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THE RADIAL DISTRIBUTION OF THERMAL NEUTRON FLUX IN
CYLINDRICAL FUEL SPECIMENS DURING NEUTRON IRRADIATION

by

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THE RADIAL DISTRIBUTION OF THERMAL NEUTRON FLUX IN CYLINDRICAL FUEL SPECIMENS DURING NEUTRON IRRADIATION

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ABSTRACT

A systematic study of the radial distribution of thermal neutrons in cylindrical fuel specimens during irradiation has been made. Natural uranium and alloys of magnesium-cadmium were used in the investigation. The alloys were used to simulate the absorption properties of reactor fuel materials. Effects of the variable cylinder radius, a , and macroscopic absorption cross section, Σ_{abs} , in the range $0 < \omega < 2.2$, where $\omega = a\Sigma_{abs}$, were studied.

It was found that the experimentally determined radial distributions of the thermal neutron flux may be expressed by an empirical formula of the type

$$n(r, \omega) = \sum_{k=0}^3 a_k(\omega) r^{2k}$$

for $r \leq a$, and $0 < \omega < 2.2$. The functions $a_k(\omega)$ are given by

$$a_k(\omega) = a^{-2k} \gamma^k / \sum_{j=0}^3 \gamma^j, \quad ,$$

where

$$\gamma = \omega / (-0.22894\omega^2 + 0.63812\omega + 1.2567)$$

The neutron flux is set equal to unity at the surface of the cylinder, i.e., $n(a, \omega) = 1$ for all ω .

INTRODUCTION

For computation of the temperature distribution within a cylindrical nuclear fuel specimen during irradiation⁽¹⁾ accurate knowledge of the neutron distribution is necessary. This information is also related to the distribution of fission events, fission damage and new atoms or isotopes. It may have other engineering uses in consideration of internal stress, of disadvantage factors and of neutron economy, also.

Neutron distributions may be deduced theoretically from diffusion or transport theory^(2,3) if the pertinent variables are accurately known. However, the problem usually must be idealized to produce a solution. This is particularly true of distributions in cylindrical rods of finite length.⁽²⁾ Thus far, the experimental support for such results has not been extensive and there are often large uncertainties in the values of parameters which must be used.

In view of an increasing need for more accurate definition of the conditions under which test specimens are irradiated, a series of studies on the neutron distribution in cylinders has been in progress at Argonne National Laboratory. A number of radial and longitudinal neutron distributions have been determined experimentally for bars of natural uranium and of magnesium-cadmium alloys. The latter were employed to simulate the absorption properties of fuel materials of various enrichments in fissionable atoms. Specimens were activated in the almost pure thermal flux of the reflector of the CP-5 reactor.

The present report deals with the radial distributions found in the central cross sections of the specimens, where end effects were deemed to be negligible. The empirical equations which have been developed from an analysis of these data, therefore, are directly applicable to long fuel specimens when first placed in a thermal-neutron flux. A companion report deals with the longitudinal distributions which were observed.⁽⁴⁾

EXPERIMENTAL METHOD

It was desired to study neutron distributions in cylinders of small, as well as of large, diameter. As the smaller specimens were also to have the greater macroabsorption properties, it was necessary to devise an experimental method which would provide a homogeneous bar and yet allow rapid and accurate sampling. This was made possible by assembling the specimens from carefully prepared disks or wafers between which thin monitor foils of the same diameter and material could be sandwiched. The assemblies could then be activated under tight longitudinal compression, and the distribution of radioactivity in each monitor foil analyzed by gamma-counting samples punched from definite radial locations. Other investigators have monitored neutron flux in large absorbing bars with embedded monitors of dissimilar metals,⁽⁵⁾ a method which could not be used in small rods without serious perturbation of flux.

Diameters of the test bars ranged from 0.635 to 2.54 cm (0.25 to 1.00 in.). Five sets of four each were of natural uranium and covered the entire range. Seven sets of four each were of the magnesium-cadmium alloys listed in Table I. Compositions were checked by duplicate chemical analyses from the top and bottom of the castings from which the specimens

Table I

MAGNESIUM-CADMIUM ALLOYS USED TO SIMULATE FUEL MATERIALS
OF VARIOUS MACROSCOPIC ABSORPTION CROSS SECTIONS

Alloy	Composition of Alloy, w/o				Density, gm/cm ³	$\Sigma_{\text{abs}},$ cm ⁻¹
	Magnesium ^a	Cadmium	Manganese	Cobalt		
A	92.564 ± 0.057	6.57 ± 0.05	0.475 ± 0.005	0.391 ± 0.002	1.868	1.921
B	87.933 ± 0.074	11.32 ± 0.08	0.410 ± 0.005	0.337 ± 0.004	1.962	3.472
C	81.210 ± 0.074	18.15 ± 0.12	0.345 ± 0.005	0.295 ± 0.002	2.012	5.704

^aBy difference.

were fabricated. Densities were obtained in duplicate by the immersion method, and macroscopic absorption cross sections were computed from the constants shown in Table II. Small amounts of manganese and cobalt (less than 1/2% each) were added to the alloys to serve as radioactive tracers after exposure to neutrons.

A detailed description of the experimental procedure employed in this investigation will now be presented.

Table II

CONSTANTS USED IN THE CALCULATION OF THE MACROSCOPIC ABSORPTION
CROSS SECTIONS OF THE TEST SPECIMEN MATERIALS

Element	Density, gm/cm ³	Atomic Wt.	Microscopic Absorption Cross Sections		g(T) Factors for Maxwell Distributions
			σ_0 , barns ^a	$\bar{\sigma}$, barns ^b	
Magnesium	1.741	24.32	0.063 ± 0.004	0.054	1.0
Manganese	7.42	54.94	13.2 ± 0.4	11.3	1.0
Cobalt	8.71	58.94	37.0 ± 1.5	31.7	1.0
Cadmium	8.65	112.41	2450 ± 50	2912	1.388
Uranium	19.0	238.07	7.68 ± 0.07	6.52	0.99

^aReference (6)

^bMaxwellian Averaged Cross Section, i.e., $\bar{\sigma}(T) = g(T)\sigma_0(\pi T_0/4T)^{1/2}$,
where $T_0 = 293.6^\circ\text{K}$ (7)

A. Assembly and Irradiation of Specimens

Each fuel specimen was constructed from a large number of carefully surface-ground disks (surface roughness about 10 microinches) which were stacked one against the other to form a solid rod. A thin monitor foil (0.0037 inch), having the same diameter and composed of the same material as that of the disks, was inserted between each pair of adjacent disks. The stack of disks and foils was placed into a graphite capsule which served to hold the pieces together. Figure 1 shows a capsule containing two identical fuel specimens separated by a graphite spacer. All irradiations were conducted in this manner so as to make possible the simultaneous study of the radial distribution and Wilkins effect.

