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COVARIANCE OF THE RARITA-SCHWINGER
FIELD WITH QUANTIZED SCALAR FIELD INTERACTION

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Covariance of the Rarita-Schwinger
Field with Quantized Scalar Field Interaction*

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ABSTRACT

The vector-spinor formulation of the mass m spin- $3/2$ field is extended to include a coupling to a scalar field ϕ through the replacement $m \rightarrow m + g\phi$. The commutation relations of the resulting system are determined and the customary absence of positive definiteness of the canonical anticommutators displayed. The generators of the transformations of the Poincaré group are determined together with their commutators with the canonical variables of the theory. It is seen that the system does indeed transform covariantly, thereby demonstrating that the appearance of indefinite metric need not imply the frequently inferred breakdown of Lorentz invariance.

INTRODUCTION

Although the theory of higher spin fields has made considerable progress since the pioneering work of Dirac¹ and Fierz and Pauli², it is also true that some very difficult problems have emerged even as some of the older ones were on their way to solution. Thus at present the question as to how an arbitrary spin field is to be described and quantized in such a way that standard (e.g. electromagnetic) couplings do not alter the number of degrees of freedom of the system is well understood³. However, as shown by Johnson and Sudarshan⁴ in the case of spin-3/2, the quantization of such fields for the case of an external coupling can introduce forms which are incompatible with the customarily assumed positive definite metric. Furthermore, even at the classical level one encounters difficulties with the spin-3/2 field since, as shown by Velo and Zwanziger⁵ there occur the so-called acausal modes of propagation in the presence of an external field.

In the example considered by Johnson and Sudarshan it was found that the breakdown of the positive definite metric occurred when the magnetic field becomes comparable to m^2 . Since the magnitude of the magnetic field, however,

is not an invariant, the sign of the offending term depends on the frame in which quantization is carried out. This observation caused those authors to argue that Lorentz covariance is lost in the presence of an interaction.

More recently the question of covariance has been revived by Mainland and Sudarshan^{6,7} who considered the same theory as that discussed in ref. 4. They showed by giving a careful definition of covariance for the case of an external field that, contrary to the assertions of ref. 4, the theory satisfies all requirements of covariance. Their result is, however, not entirely satisfying in that the complicated dependence of the spin-3/2 field on the external field variables (knowledge of which is essential in order to demonstrate covariance) is postulated and not derived directly from equations of motion. Whatever misgivings one might have on this point would, of course, be settled by consideration of the case of a coupling to a quantized field, a task which is carried out in this paper.

Rather than deal with the electromagnetic coupling itself we choose here to deal with a somewhat simpler model, namely the case of a coupling to a scalar field. This approach has the advantages of i) simplifying the calculation ii) displaying a coupling which has previously been almost totally ignored and iii) allowing (as will become clear in the subsequent calculations) a rather clear separation between the well-behaved transverse components

of the spin-3/2 field and the troublesome longitudinal components.

In the following section the Lagrangian and equations of motion are displayed and the commutation relations of the coupled system derived. Section three presents the energy-momentum tensor and a derivation of the commutators of the energy-momentum vector P^μ with the canonical variables of the system. The extension to include commutators of the generators of Lorentz transformations with these same fields is developed in the concluding section. A brief appendix is included to describe the case in which one takes the scalar field to be unquantized.

II. The Canonical Commutation Relations

In this paper the conventional description of a spin-3/2 field by means of a sixteen component vector-spinor ψ^μ will be adopted. It is a trivial step to carry out the coupling of such a field to a scalar field ϕ by means of the replacement $m \rightarrow m + g\phi$ where m is the fermion bare mass. Such a system is thus described by the Lagrangian⁸

$$\begin{aligned} \mathcal{L} = & \frac{i}{2} \psi^\alpha \beta [g_{\alpha\beta} \gamma^\mu + W(\delta_\alpha^\mu \gamma_\beta + \delta_\beta^\mu \gamma_\alpha) - \frac{1}{2}(3W^2 + 2W + 1)\delta_\alpha^\mu \gamma_\beta] \partial_\mu \psi^\beta \\ & - \frac{1}{2}(m + g\phi) \psi^\alpha \beta [g_{\alpha\beta} + (3W^2 + 3W + 1)\delta_\alpha^\mu \gamma_\beta] \psi^\beta + \phi^\mu \partial_\mu \phi \\ & + \frac{1}{2} \phi^\mu \phi_\mu - \frac{1}{2} \mu^2 \phi^2, \end{aligned}$$

