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EFFECT OF DISPERSION IN ION SLOWING DOWN
ON PLASMA ENERGY MULTIPLICATION FACTORS

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Effect of Dispersion in Ion Slowing Down on Plasma Energy Multiplication Factors

The effects of dispersion in energy of hot ions slowing down in a plasma on the plasma energy multiplication factor are estimated. Numerical results are given for plasma parameters that are expected to be attained in the TCT device to be built at Princeton Plasma Physics Laboratory. The inclusion of this effect results in an increase in the plasma energy multiplication factor of order of ten percent or less, depending on the injection energy and target temperature.

I. Introduction

The realization that appreciable fusion energy release could be attained by injecting energetic ions into a relatively cold background plasma has lead workers in this field to examine more general models for the calculation of energy multiplication factors. Previous calculations⁽¹⁾ of the energy multiplication factor, F , for two component plasmas have assumed that the distribution of injected particles is a delta-function in energy throughout the slowing down process. This assumption can be relaxed using the dispersion relation that results from the consideration of a simple binary collision model⁽²⁾.

II. Multiplication Factor Calculation

By definition, the plasma energy multiplication factor is the ratio of the energy released by fusion to the injected energy. That is,

$$\bar{F} = \frac{E_F}{E_I} = \frac{n_b Q}{E_I} \int_0^{t_{th}} \langle \sigma v \rangle_0 dt + \frac{n_b^2 Q \langle \sigma v \rangle_{th}}{j_I E_I} \quad (1)$$

or

$$\bar{F} = \frac{n_b Q}{E_I} \int_{E_t}^{E_I} \frac{\langle \sigma v \rangle_0}{\frac{dE}{dt}} dE + \frac{n_b Q \langle \sigma v \rangle_{th} \tau_c}{E_I} \quad (2)$$

where

E_I = injected particle energy

j_I = injection rate (it is assumed that the reactor is both heated and fueled by high-energy injection)

E_T = thermal energy

n_b = background-particle number density

Q = energy released per fusion reaction

τ_c = confinement time

t_{th} = ion thermalization time

$\bar{F}$ = energy multiplication factor including beam dispersion

F = energy multiplication factor excluding beam dispersion

In the above equations the first term refers to the energy released by fusion occurring between the beam particles and the background particles. The second term refers to fusion occurring between background particles. This term will typically be small in the two-component plasma concept wherein it is assumed that the background plasma is cold. Beam-beam particle interactions are neglected and, since the contribution to the energy release from fusion between background ions does not contribute to the understanding of the beam-target interaction, it too will be neglected in the following calculations. Then we have:

$$\bar{F} = \frac{n_b Q}{E_I} \int_{E_I}^{E_T} \frac{\langle \sigma v \rangle_0 dE}{\langle \frac{dE}{dt} \rangle} \quad (3)$$

In general, the reactivity involved is defined⁽³⁾ as:

$$\langle \sigma v \rangle_0 = \int \int f_1(\vec{v}_1) f_2(\vec{v}_2) (|\vec{v}_1 - \vec{v}_2|) / \sigma (|\vec{v}_1 - \vec{v}_2|) d\vec{v}_1 d\vec{v}_2 \quad (4)$$

Here $f_1(\vec{v}_1)$ is the injected-particle distribution which is assumed to be of the form

$$f_1(\vec{v}_1) = C' \delta(\theta) \delta(\phi) e^{-\frac{(\vec{v} - \vec{v}_1)^2}{\epsilon_1^2}} \quad (5)$$

where ϵ_1 is the characteristic width of the slowing-down distribution.

The use of a Gaussian dispersion during slowing down is a basic assumption in this work. This should be reasonably accurate at higher energies but, as noted later, it does not properly account for the final transition to a Maxwellian distribution. Still, it is thought that the sacrifice in accuracy is worthwhile in order to obtain a simple model that can be used for survey calculations.

