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Argonne National Laboratory

TWO AXISYMMETRIC STEADY-STATE
THERMOELASTICITY PROBLEMS FOR
TRANSVERSELY ISOTROPIC SOLIDS

by

Richard A. Valentin

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Printed in the United States of America
Available from

Clearinghouse for Federal Scientific and Technical Information
National Bureau of Standards, U. S. Department of Commerce
Springfield, Virginia 22151

Price: Printed Copy \$3.00; Microfiche \$0.65

ANL-7277
Physics (TID-4500)
AEC Research and
Development Report

ARGONNE NATIONAL LABORATORY
9700 South Cass Avenue
Argonne, Illinois 60439

CFSTI PRICES

H.C. \$ 2.00; MN 65

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Richard A. Valentin

Reactor Engineering Division

December 1966

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NOMENCLATURE

<u>Symbol</u>	<u>Description</u>
r, θ, z	cylindrical polar coordinates
ρ, ϕ, ζ	dimensionless cylindrical polar coordinates
$\epsilon_{rr}, \epsilon_{r\theta}, \dots$	strain components
$\sigma_{rr}, \sigma_{\theta\theta}, \dots$	stress components
$s_{11}, s_{12}, \dots$	material constants
u, w	displacement components
E, ν	Young's modulus and Poisson's ratio in isotropic plane
E', ν', G'	Young's modulus, Poisson's ratio, and shear modulus related to planes normal to isotropic plane
α	coefficient of thermal expansion in isotropic plane
β	coefficient of thermal expansion in axial direction
T	temperature; function of r and z
T_0	constant temperature
Φ_i, Ω_i	Singh stress functions
k_{ij}^2	roots of Eqs. (2.27) and (2.28)
L^2	ratio of conductivities in and normal to isotropic plane
$\gamma_1, \gamma_2, \dots$	dimensionless material constants of Eq. (2.39)
$\hat{\Phi}, \hat{\Omega}, \dots$	dimensionless forms of corresponding symbol
$\eta(z)$	Heavyside function
J_0, J_1	Bessel functions of first kind
I_0, I_1	modified Bessel functions of first kind
$\bar{f}$	zero-order Hankel transform of f
$\tilde{f}, s_{\tilde{f}}$	Fourier cosine and sine transform, respectively, of f
β_1	dimensionless length parameter of unit-cylinder surface temperature
$\mu_1, \mu_2, \dots$	dimensionless material constants of Eq. (4.7)
$\mathcal{I}_2, \mathcal{I}_3$	integrals defined by Eq. (3.11)

NOTE: Various incidental notations introduced during derivations are not listed above. Their definitions should be clear from context.

TWO AXISYMMETRIC STEADY-STATE THERMOELASTICITY PROBLEMS FOR TRANSVERSELY ISOTROPIC SOLIDS

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ABSTRACT

In connection with a study of the possible utility of anisotropic elastic models in the representation of the internal structure of certain reactor fuel elements, two sample problems are considered to aid in evaluating the analytical and numerical difficulties associated with the introduction of anisotropic effects. Treated are the stress fields associated with 1) a thermal inclusion in the form of a half-infinite circular cylinder in a transversely isotropic space and 2) a traction free, infinite, transversely isotropic, circular cylinder having a particular discontinuous axial surface temperature. Both the isotropic and anisotropic cases are analyzed by means of the stress function formulation of A. Singh. Numerical evaluation of representative stresses are presented for the isotropic case and for a typical transversely isotropic material (magnesium).

I. INTRODUCTION

During an investigation of problems of thermal stress associated with a particular type of reactor fuel element, it was proposed that various details of the internal structure of the elements (e.g., a regular pattern of clad coolant channels) could be incorporated into the analysis by considering the element as an anisotropic body with suitably defined equivalent elastic constants. A search of the available literature revealed that solution of relatively few anisotropic thermoelasticity problems have been attempted. For this reason it was decided that, prior to the performance of a detailed analysis of the structure in question, certain relatively simple problems of a similar nature should be attempted in the hope that these would reveal the degree of analytical and numerical difficulty associated with the introduction of anisotropic considerations. The geometry of the original problem indicated that the appropriate type of anisotropy to consider would be transverse isotropy, that is, the material possesses an

inherent axis of rotational symmetry and all directions in the planes normal to this axis are equivalent with respect to elastic properties, the body being isotropic in these planes.

In the original fuel-element problem the temperature field was assumed to be rotationally symmetric about an axis in the same direction as the material axis of symmetry and to have appreciable axial variations. This latter fact rules out the use of plane-strain or plane-stress solutions. In view of the above remarks, the sample problem class of interest may be summarized as concerning general torsionless axisymmetric thermoelasticity solutions for transversely isotropic bodies.

The two problems of the above class chosen to evaluate difficulties inherent in such analyses do not have direct bearing on the reactor applications mentioned; however, they possess sufficient interest in themselves to be formally recorded, particularly since so few anisotropic thermoelasticity solutions have been carried to the point of numerical evaluation of the results. The section immediately following presents the stress function formulation used in the solutions and is essentially that developed by A. Singh.¹ The first problem concerns the stresses in an infinite transversely isotropic space caused by the presence of a temperature inclusion in the form of a half-infinite circular cylinder of constant temperature lying along the axis of material symmetry. The second problem treats the thermal stresses in an infinite circular cylinder of transversely isotropic material having a particular axially discontinuous surface temperature. The latter problem is of the same order of difficulty as those providing the original motivation for the analyses contained herein and provides a good estimate of the amount of numerical work entailed in their treatment.

II. DERIVATION OF STRESS FUNCTIONS

In the interest of keeping the present work reasonably self-contained and because the particular set of equations to be used in describing the axisymmetric thermal stresses in a transversely isotropic body are not readily available in the literature, we include the following expository material. The contents of this section are essentially identical with the derivation presented by Singh.¹ The stress functions developed by Singh are closely analogous to the Love function or, more specifically, to the stress functions derived by Hoyle² for isotropic axisymmetrical thermal stress problems.

It is assumed that the axis of elastic symmetry coincides with the z axis and that the thermal properties in this direction differ from those in the (r, θ) plane. Due to the assumption of torsionless axisymmetry we have

$$u_r = u(r, z); \quad u_\theta = 0; \quad u_z = w(r, z) \quad (2.1)$$

and

$$\left. \begin{aligned} e_{rr} &= \frac{\partial u}{\partial r} - \alpha T; & e_{\theta\theta} &= \frac{u}{r} - \alpha T; & e_{zz} &= \frac{\partial w}{\partial z} - \beta T; \\ e_{rz} &= \frac{\partial u}{\partial z} + \frac{\partial w}{\partial r}; & e_{r\theta} &= e_{\theta z} = 0. \end{aligned} \right\} \quad (2.2)$$

We express the stress-strain equations for a transversely isotropic material in the form

$$\left. \begin{aligned} e_{rr} &= s_{11}\sigma_{rr} + s_{12}\sigma_{\theta\theta} + s_{13}\sigma_{zz}; \\ e_{\theta\theta} &= s_{12}\sigma_{rr} + s_{11}\sigma_{\theta\theta} + s_{13}\sigma_{zz}; \\ e_{zz} &= s_{13}\sigma_{rr} + s_{13}\sigma_{\theta\theta} + s_{33}\sigma_{zz}; \\ e_{rz} &= s_{44}\sigma_{rz}; \end{aligned} \right\} \quad (2.3)$$

where the elastic constants s_{ij} expressed in terms of the constants analogous to those of isotropic elasticity are given by

$$\begin{aligned} s_{11} &= \frac{1}{E}; & s_{12} &= -\frac{\nu}{E}; & s_{13} &= -\frac{\nu'}{E'}; \\ s_{33} &= \frac{1}{E'}; & s_{44} &= \frac{1}{G'}. \end{aligned} \quad (2.4)$$

In the above equations E and E' are the Young's moduli with respect to directions lying, respectively, in the plane of isotropy and perpendicular to it; ν is the Poisson's ratio which characterizes the transverse reduction in the plane of isotropy for tension in the same plane; ν' is the Poisson's ratio relating transverse reduction in the plane of isotropy to tension in a direction normal to this plane; G' is the shear modulus for planes normal to the plane of isotropy; α and β are, respectively, the coefficients of linear thermal expansion in the z plane and along the z axis.