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Figure 1. Graphite Capsule and Composite Fuel Sample

The specimens and capsule were irradiated with thermal neutrons in one of the isotope trays of the CP-5 reactor for a short interval of time. After disassembly the monitor foils were removed for study.

B. Sampling of Foils

Very small round sample ("dots") of material (about 500 micrograms each) were removed from various radial positions of most of the foils by means of a punching operation, as shown in Figures 2 and 3.

A different technique was developed for sampling foils from the specimens having the smallest diameter. Each of the foils was sectioned into several concentric rings by means of a single shearing operation, as shown in Figures 4 and 5.

C. Weighing of Samples

Each sample was accurately weighed on a microbalance and then carefully mounted at the center of an aluminum planchet. The error in the weights of the samples was determined to be no greater than $\pm 1\%$.

D. Measurement of Sample Radioactivity

A gamma-ray spectrometer having a one-inch-diameter sodium iodide crystal was used to measure the gamma-ray activities of the samples. (Counting distance was $1/4$ in. for the "dot" samples, and 3 in. for the ring samples.) In the measurement of the gross fission product activities of the irradiated uranium samples, only those gamma rays with an energy greater than 0.5 Mev were counted. This procedure eliminated much of the natural activity of the uranium and all of the gamma rays and X rays arising from the decay of U^{239} and Np^{239} . In the measurement of the gamma-ray activities induced in the irradiated samples from the various magnesium-cadmium-cobalt-manganese alloys, only those gamma rays arising from Co^{60} and Mn^{56} with an energy greater than 1.0 Mev were counted. The error in determining the counting rate was held to $\pm 1\%$ in each case by recording the time required for a total count of 10,000.

E. Determination of Specific Activities of Samples

The specific activity, i.e., the number of counts per unit weight, was determined for each sample. The specific activities of the samples taken from any one foil were then normalized to unit specific activity at the surface of the foil. The radial distribution of specific activity within the foil was assumed to be identical with the radial distribution of thermal neutron flux within the fuel sample during the irradiation.

PRELIMINARY TREATMENT OF EXPERIMENTAL DATA

A complete tabulation of the experimental data is given in Appendix A. Only those foils taken from the mid-lengths of the specimens where longitudinal gradients were absent were used in the analysis of radial neutron distributions.

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Figure 2. Device Used for Punching
Small Samples from Foils.

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Figure 3. (Bottom) Dies Used for Locating Positions
of Samples to be Punched from Foils.
(Top) Foils after Sampling Operation.

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Figure 4. Device Used for Shearing Foils into Concentric, Ring-shaped Samples.

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Figure 5. A Typical Foil after Shearing Operation.

Table III gives the sizes and neutron-absorption characteristics of specimens used in the experimental determinations. The macroscopic absorption cross sections of the specimens varied between 0.3135 cm^{-1} (natural uranium) and 5.704 cm^{-1} (alloy C). A dimensionless quantity ω , defined as the product of the radius and macroscopic absorption cross section of the specimen, i.e., $\omega = a\Sigma_{\text{abs}}$, is listed in the last column of Table III. It was suggested by the theoretical work of J. Bengston⁽³⁾ and proved to be extremely useful in the analysis of the experimental data. It varies between 0.10 and about 2.2 for the specimens listed in the table.

Table III

A DESCRIPTION OF THE SPECIMENS USED IN THE EXPERIMENTAL DETERMINATION OF THE NEUTRON FLUX DISTRIBUTIONS IN CYLINDRICAL FUEL SPECIMENS DURING IRRADIATION^a

Specimen No.	Composition of Specimen Material ^b	Diameter of Sample, 2a (cm)	Macro Cross Section $\Sigma_{\text{abs}}(\text{cm}^{-1})$	$\omega (= a\Sigma_{\text{abs}})$
1	Natural Uranium	0.635	0.3135	0.09954
2	Natural Uranium	0.898	0.3135	0.1408
3	Natural Uranium	1.270	0.3135	0.1991
4	Natural Uranium	1.796	0.3135	0.2815
5	Natural Uranium	2.540	0.3135	0.3981
6	Alloy A	0.635	1.921	0.6099
7	Alloy A	0.898	1.921	0.8625
8	Alloy B	0.635	3.472	1.102
9	Alloy A	1.270	1.921	1.220
10	Alloy B	0.898	3.472	1.559
11	Alloy C	0.635	5.704	1.811
12	Alloy B	1.270	3.472	2.205

^aQuadruplicates

^bSee Table I for alloy compositions.

Table IV gives the experimentally determined ratios of the average to surface and the axial to surface thermal neutron fluxes for the specimens listed in Table III. The values of the ratios are arranged in the table according to increasing values of ω .

