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GASEOUS FLOW

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E. L. Hull, Director

GASEOUS FLOW

C. E. Norwand

ABSTRACT

The flow of gases through pipes, orifices, and flares are investigated for conditions of laminar and molecular flow and for the transition region between laminar and molecular flow. Practical equations for the design of vacuum systems are formulated. Formulas for air flow through leaks and capillaries and the sensitivity limitations on bubble methods of leak hunting are also developed.

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27 August 1946

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EDITOR'S NOTE

Since this report involves quantities which are expressed in different dimensions, conversions between dimensions, and constants whose value depend on the dimensions, it was felt advisable in the interests of clarity to use a separate symbol for a quantity when expressed in a different dimension. Thus, e.g., P represents pressure in microns, P' represents pressure in microbars, and P'' represents pressure in lb/in². For convenient reference while reading this report, these symbols have been tabulated in appendix C (p.78) on a double page, so that this page may be unfolded and appear adjacent to the text while the report is being read.

THE FLOW OF GAS

1.1 Introduction

The formulas commonly used in dealing with problems of gaseous flow in vacuum systems are derived, in the main, from hydrodynamic and kinetic theory considerations. In reducing these basic formulas, which are often quite general in their application, to forms readily applicable to the usual vacuum system, constants of the gas involved (usually assumed to be air) have been introduced and combined with other constants; practical working units have been substituted for absolute units; and in some cases approximations have been made. As a result the full and accurate significance of the common working formulas has, in some cases, become obscure, and their application to problems other than those for which they were specifically derived is often involved. To retrace the development of a particular working formula involves a review of hydrodynamic and kinetic theory that is, at best, tedious. For convenient reference in such cases the derivations of certain important working formulas dealing with the flow of gases through tubes, pipes and ducts have been reviewed. The plan has been to start with generalized formulas growing out of theoretical considerations and to carry through to the simplified formulas commonly used. Particular attention has been given to units, constants, and conditions limiting the application of the various relations.

1.2 Modes of Gaseous Flow

Three distinct modes of gaseous flow are recognized:

1.21 Turbulent Flow

Turbulent flow, which occurs only at high pressure differences and high linear velocity, is discussed primarily to show that it seldom occurs in vacuum systems and that neglecting its effect in the solution of vacuum problems is permissible.

1.22 Viscous, or Laminar Flow

Viscous, or laminar flow, occurs at all pressures below turbulence down to the pressure at which the molecular mean free path in the gas is of the order of magnitude of the pipe diameter. Most rough vacuum lines fall within this range.

1.23 Free Molecular Flow

Free molecular flow occurs in the low pressure range where the molecular mean free path is greater than the pipe diameter. Flow through high vacuum lines is usually molecular.

Transitions from turbulent to viscous and from viscous to free molecular flow or vice versa occur gradually over considerable pressure ranges. Methods of dealing with these transitions and the conditions limiting viscous and molecular flow are dealt with.

1.3 Methods of Expressing Rate of Flow

The rate of flow of gas in a vacuum system may be variously expressed:

1.31 Mass Flow Rate (G)

This is defined as the mass of gas flowing past a point in unit time, and is constant under steady flow conditions throughout an un-branched system.

Mass flow rate is usually expressed in g/sec. In Engineering practice, lbs/sec, lbs/min, or lbs/hr may be used.

1.32 Quantity Flow Rate (Q)

This may be defined as the quantity of gas expressed as the product of its pressure by its volume flowing past a point in unit time. It may also be defined as the volume of gas flowing per second measured at unit pressure. Like mass flow rate, Q is constant throughout an un-branched system under steady flow conditions, and at constant temperature.

Quantity flow rate is usually expressed in liter-microns/sec or in cc-microbars/sec.

1.33 Volumetric Flow Rate (B)

This is defined as the volume of gas flowing past any point per unit time measured at the pressure existing at that point. Unlike mass and quantity rates of flow, B varies inversely with the pressure (assuming a perfect gas and constant temperature) throughout a system.

Volumetric flow rate is usually expressed in liters/sec or cc/sec. In Engineering practice ft³/min, gallons/min, or gal/sec may be used.

1.34 Conductance (C)

The conductance of a pipe, duct, or aperture may be defined as the quantity rate of flow per unit pressure drop. Thus defined, conductance to free molecular flow is a function of the pipe dimensions and of the density at unit pressure of the gas flowing. In the case of viscous flow, C is also a function of the average pressure. In either case, however, the reciprocal of the combined conductance of elements in series is equal to the sum of the reciprocals of the conductances of the individual elements.

1.35 Relations Between Conductance and Flow Rates Variously Expressed

If for a particular case an expression can be found for any one of the above quantities, expressions for the remaining quantities are easily derived.

Let G = Mass flow rate (g/sec)
 Q = Quantity flow rate (L-microns/sec)
 B = Volumetric flow rate (L/sec)
 C = Conductance (L/sec)
 P_1 = Inlet pressure (microns)
 P_2 = Discharge pressure (microns)
 P = Pressure at any point (microns)
 ρ_1 = Density of gas at 1 micron pressure (g/L)

Then, by definition:

$$Q = \frac{G}{\rho_1} \quad \text{L microns/sec, or L/sec at 1 micron pressure} \quad (1)$$

$$B = \frac{Q}{P} = \frac{G}{\rho_1 P} \quad \text{L/sec at pressure } P \quad (2)$$

$$C = \frac{Q}{P_1 - P_2} = \frac{PB}{P_1 - P_2} = \frac{G}{\rho_1 (P_1 - P_2)} \quad \text{L/sec} \quad (3)$$

2. TURBULENT FLOW

The laws of turbulent flow of gases and vapors in round pipes have been thoroughly worked out along hydrodynamic lines but are usually neglected as unimportant in vacuum practice. It seems desirable, however, to examine this flow regime if for no other reason than to justify its neglect.

2.1 Condition for Turbulent Flow

Whether flow is turbulent or viscous is dependent on four factors:

- (1) Pipe diameter
- (2) Gas density
- (3) Gas viscosity
- (4) Velocity of flow

These four factors have been combined to give a single dimensionless number called the Reynolds number, R_e , such that

$$R_e = \frac{D' u' \rho''}{\mu_e} \quad (4)$$

if D' is the pipe diameter in feet, u' is the mean velocity of flow in ft/sec, ρ'' is the gas density in lbs/ft³, and μ_e is the absolute viscosity in lbs/ft-sec.

Flow is viscous for $R_e < 1200$ and turbulent for $R_e > 2200$. Between these values flow may be either viscous or turbulent depending on roughness of the pipe and on end disturbances. For ordinary purposes it is advisable to consider flow as turbulent for all values of $R_e > 1200$.

Since for a gas both u and ρ vary with the pressure, these quantities vary from point to point along the pipe. Assuming a perfect gas, however, u varies inversely and ρ varies directly with the pressure so that for a steady state of flow ρu is constant along the length of the pipe and values of ρ and u may be taken at any common point.

For greater convenience in application, Eq. 4 may be written in a number of equivalent forms:

$$R_e = \frac{D' u' \rho''}{\mu_e} = \frac{124 D' u' \rho''}{\mu} = \frac{50.7 Q'' \rho''}{d \mu} = \frac{6.32 W}{d \mu} \quad (4, 4-a, 4-b, 4-c)$$

where d = pipe diameter in inches, μ = viscosity in centipoise, Q'' = mean rate of flow in gal/min, and W = rate of flow in lbs/hr. For air at 20°C (68°F), $\mu = 0.0181$ centipoise and $\rho'' = 0.00513$ p^u.

Equation 4-b may be written

$$R_e = 14.34 \frac{Q^* P^*}{d} \quad (4-d)$$

We may express the above equation in metric units by introducing

$$Q^* = \frac{60 Q}{3.785}$$

where Q = rate of flow in L/sec and

$$P^* = \frac{P}{51.85 \times 10^3}$$

where P = pressure in microns. Then

$$R_e = 4.38 \times 10^{-3} \frac{PQ}{d} \quad (4-e)$$

and $R_e > 1200$ and flow is turbulent if

$$\frac{PQ}{d} > 274 \times 10^3 \quad (5)$$

Since in vacuum practice Q is limited by the volumetric speed of the pump, it is obvious that in pipes of reasonably large diameter turbulent flow will occur only at high pressures such as those prevailing during the initial stages of roughing the system from atmospheric pressure. For example, if a 300 CFM Kinney pump (speed = 141 L/sec or less) is connected to a system by a 4 inch pipe, turbulent flow will occur only at pressures above

$$P = \frac{4 \times 274}{141} 10^3 = 7.76 \times 10^3 \text{ microns} = 7.76 \text{ millimeters}$$

Actually turbulence will cease at a pressure higher than this due to the reduced efficiency and pumping speed of the pump at this lower pressure. In the 10 mm pressure range, the speed of a 300 CFM Kinney pump averages about 110 L/sec. Therefore, a better approximation of the limiting turbulent pressure in this case is

$$P = \frac{4 \times 274}{110} = 10 \text{ mm}$$

and even this value is conservative since $R_e = 1200$ is the minimum value of Reynolds number at which turbulence occurs.

In the roughing down of systems the time required to reduce the pressure from atmospheric to 10 mm pressure (limit of turbulence in above case) is, in general, less than half but still a considerable portion of the total roughing time, and the magnitude of the error introduced by assuming that viscous flow obtains throughout the roughing period depends on the relation of turbulent flow rate to the corresponding extrapolated viscous flow rate.

2.2. Turbulent Flow (Fanning's Formula)¹

According to Fanning's formula,

$$P_1^n - P_2^n = 215 \times 10^{-6} f \frac{\rho^n H^n Q^n^2}{d^5} \quad (6)$$

in which P_1^n = inlet pressure (lbs/in²), P_2^n = discharge pressure (lbs/in²), H^n = pipe length (ft), d = pipe diameter (in), ρ^n = gas density (lbs/ft³), Q^n = mean rate of flow (gal/min), and f = friction factor.

In the realm of turbulent flow this friction factor f is a function of the Reynold's number, the pipe diameter, and the roughness of the pipe surface. Curves are available showing the value of f as a function of R_e for iron pipes of standard sizes and average roughness.

For $R < 1200$, i.e., for viscous flow, the friction factor is a function of the Reynold's number only and may be taken as

$$f = 64/R_e$$

and Eq. 6 reduces to Poisseuille's Law for viscous flow on substituting this value of f and putting

$$R_e = 50.7 \frac{Q^n \rho^n}{d \mu}$$

that is, for viscous flow ($R_e < 1200$)

$$P_1^n - P_2^n = 273 \times 10^{-6} \frac{\mu H^n Q^n}{d^4} \quad (7)$$

Comparing Eqs. 6 and 7 it is easily seen that the pressure drop is greater and the rate of flow less throughout the turbulent flow range than would have been the case had viscous flow persisted at these higher pressures. However, since it is the comparative rates of evacuation under these two conditions that is of primary interest, these have been calculated for a particular extreme case.

1. "Flow of Fluids Through Valves, Fittings, and Pipes,"
Tech Paper No 409, Crane Co., Chicago (May, 1942).

Assume a system evacuated from atmospheric pressure by a 300 CFM Kinney pumping through a 2 inch pipe 16 feet long. Since for air at 20°C (68°F) $\rho = 0.00513 \text{ lb/in}^3$ (where $\bar{P}$ = mean pressure in lb/in^2) and $\mu = 0.0131$ centipoise, these values along with the values of H and d may be substituted in Eqs. 6 and 7, giving

$$P_1 - P_2 = 55.04 \times 10^{-8} f \bar{P} Q^3 \quad \text{For turbulent flow} \quad (6-a)$$

$$P_1 - P_2 = 4.957 \times 10^{-6} Q \quad \text{For viscous flow} \quad (7-a)$$

Expressing flow rate in L/sec and pressures in microns:

$$P_1 - P_2 = 13.82 \times 10^{-5} f Q^2 \quad (6-b)$$

$$P_1 - P_2 = 4.08 Q \quad (7-b)$$

Multiplying each of these equations by $\bar{P}$, the pressure in microns at which Q is measured,

$$(P_1 - P_2) \bar{P} = 13.82 \times 10^{-5} f (Q \bar{P})^2$$

$$(P_1 - P_2) \bar{P} = 4.08 Q \bar{P}$$

But $\bar{P}$ = average pressure = $\frac{1}{2}(P_1 + P_2)$. Therefore

$$(P_1^2 - P_2^2) = 27.64 \times 10^{-5} f (Q \bar{P})^2 \quad (6-c)$$

$$(P_1^2 - P_2^2) = 8.16 Q \bar{P} \quad (7-c)$$

and since for steady flow Q is constant throughout the length of the pipe,

$$Q = P_1 Q_1 = P_2 Q_2 = \dots$$

we may write

$$(P_1^2 - P_2^2) = 27.64 \times 10^{-5} f (P_2 Q_2)^2 \quad \text{for turbulent flow} \quad (6-d)$$

$$(P_1^2 - P_2^2) = 8.16 (P_2 Q_2) \quad \text{for laminar flow} \quad (7-d)$$

Now since P_2 and Q_2 refer to conditions at the pipe discharge, their values are the same as for the pump inlet, and since the speed of the pump is known as a function of its intake pressure we may, for an assumed series of values of P_2 calculate the corresponding values of P_1 from equations 6-d, 7-d, and 7-c. Then, since $P_1 Q_1 = P_2 Q_2$, the speed of pumping from the tank, Q_1 , can be calculated for each assumed condition.

The results of such calculations are shown in figure 1 where pumping speeds in l/sec at intake pressure are plotted as a function of pump intake (pipe discharge) pressure. It will be noted that while the actual speed of pumping is less due to turbulence than is indicated if viscous flow is assumed to persist throughout the pumping range, the difference is 5% or less and is easily negligible in most practical cases. It should also be emphasized that the case here considered is an extremely unfavorable one, the ratio of pipe conductance to pump speed being much less than would normally be used. For a 4 inch to 6 inch pipe such as would normally be used with a pump of the assumed speed not only does viscous flow terminate at higher pressure but the difference between turbulent and viscous conductance becomes vanishingly small.

Figure 1 also shows the assumed variation of the Kinney pump speed with intake pressure, and it is interesting to note that for all pressures greater than 10^4 microns (10 mm) the pressure drop in the line is quite negligible if viscous flow is assumed and amounts to 6% or less in the actual case of turbulent flow. Thus no great error is introduced in the calculation of pump-down time if speed of pumping from the tank is assumed equal to the speed of the pump throughout this pressure range.

To carry the present example to its logical conclusion, one may calculate the time required to reduce the pressure in a tank of given volume (say 12,000 L) from 760 mm to about 100 microns assuming (a) that the speed of pumping is as shown in curve 1-C (figure 1) for turbulent flow and (b) that the speed of pumping is as shown in curve 1-B (figure 1).

Since the time required to reduce the pressure in a chamber of volume V from P_1 to P_2 is

$$t = \frac{V}{B} \ln \frac{P_1}{P_2}$$

where B is the speed of pumping from the system, the time required to reduce the pressure from any value to a value half as great, $P_2 = \frac{1}{2} P_1$, is

$$t = \frac{V}{B} \ln 2 = 0.70 \frac{V}{B}$$

where t = time in sec to reduce pressure by 50%, V = volume exhausted in liters, and B = speed of pumping in liter/sec.

Since B varies with pressure and therefore with time, we may use its average value over successive intervals of pressure halving and sum the resulting values of t. The resulting total times of roughing are 130.1 sec considering viscous flow and 131.9 sec assuming turbulent flow, a difference that is negligible even in this unfavorable case.

Figure 1: Effect of Pipe Resistance on Roughing Speed. (300 CFM Kinney pumping through 2 inch pipe 16 feet long. Kinney pump speed from manufacturer's curve.

