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RATIONAL APPROXIMATION OF FUNCTIONS

By
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August 1955

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LOS ALAMOS SCIENTIFIC LABORATORY
of the
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RATIONAL APPROXIMATION OF FUNCTIONS

by
Bengt Carlson
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Abstract

Rational approximations of the type derived by Cecil Hastings, Jr., formerly of the Rand Corporation, have been found to be extremely useful in numerical work. A difference method well suited for high-speed computation has been developed for obtaining such continued products (polynomials) and continued fractions. A code for the IBM Type 701 calculator, incorporating this method, is now available which rapidly and automatically computes the optimum approximation for an arbitrary degree n . Sets of polynomial and continued fraction approximations are given for the elementary functions.

Acknowledgment

We would like to thank Chester Kazek, Jr., and Josephine Powers for assistance in coding and running this problem.

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Introduction

Continued products (power series) and continued fractions play an important role in classical function theory. The study of terminated series of these types is, for instance, a basic part of approximation theory. Since the advent of the high-speed digital computer, the interest in this subject has grown considerably among numerical analysts. One reason for this is the ease and speed with which such calculators are able to perform the computations involved. For this very reason, the rational approximations due to Cecil Hastings, Jr.,⁽¹⁾ have been used extensively.

The rational expressions derived by terminating continued products and fractions may be regarded as a type of approximator for obtaining function values as needed during a calculation, or for constructing a table of such values to be used later. In general, one might define an approximator as a mathematical expression having the following properties: The expression should be readily calculable, should belong to a family which is capable of approximating a large class of functions, and should preferably have a well-defined error term relative to the given function.

In some cases, expansions of the given function, if available, may be satisfactory as approximators. In other cases, the construction of an interpolating function may serve the same purpose. But regardless of whether the function is given in discrete or analytic form, it is often worth-while to generate an optimum approximator ("best fit").

(1) Cecil Hastings, Jr., Project Rand Report R-264, Nov. 1954.

In this report we consider the problem of constructing rational approximators $\bar{y}$, along the lines indicated above, for an arbitrary single-valued function y in the case of a single independent variable x .

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Expansion into Continued Products and Fractions

A function $y(x)$, analytic at $x = x_0$, may be written in the form of a continued product as follows:

$$y(x) = a_0 + (x-x_0) \left[a_1 + (x-x_0) \left[a_2 + (x-x_0) \left[a_3 + \dots \right] \right] \right]^{(2)} \quad (1)$$

where $a_j = (1/j!) \left[d^j y / dx^j \right]_{x=x_0}$. This is the familiar Taylor series.

Similarly, $y(x)$ may be expanded into a continued fraction about $x = x_0$:

$$y(x) = \alpha_0 + (x-x_0)/\alpha_1 + (x-x_0)/\alpha_2 + (x-x_0)/\alpha_3 + \dots,^{(2)} \quad (2)$$

where α_j is expressible in terms of a_j . The coefficients α_j may be referred to as the reciprocal derivatives of $y(x)$ at $x = x_0$. It is readily verified that $\alpha_0 = a_0$, $\alpha_1 = 1/a_1$, and $\alpha_2 = -1/\alpha_1^2 a_2$. The general relationship is given by

$$\alpha_j = (-1)^{j-1} / c_{jj} \sum_{k=j_0}^j c_{jk} a_k, \quad j = 1, 2, \dots, \quad (3)$$

where j_0 is the greatest integer in $(j+2)/2$, $c_{11} = 1$, $c_{22} = \alpha_1$, $c_{21} = 0$.

and

$$c_{jk} = \begin{cases} 0, & \text{if } k \neq j_0, j_0+1, \dots, j \\ \alpha_{j-1} c_{j-1, k-1} + c_{j-2, k-1}, & j_0 \leq k \leq j \end{cases}, \quad j = 3, 4, \dots$$

(2) We have adopted this notation to represent a power series and a continued fraction of the following form respectively:

$$y(x) = a_0 + a_1(x-x_0) + a_2(x-x_0)^2 + \dots$$

$$y(x) = \alpha_0 + \frac{x - x_0}{\alpha_1 + \frac{x - x_0}{\alpha_2 + \dots}}$$

If the continued fraction expansion is given, the Taylor series may be obtained by solving (3) for a_j :

$$a_j = \left\{ \left[(-1)^{j-1} \alpha_j \right] - c_{jj} \sum_{k=j_0}^{j-1} c_{jk} a_k \right\} / c_{jj}^2, \quad j = 3, 4, \dots, \quad (4)$$

with $a_0 = \alpha_0$, $a_1 = 1/\alpha_1$, and $a_2 = -1/\alpha_1^2 \alpha_2$.

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Interpolation by Continued Products and Fractions

Let $[x_j]$ and $[y_j]$, $j=0,1,\dots,n$, represent two given sets of numbers and let the elements of $[x_j]$ be distinct. It is well known that one may then construct a polynomial $y_0(x)$ such that $y_{0j} \equiv y_0(x_j) = y_j$ for all j . This is most conveniently done with the aid of a table of divided differences obtained as follows: We write the polynomial $y_0(x)$ in the form of a continued product:

$$y_0(x) = a_0 + (x-x_0) [a_1 + (x-x_1) [a_2 + (x-x_2) [a_3 + \dots + (x-x_{n-1}) [a_n, \quad (5)$$

and note first that $a_0 = y_{00}$. We then form successively the derived functions $y_i(x) = [y_{i-1}(x) - a_{i-1}] / (x-x_{i-1})$, $i=1,2,\dots,n$, which we evaluate for $x = x_j$, $j=i,i+1,\dots,n$, thus obtaining the divided differences $y_{ij} \equiv y_i(x_j)$. Since

$$y_i(x) = a_i + (x-x_i) [a_{i+1} + \dots + (x-x_{n-1}) [a_n, \quad (6)$$

it is readily seen that $a_i = y_{ii} \equiv y_i(x_i)$.

Example 1:

	j=0	j=1	j=2	j=3
$\{x_j\}$	1	3	2	6
$\{y_{0j}\}$	<u>2</u>	6	5	4
$\{y_{1j}\}$		<u>2</u>	3	2/5
$\{y_{2j}\}$			<u>-1</u>	-8/15
$\{y_{3j}\}$				<u>7/60</u>

$$\therefore y_0(x) = 2 + (x-1) [2 + (x-3) [-1 + (x-2) [7/60 .$$

Similarly, it is also possible to construct a continued fraction $y_0(x)$ such that $y_0(x_j) = y_j$ for all j . This is done with the aid of a table of reciprocal divided differences. We write the continued fraction $y_0(x)$ as follows:

$$y_0(x) = a_0 + (x-x_0)/a_1 + (x-x_1)/a_2 + (x-x_2)/a_3 + \dots + (x-x_{n-1})/a_n. \quad (7)$$

and note that $a_0 = y_{00}$. We then form a set of derived functions

$$y_i(x) = (x-x_{i-1}) / [y_{i-1}(x) - a_{i-1}], \quad i=1,2,\dots,n, \text{ which we evaluate for } x=x_j, \quad j=i,i+1,\dots,n, \text{ obtaining } y_{ij} \equiv y_i(x_j). \text{ These are the reciprocal divided differences and again we find that } a_i = y_{ii}. \quad (3)$$

