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A POSSIBLE METHOD FOR BEAM EXTRAC-
TION FROM AN ALTERNATING GRADIENT
SYNCHROTRON

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A POSSIBLE METHOD FOR BEAM EXTRACTION FROM AN
ALTERNATING GRADIENT SYNCHROTRON*

by

Loren R. McMurray and Daniel J. Zaffarano

ABSTRACT

A mathematical analysis of the effects of perturbations upon the guide field of a circular alternating gradient synchrotron was made to determine the amount and type of perturbation which would make the horizontal motion unstable, but would not affect the vertical motion to the extent that it would become unstable. The purpose of the analysis was to determine if the technique of beam extraction developed by Hammer and Bureau¹ for a conventional synchrotron would apply to an alternating gradient synchrotron.

The machine chosen for study contained fifty sectors with each sector consisting of an alternate horizontal defocusing semi-sector with a fall off index $n_1 = +440$ and a horizontal focusing semi-sector with a fall off index $n_2 = -440$. The focusing and defocusing semi-sector were of equal length.

Two main types of perturbations were studied. The first consisted of a single n-bump, i.e. only several consecutive sectors were perturbed, and the second consisted of a several n-bumps equally spaced around the machine. For the machine studied, the first type of perturbation was found to be the better of the two, and horizontal instability could

*This report is based on an M.S. Thesis by Loren R. McMurray submitted June, 1955, to Iowa State College, Ames, Iowa. This work was done under contract with the Atomic Energy Commission.

¹C. L. Hammer and A. J. Bureau. A Method for the Prompt Destruction of the Electron Beam in a Conventional Synchrotron. Ames, Iowa. Ames Laboratory report LR-286. To be published in the Review of Scientific Instruments. May or June, 1955.

be obtained by decreasing n_1 from + 440 to - 426 and n_2 from - 440 to - 464 over four consecutive sectors.

[Most of the method of analysis and some of the results will also apply to a circular machine of fifty sectors with fall off indices between the same integral and half-integral resonances as the chosen machine. For circular machines with other fall off indices, the method may be generalized, but the numerical results would have to be recomputed.]

I. INTRODUCTION

A. General

A technique for rapid beam extraction from a conventional synchrotron was recently developed by Hammer and Bureau.¹ After hearing this technique reported at a Mid-west Universities Research Association meeting, Dr. D. W. Kerst of the University of Illinois physics department wondered if this technique could be applied to an alternating gradient synchrotron, and requested that this possibility be examined. This report was originated in response to Dr. Kerst's request.

The Hammer and Bureau technique for beam extraction from a conventional synchrotron involves exciting a half-integral resonance. At this resonance, the electrons oscillate violently and the motion becomes unstable. The half-integral resonance is obtained by changing the fall off index over part of the orbit. The fall off index is changed by discharging a condenser through wires wrapped part way around the donut of the machine.

Due to the difficulty of making the analysis for alternating gradient synchrotrons in general, parameters for a typical theoretical alternating gradient synchrotron, hereafter referred to as AGS, were chosen as described in the abstract. The azimuthal arrangement of the semi-sectors is shown in Fig. 1 where θ is the azimuthal angle. Fig. 2 shows the area of the stability diagram surrounding the operating point. As seen, the operating point is within a diamond shaped region between integral and half-integral resonance lines.

The problem is to find a perturbation which makes the horizontal motion of an AGS unstable, but does not affect the vertical motion to the extent that it becomes unstable.

¹Ibid.

FIG. 1 AZIMUTHAL ARRANGEMENT OF SEMI-SECTORS

FIG.2 REGION OF STABILITY DIAGRAM NEAR OPERATING POINT OF UNPERTURBED AGS SHOWING RESONANCE LINES

Two main types of perturbations are studied. The first consists of perturbing only several consecutive sectors, denoted as a single fall off index bump, or simply an n -bump. The second is to divide mathematically the machine into R identical groups of S consecutive sectors each and to perturb Q consecutive sectors of each group, where R , S , and Q are positive integers. In both cases, the only perturbations considered are those such that all perturbed sectors are perturbed in an identical manner, and, for reasons of mathematical convenience, are such that the fall off index remains an even function with respect to an initial azimuthal reference angle.

Before proceeding to the analysis, a review of the mathematics used in the analysis is given.

B. Mathematical Background

The mathematical tools used in this analysis are quite well known^{1,2,3} so the emphasis here is upon a brief review of the results rather than a detailed derivation of the equations used. The solution of the differential equations of motion was put into matrix algebra form and conclusions were drawn from the study of the matrices involved.

¹J. J. Stoker. Nonlinear Vibrations in Mechanical and Electrical Systems. N.Y., Interscience Publishers, Inc. 1950. Chap. 6. Hill's Equation and its Application to the Study of the Stability of Nonlinear Oscillations. P. 189-222.

²E. D. Courant and H. S. Snyder. Theory of the Alternating-Gradient Synchrotron. Brookhaven National Laboratory, Associated Universities, Inc., June 1, 1953. Unpublished Report

³M. Hammermesh. The Alternating-Gradient Synchrotron. Argonne National Laboratory. Unpublished report 1954

If $y(\theta)$ is the horizontal distance and $z(\theta)$ the vertical distance of a particle from the equilibrium orbit at an azimuthal angle θ around the machine, then the differential equations of the motion of the particle, neglecting edge effects of the pair sectors, are

$$d^2y(\theta)/d\theta^2 + K_y(\theta) y(\theta) = 0, \quad (1)$$

$$d^2z(\theta)/d\theta^2 + K_z(\theta) z(\theta) = 0, \quad (2)$$

where $K_y(\theta) = 1 - n,$ (3)

$$K_z(\theta) = n, \quad (4)$$

$$\text{fall off index } n = -(r/B) \left. \frac{\partial B}{\partial y} \right|_y = 0 \quad (5)$$

and r is the radius of curvature of the equilibrium orbit and B is the magnetic field strength at the equilibrium orbit. Throughout the rest of the report, the subscripts y and z refer to the horizontal, or radial, and vertical motion, respectively. $K(\theta)$ is a periodic function of period two π or less. In the case of the unperturbed machine, the period of $K(\theta)$ is $2\pi/50$, as shown in Fig. 1.