Thus it appears that in the design of vacuum systems and the calculation of pump-down times turbulent flow may be dismissed from consideration. This is especially true since the size of connecting piping must usually be chosen much larger than here assumed in order to avoid unduly large pressure drops in the lower viscous flow range (1 mm or less). If the size of pipe is so chosen that the pipe conductance at or below 1 mm pressure is not less than the speed of the pump, conductance at turbulent flow will obviously differ from conductance at assumed viscous flow by an amount that is entirely negligible.

3. LAMINAR FLOW

(FLOW AT MODERATE PRESSURES)

Over a wide range of moderately low pressures the flow of gas through a pipe is laminar or viscous, and the equation of flow under these conditions was derived by Jean Louis Poisseuille from hydronamic considerations.

3.1 Poisseuille's Law - Complete Form

The following derivation of Poisseuille's law follows that given by Kennard. Assume a steady state of laminar flow through a pipe of length H and radius R under an over-all pressure difference $(P_1 - P_2)$.

Figure 2

Assume further that H is so great that any departure from laminar flow near the pipe ends is of negligible importance.

In such a case the velocity u of the gas across any elementary area of cross section is a function of r , the radial distance of this elementary area from the axis of the pipe. Consider the gas inside the cylindrical volume of length dx and radius r coaxial with the pipe. For steady flow the net force due to pressures on the ends of this cylinder must be equal to the total viscous drag on its lateral surface; that is,

$$-\pi r^2 \frac{dP}{dx} dx = -2\pi r dx \cdot \eta \frac{du}{dr} \quad (8)$$

where r = the radius, dx = the length of the elementary volume, $\frac{dP}{dx}$ = the pressure gradient along x , $\frac{du}{dr}$ = the velocity gradient along r , and η = the coefficient of viscosity of the gas.

2. For derivations of Poisseuille's Law refer to: Poisseuille, Compt. Rend., 11, 961, 1041, (1840), 12, 112, (1841); 15, 1161, (1842); Loeb, Leonard B., "The Kinetic Theory of Gases," pp.193-194 and 243-252, McMillan, New York (1927); Kennard, Earle E., "Kinetic Theory of Gases," pp.293-295, McGraw-Hill, New York (1938); O'Brien, Monough P., and G. H. Hickox, "Applied Fluid Mechanics," pp.89-91, McGraw Hill, New York (1937).

Solving 8 for $\frac{du}{d\lambda}$, we get

$$\frac{du}{d\lambda} = \frac{\lambda}{2\eta} \frac{dP'}{dx} \quad (9)$$

which upon integration gives

$$u = \frac{\lambda^2}{4\eta} \frac{dP'}{dx} + K \quad (10)$$

To evaluate K, set $u = u_s$ when $\lambda = R$, then

$$K = u_s - \frac{R^2}{4\eta} \frac{dP'}{dx}$$

where u_s is the velocity of slip relative to the side wall.

Inserting this value of K in Eq. 10:

$$u = \frac{\lambda^2}{4\eta} \frac{dP'}{dx} + u_s - \frac{R^2}{4\eta} \frac{dP'}{dx} = \frac{1}{4\eta} (\lambda^2 - R^2) \frac{dP'}{dx} + u_s$$

but u_s , expressed in terms of the coefficient of slip $\int$ is given by

$$u_s = - \int \frac{du}{d\lambda}$$

and since when $\lambda = R$,

$$\frac{du}{d\lambda} = \frac{R}{2\eta} \frac{dP'}{dx} \quad (\text{from Eq. 9})$$

$$u_s = - \int \frac{R}{2\eta} \frac{dP'}{dx}$$

and

$$u = - \frac{1}{4\eta} (R^2 - \lambda^2 + 2 \int R) \frac{dP'}{dx} \quad (11)$$

Multiplying this velocity of flow u by the elementary annular area $2\pi \lambda d\lambda$ and integrating from $\lambda = 0$ to $\lambda = R$ gives for the volume of gas crossing a cross-section of the pipe per second:

$$B' = 2\pi \int_0^R u \lambda d\lambda = - \frac{2\pi}{4\eta} \left[R^2 \int_0^R \lambda d\lambda - \int_0^R \lambda^3 d\lambda + 2 \int R \lambda d\lambda \right] \frac{dP'}{dx}$$

and

$$B' = - \frac{\pi}{8\eta} R^4 \left(1 + \frac{f}{R} \right) \frac{dP'}{dx} \quad (12)$$

Since the value of U varies with pressure and, therefore, from point to point along the pipe, it is convenient, at this point, to convert the rate of flow from volume per second to quantity of gas passing a cross section per second. For a steady state this quantity is constant along the length of the pipe. This conversion can be made by multiplying Eq. 12 by P' , the pressure at which B' is measured thus giving flow in $P.V$ units. Also, since f is inversely proportional to pressure, it is convenient before integrating to set $f = f_1/P'$. Then f_1 is independent of pressure and, with the above substitutions, Eq. 12 may be integrated along the length of the pipe:

$$P'B' \int_0^H dx = -\frac{\pi}{8\eta} R^4 \left[\int_{P_1'}^{P_2'} P' dP' + 4 \frac{f_1}{R} \int_{P_1'}^{P_2'} dP' \right]$$

$$P'B'H = \frac{\pi}{8\eta} R^4 \left[\frac{P_1'^2 - P_2'^2}{2} + 4 \frac{f_1}{R} (P_1' - P_2') \right]$$

and

$$Q' = P'B' = \frac{\pi}{8\eta} \frac{R^4}{H} \left[\bar{P}' + 4 \frac{f_1}{R} \right] (P_1' - P_2') \quad (13)$$

where Q' = the quantity of gas passing any point per second in $P.V$ units (i.e., the volume of gas per second measured at unit pressure) and $\bar{P}' = \frac{1}{2} (P_1' + P_2')$ is the average pressure in the pipe.

3.2 Poisseuille's Law - Simple Form

If in the above derivation the velocity of slip is assumed to be zero, it follows that f , the coefficient of slip, is zero and Eq. 13 reduces to the simpler form

$$Q' = \frac{\pi}{8\eta} \frac{R^4}{H} \bar{P}' (P_1' - P_2') \quad (14)$$

all symbols having the meaning and units previously assigned.

In justification of the assumption of zero slip it should be noted that (a) slip velocity will differ from zero only to the extent that specular reflection of molecules occurs at the walls, and experiment shows that such reflection occurs to a negligible extent under normal conditions, and that (b) while the extent of specular reflection and, therefore, the coefficient of slip increases with decreasing pressure, the effect becomes appreciable only at pressures such that transition from laminar to molecular flow is taking place, i.e., within a region to which Poisseuille's Law is inapplicable in any case.

Equation 14 is, therefore, the basis of working formulas used in dealing with problems of flow of gas in pipes at moderate pressures.

3.21 Alternative Forms

Now Q' in Eq. 14, is the volume of gas measured at unit pressure passing any point in the pipe per unit time, and is therefore a constant along the length of the pipe when a steady state exists. Therefore, Q' divided by P' , the pressure at any point along the pipe (including the ends) is the volume of gas measured at the pressure P' passing that point per second. That is,

$$B' = \frac{\pi}{8\eta} \frac{R^4}{H} \frac{P'}{P'} (P_1' - P_2') \quad (15)$$

Also, if Q' is multiplied by ρ' , where ρ' is the density of the gas at unit pressure, we have for the mass G of gas passing any point per second

$$G = \frac{\pi}{8\eta} \frac{R^4}{H} \rho' P' (P_1' - P_2')$$

and since $\rho' = \frac{M}{R_M T}$ where M = molecular weight of the gas, R_M = gas constant per mole, and T = absolute temperature:

$$G = \frac{\pi}{8\eta} \frac{M}{R_M T} \frac{R^4}{H} P' (P_1' - P_2') \quad (16)$$

3.22 Units

In Eqs. 14, 15, and 16 c.g.s. units have been assumed throughout, that is, R = pipe radius in centimeters, H = pipe length in centimeters, P_1 , P_2 , P_1' , and P_2' = pressure in microbars (dynes/cm²), η = coefficient of viscosity in poises (dyne-sec/cm²), ρ' = density in g/cm³, Q' = quantity per second in P.V. units, i.e., cc/sec at 1 dyne/cm² pressure or cc-microbars/sec, B' = volume per unit time at pressure P' in cc/sec, and G = mass per unit time in g/sec.

3.23 Values of Constants

If we consider the particular case of the flow of air at normal (20°C) temperature, the factors η and ρ' assume the values:

$$\eta = 1.81 \times 10^{-6} \text{ poise}$$

$$\rho' = \frac{M}{R_M T} = \frac{28.8}{83.15 \times 10^6 \times 293} = 1.182 \times 10^{-9} \text{ g/cc}$$

Substituting these values in Eq. 14, 15, and 16 and combining with other constants, we get for the laminar flow of air:

$$Q' = 2168.5 \frac{d^4}{H} \bar{P} (P_1' - P_2') \quad (14-a)$$

Q' is in cc/sec at 1 microbar pressure.

$$B' = 2168.5 \frac{d^4}{H} \frac{\bar{P}}{P'} (P_1' - P_2') \quad (15-a)$$

B' is in cc/sec at pressure P microbars.

$$G = 2.563 \times 10^{-6} \frac{d^4}{H} \bar{P} (P_1' - P_2') \quad (16-a)$$

G is in g/sec.

3.24. Conversion of Units

In the practical application of the above formulas c.g.s. units are not the most convenient to use, especially where large systems are involved. In such cases the pipe length is likely to be expressed in feet, the pipe radius or its diameter in inches, and pressure in microns or mm of mercury. To make these conversions, substitute:

$$\begin{aligned} H' \text{ (ft)} &\times 12 \times 2.54 && \text{for } H \text{ (cm)} \\ d \text{ (in)} &\times 2.54 && \text{for } d \text{ (cm)} \\ P \text{ (microns)} &\times 1.33 && \text{for } P' \text{ (microbars)} \end{aligned}$$

Where results are in cc/sec (15-a), divide by 1000 to convert to L/sec. and where results are in cc/sec at 1 microbar pressure (14-a), divide by 1000 and by 1.33 to convert to L/sec at 1 micron pressure.

Then:

$$Q = 0.2146 \frac{d^4}{H'} \bar{P} (P_1 - P_2) \text{ L/sec at 1 micron pressure, or } \text{L-microns/sec} \quad (14-b)$$

$$B = 0.2146 \frac{d^4}{H'} \frac{\bar{P}}{P} (P_1 - P_2) \text{ L/sec at } P \text{ microns pressure} \quad (15-b)$$

$$G = 3.88 \times 10^{-7} \frac{d^4}{H'} \bar{P} (P_1 - P_2) \text{ g/sec} \quad (16-b)$$

pressures now being expressed in microns, diameter in inches, and length in feet.

It is in the above form that Poiseuille's Law is most readily applicable to the problems of large vacuum systems.

3.3 Conductance

In the case of free molecular flow of gas through a pipe the quantity of gas passing a point per second is proportional to the pressure drop along the pipe but is independent of the absolute pressure at any point. For this reason the ratio $\frac{Q}{P_1 - P_2}$ is a constant of the pipe and of the gas flowing where Q is the rate of flow in P V units. This has been called the conductance of the pipe for the particular gas.

By analogy, the rate of flow per unit pressure drop, $\frac{Q}{P_1 - P_2}$, is called the conductance of a pipe for viscous flow also. It should be noted, however, that in this case conductance is no longer a constant but a linear function of the average pressure.

Thus, from Eq. 14-b,

$$C = \frac{Q}{P_1 - P_2} = 0.246 \frac{d^4}{H} \bar{P} \quad (17)$$

is the conductance for air of a pipe of length H' feet and diameter d inches.

4. FREE MOLECULAR FLOW

(FLOW AT LOW PRESSURES)

For pressures such that the molecular mean free path in the gas is comparable with the pipe diameter, molecular flow occurs, and the equation of flow under these conditions can be derived from Kinetic theory consideration. Knudsen³ based such a derivation on the equating of rate of transfer of momentum to the side walls to the force arising from pressure difference at the two ends of the pipe. A little later Smoluchowski⁴ derived the same relation by considering the number of molecules passing a cross section of the pipe per unit time⁵.

In both derivations the following conditions are stated or implied:

- (a) The length of the pipe is great as compared with its diameter and end effects are negligible.
- (b) The pressure is such that the molecular mean free path is equal to or greater than the pipe diameter, i.e., impacts between molecules are rare as compared with impacts with the side wall.
- (c) The pressure gradient along the length of the pipe is small.
- (d) The average axial component of velocity of molecules reflected from the side wall is zero, i.e., molecular reflection from the side wall is in no sense directional.

Following Knudsen's reasoning, Loeb writes for G , the mass of gas flowing past a cross section of the pipe per second,

$$G = -\frac{8}{3} \sqrt{\frac{2}{\pi}} \sqrt{\frac{\rho'}{P'}} \frac{A^2}{0} \frac{dP'}{dH} \quad (18)$$

where G = mass of gas flowing (g/sec), A = cross section area (cm²), ρ' = density of gas (g/cc), P' = pressure (microbars), and dP'/dH = pressure gradient (dynes/cm²/cm).

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3. Knudsen, Ann. Physik, 29, 75 (1909)
 4. Smoluchowski, Ann. Physik, 33, 1559 (1910)
 5. Full treatments of these two derivations may be found in Loeb, "The Kinetic Theory of Gases," pp. 252-256, and Kennard, "Kinetic Theory of Gases", pp. 302-306.

Since for a uniform pipe

$$\frac{dP'}{dH} = - \frac{P'_1 - P'_2}{H}$$

and since $\frac{P'}{P'_1} = \rho'_1$, equation (13) becomes

$$Q = \frac{8}{3} \sqrt{\frac{2}{\pi}} \sqrt{P'_1} \frac{A^2}{0} \frac{P'_1 - P'_2}{H} \quad (19)$$

Dividing by P'_1

$$Q' = \frac{8}{3} \sqrt{\frac{2}{\pi}} \frac{1}{\sqrt{P'_1}} \frac{A^2}{0} \frac{P'_1 - P'_2}{H} \quad (20)$$

where Q' = the volume of gas at unit pressure flowing past a cross section of pipe per second.

Applying the reasoning of Smoluchowski, Kennard derives the relation

$$Q' = \frac{1}{2\sqrt{2\pi}} \frac{1}{\sqrt{P'_1}} \left[\int dS \int_0^{2\pi} dq \right] \frac{P'_1 - P'_2}{H} \quad (21)$$

where

dS is an element of area in the pipe cross section,
 r is the length of the line drawn from dS to the pipe wall in plane of the section,
 dQ is the angle formed by s and an arbitrary reference line in this plane.

A comparison of equations (20) and (21) and the conditions of their derivation shows that equation (20) is strictly applicable only to flow through pipes of circular cross section, but equation (21) is applicable to pipes of any non-reentrant section. Also, because of the integrals left standing in equation (21), it is obvious that equation (20) is much more easily solved.

4.1 Flow Through Uniform Circular Pipe

For the particular case of a pipe of uniform circular cross section

$$\frac{A^2}{0} = \frac{\pi^2 R^4}{2\pi R} = \frac{\pi R^3}{2}$$

and equation (20) becomes

$$Q' = \frac{4}{3} \sqrt{2\pi} \frac{1}{\sqrt{P_1'}} R^3 \frac{P_1' - P_2'}{H} \quad (20')$$

Also, for this case,

$$\int_0^{2\pi} \int_0^s ds dQ = \frac{16\pi R^3}{3}$$

and equation (21) likewise becomes

$$Q' = \frac{4}{3} \sqrt{2\pi} \frac{1}{\sqrt{P_1'}} R^3 \frac{P_1' - P_2'}{H} \quad (21')$$

It thus appears that the equations of Knudsen and Smoluchowski are identical when applied to pipes of circular cross section.