Example 2:

	j=0	j=1	j=2	j=3
$\{x_j\}$	1	3	2	6
$\{y_{0j}\}$	<u>2</u>	6	5	4
$\{y_{1j}\}$		<u>1/2</u>	1/3	5/2
$\{y_{2j}\}$			<u>6</u>	3/2
$\{y_{3j}\}$				<u>-8/9</u>

$$\therefore y_0(x) = 2 + (x-1)/\frac{1}{2} + (x-3)/6 + (x-2)/-\frac{8}{9}.$$

Now, it is sometimes desirable to present polynomials and continued fractions in a normal form, i.e.. as expansions about some fixed x , $x = \bar{x}$. For the sake of simplicity, we let $\bar{x} = x_0$ below. We, therefore, want to transform (5) into the form:

$$y_0(x) = c_0 + (x-x_0) [c_1 + (x-x_0) [c_2 + \dots + (x-x_0) [c_n, \quad (8)$$

-
- (3) The case of a "degenerate set", which can lead to zero divisors, will not be discussed here. See, for example, W. C. Milne, Numerical Calculus, Princeton Univ. Press, 1949. p. 236.

and (7) into the form:

$$y_0(x) = c_0 + (x-x_0)/c_1 + (x-x_0)/c_2 + \dots + (x-x_0)/c_n. \quad (9)$$

Although the transformation of (5) into (8) may be done in a rather simple fashion, the following procedure is applicable to both (5) and (7). The transformation is done in $n-1$ stages. Each stage k , $k=1,2,\dots,n-1$, involves the construction of a divided difference table of order $n-k+1$. The first entries of the table, $y_{kj} \equiv y_k(x_j)$, $j=0,1,2,\dots,n-k$, are all available from the previous stage except y_{k0} . This entry $y_{k0} \equiv y(x_0)$ is calculated from the $(x-x_0)$ term of the expansion of $y_{k-1}(x)$.

Applying this procedure to Example 2, we have first:

$$y_{10} \equiv y_1(x_0) = \frac{1}{2} + (x_0-3)/6 + (x_0-2)/-\frac{8}{9} = 25/114, \text{ for } x_0 = 1$$

and then:

$\{x_j\}$	1	3	2
$\{y_{1j}\}$	25/114	1/2	1/3
$\{y_{2j}\}$		57/8	114/13
$\{y_{3j}\}$			-104/171

or

$$y_1(x) = \frac{25}{114} + (x-1)/\frac{57}{8} + (x-3)/-\frac{104}{171}.$$

Similarly,

$$y_{20} = \frac{57}{8} + (x_0-3)/-\frac{104}{171} = \frac{1083}{104},$$

giving rise to this difference table

$\{x_j\}$	1	3
$\{y_{2j}\}$	$1083/104$	$57/8$
$\{y_{3j}\}$		$-104/171$

Hence

$$y_2(x) = \frac{1083}{104} + (x-1)/-\frac{104}{171} .$$

Finally, combining the results above, we have:

$$y_0(x) = 2 + (x-1)/\frac{25}{114} + (x-1)/\frac{1083}{104} + (x-1)/-\frac{104}{171} .$$

||

Optimum Rational Approximators

We consider next the problem of finding an optimum rational approximator $\bar{y}(x)$, of degree n , to a given analytic function $y(x)$ over the x -interval (a,b) . We denote the error function associated with $\bar{y}(x)$ and $y(x)$ by $\epsilon(x)$, where $\epsilon(x)$ may be defined to be equal to $(\bar{y}-y)$ (absolute error), $(\bar{y}-y)/y$ (relative error), or some other function of x , $\bar{y}$, and y .

The method, to be described below, for obtaining optimum approximators is essentially an empirical one which proceeds toward the final solution in an iterative manner. It has proven to be rapidly convergent in the many applications made to date, and it is, therefore, hoped that the method may eventually be put on a firm mathematical basis. For the present, we assume that the approximators are chosen so that the error vanishes at the end points, $\epsilon(a) = \epsilon(b) = 0$, and that $\epsilon(x)$ has precisely n relative extrema M_j , $j=1,2,\dots,n$, within the interval (a,b) at every stage of iteration. We now say that an approximator is optimized if $|M_j|$ is equal to a constant, $\bar{M}$, for all j .

We assume further that $y_k \equiv y(x_k)$, $k=0,1,\dots,K$, $x_0=a$, $x_K=b$, is given in tabular form to an accuracy exceeding that which is required of the final approximator, that K is very much greater than n , and that the x_k -values are fairly uniformly distributed over the interval (a,b) . We start the iterative procedure by choosing a subset $\{x_j\}$ and the corresponding set $\{y_j\}$, $j=0,1,\dots,n$, from the larger sets $\{x_k\}$ and $\{y_k\}$. The method may now be outlined as follows:

(A) Apply the difference methods described above to the sets $\{x_j\}$ and $\{y_j\}$ to obtain a continued product or fraction, as the case may be, and let this be the first iterate of $\bar{y}$.

(B) Calculate $\epsilon_k \equiv \epsilon(x_k)$, $k=0,1,\dots,K$, and determine the n extrema M_j of $\epsilon(x)$, $j=1,2,\dots,n$. If, for example, quadratic interpolation with equal intervals is used, then M_j is given by:

$$M_j = \epsilon_j - \left[(\epsilon_{j+1} - \epsilon_{j-1})^2 / 8(\epsilon_{j+1} - 2\epsilon_j + \epsilon_{j-1}) \right], \quad (10)$$

where ϵ_j is the apparent extremum.

(C) Determine an improved set of x 's, denoted by $\{\bar{x}_j\}$ from the empirical formula:

$$\bar{x}_j = \frac{\bar{M} + |M_j|}{2|M_j|} (x_j - x_{j-1}) + \bar{x}_{j-1}, \quad j=1,2,\dots,n-1, \quad (\bar{x}_0=x_0, \bar{x}_n=x_n) \quad (11)$$

where $\bar{M} = (x_n - x_0) / \left[\sum_j (x_j - x_{j-1}) / |M_j| \right]$. This choice of $\bar{M}$ preserves the total length $(b-a)$ of the interval.

(D) Finally, calculate the y_j -values corresponding to the new x_j -set, using the current expression for $\bar{y}(x)$ corrected by interpolation in $\epsilon(x)$. For example, if relative error has been used, then

$$y_j = \frac{\bar{y}(\bar{x}_j)}{1 + \epsilon(\bar{x}_j)}.$$

The iterative process is terminated by a simple test requiring that for a preassigned δ , $||M_j| - \bar{M}| < \delta$ for all j .

Tables of Approximations

The method outlined in the previous section was coded for the IBM Type 701 to generate continued products and fractions.