In the absence of body forces, the equations of equilibrium are

$$\left. \begin{aligned} \frac{\partial \sigma_{rr}}{\partial r} + \frac{\partial \sigma_{rz}}{\partial z} + \frac{\sigma_{rr} - \sigma_{\theta\theta}}{r} &= 0; \\ \frac{\partial \sigma_{rz}}{\partial r} + \frac{\partial \sigma_{zz}}{\partial z} + \frac{\sigma_{rz}}{r} &= 0. \end{aligned} \right\} \quad (2.5)$$

The second of (2.5) is identically satisfied if we assume

$$\sigma_{rz} = -\frac{\partial^2 \Phi}{\partial r \partial z}; \quad \sigma_{zz} = \nabla_1^2 \Phi \equiv \frac{\partial^2 \Phi}{\partial r^2} + \frac{1}{r} \frac{\partial \Phi}{\partial r}. \quad (2.6)$$

From (2.2), after forming $\frac{\partial e_{\theta\theta}}{\partial r}$ and substituting for $\frac{\partial u}{\partial r}$ in the expression for e_{rr} , one finds

$$e_{rr} - e_{\theta\theta} = r \frac{\partial e_{\theta\theta}}{\partial r} + \alpha r \frac{\partial T}{\partial r}, \quad (2.7)$$

which, in conjunction with (2.3), implies

$$(s_{11} - s_{12})(\sigma_{rr} - \sigma_{\theta\theta}) = r \frac{\partial}{\partial r} (s_{12}\sigma_{rr} + s_{11}\sigma_{\theta\theta} + s_{13}\sigma_{zz}) + \alpha r \frac{\partial T}{\partial r}. \quad (2.8)$$

Thus, upon introducing a second (as yet arbitrary) function λ through the relation

$$\sigma_{rr} = \frac{\partial^2 \Phi}{\partial z^2} + \lambda, \quad (2.9)$$

Eqs. (2.5), (2.6), and (2.9) give

$$\sigma_{rr} - \sigma_{\theta\theta} = r \frac{\partial \lambda}{\partial r} \quad (2.10)$$

and, hence, from (2.8),

$$(s_{11} - s_{12})\lambda + (s_{12}\sigma_{rr} + s_{11}\sigma_{\theta\theta} + s_{13}\sigma_{zz}) + \alpha T = 0. \quad (2.11)$$

From (2.11), (2.6), and (2.9) we obtain

$$\sigma_{\theta\theta} = -\frac{1}{s_{11}} [s_{11}\lambda + s_{12} \frac{\partial^2 \Phi}{\partial z^2} + s_{13} \nabla_1^2 \Phi + \alpha T], \quad (2.12)$$

while (2.2), (2.3), and (2.11) yield

$$u = -(s_{11} - s_{12}) r \lambda. \quad (2.13)$$

Utilizing (2.2) and the relations expressing the stresses in terms of Φ and λ , one may show

$$\frac{\partial w}{\partial r} = (s_{11} - s_{12}) \left(r \frac{\partial \lambda}{\partial z} - D \frac{\partial^2 \Phi}{\partial r \partial z} \right) \quad (2.14)$$

and

$$\frac{\partial w}{\partial z} = (s_{11} - s_{12}) \left[a \nabla_1^2 \Phi + \frac{s_{13}}{s_{11}} \frac{\partial^2 \Phi}{\partial z^2} + c T \right], \quad (2.15)$$

where

$$D = \frac{s_{44}}{s_{11} - s_{12}}; \quad a = \left[\frac{s_{11}s_{33} - s_{13}^2}{s_{11}(s_{11} - s_{12})} \right]; \quad c = \left[\frac{\beta s_{11} - \alpha s_{13}}{s_{11}(s_{11} - s_{12})} \right]. \quad (2.16)$$

Differentiating (2.14) with respect to z and (2.15) with respect to r , and equating the resulting expressions, yields

$$a \frac{\partial}{\partial r} \left(\nabla_1^2 \Phi \right) + \left(\frac{s_{13}}{s_{11}} + D \right) \frac{\partial}{\partial r} \left(\frac{\partial^2 \Phi}{\partial z^2} \right) = r \frac{\partial^2 \lambda}{\partial z^2} - c \frac{\partial T}{\partial r}. \quad (2.17)$$

If one introduces the function Ω through the definition

$$r \lambda = k_1^2 \frac{\partial \Phi}{\partial r} + \frac{\partial \Omega}{\partial r}, \quad (2.18)$$

where k_1^2 is an as yet undefined constant, (2.17) implies

$$a \nabla_1^2 \Phi + b \frac{\partial^2 \Phi}{\partial z^2} = \frac{\partial^2 \Omega}{\partial z^2} - c T \quad (2.19)$$

with $b = \left(\frac{s_{13}}{s_{11}} + D - k_1^2 \right)$. Equation (2.18) when introduced into (2.13) and (2.14) represents the displacements in the form

$$u = - (s_{11} - s_{12}) \left(k_1^2 \frac{\partial \Phi}{\partial r} + \frac{\partial \Omega}{\partial r} \right); \quad (2.20)$$

$$w = (s_{11} - s_{12}) \left(\frac{\partial \Omega}{\partial z} - (D - k_1^2) \frac{\partial \Phi}{\partial z} \right). \quad (2.21)$$

Using (2.20) and (2.21) one finds

$$\frac{\partial u}{\partial r} + \frac{u}{r} + \frac{\partial w}{\partial z} = - (s_{11} - s_{12}) \left[k_1^2 \nabla_1^2 \Phi + \nabla_1^2 \Omega - \frac{\partial^2 \Omega}{\partial z^2} + (D - k_1^2) \frac{\partial^2 \Phi}{\partial z^2} \right], \quad (2.22)$$

whereas forming the same expression using (2.2), (2.3), and the representations of the stresses in terms of Φ and Ω yields

$$\frac{\partial u}{\partial r} + \frac{u}{r} + \frac{\partial w}{\partial z} = \frac{s(s_{11} - s_{12})}{s_{11}} \frac{\partial^2 \Phi}{\partial z^2} + \left(2s_{13} + s_{33} - \frac{ss_{13}}{s_{11}} \right) \nabla_1^2 \Phi + \left(2\alpha + \beta - \frac{s\alpha}{s_{11}} \right) T, \quad (2.23)$$

where $s = s_{11} + s_{12} + s_{13}$. If we now add and subtract $k_2^2 \frac{\partial^2 \Omega}{\partial z^2}$ from (2.22), where k_2^2 is an arbitrary constant, and utilize (2.19) we obtain

$$\begin{aligned} \frac{\partial u}{\partial r} + \frac{u}{r} + \frac{\partial w}{\partial z} = & -(s_{11} - s_{12}) \left\{ [k_1^2 - (1 + k_2^2) a] \nabla_1^2 \Phi + [(D - k_1^2) - (1 + k_2^2) b] \frac{\partial^2 \Phi}{\partial z^2} \right. \\ & \left. + \nabla_1^2 \Omega + k_2^2 \frac{\partial^2 \Omega}{\partial z^2} - (1 + k_2^2) c T \right\}. \end{aligned} \quad (2.24)$$

Comparing (2.23) and (2.24) we now choose k_1^2 and k_2^2 such that

$$2s_{13} + s_{33} - \frac{ss_{13}}{s_{11}} = -(s_{11} - s_{12}) [k_1^2 - (1 + k_2^2) a]; \quad (2.25)$$

$$\frac{s(s_{11} - s_{12})}{s_{11}} = -(s_{11} - s_{12}) [(D - k_1^2) - (1 + k_2^2) b], \quad (2.26)$$

that is, k_1^2 and k_2^2 satisfy the quadratic equations

$$s_{11}(s_{12} - s_{11}) k_1^4 + s_{11}s_{44}k_1^2 + [s_{13}(2s_{13} + s_{44}) - s_{33}(s_{11} + s_{12})] = 0; \quad (2.27)$$

$$(s_{13}^2 - s_{11}s_{33}) k_2^4 - [2s_{13}(s_{12} - s_{11}) - s_{11}s_{44}] k_2^2 + (s_{12}^2 - s_{11}^2) = 0. \quad (2.28)$$

If we denote the roots of (2.27) and (2.28) as k_{11}^2 , k_{12}^2 and k_{21}^2 , k_{22}^2 , respectively, Eqs. (2.25) and (2.26) then provide the interrelationships

$$\frac{s_{13}}{s_{11}} + k_{12}^2 = ak_{22}^2; \quad \frac{s_{13}}{s_{11}} + k_{11}^2 = ak_{21}^2, \quad (2.29)$$

which serve to remove the ambiguity associated with the indices assigned to the various roots. Further, one has the following:

$$\left. \begin{aligned} D &= k_{11}^2 + k_{12}^2; \\ \frac{b_1}{a} &\equiv \frac{1}{a} \left(\frac{s_{13}}{s_{11}} + D - k_{11}^2 \right) = k_{22}^2; \\ \frac{b_2}{a} &\equiv \frac{1}{a} \left(\frac{s_{13}}{s_{11}} + D - k_{12}^2 \right) = k_{21}^2. \end{aligned} \right\} \quad (2.30)$$

Again comparing (2.23) and (2.24) one finds

$$\nabla_1^2 \Omega + k_{2i}^2 \frac{\partial^2 \Omega}{\partial z^2} = a_{1i} T; \quad i = 1, 2, \quad (2.31)$$

where

$$a_{1i} \equiv c k_{2i}^2 - \frac{\alpha}{s_{11}}.$$

From the definition of the constants k_{ij}^2 we see that there exist two consistent sets of stress functions, $\{\Phi_1, \Omega_1\}$ and $\{\Phi_2, \Omega_2\}$, and that the axially symmetric stress field in a transversely isotropic solid may be defined by either of the two sets of equations ($i = 1, j = 2$ or $i = 2, j = 1$):

$$\left. \begin{aligned} \nabla_1^2 \Phi_i + k_{2j}^2 \frac{\partial^2 \Phi_i}{\partial z^2} &= \frac{1}{a} \left[\frac{\partial^2 \Omega_i}{\partial z^2} - cT \right]; & \nabla_1^2 \Omega_i + k_{2i}^2 \frac{\partial^2 \Omega_i}{\partial z^2} &= a_{1i} T; \\ u &= - (s_{11} - s_{12}) \left(k_{1i}^2 \frac{\partial \Phi_i}{\partial r} + \frac{\partial \Omega_i}{\partial r} \right); & w &= - (s_{11} - s_{12}) \left(k_{1j}^2 \frac{\partial \Phi_i}{\partial z} - \frac{\partial \Omega_i}{\partial z} \right); \\ \sigma_{rr} &= \frac{\partial^2 \Phi_i}{\partial z^2} + \frac{1}{r} \left(k_{1i}^2 \frac{\partial \Phi_i}{\partial r} + \frac{\partial \Omega_i}{\partial r} \right); \\ \sigma_{\theta\theta} &= - \frac{1}{s_{11}} \left[s_{13} \nabla_1^2 \Phi_i + s_{12} \frac{\partial^2 \Phi_i}{\partial z^2} + \frac{s_{11}}{r} \left(k_{1i}^2 \frac{\partial \Phi_i}{\partial r} + \frac{\partial \Omega_i}{\partial r} \right) + \alpha T \right]; \\ \sigma_{zz} &= \nabla_1^2 \Phi_i; & \sigma_{rz} &= - \frac{\partial^2 \Phi_i}{\partial r \partial z}. \end{aligned} \right\} \quad (2.32)$$

