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STABILITY OF BUNCHED BEAMS
AGAINST HIGH ORDER NONLINEAR RESONANCES

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Abstract

For a given tune, beams can be made stable against the effects of isolated high order resonances. However, in a system whose betatron tune is changing, there exists a mechanism for beam loss. This mechanism is discussed and applied to the case of bunched beams, where the synchrotron motion introduces a time variation of tune. A stability criterion for a system with varying tune is presented, together with a set of suggested parameter values. The basic constraint is that the linear tune spread in the bunches (arising through the chromaticity) be smaller than the nonlinear tune spread (arising from the nonlinearities of magnetic field as well as the beam-beam interaction). This can be achieved with a small chromaticity and a relatively large nonlinear term causing a tune spread with betatron amplitude. Specific values for chromaticity and nonlinearity are suggested. A limit on the resonance excitation strength is also given. Since the beam-beam interaction introduces a tune spread with amplitude, then in some range it is found to have a stabilizing influence insofar as isolated high order nonlinear resonances are concerned. This point is discussed together with a brief discussion on the onset of stochastic phenomena, which are due to the interaction of many resonances. It is concluded that with the given set of parameters, bunched beams, stable against high order nonlinear resonances can be achieved.

1. Introduction

The effects of isolated nonlinear resonances on particle beams have been extensively studied.¹ In general, for resonances of order higher than the fourth, a static betatron system (i.e. tune independent of time) is stable for small amplitudes. One may put limits on nonlinearities by insisting that this small amplitude stable region include the entire aperture available to the beam. This can be done in establishing design criteria for accelerating or storage devices. However, as pointed out in

Ref. (1), for a system whose betatron tune is not fixed, there exists a mechanism for beam loss, a mechanism which is not directly related to the "stability limits" just referred to. Qualitatively, the mechanism is as follows: If a particle passes through resonance, meaning that its tune passes through some rational fraction, then there is a finite probability that this particle will be trapped in a stable island of phase space which is moving to phase-space regions of larger betatron amplitudes (where amplitudes are measured with respect to the beam equilibrium orbit). If the island reaches an aperture, the particle will be lost. Of course, the trapping mechanism is a complicated process and depends on, among other factors, the rate of tune variation as the resonance is crossed. Roughly, the slower the crossing, the greater the trapping probability.

Fast resonance crossing does not obviously lead to stability in the case of periodic resonance crossing, as occurs when synchrotron motion is superimposed on the betatron motion. The basic requirement is a high synchrotron frequency. But this allows a serious coupling of synchrotron to betatron motion, with the coupling leading to the introduction of satellite stopbands.² Thus, in taking this road to solve the resonance crossing problem, many new difficulties are encountered; they are dealt with in detail in Ref. (2).

We consider a different approach. We use the fact that, although the size of the moving islands is a function of the excitation width of the resonance, their movement in the phase space is a function only of the tune swing due to the synchrotron oscillations and of the nonlinear tune shift due to the 0th harmonic of the nonlinearities. By designing for a small tune swing, i.e. small chromaticity; and a large tune shift with betatron amplitude, the motion of the problematic islands can be reduced to an acceptable amount. For example, we can choose parameters so that the islands move only a given percentage across the beam, say 10%. Thus, if the beam is centered on a resonance, the islands move within the beam only, i.e. only within a small fraction, 10%, of the beam. Although there is a fluctuation of density within the beam, the mechanism for transport of particles outside the beam is effectively eliminated. If a resonance is just near the beam edge, there may be a small increase in beam emittance. Of course, in addition to its motion, the size of the islands must be limited. This gives a limit on the excitation strength of the resonance.

