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ACCELERATOR MEDICAL APPLICATIONS STUDIES

CALCULATION OF PION YIELD DISTRIBUTIONS BY KINEMATIC REFLECTION

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SUMMARY

The angle and momentum dependence of the yield $Y = d^2N/d\Omega dp$ of low energy negative pions produced in high energy nucleon-nucleon interactions was computed by kinematic reflection of the Sanford-Wang formula. The angular distributions are not forward-peaked ($Y_{\max}$ at $\theta \approx 20^\circ$ for $p_\pi = 200$ MeV/c from 28 GeV/c protons), in agreement with previous results of C.L. Wang. For 12 GeV/c protons, the calculated forward yield of 500 MeV/c π^- is twice the value measured at the ZGS by G. Marmer.

I. Method

Assume that in the initial state we have a proton of rest mass m_1 and momentum p_1 incident on a nucleon

of mass m_2 at rest. After the interaction, assume that a secondary of rest mass m_3 is produced at cm angle θ_3^* with corresponding lab angle θ_3 . We represent the other interaction products as a composite of mass m_4 at cm momentum $p_4^* = p_3^*$ and cm angle $\pi - \theta_3^*$.

In the method of kinematic reflection, which has been previously applied by Berley¹ to obtain forward production of low-momentum $\pi^\pm$ and by Zeller, et.al.²

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1. D. Berley, Accel. Dept. Internal Rept. AADD-33 (1964).
2. Zeller, Rosenson, and Lanou, 1970 Summer Study on AGS Utilization.

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to estimate $K^\pm$ and $\bar{p}$ yields, it is assumed that the secondary m_3 can be produced with equal probability at $\pi - \theta_3^*$ (i.e. symmetry about 90° in cm). Designate the reflected particle as $3'$ with cm momentum $p_{3'}^* = p_3^*$ at cm angle $\theta_{3'}^* = \pi - \theta_3^*$ and lab momentum $p_{3'}$ at angle $\theta_{3'}$. We then require

$$\frac{d^2N}{d\Omega_3^* dp_3^*} = \frac{d^2N}{d\Omega_3^* dp_3^*} \quad (1)$$

But the cm yields (or equivalently, cross sections) can be transformed from the cm to the lab using the transformation obtained by Hagedorn³

$$\frac{d^2N}{d\Omega^* dp^*} = \frac{p^* \beta^*}{p \beta} \frac{d^2N}{d\Omega dp} \quad (2)$$

final cm, reflected particle

where $p^* = mc\beta^* \gamma^*$ and $\gamma^* \equiv (1 - \beta^{*2})^{-1/2}$. We can then use Eqs. (1) and (2) to obtain the yield of particle $3'$ in terms of the yield of particle 3:

final lab, reflected particle

$$Y_{3'} = \frac{d^2N}{d\Omega_{3'} dp_{3'}} = \frac{p_{3'} \beta_{3'}}{p_3 \beta_3} \frac{d^2N}{d\Omega_3 dp_3} \quad (3)$$

3. R. Hagedorn, Relativistic Kinematics, W.A. Benjamin, Inc., N.Y. (1964), p. 43.

We use the Sanford-Wang formula⁴ to obtain the yield of particle 3:

$$Y_3 = \frac{d^2N}{d\Omega_3 dp_3} = C_1 p_3^{C_2} (1 - p_3/p_1) \exp \left[- \frac{C_3 p_3^{C_4}}{p_1^{C_5}} - C_6 \theta_3 (p_3 - C_7 p_1 \cos^{C_8} \theta_3) \right] \quad (4)$$

with $C_1 = 0.821$, $C_2 = 0.5271$, $C_3 = 3.956$, $C_4 = 1.731$, $C_5 = 1.617$, $C_6 = 4.735$, $C_7 = 0.1984$, and $C_8 = 88.75$.

II. Calculation

Given p_1 and p_3' , we wish to evaluate Eq. (3) for Y_3 , as a function of θ_3' . We then require p_3 and θ_3 for each θ_3' . To obtain p_3 we evaluate

$$p_3 = \sqrt{p_{x_3}^2 + p_{y_3}^2} \quad (5)$$

where the components can be found from the Lorentz transformation from the cm to the lab system:

$$\begin{bmatrix} p_x \\ p_y \\ p_z \\ iE/c \end{bmatrix} = \begin{bmatrix} \gamma & 0 & 0 & -i\beta\gamma \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ i\beta\gamma & 0 & 0 & \gamma \end{bmatrix} \begin{bmatrix} p_x^* \\ p_y^* \\ p_z^* \\ iE^*/c \end{bmatrix} \quad (6)$$

as

$$\begin{aligned} p_{x_3} &= \gamma(p_{x_3}^* + \beta E_3^*/c) \\ p_{y_3} &= p_{y_3}^* = p_{y_3'} = p_{y_3'} \sin \theta_{3'} \end{aligned} \quad (7)$$