The general solution of equation 1 can be written as

$$y(\theta) = y_1(\theta)y(0) + y_2(\theta)dy(0)/d\theta \quad (6)$$

so $dy(\theta)/d\theta = (dy_1(\theta)/d\theta) y(0) + (dy_2(\theta)/d\theta) dy(0)/d\theta$ (7)

where $y_1(\theta)$ and $y_2(\theta)$ are the two linear independent solutions of equation 1 and are so chosen that

$$\begin{aligned} y_1(0) &= 1 & y_2(0) &= 0 \\ dy_1(0)/d\theta &= 0 & dy_2(0)/d\theta &= 1. \end{aligned} \quad (8)$$

In matrix form

$$Y(\theta) = \begin{pmatrix} y(\theta) \\ \frac{dy(\theta)}{d\theta} \end{pmatrix} = \begin{pmatrix} y_1(\theta) & y_2(\theta) \\ \frac{dy_1(\theta)}{d\theta} & \frac{dy_2(\theta)}{d\theta} \end{pmatrix} \begin{pmatrix} y(0) \\ \frac{dy(0)}{d\theta} \end{pmatrix} = M(\theta/0)Y(0). \quad (9)$$

It can be shown¹ that

$$d |M(\theta|0)| / d\theta = 0 \quad (10)$$

and that $|M(\theta|0)| = 1, \quad (11)$

so that the four elements of the transformation matrix $M(\theta|0)$ can be represented by three parameters. The representation of $M(\theta|0)$ by these three parameters $\sigma, \alpha,$ and β was chosen to be²

$$M(\theta|0) = M(\sigma, \alpha, \beta) = \begin{pmatrix} \cos\sigma + \frac{\alpha \sin\sigma}{\beta} & \sin\sigma \\ -\frac{(1 + \alpha^2) \sin\sigma}{\beta} & \cos\sigma - \frac{\alpha \sin\sigma}{\beta} \end{pmatrix}. \quad (12)$$

In general the three parameters are complex numbers, but in this case, since the elements of $Y(\theta)$ and $Y(0)$ are real, the elements of $M(\theta|0)$ are also real. If $K(\theta)$ is an even function with respect to the initial reference angle zero, and if φ is the period of $K(\theta)$ over part of the machine, then³ $\alpha(\varphi) = 0$ and $M(\varphi|0)$ reduces to

$$M(\varphi|0) = M(\sigma, 0, \beta) = \begin{pmatrix} \cos\sigma & \sin\sigma \\ -\frac{1}{\beta} \sin\sigma & \cos\sigma \end{pmatrix} \quad (13)$$

The physical interpretation of σ , for matrix representing one period of $K(\theta)$ is that it is the change in the phase of the betatron oscillation of the particle in going through this period of the periodic structure. σ for one sector in the chosen unperturbed machine is approximately $\pi/3$.

In general⁴ $M(\theta_3|\theta_1) = M(\theta_3|\theta_2) M(\theta_2|\theta_1). \quad (14)$

If $M(\sigma, \alpha, \beta)$ represents the transformation matrix through one sector, then the transformation matrix for P consecutive

¹Stoker, op. cit., p. 200

²Courant and Snyder, op. cit., p. 6

³Hammermesh, op. cit., p. 30-32

⁴Ibid., p. 7

identical sectors is obtained¹ by replacing σ by $P\sigma$, i.e.

$$[M(\sigma, \alpha, \beta)]^P = M(P\sigma, \alpha, \beta). \quad (15)$$

If ψ is the period of $K(\theta)$ for the entire machine, the condition for instability is that the absolute value of one of the two possible eigenvalues λ in the equation

$$Y(\psi + \theta_1) = M(\psi + \theta_1 | \theta_1) Y(\theta_1) = \lambda Y(\theta_1) \quad (16)$$

be greater than one.² It is shown³ that this is equivalent to requiring that the absolute value of the trace of $M(\psi + \theta_1 | \theta_1)$ be greater than two. This is the criterion used to determine the effectiveness of the perturbation. It is shown⁴ that the trace of $M(\psi + \theta_1 | \theta_1)$ is invariant with respect to the choice of θ_1 .

II. METHOD OF PROCEDURE

A. General

The general procedure is to find the trace of $M(\psi + \theta_1 | \theta_1)$ as a function of σ , and then to find σ as a function of the changes Δn_1 and Δn_2 in the fall off indices n_1 and n_2 in the perturbed sectors. The relationship between Δn and the energy in the perturbing magnetic field is given. The relationship between the energy in the magnetic field and the energy available is assumed and the desired change in n_1 and n_2 is found.

The matrix representing one revolution is the product of fifty matrices, each representing one sector. If a perturbing field is superimposed on the guide field, the fall off index is changed in the perturbed sector. Thus, to find the matrix representing one revolution in the perturbed machine, the matrices representing the perturbed sectors replace the matrices representing these same sectors when unperturbed.

¹Ibid., p. 8

²Ibid., p. 11

³Ibid.

