenberg; preprint, Indiana University (1975)

A. W. Hendry and D. B. Lichtenberg; preprint, Indiana University (1975)

Y. Hara; see ref. 21