Multiplying equation (21') by ρ_1' , the density of the gas at unit pressure, one obtains for the mass rate of flow through round pipes

$$G = \frac{4}{3} \sqrt{2\pi} \sqrt{\rho_1'} R^3 \frac{P_1' - P_2'}{H} \quad (22)$$

Dividing equation (21') by P' , the pressure at any point, one gets for B' , the volume of gas at pressure P' flowing past that point per second,

$$B' = \frac{4}{3} \sqrt{2\pi} \frac{1}{\sqrt{P_1'}} \frac{R^3}{P'} \frac{P_1' - P_2'}{H} \quad (23)$$

It will be noted that equations (21'), (22) and (23) are equally applicable to problems of molecular flow through uniform pipes of circular cross section and the equations differ one from the other only in the units adopted for expressing rate of flow.

If Eq. 20 is written in the form

$$Q' = \frac{P_1' - P_2'}{\sqrt{P_1'} \left(\frac{3}{4} \frac{1}{\sqrt{2\pi}} \frac{H}{R^3} \right)}$$

It is obvious that Q' , the quantity of gas in P.V. units (or the volume of gas at unit pressure) flowing through the pipe per second, is:

- (a) Proportional to the pressure difference between the ends of the pipe,
- (b) Inversely proportional to a factor whose value depends on ρ_1 , the density of the gas at unit pressure, and on H and R , the dimensions of the pipe.

Knudsen called this latter factor the "resistance", E , of the pipe

$$E = \sqrt{\rho_1} \left(\frac{3}{4} \sqrt{\frac{1}{2\pi}} \frac{H}{R^3} \right) \quad (24)$$

where E is a constant of the pipe for a given gas but assumes different values for different gases.

Langmuir, choosing to define "resistance" as a function of pipe dimensions only, considered the pipe resistance as being

$$E' = \left(\frac{3}{4} \sqrt{\frac{1}{2\pi}} \frac{H}{R^3} \right) \quad (25)$$

Both Knudsen and Langmuir called the reciprocal of resistance the "conductance" of the pipe. Thus according to Knudsen

$$C' = \frac{1}{E} = \frac{1}{\sqrt{\rho_1}} \left(\frac{4}{3} \sqrt{2\pi} \frac{R^3}{H} \right) = \frac{Q'}{P_1 - P_2} \quad (24')$$

and according to Langmuir,

$$C'' = \frac{1}{E'} = \left(\frac{4}{3} \sqrt{2\pi} \frac{R^3}{H} \right) = \sqrt{\rho_1} \frac{Q'}{P_1 - P_2} \quad (25')$$

Langmuir also proposed that the quantity

$$C' = \frac{C''}{\sqrt{\rho_1}}$$

be called the "admittance" of the pipe for a particular gas.

Subsequent writers appear to be about equally divided in their interpretations of the terms "resistance" and "conductance". This is unfortunate but leads to no serious difficulty once the particular sense in which the terms are being used has been ascertained. In the application of flow formulas to specific problems a particular gas is usually assumed and $\sqrt{\rho_1}$, the density factor, combines with other constants of the working equation. Knudsen's definition of conductance is, therefore, more readily applicable to specific problems, and will be used unless otherwise indicated.

Conductance may then be defined as the quantity of a given gas, in P.V. units, flowing per second under unit driving pressure difference.

4.12 Evaluation of Constants: Let us now return to and further develop the flow formulas by the evaluation and combination of constants.

Consider particularly the flow of air at room temperature (20°C). Since

$$P_1 = \frac{M}{R_u T}$$

where M = molecular weight (28.8 for air), T = absolute temperature (293° K for 20°C), and R_u = gas constant per mole (83.15×10^6), therefore

$$\sqrt{P_1} = 3.44 \times 10^{-5}$$

Inserting this value and combining constants in Eqs. 22, 21', and 23 one gets

$$G = 11.5 \times 10^{-5} \frac{R^3}{H} (P_1 - P_2) \quad (22-a)$$

$$Q = 0.973 \times 10^5 \frac{R^3}{H} (P_1 - P_2) \quad (21'-a)$$

$$B = 0.973 \times 10^5 \frac{R^3}{H} \frac{1}{P_1} (P_1 - P_2) \quad (23-a)$$

4.13 Conversion of Units: Throughout the derivation of these equations C.G.S. units have been used. For application to the problems of large scale vacuum systems however, other units are more convenient. The following conversions are advised:

For R (cm) write $\frac{1}{2}d \times 2.54$ (d in inches)

For H (cm) write $H' \times 2.54 \times 12$ (H' in feet)

For P , P_1 , and P_2 (microbars) write P , P_1 , $P_2 \times 1.33$ (P in microns)

Divide by 1000 to reduce results from cc/sec to L/sec, and by 1000×1.33 to reduce cc-microbars/sec to L- μ /sec.

Then:

$$G = 1.03 \times 10^{-5} \frac{d^3}{H'} (P_1 - P_2) \quad (22-b)$$

$$Q = 6.54 \frac{d^3}{H'} (P_1 - P_2) \quad (21'-b)$$

$$B = 6.54 \frac{d^3}{H'} (P_1 - P_2) \frac{1}{P} \quad (23-b)$$

Where d = pipe diameter in inches, H' = pipe length in feet, P , P_1 and P_2 = pressures in microns, G = mass of gas flowing in g/sec, Q = rate of flow in liter-microns/sec = rate of flow in L/sec at 1 micron pressure, and B = rate of flow in L/sec at P microns pressure.

While Eqs. 22-b, 21'-b, and 23-b and the preceding equations from which they were derived express in various units the rate of flow of gas through a given pipe as a function of the driving pressure difference, it should be noted that a steady state of flow can exist only when some suitable pump is operating at the discharge end of the pipe, and interest is usually centered not in the rate of flow in the pipe but in the rate of exhaustion of some system or chamber attached to the pipe inlet. In other words, the problem of gas flow through pipes assumes importance when the pipe becomes a part of an operating vacuum system, and in such cases it is not the conditions in the pipe but the resulting conditions at the pump and in the region being evacuated that are of primary concern.

Consider the simple system shown in figure 3. A chamber or tank T is evacuated by a diffusion pump P through a pipe of length H' and diameter d . Let P_1 be the pressure throughout the tank and, therefore, at the pipe inlet, and let P_2 be the pressure at the head of the diffusion pump and at the discharge end of the pipe. Further, let B be the speed of the pump expressed in liters/sec measured at the pump head pressure P_2 . When a steady state exists P_1 and P_2 are constant and the quantity of air leaving the tank and entering the pipe per second is equal to the quantity leaving the pipe and removed by the pump. That is

$$P_1 B_1 = P_2 B \quad (26)$$

where B_1 = L/sec at pressure P_1 entering the pipe and is given by Eq. 21-b if $P = P_1$. Thus

$$B_1 = 6.54 \frac{d^3}{H'} \left(1 - \frac{P_2}{P_1}\right) \quad (27)$$

Figure 3.

and since

$$\frac{P_2}{P_1} = \frac{B_1}{B} \text{ by Eq. 26}$$

$$B_1 = 6.54 \frac{d^3}{H^2} (1 - \frac{B_1}{B}) \quad (28)$$

If the speed of the pump B is infinite (or very great compared with B_1) Eq. 28 reduces to

$$B_1 = C = 6.54 \frac{d^3}{H^2} \text{ L/sec} \quad (29)$$

where the right hand side of the equation is identically equal to the conductance of the pipe for air as defined by Knudsen, Eq. 24, and conductance of a pipe for a given gas may now be further defined as the volume of gas at inlet pressure entering the pipe per second when the discharge pressure is maintained at zero.

Usually, however, the pump speed is not very great as compared with the conductance of the connecting pipe. In that case combining Eq. 28 and 29 and solving for B_1 , the volume of air at tank pressure removed from the tank per second is given by

$$B_1 = C (1 - \frac{B_1}{B})$$

and

$$\frac{1}{B_1} = \frac{1}{B} + \frac{1}{C} \quad (30)$$

If a vacuum line consists of a number of sections having various lengths and diameters connected in series. The conductance of this composite line may be defined as the volume of gas entering the pipe per second when the discharge pressure is zero. If C_1, C_2, C_3 , etc., are the conductances of the various sections beginning with the discharge section, and if B_1, B_2, B_3 , etc., are the intake speeds of the respective sections then, when the discharge pressure is zero,

$$\frac{1}{B_1} = \frac{1}{C_1}$$

But B_1 is the speed at which gas is removed from the discharge of section two, so that by Eq. 30

$$\frac{1}{B_2} = \frac{1}{B_1} + \frac{1}{C_2} = \frac{1}{C_1} + \frac{1}{C_2}$$

Likewise B_2 is the speed of removal of gas from section three, so that

$$\frac{1}{B_3} = \frac{1}{B_2} + \frac{1}{C_3} = \frac{1}{C_1} + \frac{1}{C_2} + \frac{1}{C_3}$$

Repeating for any number of sections, and remembering the B_n is by definition the conductance of the composite line, we may write

$$\frac{1}{C} = \frac{1}{C_1} + \frac{1}{C_2} + \frac{1}{C_3} + \dots + \frac{1}{C_n} \quad (31)$$

4.2 Uniform pipes and ducts of non-circular section.

In dealing with problems of molecular flow through long, uniform pipes or ducts of non-circular section two approaches are possible:

- (1) Knudsen's Equation,

$$Q' = \frac{8}{3} \sqrt{\frac{2}{\pi}} \frac{1}{\sqrt{P_1}} \frac{A^2}{H} \frac{P_1 - P_2}{H} \quad (20)$$

In which the rate of flow is expressed as a function of the area and perimeter of a cross section, is easily applied to any section of regular geometric shape. The result, however, for sections other than circular is subject to question as a result of the method of derivation of the equation.

- (2) Smoluchowski's Equation,

$$Q' = \frac{1}{2\sqrt{2\pi}} \frac{1}{\sqrt{P_1}} \left[\int ds \int_0^M dq \right] \frac{P_1 - P_2}{H} \quad (21)$$

is applicable to ducts of any non-reentrant section, but evaluation of the integrals left standing is difficult, if not impossible, for all but the simplest geometric sections.

C. H. Bosanquet⁽¹⁾ has developed a relation essentially identical to that of Smoluchowski in which the unevaluated integrals are replaced by a term in $\bar{S}$, where $\bar{S}$ is the average distance of all points in the cross section from the duct wall, and can be evaluated for a number of forms of section by averaging a number of integrals.

- (1) A.S.D. Barrett and C.H. Bosanquet; Resistance of Ducts to Molecular Flow; BR-296.

This Equation takes the form

$$Q' = \sqrt{\frac{\pi}{2}} \frac{1}{\sqrt{P_1'}} \bar{A} \bar{S} \frac{P_1' - P_2'}{H} \quad (32)$$

By this method he determines the rate of flow through ducts having various rectangular, annular, and triangular cross sections, and calculates, for each section considered, a factor K by which Knudsen's equation must be multiplied to bring it into agreement with his results.

Sections considered, and the corresponding values of K are listed in the following table.

Table 1
Values of K for Particular Sections

Rectangular Section Sides X and Y		Annular Section Radii R_1 and R_2		Equilateral Triangular Section
Y/X	K	R_1/R_2	K	K
1	1.108	0 (circle)	1.000	1.24
1.5	1.126	.259	1.072	
2	1.151	.500	1.154	
3	1.198	.707	1.254	
5	1.297	.866	1.430	
8	1.400	.966	1.675	
10	1.444	1.00	∞	

From the above results it is apparent that the rate of flow through non-circular ducts is consistently greater than indicated by Knudsen's equation (Eq. 20), and that one error of this equation becomes progressively greater as the section departs from circular.

Knudsen's equation is correct to within 15% or better for rectangular ducts having a section length to width ratio of 2 or less, and for annular ducts having a ratio of outer to inner radius of 2 or less. For many practical problems this accuracy is adequate, and Eq. 20 may be applied. Where higher ratios are involved, or where a higher degree of accuracy is required we may substitute for Eq. 20

$$Q' = K \frac{8}{3} \sqrt{\frac{2}{\pi}} \frac{1}{\sqrt{P_1'}} \frac{A^2}{0} \frac{P_1' - P_2'}{H} \quad (33)$$

The value of K appropriate to the particular duct cross section is to be taken directly, or derived by interpolation, from Table 1.

All formulas derived from Knudsen's equation for the particular case of round pipes may likewise be applied to ducts of non-circular section by the introduction of this factor K . Where the pipe radius, R , occurs in these formulas as a result of substituting for $A^2/\bar{O}$ the equivalent

$\frac{(\pi R^2)^2}{2\pi R} = \frac{\pi R^3}{2}$, it is necessary to substitute for the "equivalent radius" of the non-circular section

$$R^3 = \frac{2A^2}{\pi \bar{O}} \quad (34)$$

Thus the method of dealing with problems of molecular flow through long, uniform ducts of non-circular section may be summarized:

- (1) Treat the duct as a round pipe of equivalent radius such that $R^3 = \frac{2A^2}{\pi \bar{O}}$.
- (2) If the section is far from circular, or if a high degree of accuracy is required, multiply the result by the appropriate value of K from Table 1.

It should be noted, however, that for ducts having a cross section of high length-to-width ratio, the criterion for free molecular flow as stated in terms of the ratio of molecular mean free path to pipe diameter breaks down, and there may be a considerable pressure range over which flow would be molecular in so far as the width of the duct is concerned but laminar if the length of cross section is considered.

4.3 Non-Uniform Pipes and Ducts

Problems of flow through pipes or ducts of non-uniform cross section can be solved provided it is possible to express their area A and perimeter $\bar{O}$ as functions of the length h and to integrate Eq. 18 along the whole length of the pipe. Thus Eq. 18 is to be replaced by

$$Q \int_0^H \frac{\bar{O}}{A^2} dh = - \frac{8}{3} \sqrt{\frac{2}{\pi}} \sqrt{P_1} \int_{P_1}^{P_2} dP' \quad (35)$$

and

$$Q = \frac{\frac{8}{3} \sqrt{\frac{2}{\pi}} \sqrt{P_1} (P_1 - P_2)}{\int_0^H \frac{\bar{O}}{A^2} dh}$$

In the following section,

$$\int_0^H \frac{0}{A^2} dh$$

has been evaluated for a number of simple geometric shapes.

4.31 Circular Pipe With Uniform Flare or Taper: Consider a pipe of circular section and length H flaring uniformly and symmetrically from radius a at $h = 0$ to radius b at $h = H$. Then, at any point h on the axis, the radius r is given by

$$r = a + \frac{h}{H} (b-a)$$

and at this point

$$O = 2\pi r = 2\pi \left(a + \frac{h}{H} (b-a) \right)$$

$$A = \pi r^2 = \pi \left(a + \frac{h}{H} (b-a) \right)^2$$

and

$$\frac{O}{A^2} = \frac{2}{\pi} \frac{1}{\left(a + \frac{h}{H} (b-a) \right)^3}$$

Therefore, the integral

$$\begin{aligned} \int_0^H \frac{O}{A^2} dh &= \frac{2}{\pi} \int_0^H \frac{dh}{\left(a + \frac{b-a}{H} h \right)^3} = - \frac{2}{\pi} \left[\frac{1}{2 \frac{b-a}{H} \left(a + \frac{b-a}{H} h \right)^2} \right]_0^H \\ &= - \frac{2}{\pi} \frac{H}{2(b-a)} \left[\frac{1}{b^2} - \frac{1}{a^2} \right] = - \frac{H}{\pi(b-a)} \left[\frac{a^2 - b^2}{a^2 b^2} \right] \\ &= \frac{H}{\pi} \left(\frac{a+b}{a^2 b^2} \right) \end{aligned} \quad (36)$$

but for a uniform circular pipe of radius R .

$$\int_0^H \frac{O}{A^2} dh = \frac{2\pi R}{\pi^2 R^4} H = \frac{2H}{\pi R^3} \quad (37)$$

Therefore the flared pipe is equivalent to a uniform pipe of radius R such that

$$\frac{H}{\pi} \left(\frac{a+b}{2} \right) = \frac{2H}{\pi R}$$

and

$$R^3 = 2 \frac{a^2 b^2}{a+b} \quad (38)$$

or of diameter such that

$$D^3 = 16 \frac{a^2 b^2}{a+b} \quad (39)$$

Substituting these values of R^3 or D^3 in any of the preceding formulas for round uniform pipes gives a correct expression for conductance or flow through a flared or tapered round pipe.