Our suggestion for stabilization is thus, first, to have a very small tune spread (that is, spread due to chromaticity) in the beam, say 10^{-3} ; second, to place the beam well away from low order resonances; and, third, to put some limit on the strength of nearby resonances. For example, we could place the beam between 10^{th} and 11^{th} order resonances. 0^{th} harmonic nonlinear fields can be employed to give a nonlinear tune spread, say 10^{-2} , ten times larger than the chromaticity induced linear tune spread. This provides the stability against isolated nonlinear resonance crossing. (Paranthetically, we point out that nonlinear tune spread is also an effective agent for Landau damping collective instabilities. Thus, if the transverse resistive wall instability, as well as the head-tail instability can be successfully damped with a linear tune spread of 10^{-2} , then roughly speaking, a nonlinear tune spread of this amount will also be adequate.) The island areas for the harmful resonances (order > 11) will generally be small since the excitation widths are much smaller than the nonlinear detuning available. A more detailed presentation of the stabilizing mechanism, with the parameters, constraints, and a discussion of the influence of the speed of crossing the resonances, will be given in Section 2. For simplicity, we limit ourselves to motion in only one transverse dimension. Nonlinear resonances excited by magnetic field errors as well as those excited by the beam-beam interaction will be considered. With regard to the beam-beam interaction, since it introduces a comparatively large nonlinear tune spread in the beam, we find that it has an intrinsic stabilizing tendency. On the other hand, many high order resonances are excited. This characteristic leads us to a brief discussion of the criterion for the simultaneous excitation of many resonances, or equivalently, the condition for the onset of stochastic phenomena.

We assume here that nonlinear motion is dominated by the effects of isolated resonances and any long-time blowup mechanism for the beam is correlated with the onset of stochasticity. Thus, these latter effects are avoided by designing a system sufficiently below the stochasticity limit.

2. Nonlinear Resonances

Invariant in the Case of an Isolated Nonlinear Resonance with Octupole Detuning

Using either phase-amplitude or Hamiltonian methods, we can derive the one-dimensional invariant for a particle in the presence of an isolated resonance and octupole detuning. In terms of the amplitude and phase variables γ and β , we have for the invariant,¹

$$I = \gamma \Delta_L + \frac{1}{2} \Delta_{NL} \gamma^2 + \frac{2}{p} \Delta_e \gamma^{p/2} \cos p\beta ; \quad (2.1)$$

$$\text{where } \gamma = \epsilon / \epsilon_0 ; \quad (2.2)$$

ϵ is the square of the amplitude, or the usual emittance of the particle motion (which may, of course, change in the course of its trajectory when nonlinearities are present); ϵ_0 is a reference emittance, which may be taken to be the beam emittance, ϵ_B , i.e. the emittance boundary for the beam;

β is a phase space angle (related to position and angle of a particle along its equilibrium orbit);

$$\Delta_L = \nu - n/p ; \quad (2.3)$$

ν is the linear tune, which is a function of the particle momentum through the chromaticity of the ring;

n is the harmonic of the exciting resonance;

p is the order of the exciting resonance;

$$\Delta_{NL} = -3/8 < \beta >^3 \nu \bar{O} \epsilon_0 ; \quad (2.4)$$

Δ_{NL} is the nonlinear tune shift (i.e. detuning) at the reference emittance; $< \beta > = R/\nu$ is the average β -function;

R is the average radius of the ring; $\bar{O}$ is the strength of the octupole field, related to the magnetic field $B(\theta, x)$:

$$\frac{1}{(B\rho)} B(\theta, x) = \dots + O(\theta)x^3 + \dots + g(\theta)x^{p-1} + \dots ; \quad (2.5)$$

through the octupole distribution function, $O(\theta)$. x is the particle position in betatron phase space; θ is the azimuth of the ring; $g(\theta)$ provides the resonance excitation; if the zero of the azimuth is chosen appropriately, the n^{th} harmonic of $g(\theta)$ is

$$g_n = \frac{1}{\pi} \int_0^{2\pi} g(\theta) \cos n\theta d\theta ; \quad (2.6)$$

while $\bar{0}$ is related to $0(\theta)$ by

$$\bar{0} = \frac{1}{2\pi} \int_0^{2\pi} 0(\theta) d\theta ; \quad (2.7)$$

and Δ_e is the resonance excitation width,

$$\Delta_e = -\frac{1}{2^p} < \beta >^{(p/2+1)} v_{g_n} e_o^{(p/2-1)} . \quad (2.8)$$