The remaining term in Eq. (4), $f_2(\vec{v}_2)$, is the background distribution, taken as

$$f_2(\vec{v}_2) = \frac{\beta^3}{\pi^{3/2}} e^{-\beta^2 v_2^2} \quad \text{where } \beta^2 = \frac{M_2}{2kT_2} \quad (6)$$

or

$$f_2(\vec{v}) = \frac{\beta^3}{\pi^{3/2}} e^{-\beta^2 (v^2 + v_1^2 + 2v_1 v \cos \theta)} \quad \text{where } \vec{v} = \vec{v}_2 - \vec{v}_1 \quad (7)$$

then

$$\langle \sigma v \rangle_0 = \int_u \int_v \int_\phi \int_\theta \frac{\beta^3}{\pi^{3/2}} C' \delta(\theta) \delta(\phi) e^{-(\vec{v} - \vec{v}_1)^2 / \epsilon_1^2} \sigma(v) v^3 \times e^{-\beta^2 (v^2 + v_1^2 + 2v_1 v \cos \theta)} \sin \theta d\theta d\phi dv_1 dv \quad (8)$$

$$\langle \sigma v \rangle_0(\vec{v}) = \int_{v_1} \int_v \frac{\beta^3}{\pi^{3/2}} C' e^{-(\vec{v} - \vec{v}_1)^2 / \epsilon_1^2} \sigma(v) v^3 e^{-\beta^2 (v^2 + v_1^2 + 2v_1 v \cos \theta)} dv_1 dv \quad (9)$$

or

$$\langle \sigma v \rangle_0(\vec{v}) = \int_{v_1} C' e^{-(\vec{v} - \vec{v}_1)^2 / \epsilon_1^2} \langle \sigma v \rangle(v_1) dv_1 \quad (10)$$

The reactivity $\langle \sigma v \rangle$ results from averaging σv over interactions between a Maxwellian background and a δ -function slowing-down distribution.

Here $\langle \frac{dE}{dt} \rangle$ must be evaluated for each beam particle of energy E independent of the fact that it is a part of a distribution of particles centered at $\bar{E}$. Then, neglecting the contribution to $\bar{F}$ due to background fusion,

$$\bar{F} = \frac{n_b Q}{E_I} \int_{E_I}^{E_T} \left(\frac{\int_0^\infty C e^{-(E-\bar{E})^2/\epsilon^2} \langle \sigma v \rangle (E) dE}{\langle \frac{dE}{dt} \rangle (E)} \right) d\bar{E} \quad (11)$$

or

$$\bar{F} = \frac{1}{E_I} \int_{E_I}^{E_T} \bar{G} d\bar{E} \quad (12)$$

where

$$\bar{G}(\bar{E}) = \int_0^\infty \frac{n_b Q C e^{-(E-\bar{E})^2/\epsilon^2} \langle \sigma v \rangle (E) dE}{\langle \frac{dE}{dt} \rangle} \quad (13)$$

$$= \int_0^\infty G(E) C e^{-(E-\bar{E})^2/\epsilon^2} dE \quad (14)$$

where ϵ is the counterpart of ϵ_1 as applied to the dispersion in energy.

The normalization constant C is given by

$$\frac{1}{C} = \int_0^\infty e^{-(E-\bar{E})^2/\epsilon^2} dE \quad (15)$$

However, since the particles are injected with an energy E_I , we know that the particle energies will be less than E_I at a later time, therefore the normalization of the injected-particle distribution-function can be carried out over the interval 0 to E_I .

Then
$$\frac{1}{C} = \int_0^{E_I} e^{-(E-\bar{E})^2/\epsilon^2} dE \quad (16)$$

$$\begin{aligned} &= \int_{-\bar{E}}^{E_I - \bar{E}} e^{-x^2/\epsilon^2} dx = \int_{-\bar{E}/\epsilon}^{(E_I - \bar{E})/\epsilon} e^{-y^2} \epsilon dy \\ &= \epsilon \int_0^{\frac{E_I - \bar{E}}{\epsilon}} e^{-y^2} dy + \epsilon \int_{-\bar{E}/\epsilon}^0 e^{-y^2} dy \\ &= \epsilon \frac{\sqrt{\pi}}{2} \operatorname{erf} \left(\frac{E_I - \bar{E}}{\epsilon} \right) + \epsilon \int_0^{\bar{E}/\epsilon} e^{-y^2} dy \\ &= \epsilon \frac{\sqrt{\pi}}{2} \left[\operatorname{erf} \left(\frac{E_I - \bar{E}}{\epsilon} \right) + \operatorname{erf} \left(\frac{\bar{E}}{\epsilon} \right) \right] \end{aligned} \quad (17)$$