In order to minimize figure loss during the calculations, as well as aid in the preparation of functions to be approximated, an interpretive double precision routine was built into the code. This subroutine which can perform the fundamental arithmetic operations, allows for an integral part as large as 2^{17} and a fractional part of 53 bits (approximately 16 decimals).

In the tables, which follow, each function considered was fit at the end points of the interval and only the relative error was optimized. The tables list the maximum relative error and the coefficients of the "best fit" given in normal form. The coefficients for the highest degree listed were, in some cases, derived from the interpolating function without optimization because of the lack of figures in the error curve. In the last column headed "expansion" are given the coefficients of the power series expansion for polynomials and the reciprocal derivative expansion for continued fractions.

The time required to generate an optimum rational approximator of degree n , by this method, depends to some extent on the function considered but is approximately $8n$ seconds.

$$y = x^{1/2}$$

$$\frac{1}{2} \leq x \leq 1$$

$$y = a_0 + a_1(1-x) + a_2(1-x)^2 + \dots + a_n(1-x)^n$$

	n = 2	n = 3	n = 4	n = 5	n = 6
max. rel. error	.00099	.000095	.000011	.0000013	.00000018
a ₀	1.0000000000	1.0000000000	1.0000000000	1.0000000000	1.0000000000
a ₁	-.4837687809	-.5030549907	-.4994331295	-.5001040301	-.4999811157
a ₂	-.2040353134	-.0948292282	-.1343542792	-.1224246708	-.1256546737
a ₃		-.1412673312	-.0158174420	-.0829441565	-.0549947254
a ₄			-.1217744644	.0287991569	-.0777601270
a ₅				-.1173428416	.0683445258
a ₆					-.1209862644

	n = 7	n = 8	n = 9	n = 10	EXPANSION
max. rel. error	.000000024	.0000000034	.0000000006		
a ₀	1.0000000000	1.0000000000	1.0000000000		1.0000000000
a ₁	-.5000034529	-.4999993731	-.5000000918		-.5000000000
a ₂	-.1248408855	-.1250369416	-.1249926712		-.1250000000
a ₃	-.0649710789	-.0617555849	-.0626933106		-.0625000000
a ₄	-.0210916033	-.0462695265	-.0366110378		-.0390625000
a ₅	-.0953512648	.0106562447	-.0446262628		-.0273437500
a ₆	.1126686693	-.1336559949	.0514016363		-.0205078125
a ₇	-.1308648773	.1667757100	-.1936266442		-.0161132812
a ₈		-.1460662554	.2322163982		-.0130920410
a ₉			-.1654366428		-.0109100342
a ₁₀					-.0092735291

$$y = x^{1/2}$$

$$\frac{1}{2} \leq x \leq 1$$

$$y = a_0 + \frac{1-x}{a_1 + \frac{1-x}{a_2 + \dots + \frac{1-x}{a_n}}}$$

	n=2	n=3	n=4	n=5	n=6
max. rel. error	.00050	.000019	.00000077	.000000032	.0000000014
a_0	1.0000000000	1.0000000000	1.0000000000	1.0000000000	1.0000000000
a_1	-2.0301122087	-1.9978781888	-2.0001368323	-1.9999917673	-2.0000004732
a_2	1.5479616051	2.0779105412	1.9923227253	2.0006752043	1.9999469214
a_3		-1.3952968084	-2.1490799225	-1.9818070308	-2.0020158574
a_4			1.2471683754	2.2491018637	1.9647612850
a_5				-1.1042641244	-2.3850481682
a_6					.9674273838

	n=7	n=8	EXPANSION		
max. rel. error					
a_0					1.0000000000
a_1					-2.0000000000
a_2					2.0000000000
a_3					-2.0000000000
a_4					2.0000000000
a_5					-2.0000000000
a_6					2.0000000000
a_7					
a_8					

$$y = x^{1/3}$$

$$\frac{1}{2} \leq x \leq 1$$

$$y = a_0 + a_1(1-x) + a_2(1-x)^2 + \dots + a_n(1-x)^n$$

n = 2

n = 3

n = 4

n = 5

n = 6

max. rel. error	.00098	.00010	.000012	.0000015	.00000021
a_0	1.0000000000	1.0000000000	1.0000000000	1.0000000000	1.0000000000
a_1	-.3168444464	-.3366211093	-.3326970992	-.3334540473	-.3333108165
a_2	-.1915090032	-.0783905854	-.1217049404	-.1080958229	-.1118985703
a_3		-.1471301838	-.0086220012	-.0858052928	-.0526439284
a_4			-.1351510263	.0390056736	-.0881931084
a_5				-.1363420052	.0863926703
a_6					-.1457045653

n = 7

n = 8

n = 9

n = 10

EXPANSION

max. rel. error	.0000000029	.0000000042	.0000000007		
a_0	1.0000000000	1.0000000000	1.0000000000		1.0000000000
a_1	-.3333375537	-.3333325606	-.3333334637		-.3333333333
a_2	-.1109151473	-.1111572534	-.1111010851		-.1111111111
a_3	-.0647901847	-.0607900687	-.0619889446		-.0617283951
a_4	-.0187825236	-.0502954432	-.0378740932		-.0411522634
a_5	-.1151066961	.0182521509	-.0531763563		-.0301783265
a_6	.1431065091	-.1681206392	.0718907369		-.0234720317
a_7	-.1623276113	.2151432519	-.2537467041		-.0190011685
a_8		-.1858452351	.3076055669		-.0158343071
a_9			-.2162901609		-.0134884838
a_{10}					-.0116900193

$$y = x^{1/3}$$

$$\frac{1}{2} \leq x \leq 1$$

$$y = a_0 + \frac{1-x}{a_1 + \frac{1-x}{a_2 + \dots + \frac{1-x}{a_n}}}$$

	n = 2	n = 3	n = 4	n = 5	n = 6
max. rel. error	.00039	.000017	.00000063	.000000028	.0000000012
a_0	1.0000000000	1.0000000000	1.0000000000	1.0000000000	1.0000000000
a_1	-3.0536387711	-2.9957770401	-3.0002521916	-2.9999837637	-3.0000009117
a_2	.7936788623	1.0391008461	.9964632103	1.0003342433	.9999747438
a_3		-3.0275105760	-4.8438344561	-4.4546128485	-4.5047918618
a_4			.5118822826	.9009451979	.7865736195
a_5				-2.8606601417	-6.3929786301
a_6					.3474403207

	n = 7	n = 8	EXPANSION		
max. rel. error					
a_0					1.0000000000
a_1					-3.0000000000
a_2					1.0000000000
a_3					-4.5000000000
a_4					.8000000000
a_5					-5.3571428571
a_6					.7000000000
a_7					
a_8					