The ratios were computed in the following manner: the radial distribution of specific activity within a cylindrical specimen was expressed by an equation of the form

$$n(r) = \sum_{k=0}^3 a_k r^{2k} \quad (1)$$

Table IV

EXPERIMENTALLY DETERMINED RATIOS OF AVERAGE TO SURFACE AND AXIAL TO SURFACE NEUTRON FLUXES FOR SPECIMENS HAVING VARIOUS VALUES OF ω

Specimen No.	ω	$\phi_{\text{Avg}}/\phi_{\text{Surface}}$		$\phi_{\text{Axial}}/\phi_{\text{Surface}}$	
		Average	St'd. Deviation ^a	Average	St'd. Deviation ^a
1	0.09954	0.9600 ^b	0.00478	0.9299 ^b	0.00946
2	0.1408	0.9574	0.00724	0.9093	0.01318
3	0.1991	0.8966	0.02133	0.8571	0.01204
4	0.2815	0.8850	0.00757	0.7975	0.01062
5	0.3981	0.8504	0.02557	0.7181	0.00130
6	0.6099	0.8000 ^b	0.00641	0.6428 ^b	0.00835
7	0.8625	0.7384	0.01011	0.5278	0.00656
8	1.102	0.6438 ^b	0.00957	0.4184 ^b	0.00587
9	1.220	0.6104	0.01656	0.3737	0.00481
10	1.559	0.5531	0.00975	0.2842	0.00110
11	1.811	0.4878 ^b	0.02761	0.2069 ^b	0.00375
12	2.205	0.4383	0.00698	0.1567	0.00225

$$^a \text{Standard Deviation} = \sqrt{\frac{\sum_{k=1}^N (x_k - \bar{x})^2}{N-1}}, \text{ where } N \text{ equals } 4.$$

^bThese ratios were obtained by using the sampling technique developed for small-diameter foils. See Figures 4 and 5.

The constants, a_k , were determined by the method of least squares. Hence, using equation (1), the above ratios may be written as follows:

$$\frac{\phi_{\text{axial}}}{\phi_{\text{surface}}} = \frac{n(0)}{n(a)} = \frac{a_0}{\sum_{k=0}^3 a_k a^{2k}}, \quad (2)$$

$$\frac{\phi_{\text{average}}}{\phi_{\text{surface}}} = \frac{\bar{n}}{n(a)} = \frac{1}{n(a)} \cdot \frac{2}{a^2} \int_0^a r n(r) dr = \frac{\sum_{k=0}^3 \frac{a_k}{k+1} \cdot a^{2k}}{\sum_{k=0}^3 a_k a^{2k}}. \quad (3)$$

In the analysis of the foils which were sectioned into several ring-shaped samples, it was necessary to assign a corresponding average radial position, $\bar{r}$, to each ring-shaped sample in the foil. This was accomplished in the following way: let r_1 and r_2 represent the inner and outer radii, respectively, of any one of the ring-shaped samples taken from the foil. The average neutron flux, $\bar{n}$, within the sample then becomes

$$\bar{n} = \frac{2}{r_2^2 - r_1^2} \int_{r_1}^{r_2} r n(r) dr, \quad (4)$$

where $n(r)$ is a function describing the radial distribution of flux in the sample. In order to evaluate the integral in equation (4), it is necessary to make an assumption regarding the form of the function $n(r)$. If it is assumed that the slope of $n(r)$ is proportional to r within the interval (r_1, r_2) , i.e., $dn/dr = kr$, then $n(r) = 1/2 k r^2 + c$, where c is a constant of integration. Hence, upon substitution in equation (4), $\bar{n}$ becomes

$$\bar{n} = \frac{1}{4} k (r_2^2 + r_1^2) + c. \quad (5)$$

To find $\bar{r}$, let $n(\bar{r}) = \bar{n}$. Hence,

$$\frac{1}{2} k \bar{r}^2 + c = \frac{1}{4} k (r_2^2 + r_1^2) + c,$$

and

$$\bar{r} = \left[\frac{1}{2} (r_1^2 + r_2^2) \right]^{\frac{1}{2}}. \quad (6)$$

Thus $\bar{r}$ is simply the root-mean-square of r_1 and r_2 . The $\bar{r}$'s computed from equation (6) represent a first approximation to the correct values.

A second approximation for the r 's was obtained by making use of equation (6) to determine the coefficients a_k (by the method of least squares) appearing in equation (1). The function $n(r)$ was then substituted in equation (4) and a new set of $\bar{r}$'s was computed.

In the determinations of the $\bar{r}$'s for the "dot" samples it was assumed that the neutron flux varied quadratically over the surface of the monitor foil. Upon making this assumption, the $\bar{r}$'s are given by the following formula:

$$(\bar{r})^2 = r^2 + (b^2/8), \quad (7)$$

where r is equal to the distance between the center of the "dot" sample and the geometrical center of the monitor foil, and b is equal to the diameter of the "dot" sample ($b = 1/32$ inch). The computed values of the $\bar{r}$'s are given in Appendix B.

ANALYSIS OF RESULTS

The derivation of an empirical formula for computing the radial distribution of thermal-neutron flux within a cylindrical fuel specimen will now be presented. The formula will be derived from an analysis of the experimental data shown in Table IV.

An analysis of the experimental data showed that the radial distribution of thermal-neutron flux (specific activity) within a cylindrical fuel specimen may be expressed by an equation of the form

$$n(r, \omega) = \sum_{k=0}^3 a_k(\omega) r^{2k} \quad (8)$$

The problem of deriving an empirical formula for computing the radial distribution of neutron flux within a fuel specimen is then reduced to that of finding the coefficients $a_k(\omega)$ in equation (8).

The coefficients $a_k(\omega)$ may be found as follows:

Let

$$n(r, \omega) = \sum_{k=0}^3 a_k(\omega) r^{2k} \equiv \beta_1(\omega) \sum_{k=0}^3 [\beta_2(\omega) r]^{2k} \quad (9)$$

Hence

$$a_k(\omega) = \beta_1 \beta_2^{2k} \quad (9a)$$

Now let

$$n(a, \omega) = \beta_1 \sum_{k=0}^3 (\beta_2 a)^{2k} \equiv 1 \quad (10)$$

for all values of ω , where $\underline{a}$ denotes the radius of the cylinder. (The neutron flux is normalized to unity at the surface of the cylinder, i.e., at $r = a$.) Thus,

$$\beta_1 = 1 / \sum_{k=0}^3 (\beta_2 a)^{2k} \quad (11)$$

It should be observed that, because of the relationship in equation (9) the quantity β_1 is equal to the ratio of the axial to surface neutron flux in the cylinder, that is,

$$\beta_1 = \frac{\beta_1}{\beta_1 \sum_{k=0}^3 (\beta_2 a)^{2k}} = \frac{n(0, \omega)}{n(a, \omega)} \equiv \phi_{\text{axial}} / \phi_{\text{surface}} \quad (11a)$$

where the symbol ϕ is used to denote neutron flux.