It should be noted that the equivalent radius of a flared pipe as derived by integration may differ significantly from the values yielded by various methods of averaging the dimensions of the ends of the pipe. Several possible methods of averaging and the resulting values of equivalent radius are listed briefly:

(a) By integration:

$$R_1^3 = \frac{a^2 b^2}{(a+b)} \quad (40)$$

(b) Assuming average radius $\frac{1}{2}(a+b)$ to be the equivalent radius:

$$R_2^3 = \frac{\left(\frac{a+b}{2} \right)^4}{\left(\frac{a+b}{2} \right)} \quad (41)$$

(c) Assuming that the average circumference $C = \frac{2\pi(a+b)}{2}$

and $A = \frac{\pi(a^2+b^2)}{2}$ the average of areas of ends:

$$R_3^3 = \frac{\left(\frac{a^2+b^2}{2} \right)^2}{\left(\frac{a+b}{2} \right)} \quad (42)$$

(d) Assuming $0 = \frac{2\pi(a+b)}{2}$, the average circumference, and $A^2 = \frac{\pi^2(a^4+b^4)}{2}$, the average of the squares of the ends:

$$R_4^3 = \frac{\left(\frac{a^4+b^4}{2}\right)}{\left(\frac{a+b}{2}\right)} \quad (43)$$

Since the above equations 40 to 43 differ only in their numerators, it is sufficient to compare these. Obviously, all are identical if $b = a$ and will differ little if $(b-a)$ is small compared with a , i.e., if the flare of the pipe is slight compared with its radius. If this is not the case, however,

$$a^2b^2 < \left(\frac{a+b}{2}\right)^4 < \left(\frac{a^2+b^2}{2}\right)^2 < \frac{a^4+b^4}{2}$$

That is,

$$R_1^3 < R_2^3 < R_3^3 < R_4^3$$

as illustrated in table 2.

Table 2

Variation of Equivalent Radius With Methods of Averaging

<u>a</u>	<u>b</u>	<u>R_1^3</u>	<u>R_2^3</u>	<u>R_3^3</u>	<u>R_4^3</u>
3	4	144	149	156	168.5
3	6	324	410	506	688.5
5	6	900	915	930	960

The last two approximations (R_3^3 and R_4^3) are poorer than the first and should not be used. The first approximation, in which the flared pipe is considered as equivalent to a uniform pipe having a radius equal to the average radius of the flared pipe, is easily applied and is good when the ratio of flare to radius is small (error less than 2% for $(b-a)/a < 0.2$), but poor when the ratio is large (error greater than 26% for $(b-a)/a > 1$). If $b < 1.5 a$, $R = (a+b)/2$ to within less than 8%.

4.32 Square Ducts With Uniform Flare: The case of a square duct which flares linearly and symmetrically from a section $a \times a$ at $h = 0$ to $b \times b$ at $h = H$ parallels closely that of a flared pipe.

At a distance h from the small end of the duct the cross-section has a perimeter

$$O = 4 \left(a + \frac{b-a}{H} h \right)$$

and an area

$$A = \left(a + \frac{b-a}{H} h \right)^2$$

Therefore

$$\int_0^H \frac{Q}{A^2} dh = 4 \int_0^H \frac{dh}{\left(a + \frac{b-a}{H} h \right)^3} = 2H \frac{a+b}{a^2 b^2} \quad (44)$$

and solving for the equivalent radius by setting

$$2H \frac{a+b}{a^2 b^2} = \frac{2H}{\pi R^3}$$

$$R^3 = \frac{1}{\pi} \frac{a^2 b^2}{a+b} \quad (45)$$

and

$$D^3 = \frac{4}{\pi} \frac{a^2 b^2}{a+b} \quad (46)$$

By substituting these values of R^3 or D^3 in the various formulas for a uniform round pipe, the conductance or rate of flow through a flared square duct may be determined.

If one considered the flared duct as equivalent to a uniform duct having a cross sectional area equal to the average of the flared duct, Eq. 34 gives for the equivalent radius

$$R^3 = \frac{2a^2}{\pi O} = \frac{2c^4}{4\pi o} = \frac{1}{\pi} \frac{c^4}{2o}$$

where o is the length along the edge of the average cross section.

Therefore,

$$c = \frac{1}{2} (a+b)$$

and

$$R_{0.3} = \frac{1}{\eta} \frac{\left(\frac{a+b}{2}\right)^4}{(a+b)} \quad (47)$$

Comparing Eqs. 47 and 45, it is obvious that the error introduced by such an approximation is identically the same in this case as in the case of a flared, round pipe.

The results of evaluation of the integral

$$\int_0^H \frac{0}{A^2} dh$$

for the two examples treated and for additional special shapes are given in table 2.

From the results shown it is obvious that use of these equivalent radii in the solution of flow problems is possible but tedious. Also, it is apparent that treating the flared pipe or duct as equivalent to one of uniform cross section equal to the cross section at mid-length is sufficient to give a good approximation if the flare is not excessive. Even in the case of widely flaring members the use of the average cross section introduces no greater error than is likely to arise from other causes. This is particularly true since it is quite unlikely that a widely flared pipe or duct will meet the conditions imposed by the long pipe formula (length great relative to diameter). Indeed, all flared members are likely to fall into the short pipe category. Since the formulas applied to short pipes are derived by application of a suitable end correction to the long pipe formulas, however, the relations given above can be corrected for end effects and can thus be applied to the case of short pipes and ducts as will be shown in section 7.

All equations in this section (4.32) are subject to the same corrective factor K applicable to uniform ducts.

Shape	$\int_0^L \frac{Q}{A} dl$	(Equivalent Radius)
1	$\frac{2 L}{\pi R^3}$	
2	$\frac{4 L}{\pi a^3}$	$\bar{R}^3 = \frac{1}{\pi} \frac{a^3}{2}$
3	$\frac{2 L}{a^2 b^2} (a + b)$	$\bar{R}^3 = \frac{1}{\pi} \frac{a^2 b^2}{(a+b)}$
4	$\frac{L}{\pi} \frac{a+b}{a^2 b^2}$	$\bar{R}^3 = 2 \frac{a^2 b^2}{(a+b)}$
5	$\frac{2 L}{a' a^2} \left(a + \frac{a a'}{(a' - a)} \ln \frac{a'}{a} \right)$	
6	$\frac{2 L}{a^2 b b'} \left(a + \frac{b b'}{b' - b} \ln \frac{b'}{b} \right)$	
7	$\frac{2 L}{a^2 a'^2} (a' + a)$	$\bar{R}^3 = \frac{1}{\pi} \frac{a^2 a'^2}{a + a'}$
8	$\frac{2 L}{a^2 b b'} \left(a + \frac{b' b}{b' - b} \ln \frac{b'}{b} \right)$	
9	$\frac{2 L}{a' b - a b'} \left(\frac{a' - a}{a a'} \frac{b' - b}{b b'} + \frac{(a' - a) - (b' - b)}{a' b - a b'} \ln \frac{a' b}{a b'} \right)$	

Table 2
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5. GENERAL EQUATION OF GASEOUS FLOW

(TRANSITION FROM LAMINAR TO FREE MOLECULAR FLOW)

5.1 Basis of Derivation

The general form of Poiseuille's Law gives for the quantity of gas in p.v. units flowing per second through a long pipe of radius R

$$Q' = \frac{\pi R^4}{8\eta H} \left(\bar{P} + \frac{4}{3} \frac{\eta}{R} \right) (P_1' - P_2') \quad (13)$$

while Knudsen's formula gives:

$$Q' = \frac{4}{3} \sqrt{2\pi} \frac{1}{\sqrt{P_1}} \frac{R^3}{H} (P_1' - P_2') \quad (20)$$

As was pointed out, Poiseuille's Law is applicable at pressures high enough that the molecular mean free path is small compared with the pipe diameter and Knudsen's equation applies at pressures low enough that the mean free path is large compared with the diameter. Between the pressure ranges covered by these two formulas there thus exists a pressure range to which neither formula applies. From measurements made within this pressure range Knudsen concluded that the observed rates of flow could be satisfactorily represented by an equation of the form

$$Q' = \left(a\bar{P}' + b \frac{1 + c_1 \bar{P}'}{1 + c_2 \bar{P}'} \right) (P_1' - P_2') \quad (49)$$

where $\bar{P}'$ = average pressure and where a, b, c_1 , and c_2 are constants.

It then remains to so evaluate these constants that Eq. 49 will (a) reduce to Knudsen's equation at low pressures (b) reduce to Poiseuille's Law at high pressures, and (c) agree with experimental results at pressures between these limits.

These constants have been evaluated, in part, from more or less rigorous Kinetic theory considerations. To the extent that this is true, Eq. 49 is no longer empirical. Since, however, complete and rigorous evaluation by this method is not possible, the resulting equation must be considered semi-empirical in nature.

Substituting values of the constants thus determined, Eq. 49 takes the form:

$$Q' = \frac{\pi R^4}{8\eta H} \bar{P}' (P_1' - P_2') + \frac{4}{3} \sqrt{2\pi} \frac{1}{\sqrt{P_1}} \frac{R^3}{H} \left(\frac{1 + 2.00 \frac{\sqrt{P_1}}{\eta} R \bar{P}'}{1 + 2.47 \frac{\sqrt{P_1}}{\eta} R \bar{P}'} \right) (P_1' - P_2') \quad (50)$$

5.2 Equivalence to Poiseuille's and Knudsen's Formulas at High and Low Pressures Respectively

Note that as F' approaches zero the first term in the above equation approaches zero; the term in parentheses approaches unity: and Eq. 50 becomes Knudsen's formula.

As F' becomes large the term in parentheses approaches a value $2.00/2.47 = 0.81$.

Factoring out $\frac{\pi R^4}{8\eta H} (P_1 - P_2)$, Eq. 50 becomes

$$Q' = \frac{\pi R^4}{8\eta H} \left[F' + 4 \frac{0.81 \frac{8}{3} \sqrt{\frac{2}{\pi}} \frac{\eta}{\sqrt{P_1}}}{R} \right] (P_1 - P_2) \quad (51)$$

This is the same as Poiseuille's Law if

$$0.81 \frac{8}{3} \sqrt{\frac{2}{\pi}} \frac{\eta}{\sqrt{P_1}} = \int_1$$

where $\int_1$ is the coefficient of slip at unit pressure.

Maxwell showed that $\int_1$ could be expressed as

$$\int_1 = \sqrt{\frac{\pi}{2}} \frac{\eta}{\sqrt{P_1}} \left(\frac{2-F}{F} \right)$$

where F is that fraction of the molecules striking the side wall that is absorbed and re-emitted, and $(1-F)$ is the fraction specularly reflected. F is, therefore, a function of the wall surface and of the gas. Experimentally determined values of F vary from 1.00 for He and H_2 in glass to 0.8 or less for air in tubes having oil coated surfaces.

Combining the last two equations, and solving for F gives

$$\frac{2-F}{F} = 0.81 \frac{8}{3} \frac{2}{\pi} = 1.375$$

and

$$F = 0.812$$

Eq. 51 is then equivalent to Poisseuille's Law for a system in which the coefficient of slip has this somewhat conservative value.

At very high average pressures the slip term becomes negligible compared with $\bar{P}'$ and Eq. 51 is identical with Poisseuille's Law for this case.

5.3 Evaluation of Constants for the Case of Air

Let us next evaluate and combine the constants in Eq. 50 for the particular case of air at 20°C. If at the same time R is replaced by $D/2$, the resulting equation is more directly applicable to practical problems.

For air at 20°C

$$\sqrt{P_1} = \sqrt{\frac{\eta}{R_g T}} = 34.36 \times 10^{-6} \text{ g/cc}$$

and

$$\eta = 181 \times 10^{-6} \text{ poises}$$

Eq. 50 then becomes

$$Q' = 135.5 \frac{D^4}{H} \bar{P}' (P_1' - P_2') + 12.150 \frac{D^3}{H} \frac{(1 + 0.1895 \frac{D \bar{P}'}{1 + 0.234 \frac{D \bar{P}'}{1}})}{(1 + 0.234 \frac{D \bar{P}'}{1})} \frac{(P_1' - P_2')}{1} \quad (52)$$

where Q' = cc of gas at one dyne/cm² pressure flowing past any point in the pipe per second, D = diameter of the pipe in cm, H = length of the pipe in cm, $\bar{P}'$ = average pressure, P_1' = inlet pressure, and P_2' discharge pressure (all pressures in dynes/cm²). Note that the constants appearing in Eq. 52 are functions of the viscosity and density and, therefore, of the temperature of the gas flowing. It is for this reason that the above equation is sometimes written with constants differing slightly from those given here.

Since it is more convenient in application to express the diameter in inches, the length in feet, and the pressure in microns, substitute in the equation

2.54 d (inches) for D (cm),
 2.54 $\times 12$ H' (feet) for H (cm), and
 1.33 $\times P$ (microns) for P' (dynes/cm²).

Since these substitutions give Q' in cc/sec at 1 dyne/cm², the result should be divided by 1.33 to give cc/sec at 1 micron, and if we also divide by 1000, Q will be given in L/sec at 1 micron pressure, or liter-microns/sec.

Making these changes, Eq. 52 becomes:

$$Q = 0.246 \frac{d^4}{H'} \bar{P} (P_1 - P_2) + 6.54 \frac{d^3}{H'} \left(\frac{1 + 0.64 \bar{P}}{1 + 0.79 \bar{P}} \right) (P_1 - P_2) \quad (53)$$

where all pressures are in microns, length is in feet, diameter is in inches, and Q is in liter-microns/second.

5.4 Conductance for Air

Dividing through Eq. 53 by $(P_1 - P_2)$, we obtain for the conductance

$$C = \frac{Q}{P_1 - P_2} = 0.246 \frac{d^4}{H'} \bar{P} + 6.54 \frac{d^3}{H'} \left(\frac{1 + 0.64 \bar{P}}{1 + 0.79 \bar{P}} \right) \quad (54)$$

5.5 Limits of Applicability

Let us now examine the pressure limits within which these general equations of flow and conductance rather than the limited equations of Poiseuille and Knudsen must be applied if the error introduced by use of the simpler equations is not to exceed some arbitrary value.

For air at 20°C the conductance of a round pipe of length H' feet and diameter d inches has been shown to be:

- (1) For laminar flow - Poiseuille's Law (no slip correction):

$$C_1 = 0.246 \frac{d^4}{H'} \bar{P} \quad (17)$$

- (2) For molecular flow - Knudsen formula:

$$C_2 = 6.54 \frac{d^3}{H'} \quad (29)$$

- (3) For laminar, molecular, and transition flow:

$$C_3 = 0.246 \frac{d^4}{H'} \bar{P} + 6.54 \frac{d^3}{H'} \left(\frac{1 + 0.64 \bar{P}}{1 + 0.79 \bar{P}} \right) \quad (54)$$

(all conductances in L/sec at average pressures, $\bar{P}$).

If 10% is taken as the maximum allowable error, we can determine the pressure as a function of pipe diameter above and below which C_1 and C_2 respectively differ from C_3 by less than 10% of their values.