Then

$$\bar{G} = \int_0^{E_I} C G e^{-(E-\bar{E})^2/\epsilon^2} dE \quad (18)$$

The actual integration of $\bar{G}$ is carried out from $E_T + \epsilon$ to E_I . This is done to avoid the discontinuity in G at E_T which arises from the fact that $\frac{dE}{dt}|_{E_T} = 0$ and because this model lacks the capability of smoothly transforming the injected-particle distribution into the Maxwellian background. This results in slightly underestimating $\bar{F}$, an effect that is more apparent at lower values of $\bar{E}$ for a given E_T . This happens because when $\bar{E} = E_T$, the fusion contribution due to interactions with slowing down particles having $E < E_T$ is not taken into account. This discrepancy increases with increasing background temperature.

In order to evaluate Eq. (11), the dispersion width ϵ is required.

From Butler and Buckingham⁽²⁾, assuming elastic binary collisions with both ion and electron background species, we have

$$\epsilon^2 = \sum_j \int_0^t \int dw (f(\vec{w}_j)) |\vec{v} - \vec{w}| \sigma_t \left[\langle \Delta E^2 \rangle - \langle \Delta E \rangle^2 \right] dt \quad (19)$$

where

$$f(w_j) = f_0 \exp(-(w_j/w_{tj})^2)$$

$\vec{v}$ = injected particle velocity

$\vec{w}_j$ = j^{th} background species velocity

$f(w_j)$ = distribution function of the j^{th} background species

σ_t = total scattering cross section

M = injected particle mass

$$w_{tj} = (2kT_j/M_j)^{1/2}$$

$$x_j = v/w_j$$

then

$$\begin{aligned} \epsilon^2 = & \sum_i \int_0^t 4n\Lambda_i 8\sqrt{\pi} (Z_i Z_e^2)^2 n_i w_i \left(\frac{1}{x_i} \operatorname{erf}(x_i) - e^{-x_i^2} \right) dt \\ & + \int_0^t 4n\Lambda_e 8\sqrt{\pi} (Z_e Z_e^2)^2 n_e w_e \left(\frac{1}{x_e} \operatorname{erf}(x_e) - e^{-x_e^2} \right) dt \end{aligned} \quad (20)$$

To simplify this we can use the Fokker-Planck expression for the energy loss rate given by Sivukhin⁽⁴⁾:

$$\left\langle \frac{dE}{dt} \right\rangle_j = \frac{-8\sqrt{\pi} n_j (Z_j Z_e^2)^2 4n\Lambda_j}{M_j w_{tj}} \left(\frac{1}{x_j} \operatorname{erf} x_j - \left(1 + \frac{m_j}{M}\right) e^{-x_j^2} \right) \quad (21)$$

in which case we obtain:

$$e^2(\bar{E}) = 2kT_e (E_I - \bar{E}) - \int_{\bar{E}}^{E_I} \sum_i \left[2k(T_e - T_i) \frac{\langle \frac{dE}{dt} \rangle_i}{\langle \frac{dE}{dt} \rangle} + \frac{(Z_i Z_e^2)^2 n_i \Lambda_i^2 8\sqrt{\pi} \sqrt{m_i 2kT_i}}{M \langle \frac{dE}{dt} \rangle} \right] \frac{t}{kT_i \frac{M}{M_i}} dE \quad (22)$$

Thus equations (12), (17), (18) and (20) can be solved to find $\bar{F}$.