$$y = x^{1/4}$$

$$\frac{1}{2} \leq x \leq 1$$

$$y = a_0 + a_1(1-x) + a_2(1-x)^2 + \dots + a_n(1-x)^n$$

	n = 2	n = 3	n = 4	n = 5	n = 6
max. rel. error	.00087	.000092	.000011	.0000014	.00000020
a_0	1.0000000000	1.0000000000	1.0000000000	1.0000000000	1.0000000000
a_1	-.2351855231	-.2530355854	-.2494006617	-.2501155177	-.2499781455
a_2	-.1660432932	-.0634202179	-.1037755853	-.0908511617	-.0945175653
a_3		-.1338459013	-.0043124635	-.0779052813	-.0458048338
a_4			-.1267247955	.0398255695	-.0836919161
a_5				-.1306871512	.0861514527
a_6					-.1421442983

	n = 7	n = 8	n = 9	n = 10	EXPANSION
max. rel. error	.000000028	.0000000042	.0000000008		
a_0	1.0000000000	1.0000000000	1.0000000000		1.0000000000
a_1	-.2500041209	-.2499992266	-.2500001369		-.2500000000
a_2	-.0935575953	-.0937961669	-.0937396508		-.0937500000
a_3	-.0577059556	-.0537473498	-.0549551839		-.0546875000
a_4	-.0154764103	-.0467731939	-.0342314446		-.0375976562
a_5	-.1123712924	.0204704214	-.0518275052		-.0281982422
a_6	.1429924058	-.1678243028	.0757001494		-.0223236084
a_7	-.1605451392	.2172658783	-.2595841049		-.0183372498
a_8		-.1863736520	.3165257419		-.0154720545
a_9			-.2208632687		-.0133231580
a_{10}					-.0116577633

$$y = x^{1/4}$$

$$\frac{1}{2} \leq x \leq 1$$

$$y = a_0 + \frac{1-x}{a_1 + \frac{1-x}{a_2 + \frac{1-x}{a_3 + \frac{1-x}{a_4 + \frac{1-x}{a_5 + \frac{1-x}{a_6 + \frac{1-x}{a_7 + \frac{1-x}{a_8}}}}}}}}$$

	n = 2	n = 3	n = 4	n = 5	n = 6
max. rel. error	.00031	.000014	.00000051	.000000023	.0000000010
a_0	1.0000000000	1.0000000000	1.0000000000	1.0000000000	1.0000000000
a_1	-4.0754828513	-3.9937406965	-4.0003590523	-3.9999760488	-4.0000013154
a_2	.5359768576	.6925299404	.6644286758	.6668859058	.6666504452
a_3		-4.7583547036	-7.7537023573	-7.1240039659	-7.2078552174
a_4			.3087427540	.5364921989	.4683788716
a_5				-4.9020829766	-11.1472054984
a_6					.1955461656

	n = 7	n = 8	EXPANSION		
max. rel. error					
a_0					1.0000000000
a_1					-4.0000000000
a_2					.6666666667
a_3					-7.2000000000
a_4					.4761904762
a_5					-9.3333333333
a_6					.3896103896
a_7					
a_8					

$$y = x^{1/5}$$

$$\frac{1}{2} \leq x \leq 1$$

$$y = a_0 + a_1(1-x) + a_2(1-x)^2 + \dots + a_n(1-x)^n$$

n = 2

n = 3

n = 4

n = 5

n = 6

max. rel. error	.00076	.000082	.000010	.0000013	.00000018
a_0	1.0000000000	1.0000000000	1.0000000000	1.0000000000	1.0000000000
a_1	-.1868872984	-.2027299665	-.1994545283	-.2001061414	-.1999797379
a_2	-.1440231500	-.0526588680	-.0891493316	-.0773292326	-.0807132535
a_3		-.1193578912	-.0019645722	-.0694290052	-.0397320305
a_4			-.1150282893	.0379239297	-.0765534758
a_5				-.1201816886	.0810808031
a_6					-.1320917370

n = 7

n = 8

n = 9

n = 10

EXPANSION

max. rel. error	.000000026	.0000000039	.0000000006		
a_0	1.0000000000	1.0000000000	1.0000000000		1.0000000000
a_1	-.2000038555	-.1999992849	-.2000001359		-.2000000000
a_2	-.0798196895	-.0800429160	-.0799898126		-.0800000000
a_3	-.0508324686	-.0471229813	-.0482598100		-.0480000000
a_4	-.0128213372	-.0421838145	-.0303673173		-.0336000000
a_5	-.1046508190	.0201047607	-.0480672139		-.0255360000
a_6	.1349834324	-.1571588858	.0726310448		-.0204288000
a_7	-.1505244706	.2048416644	-.2454100991		-.0169267200
a_8		-.1754101347	.2997316556		-.0143877120
a_9			-.2087946230		-.0124693504
a_{10}					-.0109730284

$$y = x^{1/5}$$

$$\frac{1}{2} \leq x \leq 1$$

$$y = a_0 + \frac{1-x}{a_1 + \frac{1-x}{a_2 + \dots + \frac{1-x}{a_n}}}$$

	n = 2	n = 3	n = 4	n = 5	n = 6
max. rel. error	.00025	.000012	.00000042	.000000020	.0000000008
a_0	1.0000000000	1.0000000000	1.0000000000	1.0000000000	1.0000000000
a_1	-5.0966502117	-4.9917400024	-5.0004622653	-4.9999685643	-5.0000016286
a_2	.4051410023	.5192435561	.4983793100	.5001620132	.4999884548
a_3		-6.5390943966	-10.7706480499	-9.8918222647	-10.0108499852
a_4			.2178613159	.3755696909	.3280122183
a_5				-7.0930071320	-16.2684656083
a_6					.1327587161

	n = 7	n = 8	EXPANSION		
max. rel. error					
a_0					1.0000000000
a_1					-5.0000000000
a_2					.5000000000
a_3					-10.0000000000
a_4					.3333333333
a_5					-13.6363636364
a_6					.2619047619
a_7					
a_8					

$$y = x^{1/6}$$

$$\frac{1}{2} \leq x \leq \frac{1}{2}$$

$$y = a_0 + a_1(1-x) + a_2(1-x)^2 + \dots + a_n(1-x)^n$$

	n = 2	n = 3	n = 4	n = 5	n = 6
max. rel. error	.00067	.000073	.0000091	.0000012	.00000017
a ₀	1.0000000000	1.0000000000	1.0000000000	1.0000000000	1.0000000000
a ₁	-.1550083867	-.1691192870	-.1661728673	-.1667633864	-.1666478761
a ₂	-.1263883543	-.0448417567	-.0777426308	-.0670063854	-.0701049010
a ₃		-.1066495937	-.0006444878	-.0620232707	-.0347900508
a ₄			-.1039780731	.0353490284	-.0697581104
a ₅				-.1095807289	.0753879533
a ₆					-.1214946487

	n = 7	n = 8	n = 9	n = 10	EXPANSION
max. rel. error	.0000 10024	.0000000036	.0000000007		
a ₀	1.0000000000	1.0000000000	1.0000000000		1.0000000000
a ₁	-.1666701786	-.1666659931	-.1666667709		-.1666666667
a ₂	-.0692794330	-.0694846955	-.0694362323		-.0694444444
a ₃	-.0450384066	-.0416174667	-.0426556820		-.0424382716
a ₄	-.0109431951	-.0380831926	-.0272817178		-.0300604424
a ₅	-.0959580974	.0195685370	-.0428053271		-.0230463392
a ₆	.1248321518	-.1461194551	.0643130353		-.0185649577
a ₇	-.1388026468	.1912233307	-.2214406018		-.0154709221
a ₈		-.1630876671	.2727189027		-.0132147460
a ₉			-.1916410857		-.0115017234
a ₁₀					-.0101598556