5.51 Pressures Above Which Poiseuille's Law Applies: For pressures above which $C_3 - C_1 = 0.1 C_3$ Poiseuille's Law applies. Inserting values of C_1 and C_3 from Eqs. 17 and 54:

$$6.54 \frac{\mu^3}{H^3} \frac{1 + 0.64 \bar{P}d}{1 + 0.79 \bar{P}d} = 0.1 \times 0.246 \frac{d^4}{H^3} \bar{P}$$

and solving for $\bar{P}d$

$$\bar{P}d = 216 \text{ micron-inches}$$

(55)

where $\bar{P}$ is the average pressure. Poiseuille's Law, uncorrected for slip, is correct to within 10% at all pressures greater than this up to the pressure at which flow becomes turbulent.

If Poiseuille's Law including the correction for slip is applied, the results are correct within the prescribed limits at pressures even lower than indicated above. This is of little practical significance, however, since the value of the slip coefficient is seldom known in actual cases.

5.52 Pressures Below Which Knudsen's Equation Applies: For pressures below which $C_3 - C_2 = 0.1 C_2$ Knudsen's equation applies. Inserting values of C_2 and C_3 from Eqs. 29 and 54 and solving for $\bar{P}d$ gives

$$\bar{P}d = 6.93 \text{ micron-inches}$$

(56)

where $\bar{P}$ is the average pressure. At all pressures lower than this Knudsen's formula is correct to within 10%.

5.6 Graphic Comparison of Conductance Formulas in the Transition Region

Figure 4 shows the conductance of a 1 foot length of 3 inch pipe as calculated by each of the three conductance formulas, Eqs. 17, 29, and 54 over the transition pressure range. Points A and B indicate the pressures at which Knudsen's and Poiseuille's formulas are in error by 10%.

Figure 4

5.7 Mean Free Path at Limiting Pressures

Since for air at 20°C the molecular mean free path

$$L = \frac{2.027}{\sqrt{P}} = \frac{2.027}{\sqrt{P \times \frac{M}{R_M T}}} = \frac{2.02 \times 181 \times 10^{-6}}{P \times 34.34 \times 10^{-6}} = \frac{10.64}{P}$$

where L is in cm and P in dynes/cm², it follows that for L in inches and P in microns

$$L' = \frac{10.64}{P} \times \frac{1}{1.33} \times \frac{1}{2.54} = \frac{3.15}{P} \text{ inches} \quad (57)$$

Combining Eq. 57 with Eqs. 55 and 56 to eliminate P we get:

$$L' = 0.0146a \quad (55-a)$$

as the maximum mean free path at average pressure for which Poiseuille's Law applies, and

$$L' = 0.455a \quad (56-a)$$

as the minimum mean free path at average pressure for which Knudsen's formula holds.

5.8 Application of the General Flow Formula

Because of its algebraic form, use of the general flow formula for calculating conductance or rate of flow is, at best, a tedious procedure. For this reason, curves showing the conductance per unit length as a function of pressure for standard pipe sizes over the pressure range to which the simple formulas of laminar or molecular flow do not apply are highly useful. Such a family of curves is given for reference in figures 5 and 6. Dotted lines indicate the boundaries between which the simple formulas for molecular or laminar flow are in error by 10% or more.

Figure 6

6. FLOW OF GAS THROUGH AN APERTURE

6.1 Viscous or Hydrodynamic Flow

At pressures such that the molecular mean free path is small compared with the dimensions of the aperture, the flow of gas through an aperture follows the established laws of hydrodynamics. From these laws it follows that:

- (1) For small back pressures, the rate of flow is proportional to the inlet pressure and is independent of the pressure drop

$$W' = \frac{P_1'' A'}{70} \quad (58)$$

where W' = rate of discharge in lbs/sec, P_1'' = inlet pressure in lbs/in², and A' = area in in².

- (2) If, however, the down stream pressure is more than 53% of the inlet pressure (for air), the rate of outflow is a function of both inlet pressure and of pressure drop.

The following formula⁶ is applicable to the flow of any gas or vapor through an orifice⁷:

$$W' = \frac{8.02 J A' P_1''}{\sqrt{RT'}} \sqrt{\frac{K}{K-1} \left[\left(\frac{P_2''}{P_1''} \right)^{2/K} - \left(\frac{P_2''}{P_1''} \right)^{\frac{K+1}{K}} \right]} \quad (59)$$

where W' = rate of discharge in lbs/sec, J = coef. of discharge (0.65 for orifice), A' = area of orifice in square inches, R = gas constant for particular gas, T' = temperature of stream in Rankine degrees ($F + 460$), K = adiabatic expansion exponent (1.4 for air), P_1'' = inlet pressure in lbs/in², and P_2'' = back pressure in lbs/in².

6. Taken from "Flow of Fluids Through Valves, Fittings, and Pipes", Tech. Paper No. 409, Crane Co., Chicago (May, 1942).

7. Where $P_2''/P_1'' < P_c''/P_1''$ (P_c'' = critical pressure), substitute P_c''/P_1'' for P_2''/P_1'' in the equation. Critical pressure ratio for air = 0.53.

Inserting appropriate values of the constants for air at 20°C (68°F)
Eq. 59 reduces to

$$W' = \frac{8.02 \times 0.65 \times A \cdot P_1^n}{\sqrt{53.34 \times 527.4}} \sqrt{\frac{1.4}{1.4 - 1} \left[\left(\frac{P_2^n}{P_1^n} \right)^{\frac{2}{1.4}} - \left(\frac{P_2^n}{P_1^n} \right)^{\frac{1.4+1}{1.4}} \right]}$$

$$= 0.0582 A \cdot P_1^n \sqrt{\left(\frac{P_2^n}{P_1^n} \right)^{1.43} - \left(\frac{P_2^n}{P_1^n} \right)^{1.71}} \quad (60)$$

Expressing area in cm² (A) and pressures in microns (P),

$$\frac{A \text{ (cm}^2\text{)}}{2.54^2} = A' \text{ (in}^2\text{)}$$

$$\frac{P_1 \text{ (microns)}}{51.85 \times 10^3} = P_1^n \text{ (lbs/in}^2\text{)}$$

and multiplying through by 454 to reduce W' (lbs/sec) to G (g/sec),

$$G = 7.9 \times 10^{-5} A P_1^n \sqrt{\left(\frac{P_2}{P_1} \right)^{1.43} - \left(\frac{P_2}{P_1} \right)^{1.71}} \text{ g/sec} \quad (61)$$

where A = area in cm² and P = pressure in microns.

Evaluating the radical in Eq. 60 for arbitrary values of P₂/P₁ between 1 and 0.53, the critical pressure ratio for air, and plotting these values as a function of the pressure ratio, figure 7 is obtained and this may be used in conjunction with Eq. 60 for calculating rate of flow under various pressure conditions.

For pressure ratios P₂/P₁ < 0.53, the value of the radical is 0.254, and equation 60 reduces to

$$G = 2.00 \times 10^{-5} A P_1 \text{ g/sec} \quad (62)$$

Dividing Eq. 61 by the density of air at unit pressure and 20°C

$$\rho_1' = \frac{M}{R \cdot T} = 15.76 \times 10^{-10} \text{ g/cc at 1 micron pressure}$$

Figure 7

P_0 = Critical Pressure Ratio for Air = 0.53

one gets for the volume of air at unit pressure passing through the orifice per second the value $12.7 \times 10^3 AP_1$ cc-microns/sec, or

$$Q = 12.7 AP_1 \text{ L-microns/sec}$$

and dividing Eq. 62 by P_1 , we have for the volume rate of out flow in L/sec measured at pressure P_1

$$B = 12.7 A \text{ L/sec}$$

This last expression also represents, by definition, the conductance of an aperture for air at 20°C under hydrodynamic flow conditions, and under small inlet to back pressure ratio

$$C = 12.7 A \text{ L/sec}$$

In the region above the critical pressure ratio, Eq. 60 may be reduced from mass rate of flow to PV or volume rate of flow as in Eqs. 64 and 65. Letting Y stand for the radical term (figure 7), its value for any given pressure ratio to be taken from the figure,

$$Q = 50 AP_1 Y \text{ L micron/sec} \quad (66)$$

$$B = 50 AY \text{ L/sec at pressure } P_1 \quad (67)$$

6.11 Correction for Velocity of Approach⁸: If the approach to the orifice is through a pipe or duct whose cross sectional area is not large compared with the area of the orifice, all of the above equations for flow rate should be multiplied by the factor f_v which corrects for the velocity of approach.

$$f_v = \frac{A_a}{A_a^2 - A^2} \quad (68)$$

A_a = Cross sectional area of pipe or duct

A = Area of orifice

If A_a is as much as $3A$, f_v differs from unity by 6% or less. Thus the factor may be neglected in most calculations.

6.2 Molecular Flow

The rate of flow of gas through an aperture at pressures such that the molecular mean free path is small compared with the dimensions of the aperture has been dealt with in the preceding section. Attention is now directed to the rate of flow through and the conductance of an aperture at pressures such that the mean free path is great compared with the dimensions of the aperture ($L > 10D$ according to Knudsen). Since rate of flow and conductance under these conditions enters directly or indirectly into the solution of a great many problems including that of flow through short pipes, the case will be considered in some detail.

Let us first consider the number of molecules passing per second in either direction through an element of area ds within a uniform body of gas of infinite extent⁹. Let N = the molecular density at any point in the gas and assume a Maxwellian distribution of molecular velocities.

The the number of molecules in a unit volume having speeds between u and $u+du$ may be written

$$N \frac{4}{\pi^{3/2}} u^2 e^{-u^2/\alpha^2} du$$

where α , the most probable speed, is equal to $\frac{1}{\sqrt{2}} \bar{u}$ ($\bar{u}$ = average speed).

8. "Flow of Fluids Through Valves, Fittings, and Pipes", Tech. Paper No 409., p. 86, Crane Co., Chicago (May, 1942).
9. Loeb, Leonard B., "The Kinetic Theory of Gases", p. 94-97, McMillan, New York (1927), Kennard, Earle H., "Kinetic Theory of Gases," p. 60-66, McGraw-Hill, New York (1938).

Take ds as the origin of a system of polar coordinates and let $d\vec{v}$ be any element of volume in the space above the plane of ds .

Since

$$dv = dr r d\phi r \sin\theta d\theta$$

$$1 = r^2 \sin\theta d\theta d\phi dr$$

The number of molecules in dv having speeds between u and $u + du$ is

$$N \frac{4}{\alpha^3 \sqrt{\pi}} u^2 e^{-u^2/\alpha^2} du r^2 \sin\theta d\theta d\phi dr \quad (70)$$

Now each of these molecules suffers an encounter and starts a new path $\bar{u}/L$ times per second where $\bar{u}$ is the average velocity and L the mean free path of all molecules. Therefore, the number of molecules having speeds within the stated range and leaving dv per second is

$$N \frac{4}{\alpha^3 \sqrt{\pi}} \frac{\bar{u}}{L} u^2 e^{-u^2/\alpha^2} du r^2 \sin\theta d\theta d\phi dr \quad (71)$$

If this number, the fraction directed toward ds will be

$$\frac{ds \cos\theta}{4\pi r^2}$$

giving

$$N \frac{1}{\alpha^3 \sqrt{\pi}} \frac{\bar{u}}{L} u^2 e^{-u^2/\alpha^2} du \sin\theta \cos\theta d\theta d\phi dr ds \quad (72)$$

as the number of molecules having speeds within the specified range leaving dv per second directed toward ds .

Of these a fraction $e^{-r/L}$ will have a free path equal to or greater than r and will, therefore, reach ds without suffering an encounter.

We then have for the number of molecules with speed between u and $u + du$ arriving at ds per second from dv ,

$$N \frac{1}{\alpha^3 \sqrt{\pi}} \frac{\bar{u}}{L} u^2 e^{-u^2/\alpha^2} du e^{-r/L} dr \sin\theta \cos\theta d\theta d\phi ds \quad (73)$$

and for the total number N' of molecules of all speeds arriving at ds per second from all points above the plane of ds , we must integrate u from 0 to ∞ , r from 0 to ∞ , θ from 0 to $\pi/2$, and ϕ from 0 to 2π .

Therefore

$$N' = \frac{N}{2\pi^{3/2}} \frac{\bar{u}}{L} ds \int_0^\infty u^2 e^{-u^2/L^2} du \int_0^\infty e^{-r/L} dr \int_0^{\pi/2} \sin\theta \cos\theta d\theta \int_0^{2\pi} d\phi \quad (74)$$

$$= \frac{N}{2\pi^{3/2}} \frac{\bar{u}}{L} ds \left[\frac{\alpha^3 \sqrt{\pi}}{4} \right] \left[L \right] \left[\frac{1}{2} \right] \left[2\pi \right]$$

$$= \frac{N\bar{u}}{4\pi L} ds \pi L$$

$$N' = \frac{1}{4} N\bar{u} ds \quad (75)$$

where N' = the number of molecules crossing ds per second in either direction.

If now we assume an impervious bounding plane placed in the plane of ds , the situation above this plane is in no way altered, since the molecules reflected or remitted from this plane are identical in number and velocity distribution with those initially crossing from below the plane of ds .

$N' = \frac{1}{4} N\bar{u} ds$ is then the rate of molecular impact on any area ds of a surface bounding a body of gas of uniform molecular density N and is also the rate of re-emission from this element of surface.

Let us now examine the situation when ds is an aperture in an infinitely thin wall of a vessel containing gas at molecular density N . Assume that the pressure outside this aperture is zero, i.e., no molecules pass back through the aperture.

6.21 Small aperture: If the aperture ds is so small that the number of molecules escaping through it produces no appreciable non-uniformity in the molecular density of the gas inside the vessel, the number of molecules passing through the aperture per second will be the same as the number initially striking ds . That is, $N' = \frac{1}{4} N\bar{u} ds$ is the rate of out-flow through

the small aperture ds , and we may proceed to express this rate of out flow in terms of measurable quantities.

If Eq. 73 is multiplied by m , the mass of a molecule

$$H'm = \frac{1}{4} H m \bar{u} ds$$

where $H'm = G$, the mass of gas escaping per second and

$Hm = \rho'$ the mass density of the gas in the vessel.

Then since

$$\bar{u} = \sqrt{\frac{8P'}{\pi\rho'}}$$

$$G = \rho' \sqrt{\frac{P'}{2\pi\rho'}} ds = \sqrt{\frac{P\rho'}{2\pi}} ds$$

and upon substituting $\rho' = P'\rho'_1$, where $\rho'_1 =$ density at unit pressure, this becomes

$$G = P' \sqrt{\frac{\rho'_1}{2\pi}} ds \quad (76)$$

and since $\rho'_1 = \frac{M}{R_M T}$,

$$G = P' \frac{1}{\sqrt{2\pi}} \frac{M}{R_M T} ds$$

where G is the rate of flow in grams per second through an aperture ds from a region of pressure P' into a region of zero pressure.