In Eqs. (6) and (7), β and γ refer to the motion of the cm relative to the lab system, i.e. $\beta = v_{cm}/c$ and $\gamma = (1 - \beta^2)^{-1/2}$. We can relate β and

4. J.R. Sanford and G.L. Wang, BNL Accel. Dept. Internal Rept. JRS/CLW-1 (1967).

γ to the lab momentum p_1 of the incident proton by use of the lab to cm Lorentz transformation, i.e. the inverse of Eq. (6)

$$\begin{bmatrix} p_x^* \\ p_y^* \\ p_z^* \\ iE^*/c \end{bmatrix} = \begin{bmatrix} \gamma & 0 & 0 & i\beta\gamma \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ -i\beta\gamma & 0 & 0 & \gamma \end{bmatrix} \begin{bmatrix} p_x \\ p_y \\ p_z \\ iE/c \end{bmatrix} \quad (8)$$

Then $p_{x1}^* = \gamma(p_{x1} - \beta E_1/c)$. But $p_{x1} = p_1 = m_1 c \beta_1 \gamma_1$ and $E_1 = m_1 c^2 \gamma_1$ where $\beta_1 = v_1/c$ and $\gamma_1 = (1 - \beta_1^2)^{-1/2}$. Then

$$p_{x1}^* = p_1^* = m_1 c [\gamma(\gamma_1^2 - 1)^{1/2} - \gamma_1(\gamma^2 - 1)^{1/2}] \quad (9)$$

But $p_{x2}^* = p_2^* = p_1^* = m_2 c (\gamma_2^2 - 1)^{1/2} = m_2 c (\gamma^2 - 1)^{1/2}$ since $\gamma_2 = \gamma$. Solving for γ

$$\gamma = \frac{(m_2 + m_1 \gamma_1)}{(m_2^2 + 2m_1 m_2 \gamma_1 + m_1^2)^{1/2}} \quad (10)$$

More simply, for $m_1 = m_2$

$$\gamma = \sqrt{(\gamma_1 + 1)/2} \quad \beta = \sqrt{(\gamma_1 - 1)/(\gamma_1 + 1)} \quad (11)$$

Eq. (11) can be expressed in terms of p_1 by noting that $p_1 = M_1 \beta_1 \gamma_1$ and $E_1 = M_1 \gamma_1$ where $p_1 \equiv p_1 c$ and $M_1 \equiv m_1 c^2$. Then

$$\beta = \frac{p_1}{E_1 + M_1} \quad \text{and, as usual,} \quad \gamma = (1 - \beta^2)^{-1/2} \quad (12)$$

Now for p_{x3}^* in Eq. (7). Note that $p_{x3}^* = -p_{x3}'$, and that p_{x3}' can be obtained directly from Eq. (8). Then

$$p_{x3}^* = -\gamma(p_{x3}' \cos \theta_3' - \beta E_3'/c) \quad (13)$$

where $E_3 = \sqrt{P_3^2 + M_3^2}$. The value of θ_3 required in Eq. (4) can then be obtained from Eqs. (5) and (7) as

$$\cos\theta_3 = p_{x_3}/p_3. \quad (14)$$

III. Results

Equation (3) was evaluated using Eqs. (4), (5), (7), (12), (13), and (14) in the attached code REFLECT. The resulting angular distributions, for $p_{\pi^-} = 100, 200, 300,$ and 400 MeV/c are given for incident proton momenta 6, 12, 18, and 28 GeV/c and are plotted in Figs. 1 through 4.* All distributions show peaking at $\theta_3 > 0^\circ$, in agreement with calculations of Wang.⁵

In Fig. 5 we show the forward yield from 29 GeV/c protons as a function of pion momentum. The present results are about 1/3 of the yields previously calculated.¹ At $p_{\pi^-} = 500$ MeV/c and incident proton momentum of $p_1 = 12$ GeV/c, the present calculation at 0° is twice the value measured by Marmer⁶ using a thin beryllium target.

IV. Acknowledgments

We thank D. Berley and M. Blecker for useful discussions. We are indebted to D. Bonar for checking the derivation and pointing out the coordinate transformation given by Hagedorn.

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*Note added in proof: It has been pointed out to us that for some range of values of incident proton momentum and pion production angles, the momentum of the reflected pion is nearly the same as that of the low momentum pion whose yield we wish to calculate. This is true. In Fig. 6 we plot the momentum of the pion which was reflected to obtain the yield of 200 MeV/c pions. Since the production data used to obtain the Sanford-Wang formula are largely for pions of $p > 1$ GeV/c, the present results at low incident proton momentum and small production angles are of questionable validity. It should be noted, however, that Marmer's data⁶ for 12.3 GeV/c protons are reasonably well represented by the Sanford-Wang expression down to pions of 500 MeV/c.