⁴Ibid., p. 15

Within an interval of θ for which $K(\theta)$ is a constant, $M(\theta | 0)$ in general is

$$M(\theta | 0) = \begin{pmatrix} \cos \sqrt{K(\theta)} \theta & (1/\sqrt{K(\theta)}) \sin \sqrt{K(\theta)} \theta \\ -\sqrt{K(\theta)} \sin \sqrt{K(\theta)} \theta & \cos \sqrt{K(\theta)} \theta \end{pmatrix} \quad (17)$$

If $K(\theta)$ is negative, $\cos \sqrt{K(\theta)} \theta$ is replaced by $\cosh \sqrt{|K(\theta)|} \theta$ and $\sin \sqrt{K(\theta)} \theta$ is replaced by $j \sinh \sqrt{|K(\theta)|} \theta$ where j is the square root of minus one. A similar change is made for $K(\theta)$ in evaluating β . For the horizontal motion,

$$M_y(\pi/2N | 0) = \begin{pmatrix} \cosh \sqrt{1-n_1} \pi/2N & 1/\sqrt{1-n_1} \sinh \sqrt{1-n_1} \pi/2N \\ \sqrt{1-n_1} \sinh \sqrt{1-n_1} \pi/2N & \cosh \sqrt{1-n_1} \pi/2N \end{pmatrix} \quad (18)$$

$$M_y(\pi/N | \pi/2N) = \begin{pmatrix} \cos \sqrt{1-n_2} \pi/2N & 1/\sqrt{1-n_2} \sin \sqrt{1-n_2} \pi/2N \\ -\sqrt{1-n_2} \sin \sqrt{1-n_2} \pi/2N & \cos \sqrt{1-n_2} \pi/2N \end{pmatrix} \quad (19)$$

where N is fifty, the total number of sectors. As magnetic fields add vectorially, the current passing through the wires can be directed so that the perturbing field adds or subtracts from the guide field. Thus n_1 and/or n_2 can be increased or decreased.

The current necessary to produce a perturbing field in an AGS that will make the motion unstable will probably be quite large as evidenced by Hammer's¹ report of using 400 amperes to change the fall off index from the original value of 0.6 to 0.786, the boundary of the stable region. From the view point of the cost of the condenser producing the current, it is desirable to find the most efficient method of applying the perturbing field, i.e., find the best way of distributing the energy stored in the condenser. The energy stored in the condenser is converted into energy in the perturbing magnetic field, into heating the wires, etc. It was assumed that the heating losses and other effects

¹C. L. Hammer, Ames, Iowa. Private communication. 1955

are small in comparison to the energy transferred to the perturbing magnetic field. The problem then reduces to finding the most efficient method of distributing the energy in the perturbing magnetic field. There are two feasible coil circuits for applying the perturbing field to the semi-sectors: one is to have the coil circuit used to change n_1 separate from the coil circuit used to change n_2 and use a dual spark gap; the other is to have a single coil circuit and use the same current to change n_1 and n_2 , the ratio between Δn_1 and Δn_2 being determined by the ratio of ampere turns. As the azimuthal length over which n_1 is changed is equal to the azimuthal length over which n_2 is changed, the total energy in the magnetic field in both of the above circuits is proportional to the azimuthal length perturbed times the quantity $i_1^2 + i_2^2$ where i_1 and i_2 are the ampere turns producing the change in n_1 and n_2 , respectively. The change in the fall off index n is proportional to the current so that the energy in the magnetic field is proportional to $(|\Delta n_1|^2 + |\Delta n_2|^2)$ times the azimuthal length perturbed. It will be shown later how to choose the best ratio between Δn_1 and Δn_2 , and a discussion is given on how to choose the number of sectors to perturb so that the machine becomes unstable for a minimum amount of energy in the perturbing magnetic field.

The two types of perturbations are discussed separately. The simpler of the two, that of a single n-bump, is analyzed first followed by an analysis of multiple n-bumps.

B. Single n-bump

In the following analysis, it is shown that for a single n-bump the trace of $M(2\pi | 0)$ can be represented by the sum of a cosine term of amplitude two plus a second oscillation term whose amplitude is small in comparison with the amplitude of the cosine term. An examination is made to determine the effects of the two terms on the trace, and especially to determine the conditions necessary for the absolute value of the trace to be greater than two.

If P consecutive sectors are perturbed, the period of $K(\theta)$ is two π . There are P consecutive identical perturbed

sectors and N-P consecutive identical unperturbed sectors, hence

$$M(2\pi - \frac{50\pi}{P} | -\frac{50\pi}{P}) = \begin{pmatrix} \cos(N-P)\sigma_u & \beta_u \sin(N-P)\sigma_u \\ -\frac{1}{\beta_u} \sin(N-P)\sigma_u & \cos(N-P)\sigma_u \end{pmatrix} \begin{pmatrix} \cos P\sigma_p & \beta_p \sin P\sigma_p \\ -\frac{1}{\beta_p} \sin P\sigma_p & \cos P\sigma_p \end{pmatrix} \quad (20)$$

The parameters with the subscripts u and p are associated with one unperturbed and one perturbed sector, respectively. It is the magnitude of the trace of this resulting matrix which must be greater than two in order that the motion be unstable. This trace, denoted $\text{tr } M$, is $\text{tr } M(2\pi - 50\pi/P | -50\pi/P)$

$$\text{tr } M(2\pi - 50\pi/P | -50\pi/P) = 2 \cos(N-P)\sigma_u \cos P\sigma_p - [\beta_u/\beta_p + \beta_p/\beta_u] \sin(N-P)\sigma_u \sin P\sigma_p. \quad (21)$$

As the trace is independent of the reference point,

$$\text{tr } M(2\pi - 50\pi/P | -50\pi/P) = \text{tr } M(2\pi | 0). \quad (22)$$

Adding and subtracting $2 \sin(N-P)\sigma_u \sin P\sigma_p$ to the terms in the right hand side of equation 21,

$$\text{tr } M(2\pi | 0) = 2 \cos[(N-P)\sigma_u + P\sigma_p] - [(\beta_u - \beta_p)^2 / \beta_u \beta_p] \sin(N-P)\sigma_u \sin P\sigma_p. \quad (23)$$

Since each of the two terms in the trace will be referred to several times, the term involving $\cos[(N-P)\sigma_u + P\sigma_p]$ will be called the first term of the trace, and the other term including the minus sign will be called the second term of the trace.