Since either of the systems (2.32) will yield the same results, we arbitrarily choose the set $i = 1, j = 2$ and drop the subscripts on the stress functions. Thus we obtain the equations

$$\left. \begin{aligned} \nabla_1^2 \Phi + k_{22}^2 \frac{\partial^2 \Phi}{\partial z^2} &= \frac{1}{a} \left[\frac{\partial^2 \Omega}{\partial z^2} - cT \right]; & \nabla_1^2 \Omega + k_{21}^2 \frac{\partial^2 \Omega}{\partial z^2} &= a_{11} T; \\ u &= - (s_{11} - s_{12}) \left(k_{11}^2 \frac{\partial \Phi}{\partial r} + \frac{\partial \Omega}{\partial r} \right); & w &= - (s_{11} - s_{12}) \left(k_{12}^2 \frac{\partial \Phi}{\partial z} - \frac{\partial \Omega}{\partial z} \right); \end{aligned} \right\} \quad \begin{array}{l} (2.33) \\ \text{(Contd.)} \end{array}$$

$$\left. \begin{aligned} \sigma_{rr} &= \frac{\partial^2 \Phi}{\partial z^2} + \frac{1}{r} \left(k_{11}^2 \frac{\partial \Phi}{\partial r} + \frac{\partial \Omega}{\partial r} \right); \\ \sigma_{\theta\theta} &= -\frac{1}{s_{11}} \left[s_{13} \nabla_1^2 \Phi + s_{12} \frac{\partial^2 \Phi}{\partial z^2} + \frac{s_{11}}{r} \left(k_{11}^2 \frac{\partial \Phi}{\partial r} + \frac{\partial \Omega}{\partial r} \right) + \alpha T \right]; \\ \sigma_{zz} &= \nabla_1^2 \Phi; \quad \sigma_{rz} = -\frac{\partial^2 \Phi}{\partial r \partial z}, \end{aligned} \right\} \quad \begin{array}{l} \text{Contd.} \\ (2.33) \end{array}$$

to which we append the equation of steady-state heat conduction

$$\nabla_1^2 T + L^2 \frac{\partial^2 T}{\partial z^2} = 0, \quad (2.34)$$

where L^2 is the ratio of coefficients of thermal conductivity along the z axis and in the z plane.

In all problems to be considered in following sections, there will exist some characteristic radial dimension (e.g., the outer radius of a cylinder or radius of a cylindrical inclusion) which we denote by d . Introducing this characteristic dimension we shall rewrite (2.33) and (2.34) in terms of the following dimensionless quantities (dimensionless forms being represented by $\hat{}$):

$$\left. \begin{aligned} \rho &= r/d; \quad \xi = z/d; \quad T = T_0 \hat{T}; \quad \sigma_{ij} = \frac{E\alpha T_0}{(1-\nu)} \hat{\sigma}_{ij}; \\ \Phi &= \frac{d^2 E\alpha T_0}{(1-\nu)} \hat{\Phi}; \quad \Omega = \frac{d^2 E\alpha T_0}{(1-\nu)} \hat{\Omega}; \\ u &= \frac{1+\nu}{1-\nu} d\alpha T_0 \hat{u}; \quad w = \frac{1+\nu}{1-\nu} d\alpha T_0 \hat{w}. \end{aligned} \right\} \quad (2.35)$$

Equations (2.33) and (2.34) now become

$$\left. \begin{aligned} \nabla_1^2 \hat{\Phi} + k_{22}^2 \frac{\partial^2 \hat{\Phi}}{\partial \xi^2} &= \frac{\gamma_2}{(1-\nu)} \frac{\partial^2 \hat{\Omega}}{\partial \xi^2} - \gamma_1 \hat{T}; \\ \nabla_1^2 \hat{\Omega} + k_{21}^2 \frac{\partial^2 \hat{\Omega}}{\partial \xi^2} &= \gamma_4 \hat{T}; \quad \nabla_1^2 \hat{T} + L^2 \frac{\partial^2 \hat{T}}{\partial \xi^2} = 0; \end{aligned} \right\} \quad (2.36)$$

$$\hat{u} = -\left(k_{11}^2 \frac{\partial \hat{\Phi}}{\partial \rho} + \frac{\partial \hat{\Omega}}{\partial \rho} \right); \quad \hat{w} = -\left(k_{12}^2 \frac{\partial \hat{\Phi}}{\partial \xi} - \frac{\partial \hat{\Omega}}{\partial \xi} \right); \quad (2.37)$$

$$\left. \begin{aligned} \hat{\sigma}_{rr} &= \frac{\partial^2 \hat{\Phi}}{\partial \xi^2} + \frac{1}{\rho} \left(k_{11}^2 \frac{\partial \hat{\Phi}}{\partial \rho} + \frac{\partial \hat{\Omega}}{\partial \rho} \right) \\ \hat{\sigma}_{\theta\theta} &= \nu \left[\gamma_3 \nabla_1^2 \hat{\Phi} + \frac{\partial^2 \hat{\Phi}}{\partial \xi^2} \right] - \frac{1}{\rho} \left(k_{11}^2 \frac{\partial \hat{\Phi}}{\partial \rho} + \frac{\partial \hat{\Omega}}{\partial \rho} \right) - (1-\nu) \hat{T}; \\ \hat{\sigma}_{zz} &= \nabla_1^2 \hat{\Phi}; \quad \hat{\sigma}_{rz} = -\frac{\partial^2 \hat{\Phi}}{\partial \rho \partial \xi}, \end{aligned} \right\} \quad (2.38)$$

where $\nabla_1^2 = \frac{\partial^2}{\partial \rho^2} + \frac{1}{\rho} \frac{\partial}{\partial \rho}$ and the various dimensionless constants have the values

$$\begin{aligned} \gamma_1 &= \frac{(1-\nu) \left[\left(\frac{\beta}{\alpha} \right) \left(\frac{E'}{E} \right) + \nu' \right]}{\left[1 - \left(\frac{E}{E'} \right) \nu'^2 \right]}; \\ \gamma_2 &= \frac{(1-\nu^2) \left(\frac{E'}{E} \right)}{\left[1 - \left(\frac{E}{E'} \right) \nu'^2 \right]}; \\ \gamma_3 &= \left(\frac{E \nu'}{E' \nu} \right); \\ \gamma_4 &= (1-\nu) \left[\left(\frac{E}{E'} \right) \frac{\left(\frac{\beta}{\alpha} \right) \left(\frac{E'}{E} \right) + \nu'}{(1+\nu)} k_{21}^2 - 1 \right] \\ &= \gamma_1 (k_{11}^2 - \nu \gamma_3) - (1-\nu). \end{aligned} \quad (2.39)$$

In the case of an isotropic material we note that

$$k_{ij}^2 \rightarrow 1; \quad \gamma_1 \rightarrow 1; \quad \gamma_2 \rightarrow 1; \quad \gamma_3 \rightarrow 1; \quad \gamma_4 \rightarrow 0; \quad \text{and } L^2 \rightarrow 1.$$

Hence, Eqs. (2.36), (2.37), and (2.38) reduce to

$$\nabla^2 \hat{\Phi} = \frac{1}{(1-\nu)} \frac{\partial^2 \hat{\Omega}}{\partial \xi^2} - \hat{T}; \quad \nabla^2 \hat{\Omega} = 0; \quad \nabla^2 \hat{T} = 0; \quad (2.40)$$

$$\hat{u} = -\left(\frac{\partial \hat{\Phi}}{\partial \rho} + \frac{\partial \hat{\Omega}}{\partial \rho}\right); \quad \hat{w} = -\left(\frac{\partial \hat{\Phi}}{\partial \xi} - \frac{\partial \hat{\Omega}}{\partial \xi}\right); \quad (2.41)$$

$$\left. \begin{aligned} \hat{\sigma}_{rr} &= \frac{\partial^2 \hat{\Phi}}{\partial \xi^2} + \frac{1}{\rho} \left(\frac{\partial \hat{\Phi}}{\partial \rho} + \frac{\partial \hat{\Omega}}{\partial \rho} \right); \\ \sigma_{\theta\theta} &= \nu \nabla^2 \hat{\Phi} - \frac{1}{\rho} \left(\frac{\partial \hat{\Phi}}{\partial \rho} + \frac{\partial \hat{\Omega}}{\partial \rho} \right) - (1-\nu) \hat{T}; \\ \hat{\sigma}_{zz} &= \nabla^2 \hat{\Phi}; \quad \hat{\sigma}_{rz} = -\frac{\partial^2 \hat{\Phi}}{\partial \rho \partial \xi}. \end{aligned} \right\} \quad (2.42)$$

Equations (2.40), (2.41), and (2.42) correspond to the dimensionless form of the equations derived by Hoyle² for the axisymmetric thermal stress problem for an isotropic solid.