$$y = x^{1/6}$$

$$\frac{1}{2} \leq x \leq 1$$

$$y = a_0 + \frac{1-x}{a_1 + \frac{1-x}{a_2 + \frac{1-x}{a_3 + \dots + \frac{1-x}{a_n}}}}$$

	n=2	n=3	n=4	n=5	n=6
max. rel. error	.00022	.000010	.00000035	.000000017	.0000000007
a_0	1.0000000000	1.0000000000	1.0000000000	1.0000000000	1.0000000000
a_1	-6.1174784835	-5.9897599802	-6.0005635088	-5.9999613749	-6.0000019898
a_2	.3258218556	.4152916977	.3987355481	.4001277506	.3999909651
a_3		-8.3485115300	-13.8486057645	-12.7160931164	-12.8711764012
a_4			.1672673284	.2867672927	.2505278397
a_5				-9.3732630500	-21.6332903096
a_6					.0991512952

	n=7	n=8	EXPANSION		
max. rel. error					
a_0					1.0000000000
a_1					-6.0000000000
a_2					.4000000000
a_3					-12.8571428571
a_4					.2545454545
a_5					-18.1318681319
a_6					.1947668607
a_7					
a_8					

$$y = x^{1/7}$$

$$\frac{1}{2} \leq x \leq 1$$

$$y = a_0 + a_1(1-x) + a_2(1-x)^2 + \dots + a_n(1-x)^n$$

	n = 2	n = 3	n = 4	n = 5	n = 6
max. rel. error	.00060	.000066	.0000082	.0000011	.00000015
a ₀	1.0000000000	1.0000000000	1.0000000000	1.0000000000	1.0000000000
a ₁	-.1324036896	-.1450724875	-.1424086253	-.1429453898	-.1428401393
a ₂	-.1122979639	-.0389763916	-.0687716047	-.0589971285	-.0618247283
a ₃		-.0959679532	.0001355815	-.0558085100	-.0309284895
a ₄			-.0943371147	.0327617613	-.0633467605
a ₅				-.1000289634	.0692215645
a ₆					-.1112335509

	n = 7	n = 8	n = 9	n = 10	EXPANSION
max. rel. error	.000000022	.0000000038			
a ₀	1.0000000000	1.0000000000			1.0000000000
a ₁	-.1428603745	-.1428564465			-.1428571429
a ₂	-.0610725470	-.0612658401			-.0612244898
a ₃	-.0402962567	-.0370652120			-.0379008746
a ₄	-.0094528597	-.0351570190			-.0270720533
a ₅	-.0881036729	.0215938691			-.0208841554
a ₆	.1153151264	-.1425904637			-.0169062210
a ₇	-.1278380856	.1870160833			-.0141460217
a ₈		-.1559252198			-.0121251615
a ₉					-.0105854584
a ₁₀					-.0093756917

$$y = x^{1/7}$$

$$\frac{1}{2} \leq x \leq 1$$

$$y = a_0 + \frac{1-x}{a_1 + \frac{1-x}{a_2 + \dots + \frac{1-x}{a_n}}}$$

	n = 2	n = 3	n = 4	n = 5	n = 6
max. rel. error	.00019	.0000088	.00000031	.000000015	.0000000028
a_0	1.0000000000	1.0000000000	1.0000000000	1.0000000000	1.0000000000
a_1	-7.1381126163	-6.9877926956	-7.0006636129	-6.9999539895	-7.0000006607
a_2	.2725456918	.3460063968	.3322992041	.3334389994	.3333300575
a_3		-10.1759037493	-16.9646467134	-15.5751545397	-15.7601526123
a_4			.1353180861	.2311075027	.2028342312
a_5				-11.7042477496	-26.2618505409
a_6					.0841921043

	n = 7	n = 8	EXPANSION	
max. error				
a_0				1.0000000000
a_1				-7.0000000000
a_2				.3333333333
a_3				-15.7500000000
a_4				.2051282051
a_5				-22.7500000000
a_6				.1538461538
a_7				
a_8				

$$y = -\frac{\ln x}{1-x}$$

$$\frac{1}{2} \leq x \leq 1$$

$$y = a_0 + a_1(1-x) + a_2(1-x)^2 + \dots + a_n(1-x)^n$$

	n = 2	n = 3	n = 4	n = 5	n = 6
max. rel. error	.0036	.00043	.000058	.0000081	.0000012
a_0	1.0000000000	1.0000000000	1.0000000000	1.0000000000	1.0000000000
a_1	.4302130173	.5158328571	.4966106095	.5006918151	.4998624352
a_2	.6847514092	.1672710286	.3927632590	.3151734306	.3383677288
a_3		.6924814022	-.0577500201	.4002538069	.1898658069
a_4			.7522719039	-.3125207264	.5197454940
a_5				.8529892900	-.6408555111
a_6					.9971601072

	n = 7	n = 8	n = 9	n = 10	EXPANSION
max. rel. error	.00000018	.000000027	.0000000041	.0000000007	
a_0	1.0000000000	1.0000000000	1.0000000000	1.0000000000	1.0000000000
a_1	.5000267575	.4999948372	.5000009805	.4999998339	.5000000000
a_2	.3320322631	.3336547885	.3332574042	.3333498856	.3333333333
a_3	.2710777216	.2432432196	.2520047500	.2494509224	.2500000000
a_4	.0414903059	.2676527259	.1743372172	.2088306252	.2000000000
a_5	.7823856409	-.1998314345	.3496802299	.0864198193	.1666666667
a_6	-1.0856223799	1.2577343433	-.6272690428	.5854799951	.1428571429
a_7	1.1914695082	-1.7051156765	2.0447631853	-1.3954192821	.1250000000
a_8		1.4498295939	-2.5600469003	3.3179887579	.1111111111
a_9			1.7827288440	-3.7653547371	.1000000000
a_{10}				2.2220565950	.0909090909

$$y = -\frac{\ln x}{1-x}$$

$$\frac{1}{2} \leq x \leq 1$$

$$y = a_0 + \frac{1-x}{a_1 + \frac{1-x}{a_2 + \frac{1-x}{\ddots + \frac{1-x}{a_n}}}}$$

	n=2	n=3	n=4	n=5	n=6
max. rel. error	.00044	.000030	.00000066	.000000041	.0000000012
a_0	1.0000000000	1.0000000000	1.0000000000	1.0000000000	1.0000000000
a_1	2.0272068273	1.9964628770	2.0001172631	1.9999891269	2.0000003822
a_2	-.6822612466	-.7692571671	-.7490701661	-.7501292207	-.7499938830
a_3		8.7533113186	17.0581362245	15.7817238912	16.0143821947
a_4			-.1052761493	-.1547097399	-.1376124861
a_5				23.44444496916	63.0507602224
a_6					-.0292661952