Let us now find an expression for the ratio of the average to surface neutron flux for a cylinder. Let the average flux, $\bar{n}$, be defined by the following equation:

$$\pi a^2 L \bar{n}(\omega) = 2\pi L \int_0^a r n(r, \omega) dr, \quad (12)$$

where L is the length of the cylinder. Hence,

$$\bar{n}(\omega) = \frac{2}{a^2} \int_0^a r n(r, \omega) dr = \frac{2\beta_1}{a^2} \int_0^a r \left[\sum_{k=0}^3 (\beta_2 r)^{2k} \right] dr.$$

To evaluate the above integral, let $z = \beta_2 r$ and $dz = \beta_2 dr$. Upon substitution, the above equation becomes

$$\bar{n}(\omega) = \frac{2\beta_1}{a^2} \int_0^\alpha \left(\sum_{k=0}^3 z^{2k+1} \right) dz, \quad (13)$$

where $\alpha = a\beta_2$. Upon integration, equation (13) becomes

$$\bar{n}(\omega) = \beta_1 \sum_{k=0}^3 \frac{\alpha^{2k}}{k+1} = \frac{\sum_{k=0}^3 \frac{\alpha^{2k}}{k+1}}{\sum_{k=0}^3 \alpha^{2k}} \quad (14)$$

It should be observed that $\bar{n}(\omega)$ is equal to the ratio of the average to surface flux in the cylinder. This may be shown by multiplying the numerator and denominator of the right-hand member of equation (14) by β_1 and noting that the denominator [equal to $n(a, \omega)$] is identically equal to unity for all values of ω . Therefore, we may write

$$\bar{n}(\omega) = \phi_{\text{average}} / \phi_{\text{surface}} \quad (14a)$$

If the substitution $\alpha(\omega) = a\beta_2(\omega)$ is made in equation (9a), then

$$a_k(\omega) = \frac{a^{-2k} [\alpha(\omega)]^{2k}}{\sum_{j=0}^3 [\alpha(\omega)]^{2j}}, \quad (15)$$

for $k = 0, 1, 2$ and 3 .

Thus, the problem of determining the coefficients $a_k(\omega)$ in equation (8), i.e., the functions $a_0(\omega)$, $a_1(\omega)$, $a_2(\omega)$ and $a_3(\omega)$, has been reduced to that of determining the single function $\alpha(\omega)$.

A method for determining a function $\alpha(\omega)$ (from the experimental data appearing in Table IV) will now be presented. Let the mathematical expression giving the ratio of the axial to surface neutron flux in a cylinder be equated to the corresponding least squares treated measured ratio, i.e.,

$$1 / \sum_{k=0}^3 \alpha^{2k} = \phi_{\text{axial}} / \phi_{\text{surface}} \quad (16)$$

Next, solve the equation for α . (It can be easily shown that there is one and only one positive value of α that satisfies the equation for any given numerical value of the right-hand member.) The values of α determined in this fashion are listed in Table V. Now substitute the various values of α into equation (14) and compute the corresponding values of $\bar{n}$. The calculated values of $\bar{n}$ are shown in Table V and are compared with the experimentally determined values. A glance at the right-hand column of the table shows that the calculated and experimental values of $\bar{n}$ are in surprisingly good agreement.

An analysis of the table of values of α shows that a relatively simple mathematical relation exists between α and the parameter ω , namely,

$$\alpha^2 = \omega / (a_2 \omega^2 + a_1 \omega + a_0) \quad , \quad (17)$$

where $a_2 = -0.22894$, $a_1 = 0.63812$, and $a_0 = 1.2567$. Figure 6 is a graph showing α^2 plotted against ω . The solid line is derived from equation (17) and the dots correspond to the values of α^2 derived from Table V.

If now we let $[\alpha(\omega)]^2 = \gamma(\omega)$, equation (15) becomes

$$a_k(\omega) = a^{-2k} \gamma^k / \sum_{j=0}^3 \gamma^j$$

for $k = 0, 1, 2, 3$ and equation (8) becomes

$$n(r, \omega) = \left(\sum_{j=0}^3 \gamma^j \right)^{-1} \sum_{k=0}^3 a^{-2k} \gamma^k r^{2k} \quad (18)$$

The pertinent results of this section may be found in the summary.

Table V

A COMPARISON BETWEEN THE EXPERIMENTAL LEAST-SQUARES-DETERMINED AND THE α -COMPUTED RATIOS OF AVERAGE TO SURFACE NEUTRON FLUX IN ABSORBING CYLINDERS FOR VARIOUS VALUES OF ω AND α

ω	α	$\phi_{\text{Avg}}/\phi_{\text{Surface}}$		$\sigma\% = \frac{\text{Exp.} - \text{Calc.}}{\text{Calc.}} \times 100$
		Experimental	Calculated from α^a	
0.09954	0.26490	0.9600	0.9641	- 0.43
0.1408	0.30121	0.9574	0.9532	+0.44
0.1991	0.37845	0.8966	0.9250	- 3.1
0.2815	0.45198	0.8850	0.8913	- 0.71
0.3981	0.53550	0.8504	0.8450	+0.64
0.6099	0.60752	0.8000	0.7988	+0.15
0.8625	0.71207	0.7384	0.7241	+2.0
1.102	0.81361	0.6438	0.6483	- 0.69
1.220	0.85788	0.6104	0.6159	- 0.89
1.559	0.95658	0.5531	0.5480	+0.93
1.811	1.0620	0.4878	0.4854	+0.49
2.205	1.1500	0.4383	0.4421	- 0.86

^aUsing Equation (14).

DISCUSSION OF RESULTS

A. Experimental Accuracy

Table VI gives the coefficient of variation for each of the experimentally determined ratios of axial to surface and average to surface neutron fluxes. The coefficient of variation, which describes the relative variation, is a dimensionless quantity defined as the ratio of the standard deviation to the arithmetic mean, i.e., $\sigma_x/\bar{x}$. (Each ratio appearing in the table represents the average value of four independent determinations.) A glance at the table shows that the coefficient of variation is 2% or less for all determinations of the ratio of axial to surface flux. For the ratio of average to surface flux, the variation is somewhat larger and varies between about 1 and 3%. Sample number 11 has a variation of about 6%.