The parameters σ and β with subscripts u are constant for a given AGS, and the parameters with subscripts p are considered functions of the changes Δn_1 and Δn_2 in the fall off indices n_1 and n_2 , respectively. If $\Delta n_1 = 0 = \Delta n_2$, then $\sigma_p = \sigma_u$ and $\beta_p = \beta_u$, so that

$$(\beta_u - \beta_p)^2 / \beta_u \beta_p = 0 \quad (24)$$

and
$$\text{tr } M(2\pi | 0) = 2 \cos N\sigma_u. \quad (25)$$

$(\beta_u - \beta_p)^2 / \beta_u \beta_p$ is the amplitude of the second term of the trace. To determine its value for small changes in the fall off index, and hence the relative importance between the first and second term of the trace, the following examination was made:

Due to the condition that $K(\theta)$ be an even function, the quantity $(\beta_u - \beta_p)^2 / \beta_u \beta_p$ depends on whether P is even or odd. If P is even, the matrix for the single sector determining β_u and β_p is $M(2\pi/N | 0)$; if P is odd, the matrix for the sector is $M(\pi/N | -\pi/N)$. The prime on a β shall denote that P is even, and a double prime shall denote that P is odd. Let $A = (\beta_u - \beta_p)^2 / \beta_u \beta_p$, and let the primes on A have the same meaning as for β .

Table 1 gives A for P odd and P even for both horizontal and vertical motion for the condition $\Delta n_1 = \Delta n_2$.

Table 1. Correlation between A and n_1 for $n_1 = n_2$

$\Delta n_1 (= \Delta n_2)$	A'_y	A''_y	A'_z	A''_z
-50	0.120	0.057	0.636	0.775
-25	0.037	0.020	0.055	0.077
-10	0.007	0.004	0.006	0.009
0	0	0	0	0
+10	0.009	0.006	0.004	0.008
+25	0.076	0.052	0.021	0.039
+50	0.692	0.570	0.059	0.125

Table 2 gives approximate A'_y and A''_z for several different values of Δn_1 and Δn_2 . As will be noticed in Table 2, the values in the column for A''_z are the same as

Table 2. Correlation between A and various values of Δn_1 and Δn_2

Δn_1 and Δn_2		A'_y	A''_z
0	0	0	0
+25	-25	0.0088	0.0088
+25	+25	0.077	0.039
-25	+25	0.0079	0.0079
-25	-25	0.039	0.077

the values in the column for A'_y except for a rearrangement of the numbers. The method for obtaining A''_z from A'_y is as follows.

Let $M(+)$ denote a focusing quarter sector, and let $M(-)$ denote a defocusing quarter sector. Then from Fig. 1 and equations 3 and 4,

$$P \text{ even} \quad \begin{cases} M_y(2\pi/N | 0) = M_y(-)M_y(+)M_y(+)M_y(-) & (26) \\ M_z(2\pi/N | 0) = M_z(+)M_z(-)M_z(-)M_z(+) & (27) \end{cases}$$

$$P \text{ odd} \quad M_y(\pi/N | -\pi/N) = M_y(+)M_y(-)M_y(-)M_y(+) \quad (28)$$

$$M_z(\pi/N | -\pi/N) = M_z(-)M_z(+)M_z(+)M_z(-) \quad (29)$$

The fall off index n equals n_1 in $M_y(-)$ and $M_z(+)$, and n equals n_2 in $M_y(+)$ and $M_z(-)$. By equation 3 and 4, if $\Delta n_1 = b$, where b is greater than zero, the absolute value of both $K_y(\theta)$ in $M_y(-)$ and $K_z(\theta)$ in $M_z(+)$ will be increased by b . Similarly, if $n_2 = a$, the absolute value of both $K_y(\theta)$ of $M_y(+)$ and $K_z(\theta)$ of $M_z(-)$ will be decreased by amount a . Using the condition $|n_1| = |n_2|$ and the approximation that

$$K_y(\theta) = -K_z(\theta) \quad (30)$$

it is found that

$$M_y(+)\bigg|_{\Delta n_2 = a} = M_z(+)\bigg|_{\Delta n_1 = -a} \quad (31)$$

$$M_y(-)\bigg|_{\Delta n_1 = b} = M_z(-)\bigg|_{\Delta n_2 = -b} \quad (32)$$

so that

$$M_y(2\pi/N|0)\bigg|_{\substack{\Delta n_1 = b \\ \Delta n_2 = a}} = M_z(\pi/N|-\pi/N)\bigg|_{\substack{\Delta n_1 = -a \\ \Delta n_2 = -b}} \quad (33)$$

$$M_y(\pi/N|-\pi/N)\bigg|_{\substack{\Delta n_1 = b \\ \Delta n_2 = a}} = M_z(2\pi/N|0)\bigg|_{\substack{\Delta n_1 = -a \\ \Delta n_2 = -b}} \quad (34)$$

As the matrices have these relationships, the corresponding differential equations and parameters σ and β have similar relationships. Thus A'' can be obtained from A' by interchanging Δn_1 with Δn_2 in Table 2 and multiplying Δn_1 and Δn_2 by minus one, e.g. A'' for $\Delta n_1 = -25 = \Delta n_2$ is equal to A' for $\Delta n_1 = +25 = \Delta n_2$. Checking back to Table 1 for which all values of A were obtained by multiplication of matrices rather than be the method used in Table 2 for A'' , it is seen that the agreement is good to within a few percent.