a structure which in turn implies the equations of motion

$$\begin{aligned} \gamma^\alpha \frac{1}{i} \partial_\alpha \psi^\mu + W \gamma^\mu \frac{1}{i} \partial_\alpha \psi^\alpha + W \frac{1}{2} \partial^\mu \gamma_\alpha \psi^\alpha - \frac{1}{2}(3W^2 + 2W + 1) \gamma^\mu \\ \gamma^\alpha \frac{1}{i} \partial_\alpha \gamma_\beta \psi^\beta + (m + g\phi) [\psi^\mu + (3W^2 + 3W + 1) \gamma^\mu \gamma_\alpha \psi^\alpha] = 0 \end{aligned} \quad (2.1a)$$

$$\partial_\mu \phi^\mu = -\mu^2 \phi - \frac{1}{2} g \psi^\alpha \beta [g_{\alpha\beta} + (3W^2 + 3W + 1) \gamma_\alpha \gamma_\beta] \psi^\beta \quad (2.1b)$$

$$\phi^\mu = -\partial^\mu \phi \quad (2.1c)$$

with symmetrization of operator products being assumed here and throughout the remainder of this paper.

As is well known under the point transformation

$$\psi'_\alpha = \psi_\alpha + \frac{1}{4} \mu \gamma_\alpha \gamma_\beta \psi^\beta \quad (2.2)$$

the matrices which occur in $\mathcal{L}$ are found to have the same structure except that W is replaced by

$$W' = W(1 - \mu) - \frac{1}{2} \mu.$$

Inasmuch as the transformation (2.2) merely mixes the spin-1/2 components while leaving the 3/2 components invariant, it follows that the parameter W is without physical significance and can be chosen at will.

In accord with the general formulation of the action principle⁹ the commutation relations of the theory under consideration are to be determined from

$$[\chi(x), G] = \frac{i}{2} \delta \chi(x)$$

where $\chi(x)$ represents any of the field operators ψ^μ, ϕ^μ , or ϕ . Making the conventional choice $W = -1$ the generator G is seen by inspection to have the form

$$G = \frac{1}{2} \int d^3x \left[i \psi_k \delta \psi_k + i \psi_k \gamma^k \delta \gamma^l \psi_l + \phi^0 \delta \phi - \phi \delta \phi^0 \right],$$

Since under a spatial rotation the eight components $\psi_k + \frac{1}{3} \gamma_k \gamma_l \psi^l$ transform among themselves and vanish upon contraction

with γ_k , one recognizes this set as being the pure spin-3/2 part of the field and defines

$$\psi_k^{3/2} \equiv \psi_k + \frac{1}{3} \gamma_k \gamma_l \psi^l.$$

Using this definition G assumes the form

$$G = \frac{1}{2} \int d^3x \left[i \psi_k^{3/2} \delta \psi_k^{3/2} + i \frac{2}{3} \psi_k \gamma^k \delta \gamma_l \psi^l + \phi^0 \delta \phi - \phi \delta \phi^0 \right]$$

where the variations of the various fields are to be carried out in such a way as to be consistent with the primary constraint

$$(m + g\phi - \frac{2}{3} \underline{\gamma} \cdot \underline{p}) \underline{\gamma} \cdot \underline{\psi} = \underline{p} \cdot \underline{\psi}^{3/2} \quad (2.3)$$

which is merely the temporal component of Eq. (2.1a).

It is of interest to note that a further decomposition of $\psi_k^{3/2}$ into four (three dimensional) longitudinal and four transverse components can be carried out such that

$$\psi_k^{3/2} = \psi_k^{3/2\tau} + \psi_k^{3/2L}$$

where

$$\underline{p} \cdot \underline{\psi}^{3/2\tau} = 0.$$

Such an approach is useful since the constraint (2.3) does not involve the transverse set $\psi_k^{3/2\tau}$ and brings the generator G to the somewhat more convenient form

$$G = \frac{1}{2} \int d^3x \left[i \psi_k^{3/2\tau} \delta \psi_k^{3/2\tau} + i \psi_k^{3/2\tau} \delta \psi_k^{3/2\tau} + i \frac{1}{3} \psi_{\mu} \gamma_{\mu} \delta \psi_{\mu} + \phi^0 \delta \phi - \phi \delta \phi^0 \right]$$