If the pressures on the two sides of the aperture are P'_1 and P'_2 respectively ($P'_1 > P'_2$ and both P'_1 and P'_2 sufficiently small that the molecular mean free path is great compared with the dimensions of ds) counter effusions from the two sides occur simultaneously and independently and

$$G_1 = P'_1 \sqrt{\frac{\rho'_1}{2\pi}} ds \quad \text{and} \quad G_2 = P'_2 \sqrt{\frac{\rho'_1}{2\pi}} ds$$

The net rate of flow, G , is then

$$G = G_1 - G_2 = \sqrt{\frac{\rho'_1}{2\pi}} (P'_1 - P'_2) ds$$

$$= \frac{1}{\sqrt{2\pi}} \sqrt{\frac{M}{R_M T}} (P'_1 - P'_2) ds$$

Dividing the mass rate of effusion G , by $P_1 = M/R_M T$, the density of the gas at unit pressure, gives for the effusion rate in volume per second at unit pressure (i.e., quantity per second in P.V. units):

$$Q' = \frac{1}{\sqrt{2\pi}} \sqrt{\frac{R_M T}{M}} (P_1' - P_2') ds \quad (78)$$

And dividing Eq. 78 by P_1' (or P_2') one gets for the rate of flow in volume/sec measured at pressure P_1' (or P_2')

$$B_1' = \frac{1}{\sqrt{2\pi}} \sqrt{\frac{R_M T}{M}} \frac{1}{P_1'} (P_1' - P_2') ds, \quad B_2' = \frac{1}{\sqrt{2\pi}} \sqrt{\frac{R_M T}{M}} \frac{1}{P_2'} (P_1' - P_2') ds \quad (79)$$

If $P_2' = 0$ (or is negligibly small compared with P_1') Eq. 79 represents the conductance of the aperture as:

$$C = \frac{1}{\sqrt{2\pi}} \frac{1}{\sqrt{\rho_1'}} ds = \frac{1}{\sqrt{2\pi}} \sqrt{\frac{R_M T}{M}} ds \quad (80)$$

For the particular case of air at 20°C,

$$\sqrt{\rho_1'} = \sqrt{\frac{M}{R_M T}} = 34.34 \times 10^{-6} \text{ g/cc at } 1 \text{ dyne/cm}^2$$

Inserting this value and combining constants in Eqs. 77, 78, and 79, and setting $ds = A$, the area of the aperture, they reduce to

$$G = 1.37 \times 10^5 (P_1' - P_2') A \quad (77a)$$

$$Q' = 0.116 \times 10^5 (P_1' - P_2') A \quad (78a)$$

$$B_1' = 0.116 \times 10^5 (1 - P_2'/P_1') A; \quad B_2' = 0.116 \times 10^5 (P_1'/P_2' - 1) A \quad (79a)$$

Where G is in g/sec, Q is in cc/sec at 1 dyne/cm² pressure = PV units/sec, B' is in cc/sec at pressures P_1' and P_2' respectively, P_1' and P_2' are in dynes/cm², and A_0 is in cm².

If pressures are expressed in microns and volumes are reduced from cc to liters, then since 1 micron = 1.33 dynes/cm², the above equations reduce to:

$$G = 1.84 \times 10^5 (P_1 - P_2) A \quad (77-b)$$

$$Q = 11.6 (P_1 - P_2) A \quad (78-b)$$

$$B_1 = 11.6 \left(1 - \frac{P_2}{P_1}\right) A; \quad B_2 = 11.6 \left(\frac{P_1}{P_2} - 1\right) A \quad (79-b)$$

where Q is in L/sec at 1 micron pressure = liter-microns/sec, and where B_1 and B_2 are in L/sec at P_1 and P_2 microns respectively.

If $P_2 = 0$ or is negligible compared with P_1 , the "conductance" C may be expressed as

$C = 11.6 A$ L/sec at inlet pressure; or per unit area of aperture:

$$C^* = C/A = 11.6 \text{ L/sec/cm}^2 \quad (81)$$

$$C_g^* = 11.6 \times 2.54^2 = 75 \text{ L/sec/in}^2 \quad (82)$$

6.22 Large Aperture:

In the preceding section dealing with the flow of gas through a small aperture it was necessary to assume that the dimensions of the aperture were (a) small relative to the molecular mean free path and (b) small relative to the dimensions of the vessel from which the gas is escaping. The first condition is necessary if the flow is to be in no sense viscous and if the rates of effusion in opposite directions are to be independent. The second condition is necessary if non-uniformity of density and mass movement in the body of gas above the aperture due to loss of the escaping molecules is to be negligible.

In considering the case of a large aperture, it is still assumed that the pressure is such that the molecular mean free path is great compared with the aperture dimensions; the non-uniformity of density arising from the loss of molecules through the aperture is, however, no longer considered negligible, and an appreciable mass motion of gas toward the aperture occurs.

From hydrodynamic reasoning it has been shown that when the cross sectional area A_a of the space above the aperture is comparable with the area A of the aperture, the rate of viscous flow or conductance is increased by a factor

$$f_v = \frac{A_a}{\sqrt{A_a^2 - A^2}}$$

which corrects for the "velocity of approach". Although this correction factor has not been demonstrated to be applicable to molecular flow, it is a correction in the proper direction and reduces to the correct values for $A_a = A$ and $A_a \gg A$.

7. FLOW THROUGH SHORT PIPES AND DUCTS

When gas flows through a pipe one may expect conditions of flow near the ends of the pipe to differ significantly from conditions existing at points removed from the ends. These nonuniformities of flow at the pipe ends may be caused by:

- (a) Limitation of flow by the action of the pipe inlet as an orifice in addition to any limitation imposed by friction with the pipe wall.
- (b) Turbulent flow at the pipe inlet becoming viscous along the length of the pipe.

If the pipe is of sufficient length or, better, if the ratio of length to diameter is great enough, each of these end effects becomes negligible. The laws of Poiseuille and Knudsen were derived on the assumption that this was the case. If, however, the ratio of length to diameter is not sufficiently great, the rate of flow may be appreciably altered due to one or both of these causes. It is of interest to note that both of these end effects act to reduce the rate of flow to a value lower than would occur if the effect were absent. Therefore, the rate of flow through a short pipe is less than indicated by application of an uncorrected long pipe formula. It is possible, however, to introduce into the long pipe formulas end-correction terms which satisfactorily account for this lowering of conductance due to end effects. This is the method commonly adopted for dealing with problems of flow in short pipes, especially in the case of limitation of flow by the orifice-action of the pipe inlet.

7.1 Molecular Flow Through Short Pipes

No rigorous derivation appears to have been made of a formula for the molecular flow of gas through a short pipe; there are, however, various empirical formulas in which Knudsen's equation of flow through a long pipe has been modified by the introduction of an end-correction term.

7.11 Dushman's Short Pipe Formula: The simplest and most widely used of these formulas is that due to Dushman. This formula is based on the assumption that the conductance of a short pipe is equivalent to two conductances in series: that of the pipe inlet considered as an aperture, and that of the pipe considered as a long pipe.

Then for a pipe of radius R and length H

$$C_a' = \frac{1}{\sqrt{2\pi}} \frac{1}{\sqrt{\rho_1}} \pi R^2 \quad (80)$$

$$C_p' = \frac{4}{3} \sqrt{2\pi} \frac{1}{\sqrt{\rho_1}} \frac{R^3}{H} \quad (24)$$

where ρ_1' is the gas density at unit pressure (dynes/cm²) and C_a' and C_p' are conductances of aperture and pipe respectively in cc/sec at inlet pressure.

Combining these conductances to find the equivalent conductance of the two in series, we get for C, the conductance of the short pipe:

$$C = \frac{C_p C_a}{C_p + C_a} = \frac{C_a}{1 + C_p/C_a} = \frac{\frac{1}{\sqrt{2\pi}} \frac{1}{\sqrt{\rho_1}} \pi R^2}{1 + \frac{3}{8} \frac{H}{R}} \quad (83)$$

which obviously reduces to Eq (1) when $L = 0$, and to Eq. (2) when $3H/8R \gg 1$

Since ρ_1 is a constant for a particular gas, and πR^2 is the area of the pipe opening, Eq. (3) may be simplified to

$$C' = \frac{KA}{1 + \frac{3H}{8R}} \quad (83-a)$$

Kennard¹⁰ points out that from Kinetic theory considerations of an approximate nature one should expect for the case of a very short pipe a factor 1/2 instead of 3/8 in the above equation, though such a factor would obviously be in error as the length of the pipe approaches a value for which the end effect becomes negligible.

7.12 Clausius's Formula: As a result of a thorough study of the problem of flow through short pipes, Clausius¹¹ proposed the following empirical formula as representing more accurately than does Dushman's simpler formula the conductance for all values of H/R :

$$C' = \frac{20 + 8 H/R}{20 + 19 H/R + 3(H/R)^2} \sqrt{\frac{\pi}{2}} \frac{1}{\rho_1} R^3 \quad (84)$$

Since this formula is more tedious to apply and since it contributes a somewhat questionable 12% or less to the accuracy of the results, it is seldom used in actual flow calculations.

7.13 Development of Dushman's Formula: We return then to a further development of Dushman's formula,

$$C' = \frac{KA}{1 + 3H/8R} \quad (83-a)$$

where all quantities are in c.g.s. units, and C' = the volume of gas at inlet pressure flowing per second. For the particular case of air at 20°C:

$$\rho_1 = 11.85 \times 10^{-10} \text{ g/cc at 1 dyne/cm}^2 \text{ pressure}$$

and

$$K = \frac{1}{\sqrt{2\pi}} \frac{1}{\sqrt{\rho_1}} = 11.6 \times 10^3$$

10. Kennard, Earle H., "Kinetic Theory of Gases", p. 306, McGraw-Hill, New York (1938).
11. Clausius, P. Physica, 9, 65, 1929. (See also Loeb, L. B., "The Kinetic Theory of Gases", p. 301 et seq.)

Substituting in Eq. 83-a and expressing C' in L/sec at inlet pressure:

$$C = \frac{11.6 A}{1 + 3H/8R} \quad \text{L/sec (A in cm}^2\text{)} \quad (83-b)$$

Converting A from cm² to in² (H and R may be expressed in any unit as long as the same unit is used for both), and replacing R by D/2, we get a form of the conductance equation that may be readily applied to large vacuum systems.

$$C = \frac{75 A'}{1 + 3H'/d} \quad \text{L/sec} \quad (83-b)$$

and

$$C = \frac{75 A'}{1 + 9H'/d} \quad \text{L/sec (H' in feet, d in inches)} \quad (83-c)$$

From this last equation it follows immediately that for a pressure difference between the ends of the pipe of (P₁ - P₂) microns the rate of flow in liter-microns per second (L/sec at 1 micron pressure) is

$$Q = \frac{75 A'}{1 + 9H'/d} (P_1 - P_2) \quad (85)$$

and the volume of air flowing per second at a point where the pressure is P microns is

$$B = \frac{75 A'}{1 + 9H'/d} \frac{1}{P} (P_1 - P_2) \quad \text{L/sec} \quad (86)$$

7.14 Criterion for Use of Short Pipe Formulas: Since each of the short pipe formulas reduces to a corresponding long pipe formula for a large ratio of pipe length to pipe diameter, we may easily establish the limiting value of the ratio H'/d above which application of the simpler long pipe relation will introduce an error less than any arbitrary per cent. For example, if it is agreed that an error of 10% or less is permissible, the long pipe formulas may be used if

$$9H'/d \geq 10$$

That is, if

$$H'/d \geq 1.1$$

As a practical working rule: any pipe whose length in feet is greater than its diameter in inches may be treated as a long pipe (error 11% or less). For smaller H'/d ratios or higher accuracy the short pipe formula should be used.

7.15 Pipe Aperture Comparable to the Cross Sectional Area of the Space Evacuated: As was pointed out in dealing with the conductance of apertures, this conductance is diminished in the ratio $A_a/A_g^2 - A^2$ (Eq. 68) when the area A of the aperture is comparable to the cross sectional area A_g of the

space being evacuated. If this correction factor for velocity of approach is employed in deriving Eqs. 83 and 83-a, one obtains

$$C' = \frac{KA}{\frac{\sqrt{A_a^2 - A^2}}{A_a} + \frac{3H}{4H}} \quad (87)$$

This equation differs significantly (say by 3% or more) from Eq. 83-a if A/A_a is greater than 0.31.

Equation 87 is awkward to apply, and in individual cases it is often simpler to:

(a) Calculate the conductance of the aperture, corrected for velocity of approach, from

$$C_a = \frac{A_a^3}{\sqrt{A_a^2 - A^2} \sqrt{2H}} \frac{1}{\sqrt{P_1}} A' \quad (88)$$

$$C_a = \frac{A_a^3}{\sqrt{A_a^2 - A^2}} 75 A' \quad (\text{for air at } 20^\circ \text{ C})$$

(b) Calculate the uncorrected conductance of the pipe from

$$C_p = \frac{4\sqrt{2H}}{3} \frac{1}{\sqrt{P_1}} \frac{R^3}{H'}$$

$$C_p = 6.54 \frac{d^3}{H'} \quad (\text{for air at } 20^\circ \text{ C}) \quad (89)$$

(c) Combine as reciprocals to find the equivalent conductance of the two in series

$$C = \frac{C_p C_a}{C_p + C_a}$$

(Note that the constants in the above equations have been evaluated for A_a and A' in square inches, d in inches, H' in feet and conductances in l./sec.)

7.16 Non-Circular Ducts: Problems of short, non-circular ducts can also be dealt with most easily by calculating aperture and duct conductances separately and combining to get their equivalent joint conductance.

Since the conductance of an aperture is independent of its shape, C_a may be found from Eq. 80 or 88.

The conductance of the duct may be calculated from Eq. 89 with d^3 replaced by the equivalent diameter term, $\frac{16 A'^2}{\pi O'}$ where A' is the cross sectional

area and O' is the perimeter as explained in section 4.2.

7.17 Non-Uniform Pipes and Ducts: In the case of flared pipes or ducts, determine the aperture conductance in the usual way, and combine with the pipe or duct conductance from Eq. 89 in which d^3 assumes the value appropriate to the particular case as given in section 4.3 and in table 3.

7.2 Viscous Flow through Short Pipes:

Corrections for end effects in the case of viscous flow are difficult to apply since pressure drops occurring at the pipe ends are functions of gas density; mean velocity along the pipe; and, at the pipe entrance, of an experimentally determined constant depending on the geometry of the pipe opening. Experiments indicate, however, that for practical purposes no serious error is introduced by neglecting end effects in such cases.

When air was allowed to flow at various measured rates through sections of standard 1 inch pipe of various lengths joining two chambers in which the pressures were measured, it was found that the effective lengths of the pipe sections, as calculated from Poisseuille's law,

$$Q = 0.246 \frac{d^4}{H} \bar{P} (P_1 - P_2) \quad (14-b)$$

exceeded the actual length by an essentially constant amount averaging about 0.25 ft (or 3 x diameter) throughout the 220 to 600 micron pressure range, as shown in figure 8. Whether or not this same correction (3d) is applicable to pipes of other sizes and to still higher pressures has not been determined. However, unless the correction is much greater at higher pressures or for pipes of other sizes, the error introduced by neglecting end effects is not serious in practical applications.

$$H' = \text{Equivalent Length} = 0.246 \frac{d^4}{Q} \bar{P} (P_1 - P_2) \text{ (ft)}$$

Figure 8

8. FLOW CHARACTERISTICS OF LEAKS

8.1 Leakage into an Evacuated Vessel

The flow of air through a leak into an evacuated system occurs under conditions not likely to be encountered in any part of a vacuum pumping system, and is of sufficient importance to warrant special consideration.

Assume that the leakage path is equivalent to a capillary tube of length H and diameter D . Assume also that leakage is from atmospheric pressure into a region of negligible pressure: i.e., $P_1 = 760 \times 10^{-3}$ microns and $P_0 = 0$.

Under these conditions air enters the leak as a viscous stream (mean free path less than D). But with decreasing pressure along the leakage path flow changes to free molecular with the air emerging as a sound velocity jet at a pressure P_2 . Any remaining pressure drop ($P_2 - P_0$) occurs outside the leak discharge and does not effect the rate of leakage.

If we assume that Knudsen's general formula of flow applies along the leakage path⁽¹⁾, and that the effective pressure drop along the path is $(P_1 - P_2)$ microns, we may write for the rate of flow through the leak,

$$Q' = \frac{\pi D^4}{256 \eta H} (P_1^2 - P_2^2) + \frac{\sqrt{2} \pi D^3}{6 \sqrt{P_1} H} (P_1 - P_2) \left[\frac{1 + \frac{\sqrt{P_1}}{2} \frac{(P_1 + P_2) D}{2}}{1 + 1.24 \frac{\sqrt{P_1}}{2} \frac{(P_1 + P_2) D}{2}} \right] \quad (91)$$

where Q' = volume per second at unit pressure (cc/sec @ 1 dyne/cm²).