Table VI

COEFFICIENT OF VARIATION FOR EACH OF THE EXPERIMENTALLY
DETERMINED RATIOS OF AXIAL TO SURFACE AND AVERAGE
TO SURFACE NEUTRON FLUX

Specimen No.	ω	$\phi_{\text{Axial}}/\phi_{\text{Surface}}$		$\phi_{\text{Avg}}/\phi_{\text{Surface}}$	
		Average	Coefficient of Variation, %	Average	Coefficient of Variation, %
1	0.09954	0.9299	1.0	0.9600	0.50
2	0.1408	0.9093	1.4	0.9574	0.76
3	0.1991	0.8571	1.4	0.8966	2.4
4	0.2815	0.7975	1.3	0.8850	0.86
5	0.3981	0.7181	0.20	0.8504	3.0
6	0.6099	0.6428	1.3	0.8000	0.80
7	0.8625	0.5278	1.2	0.7384	1.4
8	1.102	0.4184	1.4	0.6438	1.5
9	1.220	0.3737	1.3	0.6104	2.7
10	1.559	0.2842	0.40	0.5531	1.8
11	1.811	0.2069	1.8	0.4878	5.7
12	2.205	0.1567	1.4	0.4383	1.6

B. Analytical Accuracy

It should be pointed out that the method used in determining the coefficients a_k gives rise to uncertainties in the computed distributions of neutron flux. This is a consequence of the fact that the number of experimental conditions used in the determination of the coefficients are fewer (by one) than the number of coefficients to be determined. An analysis of the situation shows that the uncertainty or "analytical error" in any particular neutron distribution (as derived from the empirical formula) is given by the following formula:

$$\Delta E \leq 0.0481 a^6 \left| \epsilon_3 \right|_{\max} \quad , \quad (19)$$

where

$$-a_3 < \epsilon_3 < \frac{2}{3} a_2 a^{-2} \quad .$$

It can also be shown that the mean uncertainty over any particular neutron distribution is given by the following formula:

$$\overline{\Delta E} = 0.0275 a^6 \left| \epsilon_3 \right|_{\max} \quad . \quad (20)$$

The uncertainties in the computed distributions turn out to be of the same general magnitude as those attributed to experimental errors.

A comparison of the experimentally determined and computed (using empirical formulas presented in the summary) ratios of axial to surface and average to surface neutron fluxes within absorbing cylinders having various values of the parameter ω is shown in Table VII. Figure 7 is a graph showing the above computed ratios plotted against ω .

Table VII

A COMPARISON BETWEEN THE EXPERIMENTALLY DETERMINED AND THE EMPIRICALLY COMPUTED RATIOS OF AXIAL TO SURFACE AND AVERAGE TO SURFACE NEUTRON FLUXES WITHIN ABSORBING CYLINDERS FOR VARIOUS VALUES OF ω

Specimen No.	ω	$\phi_{\text{Axial}}/\phi_{\text{Surface}}$			$\phi_{\text{Avg}}/\phi_{\text{Surface}}^a$		
		Experimental	Formula	Deviation, %	Experimental	Formula	Deviation, %
1	0.09954	0.9299	0.9245	+ 0.58	0.9600	0.9613	-0.13
2	0.1408	0.9093	0.8952	+ 1.6	0.9574	0.9457	+1.2
3	0.1991	0.8571	0.8555	+ 0.18	0.8966	0.9241	-3.0
4	0.2815	0.7975	0.8028	- 0.66	0.8850	0.8945	-1.1
5	0.3981	0.7181	0.7339	- 2.2	0.8504	0.8544	-0.47
6	0.6099	0.6428	0.6238	+ 3.1	0.8000	0.7867	+1.7
7	0.8625	0.5278	0.5126	+ 3.0	0.7384	0.7138	+3.4
8	1.102	0.4184	0.4227	- 1.0	0.6438	0.6514	-1.2
9	1.220	0.3737	0.3829	- 2.4	0.6104	0.6226	-2.0
10	1.559	0.2842	0.2822	+ 0.71	0.5531	0.5464	+1.2
11	1.811	0.2069	0.2187	- 5.4	0.4878	0.4952	-1.5
12	2.205	0.1567	0.1366	+14.7	0.4383	0.4241	+3.4

^aSee equations in Summary.

Figure 7. Ratio of Average to Surface and Axial to Surface Thermal-Neutron Fluxes for Cylinders Having Various Values of ω ($= a \Sigma_{\text{abs}}$)

Tables VIII and IX show the experimentally determined and computed distributions of thermal-neutron flux in a natural uranium cylinder, 2.54-cm diameter, and a highly absorbing cylinder, 0.635-cm diameter. The computed distributions are based on formula (18).

Table VIII

A COMPARISON OF THE EXPERIMENTAL AND COMPUTED RADIAL NEUTRON FLUX DISTRIBUTIONS IN A NATURAL URANIUM CYLINDER OF 2.54-cm DIAMETER

r, cm	n(r)/n(a)		Variation, %
	Experi- mental	Formula	
0	0.7181	0.7339	-2.2
0.2475	0.7260	0.7415	-2.1
0.4927	0.7503	0.7650	-1.9
0.7385	0.7940	0.8076	-1.7
0.9844	0.8644	0.8754	-1.3
1.2303	0.9764	0.9789	-0.26
1.2700	None	1.000	-

Table IX

A COMPARISON OF THE EXPERIMENTAL AND COMPUTED RADIAL NEUTRON FLUX DISTRIBUTIONS IN A HIGHLY ABSORBING CYLINDER ($\Sigma_{\text{abs}} = 3.472 \text{ cm}^{-1}$) OF 0.635-cm DIAMETER

$\bar{r}$, cm	n($\bar{r}$)/n(a)		Variation, %
	Experi- mental*	Formula	
0	0.4184	0.4226	-0.99
0.09105	0.4424	0.4466	-0.96
0.1478	0.4877	0.4923	-0.93
0.2050	0.5730	0.5782	-0.89
0.2620	0.7273	0.7329	-0.76
0.3034	0.9148	0.9174	-0.28
0.3175	None	1.000	-

*This specimen was analyzed by the "ring" method developed for small diameter foils (see Figures 4 and 5).