	n=7	n=8	EXPANSION		
max. rel. error					
a_0					1.0000000000
a_1					2.0000000000
a_2					-.7500000000
a_3					16.0000000000
a_4					-.1388888889
a_5					54.0000000000
a_6					-.0486111111
a_7					
a_8					

$$y = e^{-x}$$

$$0 \leq x \leq \frac{\ln 10}{4}$$

$$y = a_0 + a_1 x + a_2 x^2 + \dots + a_n x^n$$

	n = 2	n = 3	n = 4	n = 5	n = 6
max. rel. error	.0015	.000049	.0000013	.000000030	.0000000007
a_0	1.0000000000	1.0000000000	1.0000000000	1.0000000000	1.0000000000
a_1	-.9761329399	-.9985369361	-.9999360512	-.9999978270	-.9999999333
a_2	.3749558719	.4856738913	.4989924498	.4999498088	.4999979473
a_3		-.1247264074	-.1615856633	-.1662823725	-.1666450652
a_4			.0311732783	.0403557859	.0415600350
a_5				-.0062342769	-.0080616062
a_6					.0010365431

	n = 7	n = 8	n = 9	n = 10	EXPANSION
max. rel. error					
a_0					1.0000000000
a_1					-1.0000000000
a_2					.5000000000
a_3					-.1666666667
a_4					.0416666667
a_5					-.0083333333
a_6					.0013888889
a_7					-.0001984127
a_8					.0000248016
a_9					-.0000027557
a_{10}					.0000002756

$$y = e^{-x}$$

$$0 \leq x \leq \frac{\ln 10}{4}$$

$$y = a_0 + \frac{x}{a_1 + \frac{x}{a_2 + \dots + \frac{x}{a_n}}}$$

	n=2	n=3	n=4	n=5	n=6
max. rel. error	.00076	.000016	.00000022	.0000000030	.0000000023
a_0	1.0000000000	1.0000000000	1.0000000000	1.0000000000	1.0000000000
a_1	-.9863218094	-.9994716009	-1.0000117973	-1.0000002345	-.9999999496
a_2	-1.7498758443	-1.9796637786	-2.0007788394	-2.0000222444	-1.9999927863
a_3		3.6682408655	2.9621172702	2.9983956020	3.0006450568
a_4			1.7506800347	1.9798217402	2.0099477033
a_5				-6.0012865319	-4.5437835914
a_6					-1.2912247633

	n=7	n=8	EXPANSION		
max. rel. error					
a_0					1.0000000000
a_1					-1.0000000000
a_2					-2.0000000000
a_3					3.0000000000
a_4					2.0000000000
a_5					-5.0000000000
a_6					-2.0000000000
a_7					
a_8					

$$y = e^{-x}$$

$$0 \leq x \leq \ln 2$$

$$y = a_0 + a_1 x + a_2 x^2 + \dots + a_n x^n$$

	n = 2	n = 3	n = 4	n = 5	n = 6
max. rel. error	.0027	.00010	.0000033	.000000093	.0000000023
a_0	1.0000000000	1.0000000000	1.0000000000	1.0000000000	1.0000000000
a_1	-.9664279798	-.9975088289	-.9998684066	-.9999946141	-.9999998220
a_2	.3535763634	.4798632962	.4982926186	.4998975259	.4999953413
a_3		-.1175028165	-.1595332187	-.1660178557	-.1666251408
a_4			.0293641271	.0398289503	.0414934947
a_5				-.0058730676	-.0079607704
a_6					.0009797340

	n = 7	n = 8	n = 9	n = 10	EXPANSION
max. rel. error	.0000000002				
a_0	1.0000000000				1.0000000000
a_1	-.9999999995				-1.0000000000
a_2	.4999999206				.5000000000
a_3	-.1666653019				-.1666666667
a_4	.0416573475				.0416666667
a_5	-.0083013598				-.0083333333
a_6	.0013298820				.0013888889
a_7	-.0001413161				-.0001984127
a_8					.0000248016
a_9					-.0000027557
a_{10}					.0000002756

$$y = e^{-x}$$

$$0 \leq x \leq \ln 2$$

$$y = a_0 + \frac{x}{a_1 + \frac{x}{a_2 + \dots + \frac{x}{a_n}}}$$

	n = 2	n = 3	n = 4	n = 5	n = 6
max. rel. error	.0013	.000034	.00000056	.0000000094	.0000000003
a_0	1.0000000000	1.0000000000	1.0000000000	1.0000000000	1.0000000000
a_1	-.9802494780	-.9990852016	-1.0000248168	-1.0000005898	-.9999999927
a_2	-1.7070703494	-1.9710895319	-2.0013627047	-2.0000464580	-1.9999990675
a_3		3.8299860238	2.9444808342	2.9972022281	3.0000867344
a_4			1.7084710933	1.9713380652	2.0014895904
a_5				-6.2435637685	-4.9118974390
a_6					-1.7094746586

	n = 7	n = 8	EXPANSION		
max. error					
a_0					1.0000000000
a_1					-1.0000000000
a_2					-2.0000000000
a_3					3.0000000000
a_4					2.0000000000
a_5					-5.0000000000
a_6					-2.0000000000
a_7					
a_8					

$$y = \frac{\arctan x}{x}$$

$$0 \leq x \leq 1$$

$$y = a_0 + a_1 x^2 + a_2 x^4 + \dots + a_n x^{2n}$$

	n = 2	n = 3	n = 4	n = 5	n = 6
max. rel. error	.0025	.00029	.000039	.0000055	.00000079
a_0	1.0000000000	1.0000000000	1.0000000000	1.0000000000	1.0000000000
a_1	-.3048135498	-.3271067763	-.3320414169	-.3330744193	-.3332822449
a_2	.0902117132	.1572570522	.1861657434	.1960146056	.1989352702
a_3		-.0447521125	-.0928004312	-.1219290147	-.1352204757
a_4			.0240742681	.0579833888	.0843384815
a_5				-.0135963969	-.0372934666
a_6					.0079205989

	n = 7	n = 8	n = 9	n = 10	EXPANSION
max. rel. error	.00000012	.000000018	.0000000028		
a_0	1.0000000000	1.0000000000	1.0000000000		1.0000000000
a_1	-.3333234399	-.3333314528	-.3333329796		-.3333333333
a_2	.1997316492	.1999355085	.1999850118		.2000000000
a_3	-.1403377634	-.1420889944	-.1426362737		-.1428571429
a_4	.0994378428	.1065626393	.1094847723		.1111111111
a_5	-.0597988100	-.0752896400	-.0839136437		-.0909090909
a_6	.0244108079	.0429096138	.0577315472		.0769230769
a_7	-.0047221231	-.0161657367	-.0309351227		-.0666666667
a_8		.0028662257	.0107755191		.0588235294
a_9			-.0017606673		-.0526315789
a_{10}					.0476190476