III. Results and Conclusions

The value of $\bar{F}$ was calculated numerically as a function of injection energy for plasma parameters of interest. Figures 1 and 2 indicate the results of these calculations for injection of deuterium into a pure tritium background at 3 keV. Figures 3 and 4 are analogous results at 10 keV. Results for injection of deuterium into ^3He are given in Figures 5 and 6. The dispersion of the injected beam results in more particles near the maximum of the $\langle \sigma v \rangle / \langle \frac{dE}{dt} \rangle$ curve than would otherwise be the case. That this increase is small is accounted for by the fact that, even near E_T , the width of the injected-particle distribution is such that the Gaussian distribution falls off rapidly.

The fact that a maximum occurs in the graphs of $\frac{\bar{F}}{F}$ may be accounted for by an examination of Fig. 7. As the injected particle slows down in the plasma it reaches an energy at which $\langle \frac{dE}{dt} \rangle$ is a minimum. Due to an increasing ion contribution, further reduction in the injected-particle energy increases the total $\langle \frac{dE}{dt} \rangle$. The beam dispersion results in an increase in the weight given to this region of higher $\langle \frac{dE}{dt} \rangle$ and thus results in a reduction in $\bar{F}/F$. Indeed, a comparison of Figures 2, 4 and 7 shows that the maximum of the $\bar{F}/F$ curve corresponds to the minimum of the $\langle \frac{dE}{dt} \rangle$ curves. Finally, Fig. 8 shows the effect of the background temperature

on $\bar{F}/F$. Since ϵ^2 is proportional to kT , it is expected, as shown in Fig. 8, that $\bar{F}/F$ increase with increasing kT . That this curve has a maximum is a reflection of the fact that increasing kT increases the probability that particles we ignore, i.e. those with $E < E_T = \bar{E}$, will contribute to the fusion energy released. Thus as kT increases, for a given $\bar{E}$, we increasingly underestimate $\bar{F}$. It appears that more exact calculations⁽⁵⁾ of this effect give increasing $\bar{F}/F$ values at higher background temperatures; however a detailed comparison with these calculations has not yet been possible.

Though the increase in the energy multiplication factor is small for parameters of interest, it is still important in that this contributes to, rather than detracts from, the possibility of attaining break-even fusion tests in TCT.

Figure 1: $\bar{F}$ and F values for injection of deuterium into a pure tritium background plasma with $T_e = T_i = 3$ keV and $n_T = 3.0 \times 10^{13}$ ions/cm³.

Figure 2: $\bar{F}/F$ values for injection of deuterium into a pure tritium background plasma with $T_e = T_i = 3$ keV and $n_T = 3.0 \times 10^{13}$ ions/cm³.

Figure 3: $\bar{F}$ and F values for injection of deuterium into a pure tritium background plasma with $T_e = T_i = 10$ keV and $n_t = 3.0 \times 10^{13}$ ions/cm³.

Figure 4: $\bar{F}/F$ values for injection of deuterium into a pure tritium background plasma with $T_e = T_i = 10$ keV and $n_t = 3.0 \times 10^{13}$ ions/cm³.

Figure 5: $\bar{F}$ and F values for injection of deuterium into a pure ${}^3\text{He}$ background plasma with $T_e = T_i = 25$ keV and $n_{\text{He}} = 3.0 \times 10^{13}$ ions/cm³.

Figure 6: $\bar{F}/F$ values for injection of deuterium into a pure ${}^3\text{He}$ background plasma with $T_e = T_i = 25$ keV and $n_{\text{He}} = 3.0 \times 10^{13}$ ions/cm³.

Figure 7: $\langle \frac{dE}{dt} \rangle$ for the slowing down of deuterium in a pure tritium background plasma with $T_e = T_i = 3$ keV and 10 keV, $n_T = 3.0 \times 10^{13}$ ions/cm³.

Figure 8: $\bar{F}/F$ for injection of 100-keV deuterium ions into a pure tritium background plasma at various background temperatures, $n_T = 3.0 \times 10^{13}$ ions/cm³.

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