P_1 = inlet pressure (dynes/cm²)

P_2 = discharge pressure (dynes/cm²)

D = diameter (cm)

H = length (cm)

η = viscosity (poises)

P_1 = density at unit pressure (g/cc @ 1 dyne/cm²)

If pressures are expressed in microns, and Q is in L/sec @ 1 micron pressure: then for air at normal pressure Eq 91 becomes

$$Q = 0.0917 \frac{D^4}{H} (P_1^2 - P_2^2) + 12.0 \frac{D^3}{H} (P_1 - P_2) \left(\frac{1 + 0.132 \frac{(P_1 + P_2) D}{2}}{1 + 0.163 \frac{(P_1 + P_2) D}{2}} \right) \quad (92)$$

(1) See Section 8.3 for further consideration of this assumption.

$$= 0.0917 \frac{D^4}{H} (P_1^2 - P_2^2) \left[1 + \frac{131}{D(P_1 + P_2)} \left(\frac{1 + 0.132 \frac{(P_1 - P_2)D}{P_1 + P_2}}{1 + 0.163 \frac{(P_1 - P_2)D}{P_1 + P_2}} \right) \right] \quad (93)$$

For leakage flow $P_1 = 760 \times 10^3$ microns, and if we assume for the moment that P_2 is negligible as compared with P_1 , Eq (93) becomes

$$Q = 5.30 \times 10^{10} \frac{D^4}{H} \left[1 + \frac{1.72 \times 10^{-4}}{D} \left(\frac{1 + 10.0 \times 10^4 D}{1 + 12.4 \times 10^4 D} \right) \right] \quad (94)$$

But if the gas is discharged into the evacuated space at the velocity of sound (3.44×10^4 cm/sec @ 20°C),

$$Q = 3.44 \times 10^1 \times \frac{\pi D^2}{4} P_2 \quad (95)$$

$$Q = 27 D^2 = P_2 \quad (96)$$

where Q is in L microns/sec.

Combining equations (94) and (96) to eliminate Q , one gets for the discharge pressure of the sound velocity jet

$$P_2 = 19.57 \times 10^8 \frac{D^2}{H} \left[1 + \frac{1.72 \times 10^{-4}}{D} \left(\frac{1 + 10.0 \times 10^4 D}{1 + 12.4 \times 10^4 D} \right) \right] \quad (97)$$

If a leakage path 1 cm long is assumed, and a series of diameters is assigned to this path, the resulting critical discharge pressures may be calculated from equation (97). At the same time these calculations are being made, the ratio of viscous to molecular flow for each assumed diameter of leakage path may be calculated since this is simply the ratio of unity to the value of the fractional term in the brackets.

Table 4 lists the results of such calculations. Values of Q , the flow rate in L-microns/sec as calculated from equation (95), are also given, as well as equivalent leak rate in standard cc/sec.

For leakage paths of length other than unity, the critical discharge pressure and the rate of leakage varies inversely with path length.

Table 4

Capillary Flow Into Vacuum

$d(10^{-4}\text{cm})$	Flow Ratio (visc./mol)	P_2 (microns)	Q (L microns/sec)	Standard cc/sec
1	1/1.4	47	$.0127 \times 10^{-3}$	1.67×10^{-8}
2	1/.70	133	$.144 \times 10^{-3}$	1.90×10^{-7}
3	1/.47	257	$.625 \times 10^{-3}$	8.22×10^{-7}
4	1/.35	422	1.82×10^{-3}	2.41×10^{-6}
5	1/.28	626	4.23×10^{-3}	5.57×10^{-6}
6	1/.23	865	8.40×10^{-3}	1.10×10^{-5}
8	1/.17	1470	25.4×10^{-3}	3.34×10^{-5}
10	1/.14	2230	60.2×10^{-3}	7.92×10^{-5}
15	1/.09	4810	292×10^{-3}	3.85×10^{-4}
20	1/.07	8380	905×10^{-3}	1.19×10^{-3}
25	1/.056	12,920	2180×10^{-3}	2.87×10^{-3}
50*	1/.028	50,200	$33,900 \times 10^{-3}$	4.46×10^{-2}
100*	1/.014	198,500	$536,000 \times 10^{-3}$	7.06×10^{-1}

Examining the results given in table 4, it appears that:

- (1) Flow is predominantly viscous for all but the smallest leaks considered and may be considered wholly viscous for capillary diameters greater than $.15 \times 10^{-4} \text{ cm} = .015 \text{ mm}$.
- (2) The critical discharge pressure P_2 is negligibly small compared with atmospheric pressure, as was assumed, for all but the two largest leaks considered. For leakage path diameter greater than $25 \times 10^{-4} \text{ cm}$. ($P_2 = 1.7\% P_1$), solution must be found by a series of approximations.
- (3) Obviously the rate of leakage is truly independent of P_0 as long as P_0 is less than the critical discharge pressure.
- (4) It follows that the rate of pressure rise in a blanked off system is constant until the pressure within the system exceeds the critical discharge value. Beyond this point the rate of leakage, and, therefore, the rate of rise, decreases with increasing pressure. This effect is, negligible, however, until the pressure within the system becomes appreciable relative to atmospheric pressure, say 50 microns or more.

8.2 Leakage out of a Vessel under Pressure

In pressure testing vacuum equipment for leaks, the leaks are revealed by the flow of gas outward from a region of elevated pressure P_1 into a region at essentially atmospheric pressure. Under these conditions, flow may be considered as wholly viscous, and if a capillary leakage path of length H and

* P_2 not negligible relative to P_1 . Solution must, therefore, be made by series of approximations.

radius r is assumed, the rate of leakage is given by:

$$Q = \frac{\pi}{8\eta} \frac{r^4}{H} \bar{P} (P_1 - P_2) \quad (14)$$

Where Q = Rate of flow in cc/sec at 1 microbar pressure

η = Viscosity of gas in poises

r = Radius of capillary in cm

H = Length of capillary in cm

$\bar{P}$ = Average pressure in microbars

P_1 = Inlet pressure in microbars

P_2 = Discharge pressure in microbars

For air at 20°C $\eta = 181 \times 10^{-6}$, Expressing pressures in microns and flow rate in L microns/sec, and substituting diameter for radius, equation (14) becomes

$$Q = 0.185 \frac{D^4}{H} \bar{P} (P_1 - P_2)$$

Since $\bar{P} = \frac{1}{2} (P_1 + P_2)$

$$Q = .0925 \frac{D^4}{H} (P_1^2 - P_2^2). \quad (99)$$

Expressing P_1 as a multiple of P_2

$$P_1 = a P_2$$

where a may have any value greater than unity,

$$Q = .0925 \frac{D^4}{H} P_2^2 (a^2 - 1)$$

And since $P_2 = 760 \times 10^3$ microns

$$Q = 5.34 \times 10^{10} \frac{D^4}{H} (a^2 - 1) \quad (100)$$

Table 5 shows leakage rates through capillaries of unit length and of various diameters for a series of values of the pressure ratio a . For other lengths of leakage path, rate of leakage varies inversely with length. For each assumed pressure ratio the corresponding approximate gage pressure is given since a bourdon gage is usually used in measuring the pressure applied to a vessel under test.

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Table 5 -

Flow Rate of Capillaries into Atmospheric Pressure

d (cm)	Q in (10^{-3} L microns/sec - (cc-M/sec) $a = P_1/P_2$										
	1.1	1.2	1.3	1.4	1.5	1.6	1.7	2.0	2.5	3.0	
	Equivalent Gage Pressure (lbs/in ²): (Atmosphere - 15 (lbs/in ²)).										
	1.5	3.0	4.5	6.0	7.5	9.0	10.5	15	22.5	30	105
1×10^{-4}	.00112	.00235	.00369	.00518	.00668	.00834	.0101	.0160	.0280	.0427	.105
2	.0179	.0375	.0590	.0828	.107	.133	.162	.256	.448	.683	.336
3	.0907	.190	.298	.419	.540	.674	.816	1.30	2.27	3.45	27.2
4	.288	.603	.944	1.32	1.76	2.14	2.59	4.11	7.20	11.0	86.6
5	.700	1.47	2.30	3.24	4.18	5.22	6.32	10.0	17.5	26.7	210
6	1.44	3.05	4.75	6.67	8.61	10.7	12.9	20.6	36.1	55.0	433
8	4.59	9.52	15.1	21.0	27.3	34.1	41.3	65.6	115	175	1230
10	11.2	23.5	36.9	51.8	66.8	83.3	101	160	280	427	3360
15	56.0	118	184	259	334	417	505	801	1400	2140	6850
20	179	375	589	827	1070	1330	1610	2560	4480	6820	53700
25	437	916	1440	2020	2600	3250	3930	6240	11000	16600	31000

Leakage Rate Less Than from
Atmosphere into Vacuum

Leakage Rate Greater Than from
Atmosphere into Vacuum

Referring to Table #5, it appears that the increase in leakage rate through a given capillary is more than proportional to the increase in gage pressure. A tenfold increase in gage pressure results in an eighteenfold increase in leakage rate. Also, comparing leakage rates from atmospheric pressure into an evacuated chamber, as given in table 4, with leakage rates from a chamber at two atmospheres pressure into atmosphere, it is seen that the leakage rate is greater in the latter case by a factor varying from 1.26 for the smallest leak considered to 2.85 for the largest.

If the vessel under test is filled with hydrogen in the place of air, all leakage rates will be approximately doubled since the viscosity of hydrogen is about half that of air.

Consider now the factors limiting the sensitivity of common pressure-bubble methods of leak hunting.

8.21 Pressure Limitation

Although it is obvious that any pressure method of leak hunting becomes more sensitive the higher the pressure applied, it frequently happens that structural considerations limit the pressure that can be applied to a given vessel. In the case of large systems this limitation may be quite drastic. Suppose that a given vessel cannot be safely subjected to internal pressure greater than 6 lbs/in², gage. Then from table 5, column 4 it is seen that the leakage rate will vary from .00518 to 2020 (cc-μ)/sec over the range of leak sizes considered. If it were possible to apply 30 lbs/in² gage pressure, leakage rates would be more than eight times as great.

8.22 Detection Limitations

If leakage is to be detected by observing the formation of bubbles the method becomes insensitive at leakage rates such that a bubble of easily detectable size is not formed in a reasonably short time. Assume, arbitrarily, that a spherical bubble developing to a diameter of 1 mm in 60 seconds is readily detectable, and that bubbles forming at a slower rate may easily be over-looked. Since the volume of such a bubble is 5.23×10^{-4} cc, and pressure within the bubble is essentially atmospheric (760x10³ microns), the specified rate of growth would require a leakage rate of

$$Q = \frac{5.23 \times 10^{-4} \times 760 \times 10^3}{60 \times 10^3} = 6.6 \times 10^{-3} \text{ L microns/sec} \quad (101)$$

Thus at 6 lbs/in² pressure, as assumed in the previous case, leaks of 6×10^{-4} cm diameter or larger would be found. At pressures of 105 lbs/in² leaks of about 2×10^{-4} cm diameter or larger are revealed, and to be reasonably sure of finding leaks as small as 1×10^{-4} cm diameter would require a pressure of 500 lbs/in² or higher. Returning to the case of testing at 6 lbs/in², the largest leak likely to be missed (d = 5×10^{-4} cm) would have a leakage rate into an evacuated system of 4.23 cc-μ/sec as given in table 4. In a static system having a volume of 100 L, a single leak of this size would result in a rate of pressure rise

$$R_p = \frac{4.23}{100 \times 10^3} = 4.23 \times 10^{-5} \mu/\text{sec} = 2.54 \times 10^{-3} \mu/\text{min} = 15 \mu/\text{hr.}$$

In a dynamic system pumped at a speed of 100 L/sec, such a leak would increase the equilibrium pressure by

$$\Delta P = 4.23 \times 10^{-5} \mu$$

Whether or not leaks of this magnitude, or smaller, are significant depends on operating requirements and on the number of such leaks likely to be present.

8.23 Capillarity Limitation

When the discharge end of a capillary leak is covered by a liquid, a bubble will be formed only if the force due to internal pressure exceeds that due to the surface tension of the liquid. Otherwise the liquid will effectively plug the leakage path throughout the test. Thus for a bubble to form within the body of a liquid;

$$(P_1' - P_2') \pi \frac{D^2}{4} > \gamma \pi D \quad (102)$$

where $(P_1' - P_2')$ = Pressure difference in dynes/cm²

D = Diameter of capillary in cm

γ = Surface tension of the liquid in dynes/cm.

therefore

$$P_1' - P_2' \geq 4 \frac{\gamma}{D} \quad (103)$$

and since 1 dyne/cm² = .75 microns

$$P_1' - P_2' \geq 3 \frac{\gamma}{D} \text{ microns} \quad (104)$$

if the liquid employed is water at 20°C,

γ = 72 dynes/cm and

$$P_1' - P_2' \geq \frac{216}{D} \quad (105)$$

The minimum pressure difference for bubble formation at capillaries of the diameters previously treated are listed in table 6,

Table 6

Minimum Pressure for Bubble Formation

D (10^{-4} cm)	1	2	3	4	5	6	8	10	15	20	25
$(P_1 - P_2) (10^5 \mu)$	21.6	10.8	7.2	5.4	4.3	3.6	2.7	2.16	1.04	1.08	.865
$(P_1 - P_2) (\text{lbs/in}^2)$	41.8	20.9	13.9	10.4	8.32	6.96	5.22	4.18	2.78	2.09	1.67

Thus at a gage pressure of 6 lbs/in^2 a bubble would form at a capillary of 8×10^{-4} cm diameter but would not form at a 6×10^{-4} cm capillary. That is, a capillary whose rate of leakage into a vacuum system is $25.4 \text{ cc-}\mu/\text{sec}$ could be detected, but one whose rate is $8.4 \text{ cc-}\mu/\text{sec}$ could not. It will be noted that at 6 lbs/in^2 test pressure detection is limited by capillarity rather than by the ability to detect bubbles assumed in the preceding section. At test pressures of an atmosphere or more, however, it is likely to be the ability to detect small slowly formed bubbles that limits the sensitiveness of the method.

If a soap solution is applied as a film to the leak the surface tension of the solution is only about $1/3$ that for water. This advantage is largely neutralized, however, by the presence of two bubble surfaces. The sensitiveness of immersion tests may be increased somewhat by the addition of a wetting agent to the bath.

The advantages of pressure testing at high pressure may be summarized by noting that on doubling the applied gage pressure;

- (1) All leakage rates are more than doubled, and observable bubble formation occurs at leaks of less than half the rate originally detectable.
- (2) The minimum diameter of capillary leak at which a bubble can form is halved.
- (3) The rate of leakage into a vacuum of this smaller leak is about $1/15$ that of the original limiting leak.

8.3 Correction to Leakage Flow Formula

In a recent study of leakage flow, G.W. Monk has pointed out that Knudsen's general formula of flow is not rigorously applicable since it is derived on the assumption of small pressure drop along the flow path.

Starting with the generalized formula

$$dQ' = a P' dP' + b \frac{1 + c_1 P'}{1 + c_2 P'} dP' \text{ ----- (106)}$$

where

$$a = \frac{\pi D^4}{128 \eta H}$$

$$b = \frac{\sqrt{\pi} D^3}{6 \eta \sqrt{P_1} H}$$

$$c_1 = \frac{\sqrt{P_1}}{\eta} D$$

$$c_2 = 1.24 \frac{\sqrt{P_1}}{\eta} D$$

P' = Average pressure

and all units are the same as in equation (91), and integrating between the pressures P_1 and P_2 , one obtains for the rate of flow under large pressure drop,

$$Q' = \frac{\pi}{2} (P_1^2 - P_2^2) + b \frac{c_1}{c_2} (P_1' - P_2') + \frac{b}{c_2} \left(1 - \frac{c_1}{c_2}\right) \log \frac{1 + c_2 P_1'}{1 + c_2 P_2'} \text{ (107)}$$

Assuming again that the discharge pressure is limited by the sonic velocity of the emerging jet, this equation may be approximately solved for a given gas and given leakage paths. The results of such calculations for the leakage of air through channels 1 cm long and of various diameters are shown in table 7.

