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AN ELEMENTARY CHARACTERIZATION
OF SOLUTIONS OF FREDHOLM
EQUATIONS OF THE FIRST KIND

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Date Transmitted: November 1974

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TABLE OF CONTENTS

	PAGE
ABSTRACT	iv
I. INTRODUCTION	1
II. CHARACTERIZATION OF THE SOLUTIONS OF NORM β IN TERMS OF f_1 AND $N(K)$	1
III. CHARACTERIZATION OF THE SOLUTIONS OF NORM α IN TERMS OF f_1 AND $N(K)$	2
IV. CONCLUSION	5
V. EXAMPLES	5
VI. BIBLIOGRAPHY	8

ABSTRACT

Solutions of Fredholm equations of the first kind are characterized in terms of a known solution and the null space. Several examples are given illustrating the relationship.

I. Introduction

Linear Lebesgue integral equations of the first kind of the form $(Kf)(x) = \int_0^1 k(x,y)f(y)dy = g(x)$ where $k(x,y)$ is continuous on $[0,1] \times [0,1]$ and $\int_0^1 \int_0^1 |k(x,y)|^2 dx dy < \infty$, i.e., K completely continuous, and f, g in the real Hilbert space $L_2[0,1]$ fail to have solutions or have non-unique solutions. The purpose of this paper is, when a solution exists, to characterize the solutions of $(Kf)(x) = g(x)$ in terms of a known solution and the null space of the operator K .

In general let $K: H_1 \rightarrow H_2$, K a completely continuous operator from a real separable Hilbert space H_1 into a real Hilbert space H_2 . Let $g \in R(K) \subset H_2$, $R(K)$ the range of K , and $f_1 \in H_1$ be a solution to the equation

$$Kf = g \quad (1)$$

with $\|f_1\|$ denoted by β , $g \neq 0$. Let $u \in N(K)$, the null space of K , $\|u\| = 1$, $u \neq 0$.

II. Characterization of the Solutions of Norm β in Terms of f_1 and $N(K)$

For each $u \in N(K)$ construct $f_u = f_1 + \epsilon u$, $f_u \neq f_1$ such that

$$\|f_u\| = \beta, \text{ so}$$

$$\beta^2 = \langle f_u, f_u \rangle$$

$$= \langle f_1 + \epsilon u, f_1 + \epsilon u \rangle$$

$$= \langle f_1, f_1 \rangle + 2\epsilon \langle f_1, u \rangle + \epsilon^2 \langle u, u \rangle$$

$$= \beta^2 + 2\epsilon \langle f_1, u \rangle + \epsilon^2$$

thus $\epsilon (2 \langle f_1, u \rangle + \epsilon) = 0$

so $\epsilon = 0$ and $f_1 = f_u$

or $\epsilon = -2 \langle f_1, u \rangle$

yielding $f_u = f_1 - 2 \langle f_1, u \rangle u$ (2)

is a solution of norm β .

Consider f_2 any solution of norm β , $f_2 \neq f_1$, then $f_1 - f_2 \in N(K)$ and

$$f_2 = f_1 - 2 \frac{\langle f_1, f_1 - f_2 \rangle}{\langle f_1 - f_2, f_1 - f_2 \rangle} (f_1 - f_2)$$

is a solution of form (2) of norm β .

So $\{f \mid Kf = g, \|f\| = \beta\}$ can be characterized by a solution f_1 of (1) such that $\|f_1\| = \beta$ and the null space $N(K)$.

III. Characterization of the Solutions of Norm α in Terms of f_1 and $N(K)$.

For each $u \in N(K)$ construct $f_u = f_1 + \epsilon u$ such that $\|f_u\| = \alpha$,

so $\alpha^2 = \beta^2 + 2 \epsilon \langle f_1, u \rangle + \epsilon^2$

and $0 = \beta^2 - \alpha^2 + 2 \epsilon \langle f_1, u \rangle + \epsilon^2$

so $\epsilon = \frac{-2 \langle f_1, u \rangle \pm \sqrt{4 \langle f_1, u \rangle^2 - 4(\beta^2 - \alpha^2)}}{2}$

$$= -\langle f_1, u \rangle \pm \sqrt{\langle f_1, u \rangle^2 + \alpha^2 - \beta^2}$$

take $f_u = f_1 - (\langle f_1, u \rangle \pm \sqrt{\langle f_1, u \rangle^2 + \alpha^2 - \beta^2})u$

where $\alpha^2 \geq \beta^2 - \langle f_1, u \rangle^2$ is necessary.