Figure 8 is a graph showing the manner in which the ratios of average to surface and axial to surface neutron fluxes depend upon the diameter of the cylinder (natural uranium).

SUMMARY

The results of the analysis of the experimental data may be conveniently summarized as follows:

I. The radial distribution of thermal-neutron flux within an absorbing cylinder (where the flux has been normalized to unity at the surface of the cylinder) may be computed from the empirical formula

$$n(r, \omega) = A(\gamma) [1 + \gamma (r/a)^2 + \gamma^2 (r/a)^4 + \gamma^3 (r/a)^6] \quad , \quad (19)$$

where

$$A(\gamma) = 1/(1 + \gamma + \gamma^2 + \gamma^3) \quad (19a)$$

Figure 8. Ratio of Average to Surface and Axial to Surface Thermal Neutron Fluxes for Natural Uranium Cylinders of Various Diameters

and

$$\gamma = \omega / (a_2 \omega^2 + a_1 \omega + a_0) \quad (19b)$$

The coefficients appearing in the expression for γ have the following numerical values:

$$a_2 = -0.22894, a_1 = 0.63812 \text{ and } a_0 = 1.2567$$

The above formula is valid for $0 \leq r \leq a$ and $0 < \omega < 2.2$, where a is the radius of the cylinder and $\omega = a\Sigma_{\text{abs}}$.

II. The ratio of the average to surface neutron flux within an absorbing cylinder is given by the formula

$$\frac{\bar{n}(\omega)}{n(a, \omega)} = A(\gamma) \left[1 + \frac{\gamma}{2} + \frac{\gamma^2}{3} + \frac{\gamma^3}{4} \right] \quad (20)$$

where γ and $A(\gamma)$ are defined above.

III. The ratio of the axial to surface neutron flux within an absorbing cylinder is given by the formula

$$\frac{n(0, \omega)}{n(a, \omega)} = A(\gamma) = \frac{1}{1 + \gamma + \gamma^2 + \gamma^3} \quad (21)$$

It must be emphasized that the above empirical formulae are strictly valid for $0 < \omega < 2.2$ only and should not be extrapolated beyond this region.

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APPENDIX A

Tabulation of Experimental Data - The Radial Distribution
of Specific Activity in Monitor Foils Taken from the
Mid-lengths of Specimens No. 1 through 12.

The experimental data (specific activity or counting rate divided by weight) in the following tables are given in units of 10^3 counts/min-mg. Tables X, SV, XVII and XX refer to ring samples having inner and outer radii of r_1 and r_2 (cm), respectively. The remainder of the tables refer to "dot" samples, the centers of which were r (cm) from the geometric center of the foil. In all cases the averaged data from four foils are given.

Table X

SPECIMEN NO. 1; $a = 0.3175$ cm, $\Sigma_{\text{abs}} = 0.3135$ cm⁻¹

r_1 , cm	r_2 , cm	$n(r)$ Data			
0	0.0575	1.8914	1.7983	1.7503	1.9307
0.0580	0.1145	1.9025	1.8162	1.7605	1.9263
0.1160	0.1730	1.9188	1.8261	1.7785	1.9584
0.1735	0.2310	1.9429	1.8624	1.8068	1.9749
0.2320	0.2875	1.9770	1.8872	1.8497	2.0279
0.2880	0.3175	2.0103	1.9326	1.8843	2.0476

Table XI

SPECIMEN NO. 2; $a = 0.449$ cm, $\Sigma_{\text{abs}} = 0.3135$ cm⁻¹

r , cm	$n(r)$ Data			
0	2.1189	2.0787	2.0842	2.1527
0.136	2.1303	2.1002	2.1079	2.1684
0.205	2.1545	2.1093	2.1212	2.1863
0.273	2.1857	2.1529	2.1762	2.2160
0.341	2.2525	2.2051	2.2288	2.2833
0.409	2.2991	2.2712	2.2977	2.3307

Table XII

SPECIMEN NO. 3; $a = 0.635$ cm, $\Sigma_{\text{abs}} = 0.3135$ cm⁻¹

r , cm	$n(r)$ Data			
0	1.8137	1.9079	1.8640	1.9145
0.149	1.8391	1.9192	1.8771	1.9334
0.298	1.8705	1.9558	1.9262	1.9902
0.446	1.9544	2.0436	1.9986	2.0673
0.595	2.0846	2.1597	2.1294	2.2004

Table XIII

SPECIMEN NO. 4; $a = 0.898$ cm, $\Sigma_{\text{abs}} = 0.3135$ cm⁻¹

r, cm	n(r) Data			
0	1.4273	1.4665	1.4549	1.4100
0.215	1.4499	1.4778	1.4690	1.4281
0.429	1.5052	1.5342	1.5129	1.4898
0.644	1.5906	1.6376	1.6116	1.5744
0.858	1.7578	1.7915	1.7665	1.7362

Table XIV

SPECIMEN NO. 5; $a = 1.27$ cm, $\Sigma_{\text{abs}} = 0.3135$ cm⁻¹

r, cm	n(r) Data			
0	1.3816	1.4148	1.3629	1.3294
0.246	1.3910	1.4330	1.3719	1.3371
0.492	1.4391	1.4841	1.4209	1.3875
0.738	1.5318	1.5638	1.5063	1.4600
0.984	1.6573	1.7159	1.6361	1.5897
1.230	1.8839	1.9312	1.8415	1.7982

Table XV

SPECIMEN NO. 6; $a = 0.3175$ cm, $\Sigma_{\text{abs}} = 1.921$ cm⁻¹

r ₁ , cm	r ₂ , cm	n(r) Data			
0	0.0575	3.0025	2.9527	2.9152	3.0416
0.0580	0.1145	3.0932	3.0606	3.0061	3.1180
0.1160	0.1730	3.2569	3.2095	3.1683	3.3041
0.1735	0.2310	3.5168	3.5077	3.4519	3.5626
0.2320	0.2875	3.9506	3.9185	3.9056	4.0469
0.2880	0.3175	4.4240	4.4261	4.3826	4.4887