In view of the fact that variations of the transverse set are unconstrained, it is now trivial to infer the equal time commutation relations of these components to be

$$\begin{aligned} \{ \psi_k^{3/2\tau}(x), \psi_{k'}^{3/2\tau}(x') \} &= P_{k0}^{3/2\tau} \delta(x-x') \\ \{ \psi_k^{3/2\tau}(x), \psi_{\mu}(x') \} &= [\psi_k^{3/2\tau}(x), \phi(x')] = [\psi_k^{3/2\tau}(x), \phi^0(x')] = 0 \end{aligned}$$

where we have introduced

$$P_{k0}^{3/2\tau} = S_{k0} + \frac{1}{2} \gamma_k \gamma_2 - \frac{3}{2} \nabla_k \nabla_2 \nabla^{-2} - \frac{1}{2} (\delta_k \gamma_2 \cdot \nabla \nabla_2 + \nabla_k \gamma_2 \cdot \nabla \gamma_2) \nabla^{-2}$$

which satisfies the conditions

$$\begin{aligned} \nabla_k P_{k0}^{3/2\tau} &= \gamma_k P_{k0}^{3/2\tau} = 0 \\ P_{k0}^{3/2\tau} P_{0m}^{3/2\tau} &= P_{km}^{3/2\tau} \end{aligned}$$

and is thus seen to be the projection operator onto the transverse spin-3/2 subspace.

The determination of the remaining commutation relations of the theory is somewhat more involved, being most conveniently

performed by using the Lagrange multiplier device. Since this technique has been discussed in detail in an earlier work,¹⁰ it is sufficient merely to quote the results of such a calculation. Thus one finds in terms of the longitudinal projection operator

$$P_{k2}^{3/2L} = -\frac{1}{6} \gamma_k \gamma_k + \frac{3}{2} \nabla_k \nabla_k \nabla^{-2} + \frac{1}{2} (\gamma_k \gamma_k \nabla \nabla + \nabla_k \gamma_k \nabla \gamma_k) \nabla^{-2}$$

which is defined by the condition

$$\delta_{kl} + \frac{1}{3} \gamma_k \gamma_l = P_{kl}^{3/2T} + P_{kl}^{3/2L}$$

that the equal time commutation relations assume the form

$$\begin{aligned} [\phi^0(x), \phi(x)] &= -i \delta(x-x') \\ [\gamma \cdot \psi(x), \phi^0(x')] &= -ig(m + g\phi + \frac{2}{3} \gamma \cdot p) \Delta \delta(x-x') \\ [\psi_k^{3/2L}(x), \phi^0(x')] &= -ig \frac{2}{3} P_{k2}^{3/2L} p_2 \Delta \gamma \cdot \psi \delta(x-x') \\ \{\psi_k^{3/2L}(x), \gamma \cdot \psi(x')\} &= -P_{k2}^{3/2L} p_2 \Delta (m + g\phi - \frac{2}{3} \gamma \cdot p) \delta(x-x') \\ \{\psi_k^{3/2L}(x), \psi_0^{3/2L}(x')\} &= P_{k2}^{3/2L} [\delta_{ij} + \frac{2}{3} p_i \Delta p_j] p_j \delta(x-x') \\ \{\gamma \cdot \psi(x), \gamma \cdot \psi(x')\} &= \frac{3}{2} [1 - (m + g\phi + \frac{2}{3} \gamma \cdot p) \Delta (m + g\phi - \frac{2}{3} \gamma \cdot p)] \delta(x-x') \end{aligned} \quad (2.4)$$

where

$$\Delta^{-1} = (m + g\phi)^2 - \frac{2}{3} \gamma \cdot \underline{p} g\phi$$

with all other commutators among the set of components ψ_a, ϕ^c and ϕ being equal to zero. One readily verifies from arguments of the type given in refs. 4 and 10 that the commutation relations (2.4) are incompatible with a positive definite metric in the Hilbert space. It will be seen in subsequent sections, however, that this difficulty does not have any direct bearing upon the question of the operator covariance of the theory.

III. Translational Invariance

In order to demonstrate the covariance of the theory described in the preceding section it is sufficient to show that the operators ψ^A , ϕ^μ , and ϕ transform respectively as vector-spinor, vector and scalar under the operations of the Poincaré group. Assuming that this can be shown, the covariance of the equations of motion (2.1) follows trivially.