In the sections to follow we shall work exclusively with the non-dimensional forms (2.36), (2.37), and (2.38). For convenience, we shall drop the $\hat{}$ notation for dimensionlessness. The presence of the constants γ_i should be sufficient reminder of the dimensionless nature of the results.

III. A THERMAL-INCLUSION PROBLEM FOR THE INFINITE SOLID

We shall consider the thermal stresses in a transversely isotropic space due to the presence of a discontinuous temperature field in the form of a half-infinite circular cylinder. In other words, the entire space is assumed to have zero temperature with the exception of a half infinite circular cylinder of radius d lying along the positive z axis and having a constant temperature T_0 . This somewhat unrealistic problem is being treated not because of its correspondence with any readily recognizable physical situation, but because an isotropic solution is already known (Myklestad³), and its form is of sufficient simplicity to lead one to believe that the solution in the transversely isotropic case will not be overly complex. The problem will thus provide expressions for the stresses which may be readily evaluated for several different materials and thereby qualitatively indicate how the degree of departure from isotropy may effect the thermal-stress distributions and the peak stress magnitudes. A thermal inclusion in the form of a circular cylinder rather than the physically more interesting case of a sphere is being considered since the inherent material symmetry makes the treatment of problems in spherical coordinates extremely cumbersome.

The work to follow closely parallels the isotropic solution as formulated by Nowacki.⁴ In dimensionless form, the temperature field is given by

$$T = \eta(1 - \rho) \eta(\xi), \quad (3.1)$$

where $\eta(x)$ is Heavyside's function. We consider (3.1) represented as the sum of an odd and an even function of ξ :

$$T = T^{(1)} + T^{(2)},$$

where

$$\left. \begin{aligned} T^{(1)} &= \frac{1}{2}\eta(1 - \rho)[\eta(\xi) - \eta(-\xi)]; \\ T^{(2)} &= \frac{1}{2}\eta(1 - \rho)[\eta(\xi) + \eta(-\xi)]. \end{aligned} \right\} \quad (3.2)$$

We first consider $T^{(1)}$ and proceed to solve

$$\left. \begin{aligned} \nabla_1^2 \Phi^{(1)} + k_{22}^2 \frac{\partial^2 \Phi^{(1)}}{\partial \xi^2} &= \frac{\gamma^2}{(1 - \nu)} \frac{\partial^2 \Omega^{(1)}}{\partial \xi^2} - \gamma_1 T^{(1)}; \\ \nabla_1^2 \Omega^{(1)} + k_{21}^2 \frac{\partial^2 \Omega^{(1)}}{\partial \xi^2} &= \gamma_4 T^{(1)} \end{aligned} \right\} \quad (3.3)$$

by combined use of Fourier sine and zero-order Hankel transforms. We employ the following notation: for the Fourier-sine transform,

$$\tilde{f}(p) = \sqrt{\frac{2}{\pi}} \int_0^{\infty} \sin p\zeta f(\zeta) d\zeta;$$

$$f(\zeta) = \sqrt{\frac{2}{\pi}} \int_0^{\infty} \sin p\zeta \tilde{f}(p) dp,$$

and for the zero-order Hankel transform

$$\bar{f}(q) = \int_0^{\infty} \rho f(\rho) J_0(q\rho) d\rho;$$

$$f(\rho) = \int_0^{\infty} q \bar{f}(q) J_0(\rho q) dq.$$

Using the above transforms it is an elementary matter to show that

$$\tilde{\Omega} = -\gamma_4 \frac{1}{2} \sqrt{\frac{2}{\pi}} \frac{J_1(q)}{pq(q^2 + p^2 k_{21}^2)}$$

and, hence, upon inversion,

$$\Omega^{(1)} = -\frac{\gamma_4}{\pi} \int_0^{\infty} \int_0^{\infty} \frac{\sin p\zeta}{p(q^2 + p^2 k_{21}^2)} J_0(\rho q) J_1(q) dq dp,$$

which, after integration over p , results in

$$\Omega^{(1)} = \frac{\gamma_4}{2} \int_0^{\infty} \frac{1}{q^2} \left(e^{-\frac{q}{k_{21}} \zeta} - 1 \right) J_0(\rho q) J_1(q) dq, \quad (3.4)$$

where (3.4) holds for $\zeta \geq 0$. In a similar manner, the first of (3.3) yields

$$\begin{aligned} \tilde{\Phi}^{(1)} = & -\frac{\gamma_2 \gamma_4}{2(1-\nu)} \sqrt{\frac{2}{\pi}} \frac{p J_1(q)}{q(q^2 + p^2 k_{22}^2)(q^2 + p^2 k_{21}^2)} \\ & + \frac{\gamma_1}{2} \sqrt{\frac{2}{\pi}} \frac{J_1(q)}{pq(q^2 + p^2 k_{22}^2)}, \end{aligned}$$

which, upon inversion and simplification, implies

$$\begin{aligned}\Phi^{(1)} = & -\frac{\gamma_1}{2} \int_0^\infty \frac{1}{q^2} \left(e^{-\frac{q}{k_{22}} \xi} - 1 \right) J_0(\rho q) J_1(q) dq \\ & - \frac{\gamma_2 \gamma_4}{2(1-\nu)(k_{21}^2 - k_{22}^2)} \int_0^\infty \frac{1}{q^2} \left(e^{-\frac{q}{k_{21}} \xi} - e^{-\frac{q}{k_{22}} \xi} \right) J_0(\rho q) J_1(q) dq,\end{aligned}\quad (3.5)$$

where (3.5) holds for $\xi \geq 0$. Substituting (3.4) and (3.5) into (2.38), we obtain for $\xi \geq 0$,

$$\begin{aligned}\sigma_{zz}^{(1)} = & \frac{\gamma_1}{2} \int_0^\infty \left(e^{-\frac{q}{k_{22}} \xi} - 1 \right) J_0(\rho q) J_1(q) dq \\ & + \frac{\gamma_2 \gamma_4}{2(1-\nu)(k_{21}^2 - k_{22}^2)} \int_0^\infty \left(e^{-\frac{q}{k_{21}} \xi} - e^{-\frac{q}{k_{22}} \xi} \right) J_0(\rho q) J_1(q) dq;\end{aligned}\quad (3.6)$$

$$\begin{aligned}\sigma_{rz}^{(1)} = & \frac{\gamma_1}{2k_{22}} \int_0^\infty e^{-\frac{q}{k_{21}} \xi} J_1(\rho q) J_1(q) dq \\ & + \frac{\gamma_2 \gamma_4}{2(1-\nu)(k_{21}^2 - k_{22}^2)} \int_0^\infty \left(\frac{1}{k_{22}} e^{-\frac{q}{k_{21}} \xi} - \frac{1}{k_{21}} e^{-\frac{q}{k_{22}} \xi} \right) J_1(\rho q) J_1(q) dq;\end{aligned}\quad (3.7)$$

$$\begin{aligned}\sigma_{rr}^{(1)} = & -\frac{\gamma_2 \gamma_4}{2(1-\nu)(k_{21}^2 - k_{22}^2)} \int_0^\infty J_1(q) e^{-\frac{q}{k_{21}} \xi} \left[\frac{1}{k_{21}^2} J_0(\rho q) - \frac{k_{11}^2}{\rho q} J_1(\rho q) \right] dq \\ & + \left[\frac{\gamma_2 \gamma_4}{2(1-\nu)(k_{21}^2 - k_{22}^2)} - \frac{\gamma_1}{2} \right] \int_0^\infty J_1(q) e^{-\frac{q}{k_{22}} \xi} \left[\frac{1}{k_{22}^2} J_0(\rho q) - \frac{k_{11}^2}{\rho q} J_1(\rho q) \right] dq \\ & - \frac{k_{11}^2 \gamma_1}{2} \int_0^\infty \frac{1}{\rho q} J_1(\rho q) J_1(q) dq \\ & - \frac{\gamma_4}{2} \int_0^\infty \frac{1}{\rho q} \left(e^{-\frac{q}{k_{21}} \xi} - 1 \right) J_1(\rho q) J_1(q) dq;\end{aligned}\quad (3.8)$$

$$\begin{aligned}
\sigma_{\theta\theta}^{(1)} = & \frac{\gamma_2 \gamma_4}{2(1-\nu)(k_{21}^2 - k_{22}^2)} \left\{ \int_0^\infty J_1(q) e^{-\frac{q}{k_{21}} \xi} \left[\nu \left(\gamma_3 - \frac{1}{k_{21}^2} \right) J_0(\rho q) - \frac{k_{11}^2}{\rho q} J_1(\rho q) \right] dq \right. \\
& - \int_0^\infty J_1(q) e^{-\frac{q}{k_{22}} \xi} \left[\nu \left(\gamma_3 - \frac{1}{k_{22}^2} \right) J_0(\rho q) - \frac{k_{11}^2}{\rho q} J_1(\rho q) \right] dq \Big\} \\
& + \frac{\gamma_1}{2} \int_0^\infty J_1(q) e^{-\frac{q}{k_{22}} \xi} \left[\nu \left(\gamma_3 - \frac{1}{k_{22}^2} \right) J_0(\rho q) - \frac{k_{11}^2}{\rho q} J_1(\rho q) \right] dq \\
& - \frac{\gamma_1}{2} \int_0^\infty J_1(q) \left[\nu \gamma_3 J_0(\rho q) - \frac{k_{11}^2}{\rho q} J_1(\rho q) \right] dq \\
& + \frac{\gamma_4}{2} \int_0^\infty \frac{1}{\rho q} \left(e^{-\frac{q}{k_{21}} \xi} - 1 \right) J_1(\rho q) J_1(q) dq - (1-\nu) T^{(1)}. \quad (3.9)
\end{aligned}$$