5. C.L. Wang, private communication (1972).

6. G. Marmer, Ph.D. Thesis, Ohio State University (1968), Univ. Microfilms, Inc., Ann Arbor, Mich. Also Marmer, Reibel, Schwartz, Stevens, Winston, Wolfe, Rush, Phillips, Swallow and Romanowski, Phys. Rev. 179, 1294 (1969).

FIG. 1

FIG. 2

FIG.3

PION ANGULAR DISTRIBUTION FOR
18 GeV/c INCIDENT PROTONS
KINEMATIC REFLECTION OF
SANFORD - WANG FORMULA

FIG. 4

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PROGRAM REFLECT (INPUT,OUTPUT,TAPE5=INPUT,TAPE6=OUTPUT)
C COMPUTES ANGULAR DISTRIBUTION BY KINEMATIC REFLECTION OF SANFORD-WANG,
  DIMENSION HEADER(16)
10 READ 20,HEADER,P1MIN,P3PMIN,TH3PMX,XM3,C1,C2,C3,C4,C5,C6,C7,C8
20 FORMAT(16A5/12F6.0)
  DELP1=2.0
  DELP3=1.05
  CALL DATE(DAY)
  TH3P=TH3PMX/100.
  DO 50 I=1,15
    EI=I
    P1=P1MIN+DELP1*EI
    DO 50 K=1,15
      EK=K
      P3P=P3PMIN+DELP3*EK
      E3PC=SQRT(P3P**2+XM3**2)
      BETA=P1/(SQRT(P1**2+.938**2)+.938)
      GAMMA=1.0/SQRT(1.0-BETA**2)
      BETA3P=1./SQRT(1.+(XM3/P3P)**2)
      PRINT 30,HEADER,DAY,P1,P3P,TH3PMX,XM3,C1,C2,C3,C4,C5,C6,C7,C8,
        1 TH3P,E3PC,GAMMA,BETA,BETA3P
30 FORMAT(1H1,16A5,10X,17H PROGRAM REFLECT2,5X,A10/1H0,50X,6H INPUT,
  1 5H DATA/1H0,4H P1=,F5,2.4H GEV,2X,5H P3P=,F5,3.4H GEV,2X,
  2 8H TH3PMX=,F6,4.4H RAD,2X,5H XM3=,F6,4.4H GEV,2X,4H C1=,F6,4.2X,
  3 4H C2=,F6,4.2X,4H C3=,F6,4.2X,4H C4=,F6,4.2X,4H C5=,F6,4/1H ,
  4 4H C6=,F6,4.2X,4H C7=,F6,4.2X,4H C8=,F7,4/1H0,50X,8H RESULTS/
  5 1H0,7H TH3P=,F5,4.4H RAD,2X,6H E3PC=,F5,4.4H GEV,2X,
  6 7H GAMMA=,F7,5.2X,6H BETA=,F6,5.2X,
  7 5H BETA3P=,F6,5/1H0,10H TH3P(RAD),2X,12H PX3PCM(GEV),
  8 12H E3PCMC(GEV),10H PX3(GEV),2X,10H P3(GEV),2X,7H BETA3,5X,
  9 9H TH3(RAD),3X,10H P3CM(GEV),2X,7H CTH3CM,5X,13HY3(/SR/GEV/C ,
  1 13HY3P(/SR/GEV/C//)
    TH3P=0.
    DO 50 J=1,101
      PY3=P3P*SIN(TH3P)
      CTH3P=COS(TH3P)
      PX3PCM=GAMMA*(P3P*CTH3P-BETA*E3PC)
      E3PCMC=GAMMA*(E3PC-BETA*P3P*CTH3P)
      PX3=GAMMA*(BETA+E3PCMC-PX3PCM)
      P3=SQRT(PX3**2+PY3**2)
      BETA3=1./SQRT(1.+(XM3/P3)**2)
      COSTH3=PX3/P3
      TH3=ACOS(COSTH3)
      SR=SQRT((BETA*COSTH3)**2-2.+(BETA*COSTH3/BETA3+(BETA/BETA3)**2
        1 +1./GAMMA**2)
      P3CM=GAMMA*P3*SR
      CTH3CM=(COSTH3-BETA/BETA3)/SR
      Y3=C1*P3**C2*(1.-P3/P1)*EXP(-C3*P3**C4/P1**C5-C6*TH3*(P3-C7*P1
        1 -COSTH3**C8))
      Y3P=P3P*BETA3P/(P3*BETA3)*Y3
      PRINT 40,TH3P,PX3PCM,E3PCMC,PX3,P3,BETA3,TH3,P3CM,CTH3CM,Y3,Y3P
40 FORMAT(1H ,2(E10.3,2X),E10.3,3X,E10.3)
50 TH3P=TH3P+DTH3P
    GO TO 10
  END

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FIG. 5