At this point one recognizes a certain advantage in dealing with the scalar field interaction as opposed to the minimal electromagnetic coupling. In particular since the transformation law of the electromagnetic field is known not to be a manifestly covariant one (in the absence of an indefinite metric), the proof of covariance is somewhat more involved. On the other hand the use of a manifestly covariant formulation of the electromagnetic field makes it difficult to disentangle the indefinite metric associated with the Rarita-Schwinger field from that which is required for the Maxwell field.

Since we consider the calculations described in this paper to be in some sense a test of the action principle, the energy-momentum tensor has been derived within the framework of that formalism. This is essentially a straightforward though tedious task and we consequently

merely quote the result

$$\begin{aligned}
 T^{\mu\nu} = & \phi^\mu \phi^\nu - \frac{1}{2} g^{\mu\nu} [\phi^\alpha \phi_\alpha + \mu^2 \phi^2] + \frac{m+g\phi}{2} [\psi^\mu \beta (\delta_\alpha^\nu + \gamma^\nu \gamma_\alpha) \psi^\alpha \\
 & + \psi^\alpha \beta (\delta_\alpha^\nu + \gamma^\nu \gamma_\alpha) \psi^\mu] - \frac{i}{2} \psi^\alpha A_{\alpha\mu}^\mu \partial^\nu \psi^\mu - \frac{i}{2} \psi^\nu A_{\nu\mu}^\mu \partial_\lambda \psi^\mu g^{\mu\lambda} \\
 & - \frac{i}{2} \psi^\alpha A_{\alpha\mu}^\mu \partial_\lambda \psi^\nu g^{\mu\lambda} - \frac{1}{2} \partial_\lambda [\psi^\alpha \beta \{ \beta A_{\alpha\mu}^\mu \sigma^{\lambda\nu} \} \psi^\mu]
 \end{aligned}
 \tag{3.1}$$

where symmetrization in μ and ν is assumed and use has been made of the definitions

$$\beta A_{\alpha\mu}^\mu = g_{\alpha\mu} \gamma^\mu - \delta_\alpha^\mu \gamma_\mu - \delta_\mu^\alpha \gamma_\alpha - \gamma_\alpha \gamma^\mu \gamma_\mu
 \tag{3.2}$$

and

$$\sigma^{\mu\nu} = \frac{i}{4} [\gamma^\mu, \gamma^\nu].$$

A direct verification of the validity of this expression is possible by the demonstration that the $T^{\mu\nu}$ of Eq. (3.1) is conserved. This can be carried out most readily if one notes that the equation of motion for ψ^μ can be written as

$$[\beta A_{\alpha\mu}^\mu \frac{1}{i} \partial_\mu + (m+g\phi)(g_{\alpha\mu} + \gamma_\alpha \gamma_\mu)] \psi^\mu = 0$$

and makes use of the identities

$$i [\sigma^{\mu\nu}, \beta A_{\alpha\beta}^{\lambda}] = \beta [A_{\alpha\beta}^{\mu} g^{\nu\lambda} - A_{\alpha\beta}^{\nu} g^{\mu\lambda} + A_{\alpha\beta}^{\lambda} g^{\mu\nu} - A_{\alpha\beta}^{\lambda} g^{\mu\nu} + A_{\alpha\mu}^{\lambda} g^{\nu\beta} - A_{\alpha\mu}^{\nu} g^{\lambda\beta} - A_{\alpha\nu}^{\lambda} g^{\mu\beta} + A_{\alpha\nu}^{\mu} g^{\lambda\beta}]$$

and

$$A_{\alpha\beta}^{\lambda} g^{\mu\nu} + A_{\alpha\beta}^{\mu} g^{\nu\lambda} = 0.$$

Using (3.1) one can cast the formal expression for the momentum operator into the form

$$P^{\mu} = \int d^3x \left\{ -\frac{1}{4} \psi^{\alpha} A_{\alpha\beta}^{\mu} \partial^{\mu} \psi^{\beta} + \frac{i}{2} \partial^{\mu} \psi^{\alpha} A_{\alpha\beta}^{\mu} \psi^{\beta} + \frac{1}{2} \phi^{\alpha} \phi^{\mu} + \frac{1}{2} g^{\mu\nu} [\phi^{\alpha} \dot{\phi}_{\alpha} + \dot{\phi}^{\alpha} \phi_{\alpha}] \right\}.$$