Fixed Points

The fixed points are obtained from

$$\frac{\partial I}{\partial \gamma} = \frac{\partial I}{\partial \beta} = 0 ; \quad (2.9)$$

or,

$$\Delta_L + \Delta_{NL} \gamma \pm \Delta_e \gamma^{(p/2-1)} = 0 . \quad (2.10)$$

Normal Stability Limit ($p > 4$)

We can find the usual stability limit roughly by taking $\Delta_L = 0$. So, we have

$$\gamma_{st.limit} \approx \left| \frac{\Delta_{NL}}{\Delta_e} \right|^{[2/(p-4)]} . \quad (2.11)$$

At this amplitude, we have a series of p "unstable" fixed points (unstable in the sense that detuning will result only from terms of higher order than p). At such amplitudes, particles can be drawn to much larger amplitudes and in this sense become unstable.

We will assume that $\gamma_{st.limit}$ is well outside the machine aperture, and so provides no concern for us.

Stable Islands ($p > 4$)

Equation (2.10) also gives a set of $2p$ fixed points nearer to the origin, p of them are stable fixed points, p unstable. The topology in the phase space is a set of p stable islands wrapped around the origin like a string of pearls. The position of the fixed points defining the islands can be evaluated in a series in the variables Δ_L/Δ_{NL} and Δ_e/Δ_{NL} . Keeping the first two terms in this expansion, we have for the stable and unstable fixed points, γ_s and γ_u respectively,

$$\gamma_s = \left| \frac{\Delta_L}{\Delta_{NL}} \right| + \left| \frac{\Delta_e}{\Delta_{NL}} \right| \left| \frac{\Delta_L}{\Delta_{NL}} \right|^{(p/2-1)}, \quad (2.12)$$

$$\gamma_u = \left| \frac{\Delta_L}{\Delta_{NL}} \right| - \left| \frac{\Delta_e}{\Delta_{NL}} \right| \left| \frac{\Delta_L}{\Delta_{NL}} \right|^{(p/2-1)}. \quad (2.13)$$

Note that these islands only appear on "one side of the resonance", i.e. given a sign of the detuning, Δ_{NL} , the side of the resonance where they appear is that side defined by Δ_L being of the opposite sign. Thus,

$$\frac{\Delta_L}{\Delta_{NL}} < 0. \quad (2.14)$$

For our purposes, we can neglect the second term in the expansion, and define the island position by

$$\gamma_{IS} = \left| \frac{\Delta_L}{\Delta_{NL}} \right|, \quad (2.15)$$

which is independent of the resonance excitation width, Δ_e .

Dynamic Variation of Linear Tune

If the particles in the beam are executing synchrotron oscillations, then Δ_L is a function of time. Let $\bar{\Delta}_L$ be the peak value of the linear shift from resonance. Then, for those particles having the largest synchrotron amplitude, during one synchrotron cycle, Δ_L will change by this amount. Thus, assuming the resonance tune to be the center of the tune swing, the islands will move from the origin to

$$\gamma_{\max} = \left| \frac{\bar{\Delta}_L}{\Delta_{NL}} \right| \quad (2.16)$$

in one synchrotron cycle. If $\gamma_{\max}$ is outside the beam, then any particles trapped by these stable islands will be transported outside the beam. Of course, in this model, they will be transported back also if no aperture is encountered. Thus, it would appear that $\gamma_{\max} < \gamma_{\text{aperture}}$ is a stability condition. However, in a real machine, one might expect some particle leakage while the islands are in transit. A more realistic stability criterion would be that the islands stay well within the beam. Thus, slow

transfer of particles in and out of the island area would have no consequence in terms of growth of the beam or particle loss. We thus set

$$\gamma_{\max} < \gamma_{\text{beam}} \quad , \quad (2.17)$$

or, with $\gamma_{\text{beam}} = 1$ (i.e. taking the reference amplitude as the beam amplitude),

$$\Delta_{\text{NL}} > \bar{\Delta}_{\text{L}} \quad , \quad (2.18)$$

where Δ_{NL} is the nonlinear detuning at the beam edge. We can also interpret Δ_{NL} as the nonlinear tune spread in the beam, and $\bar{\Delta}_{\text{L}}$ as the linear tune spread arising from the chromaticity.