$$y = \frac{\arctan x}{x}$$

$$0 \leq x \leq 1$$

$$y = a_0 + \frac{x^2}{a_1 + \frac{x^2}{a_2 + \dots + \frac{x^2}{a_n}}}$$

	n=2	n=3	n=4	n=5	n=6
max. rel. error	.00037	.000023	.00000060	.000000034	.0000000017
a_0	1.0000000000	1.0000000000	1.0000000000	1.0000000000	1.0000000000
a_1	-3.0496631724	-3.0054322376	-3.0002345486	-3.0000191974	-3.0000007624
a_2	-.6210681748	-.5683324887	-.5564969967	-.5556680113	-.5555619129
a_3		-27.6772007138	-16.7725764698	-15.9256035084	-15.7644793084
a_4			-.2027559258	-.1710492997	-.1614278247
a_5				-76.4566134734	-43.6311643217
a_6					-.1062968172

	n=7	n=8	EXPANSION	
max. rel. error				
a_0				1.0000000000
a_1				-3.0000000000
a_2				-.5555555556
a_3				-15.7500000000
a_4				-.1600000000
a_5				-38.6718750000
a_6				-.0754648526
a_7				
a_8				

$$y = \frac{\sin x}{x}$$

$$0 \leq x \leq \frac{\pi}{4}$$

$$y = a_0 + a_1 x^2 + a_2 x^4 + \dots + a_n x^{2n}$$

	n = 2	n = 3	n = 4	n = 5	n = 6
max. rel. error	.0000023	.0000000045	.0000000001		
a ₀	1.0000000000	1.0000000000	1.0000000000		
a ₁	-.1666288173	-.1666665309	-.1666666667		
a ₂	.0081506261	.0083320653	.0083333327		
a ₃		-.0001950233	-.0001984036		
a ₄			.0000027274		
a ₅					
a ₆					

	n = 7	n = 8	n = 9	n = 10	EXPANSION
max. error					
a ₀					1.0000000000
a ₁					-.1666666667
a ₂					.0083333333
a ₃					-.0001984127
a ₄					.0000027557
a ₅					-.0000000251
a ₆					.0000000002
a ₇					.0000000000
a ₈					.0000000000
a ₉					.0000000000
a ₁₀					.0000000000

$$y = \frac{\sin x}{x}$$

$$0 \leq x \leq \frac{\pi}{4}$$

$$y = a_0 + \frac{x^2}{a_1 + \frac{x^2}{a_2 + \dots + \frac{x^2}{a_n}}}$$

	n=2	n=3	n=4	n=5	n=6
max. rel. error	.0000026	.0000000032	.0000000001		
a_0	1.0000000000	1.0000000000	1.0000000000		
a_1	-5.9984679070	-5.9999964872	-5.9999999997		
a_2	-3.2532799082	-3.3329696138	-3.3333335179		
a_3		11.5841762677	11.4541068204		
a_4			9.5484922535		
a_5					
a_6					

	n=7	n=8	EXPANSION		
max. rel. error					
a_0					1.0000000000
a_1					-6.0000000000
a_2					-3.3333333333
a_3					11.4545454545
a_4					9.6800000000
a_5					-8.4522915101
a_6					-33.4481317285
a_7					
a_8					

$$y = \frac{\sin x}{x}$$

$$0 \leq x \leq \frac{\pi}{2}$$

$$y = a_0 + a_1 x^2 + a_2 x^4 + \dots + a_n x^{2n}$$

	n = 2	n = 3	n = 4	n = 5	n = 6
max. rel. error	.00017	.0000013	.0000000069	.0000000002	
a_0	1.0000000000	1.0000000000	1.0000000000	1.0000000000	
a_1	-.1660537570	-.1666576051	-.1666665880	-.1666666664	
a_2	.0076117733	.0083128622	.0083330455	.0083333315	
a_3		-.0001849551	-.0001980800	-.0001984090	
a_4			.0000026021	.0000027526	
a_5				-.0000000239	
a_6					

	n = 7	n = 8	n = 9	n = 10	EXPANSION
max. rel. error					
a_0					1.0000000000
a_1					-.1666666667
a_2					.0083333333
a_3					-.0001984127
a_4					.0000027557
a_5					-.0000000251
a_6					.0000000002
a_7					.0000000000
a_8					.0000000000
a_9					.0000000000
a_{10}					.0000000000

$$y = \frac{\sin x}{x}$$

$$0 \leq x \leq \frac{\pi}{2}$$

$$y = a_0 + \frac{x^2}{a_1 + \frac{x^2}{a_2 + \dots + \frac{x^2}{a_n}}}$$

	n=2	n=3	n=4	n=5	n=6
max. rel. error	.00019	.00000094	.0000000068	.0000000001	
a_0	1.0000000000	1.0000000000	1.0000000000	1.0000000000	
a_1	-5.9734931440	-5.9997569799	-6.0000028781	-6.0000000056	
a_2	-3.0213962212	-3.3273532105	-3.3334504314	-3.3333337415	
a_3		12.0034462578	11.4363323933	11.4544368439	
a_4			8.9696299723	9.6717198021	
a_5				-8.6822950335	
a_6					

	n=7	n=8	EXPANSION	
max. rel. error				
a_0				1.0000000000
a_1				-6.0000000000
a_2				-3.3333333333
a_3				11.4545454545
a_4				9.6800000000
a_5				-8.4522915101
a_6				33.4481317285
a_7				
a_8				

$$y = \cos x$$

$$0 \leq x \leq \frac{\pi}{4}$$

$$y = a_0 + a_1 x^2 + a_2 x^4 + \dots + a_n x^{2n}$$

n = 2

n = 3

n = 4

n = 5

n = 6

max. rel. error					
	.000018	.000000045	.0000000002		
a_0	1.0000000000	1.0000000000	1.0000000000		
a_1	-.4997256177	-.4999987101	-.4999999979		
a_2	.0403744911	.0416549349	.0416666240		
a_3		-.0013580656	-.0013886690		
a_4			.0000243792		
a_5					
a_6					

n = 7

n = 8

n = 9

n = 10

EXPANSION

max. error					
a_0					1.0000000000
a_1					-.5000000000
a_2					.0416666667
a_3					-.0013888889
a_4					.0000248016
a_5					-.0000002756
a_6					.0000000021
a_7					.0000000000
a_8					.0000000000
a_9					.0000000000
a_{10}					.0000000000

$$y = \cos x$$

$$0 \leq x \leq \frac{\pi}{4}$$

$$y = a_0 + \frac{x^2}{a_1 + \frac{x^2}{a_2 + \dots + \frac{x^2}{a_n}}}$$

	n=2	n=3	n=4	n=5	n=6
max. rel. error	.000028	.000000039	.0000000003		
a_0	1.0000000000	1.0000000000	1.0000000000		
a_1	-1.9982825479	-1.9999954584	-2.0000000129		
a_2	-5.7234460230	-5.9985265414	-6.0000089698		
a_3		3.3772748603	3.3328215686		
a_4			28.2789081313		
a_5					
a_6					