Considering first the spatial components P_k one verifies that upon using (3.2) one can also write

$$P^k = \int d^3x \left\{ -\frac{1}{2} \psi^{\alpha} \partial^k \psi_{\alpha} - i \frac{1}{2} \psi^{\alpha} \partial^k \psi_{\alpha} + \phi^{\alpha} \phi^k \right\}. \quad (3.3)$$

Since (3.3) is formally identical to the form of the generator G of changes in the canonical variables of the system provided only that the δ which occurs in G is replaced by $-\partial_k$ it requires a relatively modest effort to see that for all operators $\chi(x)$

$$[\chi(x), P^k] = -i \partial_k \chi(x). \quad (3.4)$$

The extension of (3.4) to the temporal component of the momentum operator is somewhat more involved. It is most conveniently carried out by recognizing that the independent components of the system can be arbitrarily taken to be $\psi_k^{3/2}, \bar{\psi}_k, \phi^0$ and ϕ with $\psi_k^{3/2}$ being defined nonlocally in terms of ϕ and $\bar{\psi}_k$ by (2.3). It is therefore sufficient to show that

$$[\chi(x), P^0] = -i \partial^0 \chi(x) \quad (3.5)$$

for $\chi(x)$ ranging over the set $\psi_k^{3/2}, \bar{\psi}_k, \phi^0$ and ϕ .

Upon writing the operator P^0 in the form

$$P^0 = \int d^3x \left\{ -\frac{i}{4} \psi_k^{3/2} \partial^0 \bar{\psi}_k - \frac{i}{4} \bar{\psi}_k \partial^0 \psi_k^{3/2} + h.c. + \frac{1}{2} (\phi_k^2 + \phi_k'^2 + g^2 \phi^2) \right\} \quad (3.6)$$

one sees that it is necessary first to obtain expressions for $\partial^0 \bar{\psi}_k$ and $\partial^0 \psi_k^{3/2}$ in terms of the canonical fields in order to calculate explicitly the required commutators. By straightforward manipulation of the equations of motion (2.1a) one finds that for $\psi_k^{3/2}$ the time development is described by

$$\frac{1}{i} \partial_0 \psi_k^{3/2} = \beta \left[P_{ke}^{3/2} + P_{ka}^{3/2} \right] \left[-\left(\frac{1}{2} k + m + g\phi \right) \psi_k^{3/2} + \frac{1}{2} \gamma_\mu \psi_k^\mu - \frac{1}{2} \gamma_0 \bar{\psi}_k \right] \quad (3.7)$$

and consequently for $\psi_k^{3/2}$ one has the form

$$\frac{1}{i} \partial_0 \psi_k^{3/2} = -\beta P_{k0}^{3/2} (\gamma \cdot p + m + g\phi) \psi_k^{3/2}. \quad (3.8)$$

In order to eliminate from (3.7) any explicit dependence upon the field ψ^0 one must use the constraint

$$\beta \partial_\mu \psi^\mu = i \frac{2}{3} g \Delta [\beta (\gamma \cdot \phi) \psi_k^{3/2} + (\phi^0 - \frac{1}{3} \beta \gamma \cdot \nabla \phi) \gamma \cdot \psi] \quad (3.9)$$

which is obtained by manipulation of the result of successive contractions of (2.1a) with γ^μ and γ^μ . A similar calculation yields for $\partial^0 \gamma \cdot \psi$ the result

$$\partial_0 \gamma \cdot \psi = -i (m + g\phi + \gamma \cdot p) \beta \gamma \cdot \psi + i \frac{3}{2} (m + g\phi + \frac{1}{3} \gamma \cdot p) \beta \sigma_\mu \psi^\mu. \quad (3.10)$$

Upon substitution of (3.8) and (3.10) into (3.6) there ensues a significant simplification in the form of P^0 which can now be written as

$$P^0 = \frac{1}{2} \int d^3x [\phi_0^2 + \phi_k^2 + \mu^2 \phi^2 + \psi_k^{3/2} \beta (\gamma \cdot p + m + g\phi) \psi_k^{3/2} - \frac{1}{3} \gamma \cdot \nabla \beta \gamma \cdot \psi] \quad (3.11)$$