Table XVI

SPECIMEN NO. 7; $a = 0.449$ cm, $\Sigma_{\text{abs}} = 1.921$ cm⁻¹

r, cm	n(r) Data			
0.	2.3894	2.3157	2.2833	2.3464
0.136	2.5097	2.4426	2.3981	2.4720
0.205	2.6088	2.5263	2.4964	2.5594
0.273	2.9458	2.8872	2.8413	2.9015
0.341	3.3587	3.2851	3.2814	3.3415
0.409	4.0301	3.9654	3.9409	3.9817

Table XVII

SPECIMEN NO. 8; $a = 0.3175$ cm, $\Sigma_{\text{abs}} = 3.472$ cm⁻¹

r ₁ , cm	r ₂ , cm	n(r) Data			
0	0.0575	1.7868	1.7501	1.8406	1.8584
0.0580	0.1145	1.9103	1.8503	1.9442	1.9567
0.1160	0.1730	2.0849	2.0397	2.1430	2.1658
0.1735	0.2310	2.4698	2.4115	2.5058	2.5299
0.2320	0.2875	3.1064	3.0792	3.1613	3.2370
0.2880	0.3175	3.9408	3.8791	3.9745	4.0331

Table XVIII

SPECIMEN NO. 9; $a = 0.635$ cm, $\Sigma_{\text{abs}} = 1.921$ cm⁻¹

r, cm	n(r) Data			
0	1.7208	1.6778	1.7247	1.7488
0.149	1.7887	1.7560	1.7978	1.8356
0.298	2.0416	1.9987	2.0685	2.0891
0.446	2.6314	2.6121	2.6953	2.7111
0.595	3.9759	3.9229	4.0850	4.1295

Table XIX

SPECIMEN NO. 10; $a = 0.449$ cm, $\Sigma_{\text{abs}} = 3.472$ cm⁻¹

r, cm	n(r) Data			
0	1.1798	1.2065	1.1554	1.1290
0.136	1.2827	1.3185	1.2652	1.2392
0.205	1.3924	1.4209	1.3639	1.3312
0.273	1.7656	1.8037	1.7513	1.7114
0.341	2.3435	2.3579	2.3227	2.2461
0.409	3.3421	3.3730	3.3296	3.2300

Table XX

SPECIMEN NO. 11; $a = 0.3175$ cm, $\Sigma_{\text{abs}} = 5.704$ cm⁻¹

r ₁ , cm	r ₂ , cm	n(r) Data			
0	0.0575	0.71789	0.69211	0.67577	0.72383
0.0580	0.1145	0.79027	0.76648	0.74519	0.79537
0.1160	0.1730	0.94085	0.90818	0.88896	0.95172
0.1735	0.2310	1.2646	1.2425	1.2143	1.2843
0.2320	0.2875	1.9626	1.9226	1.9131	2.0193
0.2880	0.3175	2.9295	2.9078	2.8669	3.0021

Table XXI

SPECIMEN NO. 12; $a = 0.635$ cm, $\Sigma_{\text{abs}} = 3.472$ cm⁻¹

r, cm	n(r) Data			
0	0.67367	0.65929	0.67691	0.67365
0.149	0.72589	0.70854	0.73110	0.72882
0.298	0.93860	0.92494	0.95746	0.94130
0.446	1.5682	1.5472	1.6208	1.5951
0.595	3.3807	3.3625	3.4973	3.4114

APPENDIX B

Least-squares-adjusted Data - The Radial Distribution of Specific
Activity in Monitor Foils Taken from the Mid-lengths of
Specimens No. 1 through 12

The quantities $n(\bar{r})/n(a)$ appearing in the following tables are the least-squares-adjusted values of the specific activity (normalized to unity at the surface of the cylinder) derived from the experimental data presented in Appendix A. Tables X, XV, XVII and XX refer to ring samples. The $\bar{r}$'s are second approximations of average radial positions of the ring samples, derived from eqs. (1) and (4). The remainder of the tables refer to "dot" samples. The $\bar{r}$'s in these tables have been computed from eq. (7). In all cases the averaged data from four foils are given.

Table XXII

SPECIMEN NO. 1; $a = 0.3175$ cm, $\Sigma_{\text{abs}} = 0.3135$ cm⁻¹

$\bar{r}$, cm	$n(\bar{r})/n(a)$ Data			
0	0.93442	0.92101	0.92308	0.94096
0.090759	0.93961	0.92827	0.92817	0.94359
0.14728	0.94783	0.93824	0.93753	0.95111
0.20428	0.95988	0.95066	0.95316	0.96682
0.26123	0.97648	0.96770	0.97507	0.98827
0.30311	0.99306	0.98917	0.99365	0.99948
0.31750	1.0000	1.0000	1.0000	1.0000

Table XXIII

SPECIMEN NO. 2; $a = 0.449$ cm, $\Sigma_{\text{abs}} = 0.3135$ cm⁻¹

r , cm	$\bar{r}$, cm	$n(\bar{r})/n(a)$ Data			
0	0	0.92027	0.90117	0.89497	0.92069
0.136	0.13862	0.92431	0.90675	0.90225	0.92438
0.205	0.20675	0.93355	0.91586	0.91343	0.93293
0.273	0.27432	0.95130	0.93158	0.93169	0.94953
0.341	0.34206	0.97635	0.95486	0.95717	0.97347
0.409	0.40988	0.99821	0.98357	0.98587	0.99589
0.449	0.44900	1.0000	1.0000	1.0000	1.0000

Table XXIV

SPECIMEN NO. 3; $a = 0.635$ cm, $\Sigma_{\text{abs}} = 0.3135$ cm⁻¹

r , cm	$\bar{r}$, cm	$n(\bar{r})/n(a)$ Data			
0	0	0.85327	0.87470	0.85317	0.84737
0.149	0.15140	0.85969	0.87891	0.86085	0.85686
0.298	0.29921	0.87958	0.89659	0.88176	0.88096
0.446	0.44681	0.91670	0.93643	0.91562	0.91556
0.595	0.59561	0.97837	0.98974	0.97536	0.97439
0.635	0.63500	1.0000	1.0000	1.0000	1.0000