To obtain the stresses for $\xi < 0$, one replaces ξ by $-\xi$ and changes the signs in (3.6) through (3.9). The temperature field $T^{(2)}$ is zero everywhere in the space with the exception of the cylinder of unit radius lying along the entire ξ axis, where the temperature has the constant value $\frac{1}{2}$. The stress field caused by $T^{(2)}$ is independent of ξ and may be most conveniently obtained by considering the limit $\xi \rightarrow \infty$ in equations (3.6) through (3.9). After noting that

$$\int_0^\infty \frac{J_1(q)}{\rho} J_1(\rho q) dq = \frac{1}{2} \left[\eta(1-\rho) + \frac{1}{\rho^2} \eta(\rho-1) \right];$$

$$\int_0^\infty J_0(\rho q) J_1(q) dq = \eta(1-\rho),$$

one obtains as the symmetric components of the stress field

$$\left. \begin{aligned} \sigma_{zz}^{(2)} &= -\frac{\gamma_1}{2} \eta(1-\rho); \\ \sigma_{rz}^{(2)} &= 0; \end{aligned} \right\} \quad (3.10)$$

Contd.

$$\left. \begin{aligned} \sigma_{rr}^{(2)} &= \frac{1}{4}(\gamma_4 - k_{11}^2 \gamma_1) \left[\eta(1 - \rho) + \frac{1}{\rho^2} \eta(\rho - 1) \right]; \\ \sigma_{\theta\theta}^{(2)} &= -\frac{1}{4} \left[(1 - \nu + \gamma_1 \gamma_3 \nu) \eta(1 - \rho) - (k_{11}^2 \gamma_1 - \gamma_4) \frac{1}{\rho^2} \eta(\rho - 1) \right]. \end{aligned} \right\} \text{Contd. (3.10)}$$

The integrals appearing in (3.6) through (3.9) are well known and may be represented in terms of elliptic integrals and Legendre functions. We introduce the following notation:

$$\left. \begin{aligned} \mathcal{J}_2(z) &= \int_0^\infty e^{-zq} J_1(q) J_1(\rho q) dq; \\ \mathcal{J}_3(a, b; z) &= \int_0^\infty e^{-zq} J_1(aq) J_0(bq) dq, \end{aligned} \right\} \quad (3.11)$$

and note that (Nowacki,⁴ p. 58):

$$\mathcal{J}_2(z) = (\pi^2 \rho)^{-1/2} Q_{1/2} \left(\frac{\rho^2 + z^2 + 1}{2\rho} \right);$$

$$\mathcal{J}_3(1, \rho; z) = \frac{1}{\pi} G(\rho, z);$$

$$\mathcal{J}_3(\rho, 1; z) = \frac{1}{\pi \rho} G(\rho, z);$$

$$\int_0^\infty e^{-zq} \frac{1}{q} J_1(q) J_1(\rho q) dq = \frac{1}{2} [\mathcal{J}_3(\rho, 1; z) + \rho \mathcal{J}_3(1, \rho; z) - z \mathcal{J}_2(z)];$$

$$G(\rho, z) = \{K'E(k, \theta) + (E' - K') F(k, \theta) - z[(1 + \rho)^2 + z^2]^{-1/2} K'\},$$

where $Q_{1/2}$ is the Legendre function of the second kind, F and E are incomplete elliptic integrals of the first and second kind, respectively, with modulus $k = [(1 - \rho)^2 + z^2]^{1/2} [(1 + \rho)^2 + z^2]^{-1/2}$ and argument $\theta = \sin^{-1} z[(1 - \rho)^2 + z^2]^{-1/2}$, ($0 \leq \theta \leq \pi$), K' and E' are complete elliptic integrals of the first and second kind, respectively, with complementary modulus $k' = (1 - k^2)^{1/2}$.

Upon adding the stress fields due to $T^{(1)}$ and $T^{(2)}$ we obtain, in terms of the notation of (3.11),

$\xi \geq 0$:

$$\begin{aligned}\sigma_{zz} &= \frac{\gamma_1}{2} \mathcal{J}_3\left(1, \rho; \frac{\xi}{k_{22}}\right) - \gamma_1 \eta(1 - \rho) \\ &+ \frac{\gamma_2 \gamma_4}{2(1 - \nu)(k_{21}^2 - k_{22}^2)} \left[\mathcal{J}_3\left(1, \rho; \frac{\xi}{k_{21}}\right) - \mathcal{J}_3\left(1, \rho; \frac{\xi}{k_{22}}\right) \right].\end{aligned}$$

$\xi < 0$:

$$\begin{aligned}\sigma_{zz} &= -\frac{\gamma_1}{2} \mathcal{J}_3\left(1, \rho; -\frac{\xi}{k_{22}}\right) \\ &- \frac{\gamma_2 \gamma_4}{2(1 - \nu)(k_{21}^2 - k_{22}^2)} \left[\mathcal{J}_3\left(1, \rho; -\frac{\xi}{k_{21}}\right) - \mathcal{J}_3\left(1, \rho; -\frac{\xi}{k_{22}}\right) \right].\end{aligned}\quad (3.12)$$

$\xi \geq 0$:

$$\sigma_{rz} = \frac{\gamma_1}{2k_{22}} \mathcal{J}_2\left(\frac{\xi}{k_{22}}\right) + \frac{\gamma_2 \gamma_4}{2(1 - \nu)(k_{21}^2 - k_{22}^2)} \left[\frac{1}{k_{21}} \mathcal{J}_2\left(\frac{\xi}{k_{21}}\right) - \frac{1}{k_{22}} \mathcal{J}_2\left(\frac{\xi}{k_{22}}\right) \right].$$

$\xi < 0$:

$$\begin{aligned}\sigma_{rz} &= -\frac{\gamma_1}{2k_{22}} \mathcal{J}_2\left(-\frac{\xi}{k_{22}}\right) - \frac{\gamma_2 \gamma_4}{2(1 - \nu)(k_{21}^2 - k_{22}^2)} \left[\frac{1}{k_{21}} \mathcal{J}_2\left(-\frac{\xi}{k_{21}}\right) - \frac{1}{k_{22}} \mathcal{J}_2\left(-\frac{\xi}{k_{22}}\right) \right].\end{aligned}\quad (3.13)$$

$\xi \geq 0$:

$$\begin{aligned}\sigma_{rr} &= -\frac{\gamma_1}{2} \left\{ \left(\frac{1}{k_{22}^2} - \frac{k_{11}^2}{2} \right) \mathcal{J}_3\left(1, \rho; \frac{\xi}{k_{22}}\right) - \frac{k_{11}^2}{\rho} \left[\mathcal{J}_3\left(\rho, 1; \frac{\xi}{k_{22}}\right) - \frac{\xi}{k_{22}} \mathcal{J}_2\left(\frac{\xi}{k_{22}}\right) \right] \right. \\ &\quad \left. + k_{11}^2 \left[\eta(1 - \rho) + \frac{1}{\rho^2} \eta(\rho - 1) \right] \right\} \\ &- \frac{\gamma_2 \gamma_4}{2(1 - \nu)(k_{21}^2 - k_{22}^2)} \left\{ \left[\left(\frac{1}{k_{21}^2} - \frac{k_{11}^2}{2} \right) \mathcal{J}_3\left(1, \rho; \frac{\xi}{k_{21}}\right) - \frac{k_{11}^2}{\rho} \left[\mathcal{J}_3\left(\rho, 1; \frac{\xi}{k_{21}}\right) - \frac{\xi}{k_{21}} \mathcal{J}_2\left(\frac{\xi}{k_{21}}\right) \right] \right] \right. \\ &\quad \left. - \left[\left(\frac{1}{k_{22}^2} - \frac{k_{11}^2}{2} \right) \mathcal{J}_3\left(1, \rho; \frac{\xi}{k_{22}}\right) - \frac{k_{11}^2}{\rho} \left[\mathcal{J}_3\left(\rho, 1; \frac{\xi}{k_{22}}\right) - \frac{\xi}{k_{22}} \mathcal{J}_2\left(\frac{\xi}{k_{22}}\right) \right] \right] \right\}\end{aligned}\quad (3.14)$$

Contd.

$$\begin{aligned}
& - \frac{\gamma_4}{4\rho} \left[\mathcal{J}_3\left(\rho, 1; \frac{\xi}{k_{21}}\right) + \rho \mathcal{J}_3\left(1, \rho; \frac{\xi}{k_{21}}\right) - \frac{\xi}{k_{21}} \mathcal{J}_2\left(\frac{\xi}{k_{21}}\right) \right] \\
& + \frac{\gamma_4}{2} \left[\eta(1 - \rho) + \frac{1}{\rho^2} (\rho - 1) \right]
\end{aligned}$$