Using Eq. (3.11) and the equal time commutation relations it follows by inspection that the commutator of P^0 with $\psi_k^{3/2}$ and ϕ yields the time derivatives of these components as given by (3.8) and (2.1c) respectively. In order to verify that (3.5) holds for ϕ^0 one notes that the equation of motion implies

$$\mathcal{F}\phi^0 = \mu^2 \phi - \nabla^2 \phi - \frac{1}{2} g \psi^\alpha \beta (g_{\alpha\beta} + \partial_\alpha \partial_\beta) \psi^\beta \quad (3.12)$$

while straightforward commutation yields

$$i[\phi^0, P^0] = \mu^2 \phi - \nabla^2 \phi + \frac{1}{2} g \left[\frac{1}{2} \psi_\alpha \cdot \partial_\alpha \Delta \phi_k (P_{k1}^{\frac{3}{2}} + P_{k2}^{\frac{3}{2}}) \beta \right. \\ \left. (\partial \cdot k + m + g \phi) \psi_k^{\frac{3}{2}} + \frac{1}{2} \psi_\alpha \cdot \partial_\alpha \Delta (m + g \phi - \frac{1}{2} \partial \cdot k) \beta \right] \frac{1}{2} \psi_\alpha \cdot \partial_\alpha \psi_k + h.c. \quad (3.13)$$

While the equality of the last terms in Eqs. (3.12) and (3.13) is by no means obvious, it requires only straightforward manipulation of the constraint equations to establish their identity. Finally one considers the case $\mathcal{K}(x) = \gamma_\mu \psi_\mu$. This time the commutator is

$$[\gamma_\mu \psi_\mu, P^0] = -ig\phi^0 (m + g\phi + \frac{1}{2} \partial \cdot k) \Delta \gamma_\mu \psi_\mu + \frac{1}{2} \{ (m + g\phi + \frac{1}{2} \partial \cdot k) \Delta \\ \phi_k (P_{k1}^{\frac{3}{2}} + P_{k2}^{\frac{3}{2}}) \beta (\gamma_\mu \cdot k + m + g\phi) \psi_k^{\frac{1}{2}} - \frac{1}{2} [1 - (m + g\phi + \frac{1}{2} \partial \cdot k) \\ \Delta (m + g\phi - \frac{1}{2} \partial \cdot k)] \beta \gamma_\mu \cdot k \psi_k + h.c. \},$$

a result which after some minor rearrangements yields the result given in Eq. (3.10). Since it has already been remarked that the translational invariance of the theory depends only upon the verification of (3.4) and (3.5) for the set $\psi_k^{\frac{3}{2}T}$, $\gamma_\mu \psi_\mu$, ϕ , and ϕ^0 one can now consider such invariance as having been established and focus subsequent consideration of this theory purely on

the question of Lorentz invariance.

IV. Lorentz Invariance

In order to establish the invariance of the theory under rotations in Minkowski space one considers the result of commuting the conserved generators

$$J^{\mu\nu} = \int d^3x (x^\nu T^{\mu 0} - x^\mu T^{\nu 0})$$

with the complete set of basic field variables. Thus covariance will clearly be established provided that one obtains

$$[\phi, J^{\mu\nu}] = i(x^\mu \partial^\nu - x^\nu \partial^\mu) \phi \quad (4.1a)$$

$$[\phi^\alpha, J^{\mu\nu}] = i(x^\mu \partial^\nu - x^\nu \partial^\mu) \phi^\alpha + (g^{\nu\alpha} \phi^\mu - g^{\mu\alpha} \phi^\nu) \quad (4.1b)$$

$$[\psi^\alpha, J^{\mu\nu}] = i(x^\mu \partial^\nu - x^\nu \partial^\mu) \psi^\alpha - i(g^{\nu\alpha} \psi^\mu - g^{\mu\alpha} \psi^\nu) - \sigma^{\mu\nu} \psi^\alpha \quad (4.1c)$$

Since, however, (4.1b) is an immediate consequence of (4.1a) and (2.1c) it is sufficient merely to establish (4.1a) and (4.1c). Further simplification ensues upon recognition of the fact that verification of Eqs. (4.1) for the case $\mu=0, \nu=k$ implies the desired result for $\mu=k, \nu=0$. This is established merely by combining the results of two successive Lorentz transformations along mutually orthogonal directions.