Comparing the values of A in Table 2, it is seen that A is much larger for $\Delta n_1 = \Delta n_2$ than for $\Delta n_1 = -\Delta n_2$, in fact, the change in A for $\Delta n_1 = \Delta n_2$ is probably near a maximum. Comparing the values of A in Table 1, it is seen that for $-25 \leq \Delta n_1 = \Delta n_2 \leq +25$, A is quite small in comparison with two, the amplitude of the first term of the trace of $M(2\pi|0)$ as represented in equation 23. It will be shown later that horizontal instability can be obtained without using larger changes in n, and in fact, it would be difficult to make a large change in n without first reaching vertical or horizontal instability.

For a particular perturbation of the chosen machine, N, P and σ_u are constant so that σ_p is the variable in the first term of the trace as represented in equation 23.

Thus the trace with respect to σ_p , for σ_p a real number, is a real valued cosine curve with amplitude two plus a small oscillation of amplitude A. Therefore, in order that the absolute value of the trace be greater than two, it will be necessary for the $\cos[N\sigma_u + P(\sigma_p - \sigma_u)]$ to be approximately equal to plus or minus one. If given $N\sigma_u$, it is easy to determine whether $\sigma_p - \sigma_u$ should be positive or negative for a minimum perturbation. For the chosen machine, $N\sigma_{uy} = 17\pi - 0.6626$ and $N\sigma_{uz} = 16\pi + 1.9244$. Their polar coordinate representation on the unit circle is shown in Fig. 3. As can be seen $\sigma_{py} - \sigma_{uy}$ should be positive. For a given P, the value of σ_p for which the first term of the trace is plus or minus two is readily computed. For the AGS chosen with P equal to four, one requires $\sigma_{py} = \sigma_{uy} + 0.6626/4 = 1.0549 + 0.1656 = 1.2205$.

There is a possibility that for a given P, the second term of the trace would decrease the absolute value of the trace when the first term of the trace is going through a maximum, or minimum as the case may be, so that the motion would remain stable. If this occurs, a slightly different value of P can be found from equation 23 such that the second term of the trace adds in the right direction to make the absolute value of the trace greater than two.

To show the variation of σ_{py} for various values of Δn_1 and Δn_2 , contours of constant σ_{py} as a function of Δn_1 and Δn_2 are plotted in Fig. 4. For large values of Δn_1 and Δn_2 , both greater than zero, σ_{py} approaches zero and then becomes a pure imaginary number. As mentioned before, values of Δn this large need not be considered.

It is desirable to find the most efficient method of maximizing the absolute value of $\sigma_{py} - \sigma_{uy}$ for a given amount of energy in the perturbing magnetic field. To do this, it would be necessary to know exactly how the fall off index n changed as a function of the available energy. The change in n which would be most efficient is found by drawing a curve on Fig. 4 for which the available energy is a constant. The best change in n_1 and n_2 is then the coordinates of the maximum, or minimum, value of σ_{py} on this curve. Using the criterion discussed previously, the curve for constant energy is the circle $|\Delta n_1|^2 + |\Delta n_2|^2 = c^2$, the constant c being fifty for the circle shown in Fig. 4. As can be seen the maximum value of σ_{py} occurs at approximately $\Delta n_1 = 1/2 \Delta n_2 < 0$.

FIG. 3 POLAR COORDINATE REPRESENTATION OF $N\sigma_U$
ON THE UNIT CIRCLE

FIG.4 CONTOUR LINES OF CONSTANT σ_{py} FOR VARIOUS PERTURBATIONS

It is now necessary to show that the vertical motion is stable when a perturbation of the type described above is applied. Due to the inductance of the wires, the change in n and consequently the change in σ_p will not occur instantaneously, but will rise as a continuous function from the initial value of zero. In order that the perturbation be uable, the vertical motion must not only be stable at the time the horizontal motion becomes unstable, but must remain stable throughout the rising perturbing field until the horizontal motion becomes unstable. As can be seen from Fig. 3, if $\sigma_{pz} - \sigma_{uz} > \sigma_{py} - \sigma_{uy} > 0$, or if $\sigma_{pz} - \sigma_{uz}$ is sufficiently negative, then there is a possibility that the vertical motion will become unstable before the horizontal motion becomes unstable.

The value of σ_{pz} can be found by one of two equivalent methods. One method is to construct another contour graph for constant σ_{pz} similar to Fig. 4. By the argument given after Table 2 and the fact that the trace is invariant with respect to θ , this can be done by reflecting the contours of constant σ_{py} through the line $\Delta n_1 = -\Delta n_2$. The second equivalent method is that σ_{pz} evaluated at $\Delta n_1 = a, \Delta n_2 = b$ is equal to σ_{py} evaluated at $\Delta n_1 = -b, \Delta n_2 = -a$. As will be noticed, $\sigma_{py} - \sigma_{uy}$ and $\sigma_{pz} - \sigma_{uz}$ usually have opposite signs.