	n=7	n=8	EXPANSION	
max. rel. error				
a_0				1.0000000000
a_1				-2.0000000000
a_2				-6.0000000000
a_3				3.3333333333
a_4				29.0769230769
a_5				-1.9096045198
a_6				-139.1556632344
a_7				
a_8				

$$y = \cos x$$

$$0 \leq x \leq \frac{\pi}{2}$$

$$y = a_0 + a_1 x^2 + a_2 x^4 + \dots + a_n x^{2n}$$

	n = 2	n = 3	n = 4	n = 5	n = 6
max. rel. error	.00088	.0000088	.000000057	.0000000005	
a_0	1.0000000000	1.0000000000	1.0000000000	1.0000000000	
a_1	-.4967046550	-.4999446545	-.4999994617	-.4999999963	
a_2	.0370510982	.0415151450	.0416642861	.0416666418	
a_3		-.0012770216	-.0013857673	-.0013888397	
a_4			.0000232242	.0000247609	
a_5				-.0000002605	
a_6					

	n = 7	n = 8	n = 9	n = 10	EXPANSION
max. error					
a_0					1.0000000000
a_1					-.5000000000
a_2					.0416666667
a_3					-.0013888889
a_4					.0000248016
a_5					-.0000002756
a_6					.0000000021
a_7					.0000000000
a_8					.0000000000
a_9					.0000000000
a_{10}					.0000000000

$$y = \cos x$$

$$0 \leq x \leq \frac{\pi}{2}$$

$$y = a_0 + \frac{x^2}{a_1 + \frac{x^2}{a_2 + \dots + \frac{x^2}{a_n}}}$$

	n=2	n=3	n=4	n=5	n=6
max. rel. error	.0014	.0000079	.000000085	.0000000004	
a_0	1.0000000000	1.0000000000	1.0000000000	1.0000000000	
a_1	-1.9771857295	-1.9997969716	-2.0000032708	-2.0000000114	
a_2	-5.0333001531	-5.9805680310	-6.0005194716	-6.0000026702	
a_3		3.5054279795	3.3255046137	3.3332745680	
a_4			26.1640132728	29.0376247421	
a_5				-1.9656456651	
a_6					

	n=7	n=8	EXPANSION		
max. rel. error					
a_0					1.0000000000
a_1					-2.0000000000
a_2					-6.0000000000
a_3					3.3333333333
a_4					29.0769230769
a_5					-1.9096045198
a_6					-139.1556632344
a_7					
a_8					

$$y = \frac{\tan x}{x}$$

$$0 \leq x \leq \frac{\pi}{4}$$

$$y = a_0 + a_1 x^2 + a_2 x^4 + \dots + a_n x^{2n}$$

	n = 2	n = 3	n = 4	n = 5	n = 6
max. rel. error	.00095	.000061	.0000041	.00000028	.000000020
a ₀	1.0000000000	1.0000000000	1.0000000000	1.0000000000	1.0000000000
a ₁	.3175541106	.3352141866	.3331313396	.3333535488	.3333314036
a ₂	.2032991796	.1164174299	.1363364980	.1328869332	.1333923995
a ₃		.0944350928	.0405161308	.0571091050	.0533740603
a ₄			.0439349681	.0125423299	.0245650893
a ₅				.0204460818	.0029005250
a ₆					.0095168091

	n = 7	n = 8	n = 9	n = 10	EXPANSION
max. rel. error	.0000000014	.0000000002			
a ₀	1.0000000000	1.0000000000			1.0000000000
a ₁	.3333335190	.3333333268			.3333333333
a ₂	.1333259638	.1333338651			.1333333333
a ₃	.0540672317	.0539568647			.0539682540
a ₄	.0212449852	.0219759152			.0218694885
a ₅	.0109317770	.0083446781			.0088632355
a ₆	-.0000322541	.0049960513			.0035921280
a ₇	.0044366415	-.0006245004			.0014558344
a ₈		.0020618825			.0005900274
a ₉					.0002391291
a ₁₀					.0000969154

$$y = \frac{\tan x}{x}$$

$$0 \leq x \leq \frac{\pi}{4}$$

$$y = a_0 + \frac{x^2}{a_1 + \frac{x^2}{a_2 + \dots + \frac{x^2}{a_n}}}$$

	n = 2	n = 3	n = 4	n = 5	n = 6
max. rel. error	.00000089	.0000000049	.00000000001		
a_0	1.00000000000	1.00000000000	1.00000000000		
a_1	3.0012017257	2.9999872976	3.00000000111		
a_2	-.8294813512	-.8334128270	-.8333332131		
a_3		239.5834592777	252.0353348962		
a_4			-.0993504216		
a_5					
a_6					

	n = 7	n = 8	EXPANSION		
max. rel. error					
a_0					1.00000000000
a_1					3.00000000000
a_2					-.83333333333
a_3					252.00000000000
a_4					-.10000000000
a_5					2475.00000000000
a_6					-.0309523810
a_7					
a_8					

$$y = x \cot x$$

$$0 \leq x \leq \frac{\pi}{4}$$

$$y = a_0 + a_1 x^2 + a_2 x^4 + \dots + a_n x^{2n}$$

	n = 2	n = 3	n = 4	n = 5	n = 6
max. rel. error	.000031	.00000044	.0000000066	.0000000004	
a_0	1.0000000000	1.0000000000	1.0000000000	1.0000000000	
a_1	-.3328671922	-.3333459249	-.3333330271	-.3333333410	
a_2	-.0243692868	-.0221085762	-.0222267401	-.0222220287	
a_3		-.0024067727	-.0020956370	-.0021177168	
a_4			-.0002488001	-.0002078504	
a_5				-.0000262619	
a_6					

	n = 7	n = 8	n = 9	n = 10	EXPANSION
max. rel. error					
a_0					1.0000000000
a_1					-.3333333333
a_2					-.0222222222
a_3					-.0021164021
a_4					-.0002116402
a_5					-.0000213778
a_6					-.0000021644
a_7					-.0000002193
a_8					-.0000000222
a_9					-.0000000023
a_{10}					-.0000000002

$$y = x \cot x$$

$$0 \leq x \leq \frac{\pi}{4}$$

$$y = a_0 + \frac{x^2}{a_1 + \frac{x^2}{a_2 + \dots + \frac{x^2}{a_n}}}$$

	n=2	n=3	n=4	n=5	n=6
max. rel. error	.0000089	.000000020	.0000000001		
a_0	1.0000000000	1.0000000000	1.0000000000		
a_1	-3.0012017243	-2.9999950248	-3.0000000140		
a_2	4.8644612818	5.0011164012	4.9999952948		
a_3		-6.8599737301	-7.0010169434		
a_4			8.8548755195		
a_5					
a_6					

	n=7	n=8	EXPANSION	
max. error				
a_0				1.0000000000
a_1				-3.0000000000
a_2				5.0000000000
a_3				-7.0000000000
a_4				9.0000000000
a_5				-11.0000000000
a_6				13.0000000000
a_7				
a_8				