As a final step toward the reduction of the problem

one observes that if (4.1c) is valid for the spin-3/2 part of the field, i.e.

$$[\psi_0^{3/2}, J^{0k}] = i(x^0 \partial^k - x^k \partial^0) \psi_0^{3/2} - i(\delta_{0k} + \frac{1}{2} \gamma_k \gamma_0) \psi_0^0 \quad (4.2)$$

$$- \frac{1}{2} (\delta_{0m} + \frac{1}{2} \gamma_0 \gamma_m) \beta \gamma_k \psi_m,$$

then upon contracting with $\frac{1}{2} \beta$ and using (2.3) one obtains

$$(m + \gamma^0 \partial^0 - \frac{1}{2} \gamma_k \partial^k) [\frac{1}{2} \beta \psi_0^{3/2}, J^{0k}] = i(x^0 \partial^k - x^k \partial^0) (\frac{1}{2} \beta \psi_0^{3/2} - \frac{1}{2} \gamma_k \partial^k \psi_0^{3/2})$$

$$- \partial^0 \psi_0^k - (\partial^0 + \frac{1}{2} \gamma_k \partial^k) (\delta_{0k} + \frac{1}{2} \gamma_k \gamma_0) \psi_0^0$$

$$+ i \gamma (x^k \partial^0 - x^0 \partial^k) \phi \beta \psi_0^{3/2}.$$

Somewhat lengthy but essentially straightforward manipulations of the equations of motion allow one to extract a factor of $(m + \gamma^0 \partial^0 - \frac{1}{2} \gamma_k \partial^k)$ from both sides and thereby obtain

$$[\frac{1}{2} \beta \psi_0^{3/2}, J^{0k}] = i(x^0 \partial^k - x^k \partial^0) \frac{1}{2} \beta \psi_0^{3/2} - i \beta \psi_0^k - \frac{i}{2} \beta \gamma^k \gamma_0 \psi_0^0 - (\delta^k_0 \psi_0^0)$$

which is clearly a special case of (4.1c). A similar calculation based on Eq. (3.9) can be carried out for ψ^0 so that in analogy to the approach of the preceding section one finds it to be sufficient to consider only the commutators $[\phi, J^{0k}]$ and $[\psi_0^{3/2}, J^{0k}]$.

The evaluation of these commutators requires now that utilization be made of the explicit form of J^{0k} .

Referring to (3.1) and eliminating all time derivatives by means of the equations of motion one finds after some manipulation that J^{sh} can be cast into the form

$$J^{0k} = -\lambda^k P_k + \int d^3x \left\{ x^k \left[\frac{1}{2} (\dot{\phi}_0^2 + \dot{\phi}_k^2 + \rho^2 \dot{\phi}^2) - \frac{1}{2} \psi_2^2 \beta \left(\frac{1}{2} k + m + g\phi \right) \psi_2^{3/2} \right. \right. \\ \left. \left. - \frac{1}{9} \psi_2 \cdot \psi_2 \beta \left(\frac{1}{2} k + m + g\phi \right) \right] - i \frac{1}{3} \psi_2^2 \beta \left(\frac{1}{2} k + m + g\phi \right) \psi_2^{3/2} \right\}.$$

This structure allows one to verify by inspection the validity of (4.1a) and consequently leaves only the task of verifying (4.2). Upon direct calculation that commutator with some rearrangement takes the form

$$\begin{aligned} [\psi_2^{3/2}, J^{0k}] = & i (\kappa^0 \partial^k - \gamma^k \partial^0) \psi_2^{3/2} - i (\delta_{\kappa 0} + \frac{1}{3} \gamma_2 \gamma_k) (\psi_2^0 + \frac{1}{2} \rho \gamma \cdot \psi_2) \\ & - \frac{1}{3} (\delta_{ij} + \frac{1}{3} \gamma_i \gamma_j) \left\{ \frac{2}{3} \beta_i \Delta \beta_j (\delta_{jk} + \frac{1}{3} \gamma_j \gamma_k) \rho \gamma \cdot \psi_2 \right. \\ & - \beta_i \Delta (m + g\phi - \frac{2}{3} \gamma \cdot \beta) \rho \psi_2^2 + \beta_i \Delta (\delta_{km} + \frac{1}{3} \gamma_k \gamma_m) \beta \left(\frac{1}{2} k + m + g\phi \right) \psi_2^{3/2} \\ & + \frac{2}{9} (\delta_{ij} + \frac{2}{3} \beta_i \Delta \beta_j) (\delta_{jm} + \frac{1}{3} \gamma_j \gamma_m) \rho \gamma_k \psi_2^{3/2} - \frac{2}{9} \beta_i \Delta \gamma_i \rho \gamma \cdot \beta \gamma \cdot \psi_2 \\ & \left. + \frac{1}{6} \beta_i \Delta (m + g\phi - \frac{2}{3} \gamma \cdot \beta) \rho \gamma_k \gamma \cdot \psi_2 - h.c. \right\}. \end{aligned}$$