Particle Loss and Speed of Resonance Crossing

Thus, we see that high order nonlinear resonances can provide beam loss even though the "normal stability limits" are well outside the aperture.¹ Particles are simply transported from the beam to the aperture in stable islands formed within the beam. This will occur in the case where synchrotron oscillations are present if

$$\gamma_{\max} > \gamma_{\text{aperture}} \quad . \quad (2.19)$$

The question of how many particles will be transported, i.e. the loss rate, is in general a very difficult one. Clearly, however, the maximum loss rate is related to the area of the pearls relative to the phase space area of the beam. Loss is also obviously related to the synchrotron frequency, or the rate of resonance crossing. In the limit of fast crossing, the mechanism is not applicable in that presumably no particles are actually trapped in the islands. However, other more subtle effects are introduced, i.e. the formation of satellite stopbands. In the slow crossing limit, the transport will be fully effective, and assuming a uniform particle density in the phase space, the loss rate is determined by the area of phase space, relative to the beam area, transported out of the beam. The fraction of particles transported per crossing is approximately

$$r = \frac{\text{Area of pearls at beam edge}}{\text{Beam Area}} = \frac{2/2 \pi}{\sqrt{p}} \left| \frac{\Delta_e}{\Delta_{\text{NL}}} \right|^{\frac{1}{2}} \quad , \quad (2.20)$$

where Δ_e and Δ_{NL} are the excitation width and the detuning at the beam edge. The efficiency of the process will diminish in a complicated way as the speed of resonance crossing increases. In the limit of fast crossing the transport process is conceptually inefficient. However, the whole picture of fixed points and transport through stable islands is inapplicable and a different formulation is required.² A quantitative criterion for fast crossing is¹

$$\nu_s \gg 8\pi \left(\frac{\Delta_{NL}}{\bar{\Delta}_L} \right) \Delta_e, \quad (2.21)$$

where ν_s is the synchrotron number (number of synchrotron oscillations per particle revolution); while Δ_{NL} and Δ_e are maximum nonlinear detuning and resonance excitation widths, respectively, i.e. at the beam edge. Since our approach to stability requires $\Delta_{NL} > \bar{\Delta}_L$, we are led to the situation where the picture of stable islands and fixed points are valid.

Stability in the Presence of Dynamic Tune Variation

Our stability requirement that the nonlinear tune spread be much larger than the chromaticity induced linear tune spread can be achieved with sextupoles controlling the chromaticity and octupoles providing the necessary nonlinear detuning. The criterion, $\Delta_{NL} > \bar{\Delta}_L$ is independent of the excitation width, Δ_e . This has occurred only because we have presumed terms in Δ_e/Δ_{NL} to be small in going from (2.13) and (2.14) to (2.15). We can include a limit on the size of Δ_e using (2.20). From the condition that the ratio of pearl area to beam area be less than one, we obtain

$$\left| \frac{\Delta_e}{\Delta_{NL}} \right| < \frac{p}{8\pi^2}, \quad (2.22)$$

where in this equation, Δ_e and Δ_{NL} are to be evaluated at the maximum amplitude of the islands, i.e. at

$$\gamma_{\max} = \frac{\bar{\Delta}_L}{\Delta_{NL}}.$$

Resonances Excited by Magnetic Field Errors

From (2.4) and (2.18), to achieve stability, the octupole field, $\bar{O}$, must satisfy

$$\bar{O} > \frac{8 \bar{\Delta}_L}{3 < \beta >^3 v_{eB}}, \quad (2.23)$$

where e_B is the transverse beam emittance. Using the parameters for a bunched beam storage ring design, given in Ref. (3), $R = 385$ m, $\nu = 20$, $e_B = 0.4 \times 10^{-6}$ m-rad (for ~ 200 GeV particle energy), we obtain