$\xi < 0$:

$$\begin{aligned}
\sigma_{rr} = & \frac{\gamma_1}{2} \left\{ \left(\frac{1}{k_{22}^2} - \frac{k_{11}^2}{2} \right) \mathcal{J}_3\left(1, \rho; -\frac{\xi}{k_{22}}\right) - \frac{k_{11}^2}{\rho} \left[\mathcal{J}_3\left(\rho, 1; -\frac{\xi}{k_{22}}\right) + \frac{\xi}{k_{22}} \mathcal{J}_2\left(-\frac{\xi}{k_{22}}\right) \right] \right\} \\
& + \frac{\gamma_2 \gamma_4}{2(1 - \nu)(k_{21}^2 - k_{22}^2)} \left\{ \left(\frac{1}{k_{21}^2} - \frac{k_{11}^2}{2} \right) \mathcal{J}_3\left(1, \rho; -\frac{\xi}{k_{21}}\right) \right. \\
& - \frac{k_{11}^2}{\rho} \left[\mathcal{J}_3\left(\rho, 1; -\frac{\xi}{k_{21}}\right) + \frac{\xi}{k_{21}} \mathcal{J}_2\left(-\frac{\xi}{k_{21}}\right) \right] \\
& - \left[\left(\frac{1}{k_{22}^2} - \frac{k_{11}^2}{2} \right) \mathcal{J}_3\left(1, \rho; -\frac{\xi}{k_{22}}\right) \right. \\
& \left. \left. - \frac{k_{11}^2}{\rho} \left[\mathcal{J}_3\left(\rho, 1; -\frac{\xi}{k_{22}}\right) + \frac{\xi}{k_{22}} \mathcal{J}_2\left(-\frac{\xi}{k_{22}}\right) \right] \right] \right\} \\
& + \frac{\gamma_4}{4\rho} \left[\mathcal{J}_3\left(\rho, 1; -\frac{\xi}{k_{21}}\right) + \rho \mathcal{J}_3\left(1, \rho; -\frac{\xi}{k_{21}}\right) + \frac{\xi}{k_{21}} \mathcal{J}_2\left(-\frac{\xi}{k_{21}}\right) \right].
\end{aligned}$$

Contd. (3.14)

$\xi \geq 0$:

$$\begin{aligned}
\sigma_{\theta\theta} = & \frac{\gamma_1}{2} \left\{ \nu \left(\gamma_3 - \frac{1}{k_{22}^2} \right) \mathcal{J}_3\left(1, \rho; \frac{\xi}{k_{22}}\right) - \frac{k_{11}^2}{2\rho} \left[\mathcal{J}_3\left(\rho, 1; \frac{\xi}{k_{22}}\right) \right. \right. \\
& \left. \left. + \rho \mathcal{J}_3\left(1, \rho; \frac{\xi}{k_{22}}\right) - \frac{\xi}{k_{22}} \mathcal{J}_2\left(\frac{\xi}{k_{22}}\right) \right] \right\} \\
& + \frac{\gamma_2 \gamma_4}{2(1 - \nu)(k_{21}^2 - k_{22}^2)} \left\{ \nu \left(\gamma_3 - \frac{1}{k_{21}^2} \right) \mathcal{J}_3\left(1, \rho; \frac{\xi}{k_{21}}\right) \right.
\end{aligned}$$

(3.15)
Contd.

$$\begin{aligned}
& - \frac{k_{11}^2}{2\rho} \left[\mathcal{J}_3\left(\rho, 1; \frac{\xi}{k_{21}}\right) + \rho \mathcal{J}_3\left(1, \rho; \frac{\xi}{k_{21}}\right) - \frac{\xi}{k_{21}} \mathcal{J}_2\left(\frac{\xi}{k_{21}}\right) \right] \\
& - \left[\nu \left(\gamma_3 - \frac{1}{k_{22}^2} \right) \mathcal{J}_3\left(1, \rho; \frac{\xi}{k_{22}}\right) \right. \\
& \left. - \frac{k_{11}^2}{2\rho} \left[\mathcal{J}_3\left(\rho, 1; \frac{\xi}{k_{22}}\right) + \rho \mathcal{J}_3\left(1, \rho; \frac{\xi}{k_{22}}\right) - \frac{\xi}{k_{22}} \mathcal{J}_2\left(\frac{\xi}{k_{22}}\right) \right] \right] \Bigg\} \\
& + \frac{\gamma_4}{2\rho} \left[\mathcal{J}_3\left(\rho, 1; \frac{\xi}{k_{21}}\right) + \rho \mathcal{J}_3\left(1, \rho; \frac{\xi}{k_{21}}\right) - \frac{\xi}{k_{21}} \mathcal{J}_2\left(\frac{\xi}{k_{21}}\right) \right] \\
& + \frac{1}{2} (\gamma_1 k_{11}^2 - \gamma_4) [\eta(1-\rho) + \frac{1}{\rho^2} \eta(\rho-1)] - \nu \gamma_1 \gamma_3 \eta(1-\rho) - (1-\nu).
\end{aligned}$$

$\xi < 0$:

$$\begin{aligned}
\sigma_{\theta\theta} = & - \frac{\gamma_1}{2} \left\{ \nu \left(\gamma_3 - \frac{1}{k_{22}^2} \right) \mathcal{J}_3\left(1, \rho; -\frac{\xi}{k_{22}}\right) - \frac{k_{11}^2}{2\rho} \left[\mathcal{J}_3\left(\rho, 1; -\frac{\xi}{k_{22}}\right) \right. \right. \\
& \left. \left. + \rho \mathcal{J}_3\left(1, \rho; -\frac{\xi}{k_{22}}\right) + \frac{\xi}{k_{22}} \mathcal{J}_2\left(-\frac{\xi}{k_{22}}\right) \right] \right\} \\
& - \frac{\gamma_2 \gamma_4}{2(1-\nu)(k_{21}^2 - k_{22}^2)} \left\{ \nu \left(\gamma_3 - \frac{1}{k_{21}^2} \right) \mathcal{J}_3\left(1, \rho; -\frac{\xi}{k_{21}}\right) \right. \\
& - \frac{k_{11}^2}{2\rho} \left[\mathcal{J}_3\left(\rho, 1; -\frac{\xi}{k_{21}}\right) + \rho \mathcal{J}_3\left(1, \rho; -\frac{\xi}{k_{21}}\right) + \frac{\xi}{k_{21}} \mathcal{J}_2\left(-\frac{\xi}{k_{21}}\right) \right] \\
& - \left[\nu \left(\gamma_3 - \frac{1}{k_{22}^2} \right) \mathcal{J}_3\left(1, \rho; -\frac{\xi}{k_{22}}\right) - \frac{k_{11}^2}{2\rho} \left[\mathcal{J}_3\left(\rho, 1; -\frac{\xi}{k_{22}}\right) \right. \right. \\
& \left. \left. + \rho \mathcal{J}_3\left(1, \rho; -\frac{\xi}{k_{22}}\right) + \frac{\xi}{k_{22}} \mathcal{J}_2\left(-\frac{\xi}{k_{22}}\right) \right] \right] \Bigg\} \\
& - \frac{\gamma_4}{2\rho} \left[\mathcal{J}_3\left(\rho, 1; -\frac{\xi}{k_{21}}\right) + \rho \mathcal{J}_3\left(1, \rho; -\frac{\xi}{k_{21}}\right) + \frac{\xi}{k_{21}} \mathcal{J}_2\left(-\frac{\xi}{k_{21}}\right) \right].
\end{aligned}$$

Contd. (3.15)

Equations (3.12) through (3.15) assume a much simpler form when evaluated along the axis of the cylinder ($\rho = 0$). One finds for $\xi \geq 0$ and $\rho = 0$:

$$\sigma_{zz} = -\frac{\gamma_1}{2} \left[1 + \frac{\xi}{(\xi^2 + k_{22}^2)^{1/2}} \right] - \frac{\gamma_2 \gamma_4 [(\xi^2 + k_{22}^2)^{1/2} + (\xi^2 + k_{21}^2)^{1/2}] \xi}{2(1-\nu)(k_{21}^2 - k_{22}^2)(\xi^2 + k_{22}^2)^{1/2}(\xi^2 + k_{21}^2)^{1/2}}; \quad (3.16)$$

$$\begin{aligned} \sigma_{rr} = \sigma_{\theta\theta} = & -\frac{\gamma_1}{2} \left\{ \left(\frac{1}{k_{22}^2} - \frac{k_{11}^2}{2} \right) \left[1 - \frac{\xi}{(\xi^2 + k_{22}^2)^{1/2}} \right] + k_{11}^2 \right\} \\ & + \frac{\gamma_4}{2(1-\nu)} + \frac{\gamma_2 \gamma_4}{2(1-\nu)(k_{21}^2 - k_{22}^2)} \left[\frac{\xi \left(\frac{1}{k_{21}^2} - \frac{k_{11}^2}{2} \right)}{(\xi^2 + k_{21}^2)^{1/2}} - \frac{\xi \left(\frac{1}{k_{22}^2} - \frac{k_{11}^2}{2} \right)}{(\xi^2 + k_{22}^2)^{1/2}} \right] \\ & + \frac{\gamma_4}{4} \left[1 + \frac{\xi}{(\xi^2 + k_{21}^2)^{1/2}} \right]; \end{aligned}$$

$$\sigma_{r\theta} = 0,$$

and for $\xi < 0$ and $\rho = 0$:

$$\begin{aligned} \sigma_{zz} = & -\frac{\gamma_1}{2} \left[1 + \frac{\xi}{(\xi^2 + k_{22}^2)^{1/2}} \right] - \frac{\gamma_2 \gamma_4 [(\xi^2 + k_{22}^2)^{1/2} - (\xi^2 + k_{21}^2)^{1/2}] \xi}{2(1-\nu)(k_{21}^2 - k_{22}^2)(\xi^2 + k_{21}^2)^{1/2}(\xi^2 + k_{22}^2)^{1/2}}; \\ \sigma_{rr} = \sigma_{\theta\theta} = & \frac{\gamma_1}{2} \left(\frac{1}{k_{22}^2} - \frac{k_{11}^2}{2} \right) \left[1 + \frac{\xi}{(\xi^2 + k_{22}^2)^{1/2}} \right] \\ & - \frac{\gamma_4}{2(1-\nu)} + \frac{\gamma_2 \gamma_4}{2(1-\nu)(k_{21}^2 - k_{22}^2)} \left[\frac{\xi \left(\frac{1}{k_{21}^2} - \frac{k_{11}^2}{2} \right)}{(\xi^2 + k_{21}^2)^{1/2}} - \frac{\xi \left(\frac{1}{k_{22}^2} - \frac{k_{11}^2}{2} \right)}{(\xi^2 + k_{22}^2)^{1/2}} \right] \\ & + \frac{\gamma_4}{4} \left[1 + \frac{\xi}{(\xi^2 + k_{21}^2)^{1/2}} \right]; \quad (3.17) \end{aligned}$$

$$\sigma_{r\theta} = 0.$$

In the limiting case of isotropic behavior, (3.16) and (3.17) reduce to

$\xi \geq 0$:

$$\begin{aligned}\sigma_{zz} &= -\frac{1}{2} \left[1 + \frac{\xi}{(\xi^2 + 1)^{1/2}} \right]; \\ \sigma_{rr} = \sigma_{\theta\theta} &= -\frac{1}{4} \left[3 - \frac{\xi}{(\xi^2 + 1)^{1/2}} \right]; \\ \sigma_{r\theta} &= 0;\end{aligned}\tag{3.18}$$