Comparing the leakage rates and critical discharge pressures as calculated from this formula with those calculated from the usual Knudsen Formula it is seen that the results are not significantly different within the limits of accuracy usually required.

Table 7

Comparative Values of Critical Discharge Pressure and Leakage Rate
as Computed by Knudsen's Equation and the More General Integrated
Equation

Leakage Path Diameter (cm)	Critical Discharge Pressure (μ)		Leakage Rate cc- μ /sec	
	Knudsen's Eq.	Integrated Eq.	Knudsen's Eq.	Integrated Eq.
0.2×10^{-4}	6.58	7.0	.000071	.000075
2×10^{-4}	133	133	.144	.143
20×10^{-4}	8.38×10^3	7.12×10^3	905	873

Appendix A. BASIC RELATIONS OF KINETIC THEORY

Throughout the preceding derivations certain relations based on the kinetic theory of gases have been accepted as true and applied without proof or derivation. These relations are here briefly presented.

A.1. Density of a Gas

(a) For any gas at 0°C and 760 mm pressure, the volume of a gram molecule is

$$V_0 = 22,414 \text{ cc}$$

and for a gas of molecular weight M , the density at standard temperature and pressure is

$$\rho' = M/V_0 = 4.461 \times 10^{-5} M \text{ g/cc}$$

(b) For a perfect gas, $PV = RT$, and the density at any pressure P' and any temperature T is

$$\rho' = \frac{M}{V} = \frac{MP'}{P_0 V_0} = \frac{MP'}{R_M T} \text{ g/cc}$$

where M = molecular weight of gas, P' = pressure in dynes/cm², R_M = gas constant per gram molecule = 83.15×10^6 dyne cm/degree, and T = temperature in degrees Kelvin.

(c) Molecular density = number of molecules/cc:

$$N_0 = 2.686 \times 10^{19} \text{ molecules/cc at } 0^\circ\text{C and } 760 \text{ mm pressure}$$

and

$$N = \rho'/m$$

at any density ρ' , where m = weight per molecule in grams.

A.2. Molecular Speeds

Let α = most probable speed, $\bar{u}$ = average speed, and U = root mean square speed, i.e.,

$$\alpha : \bar{u} : U = 1 : 2/\sqrt{\pi} : \sqrt{3/2}$$

Since

$$P' = \frac{1}{3} N m \bar{u}^2 = \frac{1}{3} \rho' \bar{u}^2$$

Then

$$u = \sqrt{\bar{u}^2} = \sqrt{\frac{3P'}{\rho'}} = \sqrt{\frac{3}{\rho_1}}$$

$$\bar{u} = \sqrt{\frac{8}{\pi \rho_1}}$$

$$\alpha = \sqrt{\frac{2}{\rho_1}}$$

where ρ_1 = gas density in g/cc at 1 micobar pressure.

A.3. Distribution of Molecular Speeds

If n' = number of molecules per cc having speed between u and $u+du$,

$$n' = \frac{4N}{\alpha^3 \sqrt{\pi}} u e^{-u^2/\alpha^2} du$$

A.4. Molecular Mean Free Path- Maxwellian Gas

The mean free path L is given by

$$L = \frac{1}{\sqrt{2} \pi \sigma^2 N}$$

where σ = molecular diameter. Also, since viscosity

$$\eta = 1/3 N m \bar{u} L \quad \text{and} \quad N m = \rho'$$

Then

$$L = \frac{3\eta}{\rho' \bar{u}} = \frac{3\sqrt{\pi}}{\sqrt{8}} \frac{\eta}{\sqrt{P' \rho'}} = \frac{1.88 \eta}{\sqrt{P' \rho'}}$$

η and ρ' having values corresponding to P' .

A.5. Collision Frequency- Maxwellian Gas

If Z = collision frequency;

$$Z = \frac{\bar{u}}{L} = \sqrt{2} n \bar{u} \sigma^2$$

A.6. Viscosity

$$\eta = \frac{1}{3} L N \bar{u} m = \frac{1}{3\sqrt{2}} \frac{\bar{u} N}{\pi \sigma^2 N_A}$$

Appendix B. EQUIVALENTS AND CONSTANTS

Table 1. Pressure Equivalents

UNIT	Microns of Mercury	Dynes/Cm ² (Microbars)	MM of Mercury	Inches of Hg	Lbs/In ²	Atmospheres
Microns Hg	1	1.333	0.001	3.937×10^{-5}	1.934×10^{-5}	1.316×10^{-6}
Dyne/Cm ² (Microbar)	0.75	1	7.50×10^{-4}	3.953×10^{-5}	1.45×10^{-5}	9.869×10^{-7}
MM of Hg	1000	1333	1	3.937×10^{-2}	1.934×10^{-2}	1.316×10^{-3}
In of Hg	25400	33860	25.4	1	0.491	0.0334
Lbs/In ²	51715	68950	51.715	2.036	1	0.0680
Atmos.	7.6×10^5	1.013×10^6	760	29.921	14.696	1

Table 2. Volume Equivalents

UNIT	CC	Liter	Cu In	Cu Ft	Qt	Gal
CC	1	0.001	0.0608	3.5×10^{-5}	1.057×10^{-3}	2.642×10^{-4}
Liter	1000	1	60.8	0.0353	1.057	0.2642
Cu In	16.387	0.0164	1	5.79×10^{-4}	1.732×10^{-2}	4.33×10^{-4}
Cu Ft	28300	28.3	1728	1	29.92	7.48
Qt	946.4	0.9464	57.75	0.0334	1	0.25
Gal	3785	3.785	231	0.1336	4	1

Table 3. Volumetric Speed Equivalents

UNIT	CC/Sec	L/Sec	Cu Ft/Min	Gal/Min
CC/Sec	1	0.001	2.13×10^{-3}	1.585×10^{-2}
L/Sec	1000	1	2.13	15.85
Cu Ft/Min	470	0.47	1	7.48
Gal/Min	63.1	0.0631	0.1336	1

Table 4. Constants of Selected Gases and Vapors

Gas or Vapor	Mol. Wt. M	Wt./Molecule (10 ⁻²⁴ Gr)	Mean Mol. Diam. (10 ⁻⁸ cm)	Sp. Gr. (Air = 1)	Viscosity @ 20 C (10 ⁻⁶ Poise)	Gas Constant (10 ⁶ ergs/°C)
Air	28.96	48.11	3.72	1	181	2.871
Chlorine	70.91			2.486	147	1.172
Helium	4.002	6.648	2.18	.138	196	20.78
Hydrogen	2.016	3.349	2.74	.0695	89	41.25
Nitrogen	28.02	46.54	3.75	.969	184	2.968
Oxygen	32.00	53.16	3.61	1.155	209	2.598
Mercury	200.6	333.2	4.26		494 @ 273 °C	.4145
Water	36	59.86	4.60	.622	98	

Table 5. Absolute Density of Air

$$P = \frac{0.001293}{1-0.00367T} \frac{H}{76}, \quad H = \text{Pres. (cm)}. \quad T = \text{Temp. (°C)}$$

PRESSURE	TEMPERATURE	DENSITY		
		Gr/CC	Gr/L	Lbs/ Cu Ft
1 Atmos. = 760 mm = 29.86 In. Hg. = 14.7 Lbs/In	0 °C = 32 °F	1.293x10 ³	1.293	.0807
1 Atmos.	20 °C = 68 °F	1.205x10 ³	1.205	.0752
1 Dyna/cm ² = 1 Microbar	20 °C	1.182x10 ⁹	1.182x10 ⁶	7.38x10 ⁸
1 Micron Hg	20 °C	1.5756x10 ⁹	1.5756x10 ⁶	9.83x10 ⁸
1 Lb/ In ²	68 °F	8.21x10 ⁵	.0821	.00512

Appendix C. SYMBOLS

A	- cross sectional area - cm^2	R_M	- gas constant/mole
A'	- cross sectional area - in^2	R_e	- Reynold's number
a	- coordinate, constant	R_r	- rate of rise
A_a	- cross sectional area of approach region - cm^2	r	- coordinate
A'_a	- cross sectional area of approach region - in^2	s	- coordinate
B	- volumetric flow rate - L/sec	t	- time - sec
B'	- volumetric flow rate - cc/sec	T	- absolute temperature - kelvin
b	- coordinate, constant	T'	- absolute temperature - Rankine
C	- conductance - L/sec	U	- velocity - cm/sec
C'	- Knudsen conductance - cc/sec	u	- velocity - cm/sec
C''	- Langmuir conductance - cc/sec	u'	- velocity - ft/sec
C_a	- conductance of aperture - L/sec	u_s	- velocity of slip
C'_a	- conductance of aperture - cc/sec	V	- volume - liters
C_p	- conductance of pipe - L/sec	v	- volume - liters
C'_p	- conductance of pipe cc/sec	W	- rate of flow - lb/hr
c	- coordinate, constant	x	- coordinate
d	- diameter - in	Y	- radical term
D	- diameter - cm	y	- coordinate
D'	- diameter - ft	z	- coordinate
E	- Knudsen resistance	α	- most probable speed
E'	- Langmuir resistance	ρ	- density - g/L
f	- friction factor	ρ'	- density - lb/cc
f_v	- correction factor for velocity of approach	ρ_j	- density at unit pressure g/cc @ 1 dyne/cm ²
F	- fraction of molecules	η	- coefficient of viscosity - poises (dyne sec/cm ²)
Q	- mass flow rate - g/sec	μ	- absolute dynamic viscosity - centipoise
H	- length - cm	μ_s	- absolute dynamic viscosity lbs/ft-sec
H'	- length - ft	$\int$	- coefficient of slip
h	- length	Θ	- coordinate
J	- coefficient of discharge	Φ	- coordinate
K	- constant, adiabatic expansion coefficient	γ	- surface tension - dynes/cm
L	- mean free path - cm		
L'	- mean free path - in		
M	- molecular weight		
m	- mass of molecule		
N	- number of molecules		
O	- perimeter - cm		
O'	- perimeter - in		
P	- pressure - microns		
P'	- pressure - microbars (dynes/cm ²)		
P''	- pressure - lbs/in ²		
Q	- quantity flow rate - L microns/sec		
Q'	- quantity flow rate - cc microbars/sec		
Q''	- quantity flow rate - gal/min		
R	- gas constant		
R	- radius - cm		
r	- radius - cm		
R'	- radius - in		

APPENDIX D

Practical working formulas are tabulated in the following pages.

- (1) Formulas apply to air at 20°C.
- (2) Pressures are expressed in microns;
- (3) Cross section dimensions are in inches,
- (4) Distances along path of flow are in feet.
- (5) Conductances are in L/sec.

Only conductances are given. Rates of flow in various units can be derived from:

$$(a) \quad Q = C (P_1 - P_2) \quad \text{L-microns/sec.}$$

$$(b) \quad B = C \frac{P_1 - P_2}{P} \quad \text{L/sec at } P \text{ microns.}$$

$$(c) \quad G = CP_1 (P_1 - P_2) \quad \text{g/sec.}$$

1.1 Turbulent Flow

Flow is turbulent if

$$Q > 274 \times 10^3 d$$

For systems of practical proportions this is true only at pressures of several cm Hg or more, and turbulent flow may be neglected.

Assuming laminar flow when turbulence actually exists introduces a positive, but usually negligible, error.

1.2 Laminar Flow

$$C = .246 \frac{d^4}{H} \bar{P}$$

Flow is laminar, within 10%, if

$$\bar{P} d > 216$$

1.3 Molecular Flow

$$C = 6.54 \frac{d^3}{H}$$

Flow is molecular, within 10%, if

$$\bar{P} d < 6.93$$

1.4 Laminar, Molecular, and Transition

$$C = .246 \frac{d^4}{H^4} \bar{P} + 6.54 \frac{d^3}{H^3} \frac{(1 + .64 \frac{d}{\bar{P}})}{(1 + .79 \frac{d}{\bar{P}})}$$

Reduces to (1.2), within 10%, if $\bar{P} d > 216$

Reduces to (1.3), within 10%, if $\bar{P} d < 6.93$

Apply in complete form, for 10% accuracy or better, or if $216 > \bar{P} d > 6.93$

2. Long Uniform Ducts of Non-Circular Section

2.1 Turbulent Flow - Treat as laminar.

2.2 Laminar Flow - No rigorous solution. Occurrence unlikely. As approximation, treat as pipe of circular section having equal area.

2.3 Molecular Flow

(a) First approximation: Treat as circular pipe having $d^3 = \frac{16A^2}{\pi^3}$.

Error: - 11% for square section; - 15% for 1x2 rectangular section; - 30% for 1x5 rectangular section.

(b) Second approximation: Multiply (2.3(a)) by K from table 1, p. 30.

2.4 Transition Flow

Approximation: Treat first term as in (2.2), second term as in (2.3).

3. Non-Uniform Pipes and Ducts. (Uniformly Flaring)

Approximation, good if ratio of flare to small end dimension is small. Treat as uniform pipe or duct having section equal to that at the mid-point.

4. Flow Through Small Aperture

An aperture may be considered small if its area is less than 1/3 that of the section through which the gas approaches the aperture.

4.1 Laminar

Flow is laminar if $L < d$, i.e. if $P_1 d > 3.15$

$$C = 323 A^4 Y$$

$$Y = .254 \text{ if } P_2/P_1 < .53 \text{ (.53 = critical pressure ratio)}$$

Y obtained from graph, p. 49, if

$$P_2/P_1 > .53.$$

4.2 Molecular Flow

$$C = 75 A'$$

Flow is molecular if $L \geq 10 d$, i.e. if $P_1 d \leq .315$

5. Flow Through Large Aperture

If the area of an aperture is greater than 1/3 the sectional area (A_g) through which the gas approaches the aperture, it is considered large.

5.1 Laminar Flow

Multiply small aperture conductance by $f_v = \frac{A_a}{\sqrt{A_g^2 - A_a^2}}$

5.2 Molecular Flow

Application of the same correction (5.1) has not been fully justified. It corrects in the right direction, however, and approaches the proper limits.

6. Short Circular Pipe

A pipe is considered short if

$$\frac{H'}{d} \leq 1.1 \quad (\text{Error } 10\% \text{ or less})$$

6.1 Molecular Flow

- (a) Determine C_1 as for long pipe (1.3)
- (b) Determine C_2 of aperture at pipe entrance (4.2 or 5.2)
- (c) Combine C_1 and C_2 for short pipe conductance

$$\frac{1}{C} = \frac{1}{C_1} + \frac{1}{C_2}$$

or, combining (1.3) and (4.2) analytically

$$C = \frac{75A'}{1 + 9 H'/d} \quad \text{may be applied directly.}$$

6.2 Viscous Flow

No reliable treatment of this case is given. The over-correction resulting from applying (6.1) introduces a larger error than the simple neglect of all end effect (see pp. 60-61).

7. Short non-circular and non-uniform pipes and ducts.

7.1 Molecular Flow

Treat as long pipes or ducts, (2.3 and 3) and apply aperture correction as in 6.1.

7.2 Viscous Flow

Neglect end effect.

8. Series Conductances

$$\frac{1}{C} = \frac{1}{C_1} + \frac{1}{C_2} + \text{etc.}$$

9. Evacuation Rate

$$t = \frac{V}{S_p} \log_e \frac{P_1}{P_2}$$

V = volume of system evacuated (L)

S_p = volumetric speed of pumping from the system (L/sec)

P_1 = initial pressure (μ)

P_2 = final pressure (μ)

t = time required to reduce pressure from P_1 to P_2 (sec)

To reduce pressure from any value to half that value requires

$$t_{1/2} = 0.7 \frac{V}{S_p} \text{ sec.}$$