Comparing Fig. 4 and Fig. 3, it is seen that if $\Delta n_1 = 1/2 \Delta n_2$ approximately, and if $\Delta n_1 < 0$ is the type of perturbation applied, then the angle $N\sigma_{uy} + P(\sigma_{py} - \sigma_{uy})$ increases and the angle $N\sigma_{uz} + P(\sigma_{pz} - \sigma_{uz})$ decreases. Furthermore, $|\sigma_{py} - \sigma_{uy}| = |\sigma_{pz} - \sigma_{uz}|$ approximately so that the vertical motion is still very stable at the time that the horizontal motion becomes unstable. Thus this is the desired type of perturbation. Using the value of $\sigma_{py} = 1.2205$ for $P = 4$ arrived at previously, $\Delta n_1 = -14$ and $\Delta n_2 = -24$ will make the first term of the trace equal to minus two. Both sine terms are negative and the amplitude of the product of the sine terms is positive so the second term of the trace is negative and the horizontal motion is unstable.

The choice of P used here is to give an estimate of the necessary change in n , and does not necessarily represent the best choice of P . For the machine to be unstable for a single n -bump, it was shown that it was necessary for $N\sigma_u - P(\sigma_p - \sigma_u)$ to be near an integral

number of pi. From Fig. 4, $\sigma_p - \sigma_u$ is approximately proportional to Δn , which is proportional to the current. Thus if P is doubled, the current is halved and the necessary energy in the perturbing magnetic field, which is proportional to P times the current squared, is decreased by one-half. It is to be noted that increasing P increases the resistance and inductance of the wires. A large value of P would probably be favored, but it will be necessary to know the coil configuration, size of the wire used, etc., before the best value of P can be determined.

C. Periodic n-bumps

It was not obvious that a single n-bump was the best perturbation. Perhaps several n-bumps equally distributed around the machine would be better. An example of this type of perturbation would be to perturb every tenth sector. The probability that the beam would emerge at several different locations, and the engineering problems of distributing the current to several different places around the machine were not considered.

Consider mathematically grouping the fifty sectors into R groups of S consecutive sectors each with S being a positive integer. R can have the values of one, two, five, ten, twenty five, or fifty. If Q consecutive sectors out of each group are perturbed and if these Q sectors occupy the same position within each group, then the equation for the trace of the matrix for one group is, using the invariance of the trace with respect to θ ,

$$\begin{aligned} \text{tr } M(2\pi S/50 | 0) = & 2 \cos (S-Q) \sigma_u \cos Q \sigma_p \quad (35) \\ & - (\beta_u/\beta_p + \beta_p/\beta_u) \sin (S-Q) \sigma_u \sin Q \sigma_p \end{aligned}$$

which is similar to equation 21. The $\text{tr } M(2\pi S/50 | 0)$ has a representation corresponding to equation 23. As the R groups are identical, $K(\theta)$ has a period of $2\pi S/50$, and by the statement after equation 15, the stability of the entire machine is the same as the stability of one group of sectors. Thus as before the perturbation must be such that $\cos [S\sigma_u + Q(\sigma_p - \sigma_u)]$ must be near plus or minus one in order that the motion become unstable.

In order that this type of perturbation be better, it will be necessary that the required perturbation be less than the perturbation needed if QR consecutive sectors are perturbed. For QR consecutive perturbed sectors, the necessary perturbation is that for which

$$\sigma_{py} - \sigma_{uy} = (17\pi - N\sigma_{uy})/QR = 0.6626/QR: \quad (36)$$

for periodic perturbation, the necessary perturbation is that for which

$$\sigma_{py} - \sigma_{uy} = (q_y \pi - S\sigma_{ut})/Q \quad (37)$$

where q_y is a positive integer such that the quantity $|q_y \pi - S\sigma_{uy}|$ is a minimum. The quantity $|q_y \pi - S\sigma_{uy}|$ is then the radian measure of the distance from the operating point of the unperturbed machine to the nearest horizontal integral resonance if q_y is even, or to the nearest horizontal half-integral resonance if q_y is odd. The value of q_y is a step function of $S\sigma_{uy}$. As can be seen from Fig. 4, σ_{py} changes approximately proportionally to Δn for small Δn . Thus for the periodic perturbation to be an improvement over continuous perturbation,

$$|(q_y \pi - S\sigma_{uy})/Q| < |0.6626/QR| \quad (38)$$

$$|(Rq_y \pi - RS\sigma_{uy})| < 0.6626 \quad (39)$$

$$|(Rq_y \pi - N\sigma_{uy})| < 0.6626 \quad (40)$$

$$|(Rq_y \pi - 17\pi + 0.6626)| < 0.6626, \quad (41)$$

$$\text{hence} - (2 \times 0.6626)/\pi < (Rq_y - 17) < 0. \quad (42)$$

Now R and q_y are integers and by equation 42 must differ from seventeen by less than one and more than zero. Therefore, no solutions to equation 45 exist. This means that no improvement results from grouping this machine into more than one group of sectors. For the periodic perturbation to be as good as a single n -bump, the right and left hand side of equation 38 are set equal to each other. This changes equation 42 to $Rq_y - 17$ equals zero. If $R = 1$, then $q_y = 17$, and the quantity $Rq_y - 17$ equals zero. This being the case of a single n -bump considered previously in section B. Now 17 is a prime number and R and q_y are integers. Therefore, for any value of R except unity, $|Rq_y - 17| \geq 1$ so that R equal unity is the best grouping.