$\xi \leq 0$:

$$\begin{aligned}\sigma_{zz} &= -\frac{1}{2} \left[1 + \frac{\xi}{(\xi^2 + 1)^{1/2}} \right]; \\ \sigma_{rr} = \sigma_{\theta\theta} &= \frac{1}{4} \left[1 + \frac{\xi}{(\xi^2 + 1)^{1/2}} \right]; \\ \sigma_{r\theta} &= 0,\end{aligned}\tag{3.19}$$

which are the dimensionless forms of the isotropic results quoted by Nowacki,⁴ p. 59. The qualitative differences between the transversely isotropic results (3.16) and (3.17), and the isotropic relations (3.18) and (3.19), are to some extent revealed by the following limit values ($\rho = 0$):

ξ	Transversely Isotropic		Isotropic	
	$\sigma_{rr}, \sigma_{\theta\theta}$	σ_{zz}	$\sigma_{rr}, \sigma_{\theta\theta}$	σ_{zz}
$+\infty$	$-\frac{1}{2} + \frac{\nu}{2} (1 - \gamma_1 \gamma_3)$	$-\gamma_1$	$-\frac{1}{2}$	-1
$-\infty$	0	0	0	0
$0+$	$-\frac{3}{4} + \frac{\nu}{4} (1 - \gamma_1 \gamma_3)$	$-\frac{\gamma_1}{2}$	$-\frac{3}{4}$	$-\frac{1}{2}$
$0-$	$\frac{1}{4} + \frac{\nu}{4} (1 - \gamma_1 \gamma_3)$	$-\frac{\gamma_1}{2}$	$\frac{1}{4}$	$-\frac{1}{2}$

We note from the above table, for example, that the unit discontinuity in σ_{rr} and $\sigma_{\theta\theta}$ at $\xi = 0$ is unchanged by the assumption of transverse isotropy even though the magnitudes of the stresses are altered.

IV. A THERMAL-STRESS PROBLEM FOR A TRANSVERSELY ISOTROPIC CYLINDER

We consider a transversely isotropic infinite right circular cylinder of radius d whose axis of elastic symmetry coincides with the axis of the cylinder and assume that a band of the cylinder surface of width $2b$ is kept at temperature T_0 while the remainder of the surface is at temperature zero. If we introduce the dimensionless constant $\beta_1 = b/d$, the boundary value problem for the dimensionless temperature is

$$\left. \begin{aligned} \nabla_1^2 T + L^2 \frac{\partial^2 T}{\partial \xi^2} &= 0; \quad |\xi| < \infty, 0 \leq \rho < 1; \\ T(1, \xi) &= \eta(\beta_1 - |\xi|); \\ T &\rightarrow 0 \text{ as } |\xi| \rightarrow \infty. \end{aligned} \right\} \quad (4.1)$$

Denoting the Fourier cosine transform of $f(\xi)$ by $\tilde{f}(p)$, we transform (4.1) with respect to ξ and obtain

$$\left. \begin{aligned} \nabla_1^2 \tilde{T} - p^2 L^2 \tilde{T} &= 0; \\ \tilde{T}(1; p) &= \left(\frac{2}{\pi}\right)^{1/2} \frac{\sin \beta_1 p}{p}, \end{aligned} \right\} \quad (4.2)$$

and hence

$$\left. \begin{aligned} \tilde{T}(\rho; p) &= A(p) I_0(pL\rho), \\ \text{where} \\ A(p) &= \left(\frac{2}{\pi}\right)^{1/2} \frac{\sin \beta_1 p}{p I_0(pL)}. \end{aligned} \right\} \quad (4.3)$$

Thus, the temperature is given by

$$T(\rho, \xi) = \frac{2}{\pi} \int_0^\infty \frac{\sin \beta_1 p \cos p\xi}{p} \frac{I_0(pL\rho)}{I_0(pL)} dp. \quad (4.4)$$

Applying the Fourier cosine transform to the remaining equations of (2.36), we have for the transformed equations defining the stress functions

$$\left. \begin{aligned} \nabla_1^2 \tilde{\Omega} - p^2 k_{21}^2 \tilde{\Omega} &= \gamma_4 \tilde{T}; \\ \nabla_1^2 \tilde{\Phi} - p^2 k_{22}^2 \tilde{\Phi} &= -p^2 \frac{\gamma_2}{(1-\nu)} \tilde{\Omega} - \gamma_1 \tilde{T}. \end{aligned} \right\} \quad (4.5)$$

It is an elementary matter to verify that an appropriate form for the general solution of (4.5) is given by

$$\left. \begin{aligned} \tilde{\Omega} &= B(p) I_0(pk_{21}\rho) + \mu_1 \frac{A(p)}{p^2} I_0(pL\rho); \\ \tilde{\Phi} &= D(p) I_0(pk_{22}\rho) + \mu_2 B(p) I_0(pk_{21}\rho) + \mu_3 \frac{A(p)}{p^2} I_0(pL\rho), \end{aligned} \right\} \quad (4.6)$$

where

$$\left. \begin{aligned} \mu_1 &= \gamma_4 / (L^2 - k_{21}^2); \quad \mu_2 = -\gamma_2 / (1 - \nu)(k_{21}^2 - k_{22}^2); \\ \mu_3 &= -\frac{\gamma_2 \gamma_4 + (1 - \nu) \gamma_1 (L^2 - k_{21}^2)}{(1 - \nu)(L^2 - k_{21}^2)(L^2 - k_{22}^2)}, \end{aligned} \right\} \quad (4.7)$$

and where $B(p)$ and $D(p)$ are as yet undetermined functions of p .

The boundary conditions which serve to define $B(p)$ and $D(p)$ are derived from the condition that the exterior surface of the cylinder is traction free, that is,

$$\sigma_{rr} = \sigma_{rz} = 0 \text{ at } \rho = 1. \quad (4.8)$$

If we take the Fourier cosine transform of σ_{rr} and the Fourier sine transform of σ_{rz} , (4.8) implies

$$\tilde{\sigma}_{rr}(1; p) = 0; \quad {}^s\tilde{\sigma}_{rz}(1; p) = 0, \quad (4.9)$$

where we employ a superscript s to denote the sine transform. From (2.38) we then have

$$\left. \begin{aligned} \frac{\partial \tilde{\Phi}}{\partial \rho} &= 0, \quad (\rho = 1); \\ -p^2 \tilde{\Phi} + \frac{\partial \tilde{\Omega}}{\partial \rho} &= 0, \quad (\rho = 1). \end{aligned} \right\} \quad (4.10)$$

Equations (4.6) and (4.10) thus yield

$$B(p) = \frac{A(p)}{p^2 \Delta(p)} B_1(p);$$

$$D(p) = \frac{A(p)}{p^2 \Delta(p)} D_1(p),$$

where

$$\left. \begin{aligned}
 B_1(p) &= -L\mu_1 k_{22} I_1(pk_{22}) I_1(pL) + \mu_3 [k_{22} p I_1(pk_{22}) I_0(pL) \\
 &\quad - L p I_1(pL) I_0(pk_{22})]; \\
 D_1(p) &= L k_{21} (\mu_1 \mu_2 - \mu_3) I_1(pL) I_1(pk_{21}) + \mu_2 \mu_3 [L p I_1(pL) I_0(pk_{21}) \\
 &\quad - k_{21} p I_1(pk_{21}) I_0(pL)]; \\
 \Delta(p) &= k_{21} k_{22} I_1(pk_{21}) I_1(pk_{22}) + \mu_2 [k_{21} p I_1(pk_{21}) I_0(pk_{22}) \\
 &\quad - k_{22} p I_1(pk_{22}) I_0(pk_{21})].
 \end{aligned} \right\} \quad (4.11)$$