$$\bar{O} > 46.7 \bar{\Delta}_L m^{-4} ;$$

or, taking $\bar{\Delta}_L = 10^{-3}$,

$$\bar{O} > 0.05 m^{-4} .$$

If the octupole field is distributed over the entire circumference, this corresponds to a field at the pole tip of

$$B_{\text{pole}}(\text{octupole}) > \bar{O} p r^3 = 5 \text{ gauss} ;$$

where the momentum, $p = 2/3 \times 10^4$ kG-m and the radius to the pole, $r = 2.5$ cm. More realistically, the field should be taken to be distributed over, say, 10% of the circumference. So, with $\bar{O}$ chosen to be 10 times above the limiting value ($\bar{O} = 0.5 m^{-4}$),

$$B_{\text{pole}}(\text{octupole}) \geq 0.5 \text{ kG} .$$

As implied, the nonlinear detuning is 10 times larger than the assumed linear tune spread, $\bar{\Delta}_L$; that is,

$$\Delta_{NL} = 0.01 .$$

For any amplitude, denoted e_o ,

$$\Delta_{NL} = 0.01 \left(\frac{e_o}{e_B} \right) .$$

We have under these circumstances that the islands move only 1/10 (in the emittance) to the beam edge. At this point ($e_o = 1/10 e_B$), $\Delta_{NL} = 10^{-3}$. The condition on the resonance width, from (2.22), is, for an 8th order resonance,

$$\left| \frac{\Delta_e}{\Delta_{NL}} \right| < \frac{1}{\pi^2} ,$$

where Δ_e is given in (2.8),

$$\Delta_e = \frac{1}{2^8} < \beta >^5 e_o^3 \nu g_n . \quad (2.24)$$

the constraint on the azimuthal harmonic, g_n is

$$g_n < \frac{96}{\pi^2} \frac{\bar{0}}{< \beta >^2 e_o^2} \quad , \quad (2.25)$$

or

$$g_n < 8.2 \times 10^{12} \text{ m}^{-8} = 1.4 \text{ in.}^{-8} \quad .$$

Such a constraint is in reality no constraint at all as can be seen by comparing this result with tolerances quoted for a p^{th} order resonance in Ref. (4).

Beam-Beam Interaction

For the beam-beam interaction, we will use the calculations in Ref. (5). The nonlinear detuning at the beam edge due to the octupole term, for a circular beam, and assuming head-on collisions, is⁵

$$\Delta_{NL} = \frac{3}{8} \Delta_{bb} \quad , \quad (2.26)$$

where Δ_{bb} is the linear beam-beam tune shift for head-on collisions. The condition for maintaining the moving phase space islands within the beam is

$$\Delta_{bb} > \frac{8}{3} \bar{\Delta}_L \quad . \quad (2.27)$$

With $\bar{\Delta}_L = 10^{-3}$, we have $\Delta_{bb} > 2.7 \times 10^{-3}$. For the islands to move only through 10% of the beam area, we require $\Delta_{bb} > 2.7 \times 10^{-2}$. This corresponds to a nonlinear detuning, $\Delta_{NL} = 0.01$.

The condition on the resonance excitation width, Eq. (2.22), becomes, using (2.26)

$$\left| \frac{\Delta_e}{\Delta_{bb}} \right| < \frac{3p}{64\pi^2} \quad . \quad (2.28)$$

The limit on the resonance width, (2.28), is, in essence, independent of the strength of the interaction, which is represented by the linear beam-beam tune shift, Δ_{bb} . This is so since Δ_e itself is proportional to Δ_{bb} , the latter dropping out of the condition (2.28). Thus, the strength, when considering isolated resonances, primarily determines the amount of detuning, that is, the distance the islands will move within the beam during a synchrotron period.