III. GENERALITY OF METHOD OF PROCEDURE FOR CIRCULAR MACHINES

A. For Fifty Sector Machines Operating in the Same Diamond as the Chosen Machine

As it will be impossible to construct a machine that has the exact design values of n_1 and n_2 , it is desirable to know how generally the results and methods of the foregoing can be applied to some other value of n_1 and n_2 . Although the exact value of the fall off indices can not be predicted, the value should lie somewhere within a diamond as shown in Fig. 2 to avoid the integral and half-integral resonances.

To determine the variation of $\sigma_p - \sigma_u$ for other operating points within the diamond, two other operating points were chosen so that they were about as far from the original operating point $n_1 = +440$, $n_2 = -440$ as practical and still lie within the diamond. One point was chosen on the diagonal line $n_1 = n_2$, and the other was chosen off the diagonal. Their actual positions are shown in Fig. 2 as a square and a triangle. Table 3 gives a summary of the results. Within the accuracy of the results, $(\sigma_{py} - \sigma_{uy}) = (\sigma_{pz} - \sigma_{uz})$

$\Delta n_1 = a$ $\Delta n_2 = b$	=	$\Delta n_1 = -b$ $\Delta n_2 = -a$
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As can be seen from Table 3, $\sigma_{py} - \sigma_{uy}$ is almost independent of the operating point in the diamond. Hence, Fig. 4 would retain its general pattern except that the value of σ_y on all the contour curves would be raised or lowered by the same amount.

A similar study of $(\beta_u^i - \beta_p^i)^2 / \beta_u^i \beta_p^i (= A_y^i)$ was made. To aid computation, the approximation $K_y^i(\theta) = -n$ was used. The results are given in Table 4. As seen, A_y^i is approximately independent of the operating point within the diamond. It is to be noted that the values of Δn_1 and Δn_2 for which A_y^i is a maximum are the same values of Δn_1 and Δn_2 for which the absolute value of $\sigma_{py} - \sigma_{uy}$ of Table 3 is a maximum.

Table 3. $\sigma_{py} - \sigma_{uy}$ for different operating points and various perturbations

Δn_1 and Δn_2		$\sigma_{py} - \sigma_{uy}$ ($\sigma_{uy} = 1.0549$)		
		$n_1 = + 440$ $n_2 = - 440$	$n_1 = + 427$ $n_2 = - 427$	$n_1 = + 438$ $n_2 = - 431$
0	0	0	0	0
+ 25	- 25	+0.07	+0.07	+0.07
+ 25	+ 25	-0.23	-0.24	-0.25
- 25	+ 25	-0.06	-0.06	-0.06
- 25	- 25	+0.20	+0.20	+0.20

Table 4. A'_y at different operating points and various perturbations

Δn_1 and Δn_2		A'_y		
		$n_1 = + 440$ $n_2 = - 440$	$n_1 = + 427$ $n_2 = - 427$	$n_1 = + 438$ $n_2 = - 431$
0	0	0	0	0
+ 25	- 25	+ 0.0088	+ 0.0083	+ 0.0087
+ 25	+ 25	0.077	0.090	0.094
- 25	+ 25	0.0079	0.0087	0.0092
- 25	- 25	+ 0.039	+ 0.041	+ 0.042

It remains to be shown whether $\sigma_{py} - \sigma_{uy}$ should be positive or negative in order that the argument of the cosine term of the trace approaches the nearest integral number of π for the minimum perturbation, and whether a single n -bump is still preferable. To determine the value of σ_{py} and σ_{uy} at which the machine becomes unstable, it

is necessary to know the argument of the cosine term of the trace for the unperturbed machine. For a single n-bump, this argument is $N\sigma_{uy}$; for periodic n-bumps, i.e. a machine of R groups of S sectors each, this argument is $S\sigma_{uy}$. If the AGS operates within the diamond of Fig. 2, then

$$16\pi/50 < \sigma_u < 17\pi/50 \quad (43)$$

so
$$\sigma_u = f + \pi/3 \quad (44)$$

where f is the small variation of σ_u from $\pi/3$ and has the limitations

$$-\pi/75 < f < \pi/150. \quad (45)$$

Thus
$$S\sigma_u = S\pi/3 + Sf. \quad (46)$$

For a single n-bump, $S = N$ so that

$$N\sigma_u = N\pi/3 + Nf = 17\pi - g \quad (47)$$

where due to the limitation on f, and by equation 47

$$0 < g = \pi/3 - Nf < \pi. \quad (48)$$

Thus $N\sigma_u$ as represented by Fig. 3 is on the upper half of the unit circle. If σ_{uy} is known, it is easy to determine whether $\sigma_{py} - \sigma_{up}$ should be positive or negative by plotting $N\sigma_{uy}$ on Fig. 3.

To determine the best grouping of the machine, the same type of analysis as used before applies, and the equations corresponding to equation 41 and 42 are

$$|Rq_y\pi - 17\pi + g| < |g|, \quad (49)$$

hence
$$-2g/\pi < (Rq_y - 17) < 0. \quad (50)$$

By the argument given after equation 42, g must be greater than $\pi/2$ in order that equation 50 have a solution. This means that f must be less than $-\pi/300$. Table 5 gives the value of Rq_y for $-\pi/75 < f < -\pi/300$. As before, q_y is a positive integer such that $|q_y\pi - S\sigma_{uy}|$ is a minimum. As can be seen from Table 5 and equation 50, the only possible value of S that might be better than S equal to fifty for applying a periodic perturbation is for S equal to twenty five. Consider an AGS of R_0 groups each with an even number of sectors S_0 of which an even number Q_0 are perturbed.