It is now possible to show by making use of the constraint equations (2.3) that all terms in the curly brackets proportional to β_i add up to zero. After incorporating this cancellation the few remaining terms readily allow themselves to be brought to the form given by (4.2),

thereby completing the proof of covariance.

One thus has the remarkable result that the theory of a spin-3/2 field with scalar interaction is fully covariant despite the fact that it requires an indefinite metric and possesses acausal modes of propagation.¹¹ This provides on the one hand a remarkable demonstration of the strength of the action principle insofar as it shows the possibility of having an otherwise formally consistent formulation despite the remarkable difficulties already noted. On the other hand it seems to remove once and for all any hope that the indefinite nature of the canonical commutation relations could be the result of an incorrect technique of calculation which might be eliminated within the framework of some alternative, but as yet unspecified dynamical approach.

Appendix

In this appendix we consider the special case in which ϕ is unquantized thereby allowing one to verify the close correspondence between the results of this paper and those obtained in ref. 6 for the case of the electromagnetic interaction. In considering the set of commutators (3.4) and (3.5) one can, of course, drop all terms in P^μ which are bilinear in ϕ and ϕ' . Furthermore, since $\psi_1^{\gamma_1 \tau}$ commutes with ϕ and ϕ' even for a quantized ϕ , it is easily seen that the commutators of $\psi_1^{\gamma_1 \tau}$ with P^μ continue to satisfy Eqs. (3.4) and (3.5) thereby leaving only ψ_1 and $\psi_1^{\gamma_1}$ for special treatment.

In considering the translational properties of a theory with an unquantized interaction one, of course, does not expect the entire space-time dependence to be generated by P^μ . Thus the total derivative (which we continue to denote by ∂_μ) is expressed by

$$\frac{1}{i} \partial_\mu \chi = [\chi, P_\mu] + \frac{1}{i} \bar{\partial}_\mu \chi \quad (A.1)$$

where we have introduced an overbar notation to denote differentiation with respect to explicit dependence on the space-time coordinates. The last term in (A.1) arises from the fact that constraints such as (2.3) generally imply such dependence upon the coordinates.

Taking (A.1) as given one computes for $\chi = \pi \cdot \psi$

$$\bar{\partial}_\mu \psi = -g(\phi + g\phi + \frac{1}{2}g^2\phi) \Delta \bar{\partial}_\mu \phi \psi \quad (A.2)$$

If on the other hand one chooses $\psi = \psi_{\kappa}^{\chi_1}$ there obtains

$$\bar{\partial}_\mu \psi_{\kappa}^{\chi_1} = -\frac{1}{2}g p_{\kappa}^{\chi_1} \beta_2 \Delta \bar{\partial}_\mu \phi \psi \quad (A.3)$$

Since there must be internal consistency between (A.2) and (A.3) one displays this by considering ψ by means of (2.3) to be a function of $\psi_{\kappa}^{\chi_1}$ and ϕ and writing

$$\bar{\partial}_\mu \psi = \int \left[\frac{\delta \psi}{\delta \phi(x)} \bar{\partial}_\mu \phi(x) + \frac{\delta \psi}{\delta \psi_{\kappa}^{\chi_1}(x)} \bar{\partial}_\mu \psi_{\kappa}^{\chi_1}(x) \right] \beta_{\chi_1} \quad (A.4)$$

Upon using (A.3) and the constraint (2.3), Eq. (A.4) is seen by elementary calculation to yield the form (A.2) for $\bar{\partial}_\mu \psi$. Despite this result it is to be noted that unlike the quantized interaction the case discussed here requires the assumption of a form such as (A.1) for the commutators of P^μ which can then merely be shown by a relation of the type (A.4) to generate internally consistent results.

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