Consider the case of a 10^{th} order resonance, $p = 10$. Equation (2.28) gives

$$\left| \frac{\Delta_e}{\Delta_{bb}} \right| < 0.05 \quad .$$

From Ref. (5), at an amplitude $\frac{1}{2}$ of the beam amplitude, which corresponds to about 7% of the beam in the emittance (or squared amplitude),

$$\left| \frac{\Delta_e}{\Delta_{bb}} \right| \approx 0.07 \times 10^{-2} \quad ,$$

for the worst 10^{th} order resonance. However, the ratio increases rapidly, and at the beam edge (i.e. 2 rms amplitude), we have

$$\left| \frac{\Delta_e}{\Delta_{bb}} \right| \approx 0.06 \quad ,$$

which means that the stable islands have reached the betatron size of the beam itself. Thus, the importance of staying away from even a 10^{th} order resonance is obvious. We point out that these results should be understood as somewhat qualitative in that the calculations in Ref. (5) are given, not for a circular beam, but for a beam in which the vertical radius is $\frac{1}{2}$ the horizontal radius, and further, because the "worst" resonance, cited above, is actually a two-dimensional resonance.

Thus, we can conclude that a nonlinear detuning of the order of 10^{-2} or higher is required if the linear beam tune spread is 10^{-3} . This could be achieved by magnetic fields, which we have seen could provide such a magnitude of detuning. Since the beam-beam interaction could also provide this detuning, it is important to keep the sign of the magnetic detuning the same as that arising from the beam-beam interaction. For proton-proton collisions, the beam-beam detuning is negative, which means a negative detuning from the ring octupoles.

The limit on the beam-beam interaction must come, not from the consequences of isolated resonances, but from phenomena related to the interaction of many resonances. In particular, if we can estimate a strength required for the onset of nonlinear stochastic phenomena, then we can set this as an upper limit to the beam-beam linear tune shift. The stochastic limit can be estimated as a ratio of the sum of

resonance excitation widths to the available tune space.⁵ In Ref. (5), the stochastic limit is estimated by taking the sum of resonance widths over a unit square (in the space horizontal tune x vertical tune). However, since we conceive of placing the beam in a small portion of this unit square, say, between 10th and 11th order resonances, all resonances up to 11th order must be removed. Now the partial sum of resonance widths in the full unit square, including only resonances of order higher than 11, can be obtained from Ref. (5), Fig. 1,

$$\Sigma_{12} = 2\Delta_{bb} \quad , \quad (2.29)$$

as compared to

$$\Sigma_{TOTAL} = 16\Delta_{bb} \quad . \quad (2.30)$$

If we assume that the average density of resonances above 11 is the same in our presumed operating region as in the whole tune square, then the stochastic limit condition

$$\Sigma_{12} < 1 \quad ,$$

can be written as a limit on the linear b-b tune shift,

$$\Delta_{bb} < 0.5 \quad . \quad (2.31)$$

This should be compared with the beam-beam limit obtained from (2.30),

$$\Delta_{bb} < 0.06 \quad . \quad (2.32)$$

The point is, of course, that by removing low order resonances, we can raise the stochastic limit, since the resonance widths diminish with order, the higher order resonances contributing less to the stochastic limit. Thus, with a beam-beam strength, $\Delta_{bb} = 2.5 \times 10^{-2}$, we are a factor of 20 below the stochastic limit as given in (2.31).

3. Conclusions

Consider the following set of design parameters:

1. linear beam tune spread, $\bar{\Delta}_L = 10^{-3}$,
2. nonlinear octupole detuning, $\Delta_{NL} = 10^{-2}$ (provided by octupole magnetic fields),
3. beam-beam linear tune shift, $\Delta_{bb} = 2.5 \times 10^{-2}$, and
4. synchrotron number, $\nu_s = 5 \times 10^{-4}$.

Under these conditions, we conclude that bunched beams can be made stable against the combination of high order nonlinear resonances and synchrotron oscillations.

References

1. A Schoch, CERN Report, CERN 57-23 (1958). We point out that there is an extensive literature on this subject, some work being referred to in this reference.
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