Table 5. Rq_y for the allowed values of S

S	R	q	Rq
50	1	16	16
25	2	8	16
10	5	3	15
5	10	2	20
2	25	1	25
1	50	0	0

Dividing the machine into two groups will change the argument of the cosine term of the trace of the unperturbed machine from $S_0\sigma_u$ to $(S_0/2)\sigma_u$. The angular distance of $(S_0/2)\sigma_u$ from the first integral number of π less than $(S_0/2)\sigma_u$ is one-half of the angular distance of $S_0\sigma_u$ from the first even integral number of π less than $S_0\sigma_u$. However, $S_0\sigma_u + Q_0(\sigma_p - \sigma_u)$ decreases twice as fast as $(S_0/2)\sigma_u + (Q_0/2)(\sigma_p - \sigma_u)$ for a given perturbation.

Hence in any case subdividing the machine does not help any and furthermore, if $S_0\sigma_u$ is closer to an odd than an even integral number of π , then it would be better not to double the number of groups R_0 . Thus S equal to twenty five would not be any better than S equal to fifty, and it could be much worse.

As $\sigma_p - \sigma_u$ and A are nearly invariant with respect to different operating points in the diamond, and as the better of the two types of perturbations studied is the single perturbation over consecutive sectors, the general analysis arising from the $\text{tr } M(2\pi | 0)$ as represented in equations 21 through 23 is still valid.

The position of $N\sigma_{uy}$ and $N\sigma_{uz}$ should be known to determine whether n_1 and n_2 should be positive or negative. If the change in the fall off index n is proportional to the current, $\Delta n_1 = 1/2 \Delta n_2$ is usually the approximate desired change in the fall off indices; however, there exists one exception to this choice of Δn_1 and Δn_2 .

Let $\phi = N\sigma_u + P(\sigma_p - \sigma_u)$, and define k to be a positive integer such that $|\phi - k\pi|$ is a minimum. ϕ is then the argument of the cosine term of the trace, and $k\pi$ is the integral number of π nearest to ϕ . There is a possibility that $N\sigma_{uy}$ and $N\sigma_{uz}$ will be so located that ϕ_y approaches $k_y\pi$ and ϕ_z approaches $k_z\pi$ at a rate approximately proportional to their angular distance from $k_y\pi$ and $k_z\pi$, respectively. In this case, the vertical and horizontal motion may become unstable at the same time. There are two changes which might make the horizontal motion unstable before the vertical motion becomes unstable: reverse the sign of Δn_1 and Δn_2 , and/or change the ratio of Δn_1 to Δn_2 . Reversing the sign of Δn_1 and Δn_2 reverses the sign of $(\sigma_p - \sigma_u)$; changing the ratio of Δn_1 to Δn_2 changes the rate at which ϕ approaches $k\pi$. Either one of these changes will increase the amount of perturbation needed to make the motion unstable, but ϕ_y will not reach $k_y\pi$ at the same time that ϕ_z reaches $k_z\pi$. Further examination of both terms of the trace would be necessary to determine if either one of these changes will give the desired instability.

B. Alternating Gradient Machines in General

To analyze a circular machine similar to the machine chosen for the above analysis, the procedure used is outlined below and is similar to the above procedure.

Determine $N\sigma_{uy}$ and $N\sigma_{uz}$. Construct a contour graph for σ_{py} and σ_{pz} as a function of Δn_1 and Δn_2 . On this graph, draw a second family of contour lines giving the variation of Δn_1 and Δn_2 for a constant perturbing magnetic field energy. From $N\sigma_{uy}$, $N\sigma_{uz}$ and the contour curves, determine the best method of applying the perturbation. Examine $(\beta_u - \beta_p)^2 / \beta_u \beta_p$ for the range of Δn of interest to determine if the amplitude of the second term of the $\text{tr } M(2\pi | 0)$ is still small. Group the machine up into R groups of S sectors each. Use equation 37 to determine σ_{py} and hence the change in n necessary for each value of R . Using the criterion that for a given length RQ of the machine perturbed, the perturbing magnetic

field energy is to be a minimum, the best grouping of the machine can be found. As before, doubling R does not help any. If R is not unity, determine if the engineering difficulties and beam scatter would offset the advantages of applying the perturbing field at more than one place.

After R has been determined, use equation 23 or its counterpart if R is not unity to determine P such that the trace is greater than two. Within the engineering simplicity assumed, this gives the desired perturbation.

IV. DISCUSSION

There were three basic known quantities used in the analysis; a circular machine, the azimuthal arrangement of the focusing and defocusing semi-sectors, and the initial fall off indices in the semi-sectors. To carry out the analysis, many simplifications were made to reduce the complexity of the problems involved. In some cases, the simplifications were made to reduce the mathematical computations. In other cases, there were more situations to examine than was practical so that a particular typical example was chosen for study. In still other cases, more parameters such as the size of the donut would have to be known before certain results could be obtained. Many of these simplifications were pointed out in the report, but a few were not.

One of the main simplifications was choosing a circular machine. It is very probable that an actual AGS will have some straight sectors for which the fall off index equals zero. A second main simplification was using wires and a condenser as the mechanism for applying the perturbation. For an AGS, other types of mechanisms such as using frequency modulation to move the beam over to the side of the donut before applying the perturbation might be better. In regard to using wires for the perturbation mechanism, it was assumed that the perturbations did not change the equilibrium radius.

Five significant figure were used in evaluating the elements of the matrices. The determinants of all matrices were checked to be equal to unity to within round off error. In multiplying the matrices, this round off error

accumulated so that five place accuracy could not always be obtained if several matrices were multiplied together.

Possible future continuation of the analysis would be to examine the path of the beam to determine where and when the beam would be ejected. This work would be quite important in determining the best position for the target location.