From (2.38), (4.6), and (4.11), and from the inversion formula for the appropriate Fourier transforms, we obtain the stresses

$$\begin{aligned}
 \sigma_{zz} &= \frac{2}{\pi} \int_0^\infty \frac{\sin p\beta_1 \cos p\xi}{p\Delta(p) I_0(pL)} [k_{22}^2 D_1(p) I_0(pk_{22}\rho) + k_{21}^2 \mu_2 B_1(p) I_0(pk_{21}\rho) \\
 &\quad + L^2 \mu_3 \Delta(p) I_0(pL\rho)] dp; \quad (4.12)
 \end{aligned}$$

$$\begin{aligned}
 \sigma_{rz} &= \frac{2}{\pi} \int_0^\infty \frac{\sin p\beta_1 \sin p\xi}{p\Delta(p) I_0(pL)} [k_{22} D_1(p) I_1(pk_{22}\rho) + \mu_2 k_{21} B_1(p) I_1(pk_{21}\rho) \\
 &\quad + \mu_3 L \Delta(p) I_1(pL\rho)] dp; \quad (4.13)
 \end{aligned}$$

$$\begin{aligned}
 \sigma_{rr} &= \frac{2}{\pi} \int_0^\infty \frac{\sin p\beta_1 \cos p\xi}{p\Delta(p) I_0(pL)} \left\{ -D_1(p) I_0(pk_{22}\rho) - \mu_2 B_1(p) I_0(pk_{21}\rho) \right. \\
 &\quad - \mu_3 \Delta(p) I_0(pL\rho) + \frac{1}{p\rho} [D_1(p) k_{11}^2 k_{22} I_1(pk_{22}\rho) \\
 &\quad + B_1(p) k_{21} (\mu_2 k_{11}^2 + 1) I_1(pk_{21}\rho) \\
 &\quad \left. + \Delta(p) L (\mu_3 k_{11}^2 + \mu_1) I_1(pL\rho) \right\} dp; \quad (4.14)
 \end{aligned}$$

$$\begin{aligned}
 \sigma_{\theta\theta} &= \frac{2}{\pi} \int_0^\infty \frac{\sin p\beta_1 \cos p\xi}{p\Delta(p) I_0(pL)} \left\{ \nu (\gamma_3 k_{22}^2 - 1) D_1(p) I_0(pk_{22}\rho) \right. \\
 &\quad + \nu \mu_2 (\gamma_3 k_{21}^2 - 1) B_1(p) I_0(pk_{21}\rho) \\
 &\quad \left. + [\nu \mu_3 (\gamma_3 L^2 - 1) - (1 - \nu)] \Delta(p) I_0(pL\rho) \right\} dp; \quad (4.15)
 \end{aligned}$$

Contd.

$$\begin{aligned}
& - \frac{1}{p\rho} [D_1(p) k_{11}^2 k_{22} I_1(pk_{22}\rho) + B_1(p) k_{21}(\mu_2 k_{11}^2 + 1) I_1(pk_{21}\rho) \\
& + \Delta(p) L(\mu_3 k_{11}^2 + \mu_1) I_1(pL\rho)] \} dp. \quad \text{Contd. (4.15)}
\end{aligned}$$

The complexity of the expressions for the stresses, (4.12) through (4.15), makes it impractical to recover the isotropic solution directly from these equations by any straightforward limiting process. For this reason, the isotropic solution to be used for comparison purposes will be derived directly from the isotropic equations (2.40), (2.41), and (2.42). If we denote isotropic results by a left subscript i , it is an elementary matter to repeat the first portion of this section for the isotropic case and thereby obtain

$$\begin{aligned}
& i\tilde{\Omega} = iB(p) I_0(p\rho); \\
& i\tilde{\Phi} = iD(p) I_0(p\rho) + iC(p) \rho I_1(p\rho),
\end{aligned}$$

where

$$\begin{aligned}
iB(p) &= \frac{(1-\nu)}{p i\Delta(p)} iA(p) p[I_0^2(p) - I_1^2(p)]; \\
iD(p) &= \frac{(1-\nu)}{p i\Delta(p)} iA(p) I_0(p) I_1(p); \\
iC(p) &= -\frac{(1-\nu)}{p i\Delta(p)} iA(p) I_1^2(p); \\
iA(p) &= \left(\frac{2}{\pi}\right)^{1/2} \frac{\sin p\beta_1}{pI_0(p)}; \\
i\Delta(p) &= [p^2 + 2(1-\nu)] I_1^2(p) - p^2 I_0^2(p).
\end{aligned} \quad (4.16)$$

Equations (4.16) and (2.42), and the inversion formula for the appropriate Fourier transforms thus yield the isotropic solution

$$\begin{aligned}
i\sigma_{zz} &= \frac{2(1-\nu)}{\pi} \int_0^\infty \frac{\sin \beta_1 p \cos \rho\xi}{p i\Delta(p) I_0(p)} I_1(p) \left\{ pI_0(p) I_0(p\rho) \right. \\
& \quad \left. - I_1(p)[2I_0(p\rho) + p\rho I_1(p\rho)] \right\} dp; \quad (4.17)
\end{aligned}$$

$$i\sigma_{rz} = \frac{2(1-\nu)}{\pi} \int_0^\infty \frac{\sin \beta_1 p \sin \zeta p}{i\Delta(p) I_0(p)} I_1(p) \left\{ I_0(p) I_1(p\rho) - \rho I_1(p) I_0(p\rho) \right\} dp; \quad (4.18)$$

$$i\sigma_{rr} = \frac{2(1-\nu)}{\pi} \int_0^\infty \frac{\sin \beta_1 p \cos \zeta p}{p i\Delta(p) I_0(p)} \left\{ -[pI_0(p) + I_1(p)] I_1(p) I_0(p\rho) + p\rho I_1^2(p) I_1(p\rho) + \frac{1}{\rho} [I_0(p) I_1(p) + pI_0^2(p) - pI_1^2(p)] I_1(p\rho) \right\} dp; \quad (4.19)$$

$$i\sigma_{\theta\theta} = \frac{2(1-\nu)}{\pi} \int_0^\infty \frac{\sin \beta_1 p \cos \zeta p}{p i\Delta(p) I_0(p)} \left\{ I_0(p\rho) [p^2 I_0^2(p) - (p^2 + 1) I_1^2(p)] + \frac{1}{\rho} [pI_1^2(p) - pI_0^2(p) - I_0(p) I_1(p)] I_1(p\rho) \right\} dp. \quad (4.20)$$

The stresses (4.17) through (4.20) could also be obtained from a paper of Youngdahl and Sternberg⁵ by considering only the steady-state portion of their solution. The above equations were rederived and included for completeness of presentation.

V. DISCUSSION OF NUMERICAL RESULTS

Since the purpose of the two problems developed above was to provide an indication of the analytical difficulty associated with their solution, extensive numerical computations were not performed. The desired objectives could be met by evaluating selected stress expressions for a typical transversely isotropic solid and comparing these results with the equivalent isotropic solution. The material chosen was magnesium, since its material properties are such that the derived constants k_{ij}^2 are real and positive, thereby making unnecessary the evaluation of various Bessel functions for complex arguments. The values used for the material constants of magnesium were (in units of 10^{-12} cm²/dyne):

$$s_{11} = 2.21; \quad s_{33} = 1.97;$$

$$s_{12} = -0.77; \quad s_{44} = 6.03,$$

$$s_{13} = -0.49;$$

and the thermal expansion coefficients were assumed equal, as were the conductivities in the two directions (i.e., $\alpha/\beta = 1.0$; $L = 1.0$).

Single crystals of magnesium, while transversely isotropic, are, in a rough sense, "not very anisotropic;" and it is to be expected that the stress distributions closely approximate those for the isotropic material even though their defining equations may differ radically from the isotropic expressions. It should be noted that comparisons between anisotropic and isotropic results plotted on the same graph must be made with care since there is a degree of arbitrariness involved in the definition of the non-dimensional form of the results. For example, in the particular problem under consideration, one could, with equal justification, define the non-dimensional stresses in terms of E , ν , α , or E' , ν' , β , or in terms of suitably averaged constants. Thus, depending on the definition, the plotted results relative to the isotropic solution can change character appreciably.

Since the transversely isotropic solution for the thermal-inclusion problem of Sect. III is expressed in terms of the same tabulated functions occurring in the isotropic case, there is no appreciable difference in numerical difficulty between the two cases and, hence, only one representative set of results has been plotted. Figure 1 shows the stress state along the axis of the inclusion ($\rho = 0$) as expressed in Eqs. (3.16) and (3.17) in the case of magnesium, and Eqs. (3.18) and (3.19) in the isotropic case.

Figures 2 through 5 refer to the cylinder problem of Sect. IV and were obtained from an evaluation of the integrals (4.12) through (4.15) in the case of magnesium, and from integrals (4.17) through (4.20) in the

isotropic case. In all instances the integrals were sufficiently well behaved that their numerical evaluation provided no outstanding problems. The decays of all of the integrands are extremely rapid with large values of the integration variable, and it was not necessary to perform an exact evaluation of the asymptotic "tails" in order to achieve results of an accuracy sufficient for practical application. For example, for the value $\beta_1 = 1.0$ as the length parameter of the unit surface temperature segment, stresses were computed by truncating the integration at $p = 8.0$, using a step length of $\Delta p = 0.1$, and evaluating with a modified Simpson's rule. By examining the decay of the stresses for large ζ and noting when an oscillating pattern begins to develop, one finds that this crude method of evaluation yields 4-place accuracy in the range of ζ of interest (i.e., near $\zeta = \beta_1$). The computation of $\sigma_{\phi\phi}$ at the cylinder surface (see Fig. 4) is the only stress requiring any particular care, and this only amounts to subtracting out the discontinuous portion of the integral before evaluation.

Fig. 1. Half-infinite Cylindrical Thermal Inclusion; Stresses at $\rho = 0$

Fig. 2. Axial Stress at Centerline ($\beta_1 = 1.0$)

Fig. 3
Radial Stress at Centerline
($\beta_1 = 1.0$)

Fig. 4
Circumferential Stress at
Cylinder Surface ($\beta_1 = 1.0$)

Fig. 5
Shear Stress at $\rho = 0.5$
($\beta_1 = 1.0$)

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