Table XXV

SPECIMEN NO. 4; $a = 0.898$ cm, $\Sigma_{\text{abs}} = 0.3135$ cm⁻¹

r , cm	$\bar{r}$, cm	$n(\bar{r})/n(a)$ Data			
0	0	0.78844	0.80410	0.80903	0.78858
0.215	0.21667	0.80053	0.81253	0.81594	0.80117
0.429	0.42984	0.83144	0.84141	0.84120	0.83343
0.644	0.64456	0.87845	0.89907	0.89569	0.88179
0.858	0.85842	0.97083	0.98332	0.98188	0.97214
0.898	0.89800	1.0000	1.0000	1.0000	1.0000

Table XXVI

SPECIMEN NO. 5; $a = 1.27$ cm, $\Sigma_{\text{abs}} = 0.3135$ cm⁻¹

r , cm	$\bar{r}$, cm	$n(\bar{r})/n(a)$ Data			
0	0	0.71254	0.71766	0.72212	0.72003
0.246	0.24746	0.72110	0.72521	0.73012	0.72763
0.492	0.49273	0.74635	0.74923	0.75467	0.75091
0.738	0.73849	0.78962	0.79443	0.79884	0.79304
0.984	0.98437	0.85834	0.86788	0.86910	0.86209
1.230	1.2303	0.97417	0.97815	0.97763	0.97557
1.270	1.2700	1.0000	1.0000	1.0000	1.0000

Table XXVII

SPECIMEN NO. 6, $a = 0.3175$ cm, $\Sigma_{\text{abs}} = 1.921$ cm⁻¹

$\bar{r}$, cm	$n(\bar{r})/n(a)$ Data			
0	0.64826	0.63504	0.63642	0.65166
0.090759	0.66801	0.65699	0.65587	0.66951
0.14728	0.70238	0.69327	0.69152	0.70423
0.20428	0.76004	0.75142	0.75357	0.76693
0.26123	0.85248	0.84421	0.85231	0.86523
0.30311	0.95521	0.95151	0.95657	0.96219
0.31750	1.0000	1.0000	1.0000	1.0000

Table XXVIII

SPECIMEN NO. 7; $a = 0.449$ cm, $\Sigma_{\text{abs}} = 1.921$ cm⁻¹

r , cm	$\bar{r}$, cm	$n(\bar{r})/n(a)$ Data			
0	0	0.53066	0.52151	0.52350	0.53573
0.136	0.13862	0.55154	0.54299	0.54283	0.55490
0.205	0.20675	0.58576	0.57795	0.57820	0.58992
0.273	0.27432	0.64717	0.64044	0.64465	0.65550
0.341	0.34206	0.74666	0.74158	0.75193	0.76074
0.409	0.40988	0.89290	0.89052	0.90075	0.90514
0.449	0.44900	1.0000	1.0000	1.0000	1.0000

Table XXIX

SPECIMEN NO. 8; $a = 0.3175$ cm, $\Sigma_{\text{abs}} = 3.472$ cm⁻¹

$\bar{r}$, cm	$n(\bar{r})/n(a)$ Data			
0	0.41346	0.41315	0.42304	0.42385
0.090759	0.43925	0.43634	0.44757	0.44628
0.14728	0.48519	0.48171	0.49274	0.49136
0.20428	0.56803	0.56891	0.57607	0.57912
0.26123	0.71908	0.72675	0.72720	0.73631
0.30311	0.91036	0.91544	0.91400	0.91948
0.31750	1.0000	1.0000	1.0000	1.0000

Table XXX

SPECIMEN NO. 9; $a = 0.635$ cm, $\Sigma_{\text{abs}} = 1.921$ cm⁻¹

r , cm	$\bar{r}$, cm	$n(\bar{r})/n(a)$ Data			
0	0	0.37798	0.37768	0.36939	0.36961
0.149	0.15140	0.39364	0.39295	0.38569	0.38571
0.298	0.29921	0.44855	0.44961	0.44311	0.44123
0.446	0.44681	0.57849	0.58643	0.57770	0.57149
0.595	0.59561	0.87396	0.88105	0.87547	0.87082
0.635	0.63500	1.0000	1.0000	1.0000	1.0000

Table XXXI

SPECIMEN NO. 10; $a = 0.449$ cm, $\Sigma_{\text{abs}} = 3.472$ cm⁻¹

r , cm	$\bar{r}$, cm	$n(\bar{r})/n(a)$ Data			
0	0	0.28643	0.28794	0.28113	0.28136
0.136	0.13862	0.30499	0.30818	0.30004	0.30144
0.205	0.20675	0.34302	0.34600	0.33838	0.33963
0.273	0.27432	0.42234	0.42240	0.41800	0.41726
0.341	0.34206	0.56766	0.56286	0.56391	0.55990
0.409	0.40988	0.80731	0.80131	0.80523	0.80069
0.449	0.44900	1.0000	1.0000	1.0000	1.0000

Table XXXII

SPECIMEN NO. 11; $a = 0.3175$ cm, $\Sigma_{\text{abs}} = 5.704$ cm⁻¹

$\bar{r}$, cm	$n(\bar{r})/n(a)$ Data			
0	0.21125	0.20400	0.20350	0.20870
0.090759	0.23259	0.22607	0.22367	0.22884
0.14728	0.27666	0.27015	0.26748	0.27286
0.20428	0.37230	0.36446	0.36538	0.37117
0.26123	0.57740	0.56895	0.57522	0.58042
0.30311	0.86203	0.85819	0.86228	0.86439
0.31750	1.0000	1.0000	1.0000	1.0000

Table XXXIII

SPECIMEN NO. 12; $a = 0.635$ cm, $\Sigma_{\text{abs}} = 3.472$ cm⁻¹

r , cm	$\bar{r}$, cm	$n(\bar{r})/n(a)$ Data			
0	0	0.15862	0.15523	0.15430	0.15859
0.149	0.15140	0.17118	0.16809	0.16709	0.17067
0.298	0.29921	0.22105	0.21804	0.21834	0.22141
0.446	0.44681	0.36949	0.36550	0.36988	0.37464
0.595	0.59561	0.79648	0.79408	0.79803	0.80141
0.635	0.63500	1.0000	1.0000	1.0000	1.0000