Case 1. Suppose $\langle f_1, u \rangle = 0$ for all $u \in N(K)$, i.e., $f_1 \in N(K)$, then

$$f_u = f_1 \pm \sqrt{\alpha^2 - \beta^2} u$$

and for any solution f_2 of norm α

$$f_2 = f_1 \pm \frac{\sqrt{\alpha^2 - \beta^2} (f_1 - f_2)}{\|f_1 - f_2\|} \quad (3)$$

(3) is clear since

$$\begin{aligned} \langle f_1 - f_2, f_1 - f_2 \rangle &= \langle f_1, f_1 - f_2 \rangle - \langle f_2, f_1 - f_2 \rangle \\ &= 0 - \langle f_2, f_1 \rangle + \langle f_2, f_2 \rangle \\ &= -\langle f_2, f_1 \rangle + \alpha^2 \end{aligned}$$

$$\begin{aligned} \text{and } \langle f_1 - f_2, f_1 - f_2 \rangle &= \langle f_1, f_1 \rangle - 2\langle f_1, f_2 \rangle + \langle f_2, f_2 \rangle \\ &= \beta^2 - 2\langle f_1, f_2 \rangle + \alpha^2 \end{aligned}$$

$$\text{so } \beta^2 - 2\langle f_1, f_2 \rangle + \alpha^2 = -\langle f_2, f_1 \rangle + \alpha^2$$

$$\text{implies } \langle f_1, f_2 \rangle = \beta^2$$

$$\text{so } \langle f_1 - f_2, f_1 - f_2 \rangle = \alpha^2 - \beta^2.$$

Note $f_1 \in N(K)^\perp$ is unique since if f_3 is another solution such that

$f_3 \in N(K)^\perp$ then $f_1 - f_3 \in N(K)^\perp$ and $f_1 - f_3 \in N(K)$. So $f_1 - f_3 = 0$

implies $f_1 = f_3$. Clearly f_1 is the solution of (1) of minimum norm.

So $\{f \mid Kf = g, \|f\| = \alpha\}$ can be characterized by a solution f_1 of (1)

such that $\|f_1\| = \beta$, $f_1 \in N(K)^\perp$ and the $N(K)$.

Case 2. For $f_1 \notin N(K)^\perp$, $u \in N(K)$ and $\|u\| = 1$.

$$\text{Let } f_u = f_1 - (\langle f_1, u \rangle \pm \sqrt{\langle f_1, u \rangle^2 + \alpha^2 - \beta^2})u$$

$$\text{for } \alpha^2 \geq \beta^2 - \langle f_1, u \rangle^2$$

then f_u is a solution of norm α , $\langle f_1, u \rangle \neq 0$ for some $u \in N(K)$ since $f_1 \notin N(K)^\perp$, also. There exists a minimum choice for α from Case 1.

$$\text{Note that } \langle f_1, u \rangle \leq \langle f_1, f_1 \rangle^{\frac{1}{2}} \langle u, u \rangle^{\frac{1}{2}} \leq \langle f_1, f_1 \rangle^{\frac{1}{2}}$$

$$\text{so } \langle f_1, u \rangle^2 \leq \langle f_1, f_1 \rangle = \beta^2$$

$$\text{so } \alpha^2 \geq \beta^2 - \beta^2 = 0$$

If $\alpha^2 = 0$ then $\alpha = 0$ so $\|f_2\| = 0$ implies $f_2 = 0$ implies $g = 0$ a contradiction.

Suppose f_2 is a solution of norm α , $f_1 \neq f_2$ then $f_1 - f_2 \in N(K)$

$$\text{so } \frac{f_1 - f_2}{\|f_1 - f_2\|} \in N(K) \text{ of norm 1.}$$

$$f_2 = f_1 - (\langle f_1, \frac{f_1 - f_2}{\|f_1 - f_2\|} \rangle \pm \sqrt{\langle f_1, \frac{f_1 - f_2}{\|f_1 - f_2\|} \rangle^2 + \alpha^2 - \beta^2}) \frac{f_1 - f_2}{\|f_1 - f_2\|}$$

is a solution of (1) of norm α .

Note that

$$\begin{aligned} \langle f_1, \frac{f_1 - f_2}{\|f_1 - f_2\|} \rangle^2 + \alpha^2 - \beta^2 &= \frac{\langle f_1, f_1 - f_2 \rangle^2 + (\alpha^2 - \beta^2) \langle f_1 - f_2, f_1 - f_2 \rangle}{\|f_1 - f_2\|^2} \\ &= \frac{(\langle f_1, f_1 \rangle - \langle f_1, f_2 \rangle)^2 + (\alpha^2 - \beta^2)(\langle f_1, f_1 \rangle - 2\langle f_1, f_2 \rangle + \langle f_2, f_2 \rangle)}{\|f_1 - f_2\|^2} \end{aligned}$$

$$\begin{aligned}
&= \frac{\beta^4 - 2\beta^2 \langle f_1, f_2 \rangle + \langle f_1, f_2 \rangle^2 + (\alpha^2 - \beta^2)(\beta^2 - 2\langle f_1, f_2 \rangle + \alpha^2)}{\|f_1 - f_2\|^2} \\
&= \frac{\alpha^4 - 2\alpha^2 \langle f_1, f_2 \rangle + \langle f_1, f_2 \rangle^2}{\|f_1 - f_2\|^2} \\
&= \frac{(\alpha^2 - \langle f_1, f_2 \rangle)^2}{\|f_1 - f_2\|^2} \geq 0
\end{aligned}$$

So Case 2 says α is a possible norm for a solution if there is a $u \in N(K)$ such that $\alpha^2 \geq \beta^2 - \langle f_1, u \rangle^2$ and then such a solution is

$$f_u = f_1 - (\langle f_1, u \rangle \pm \sqrt{\langle f_1, u \rangle^2 + \alpha^2 - \beta^2})u.$$

Now then using Part II and the particular f_u found just above as the f_1 in Part II all solutions of norm α , given a solution of norm β , can be found.

IV. Conclusion

In general the solutions of (1) can be characterized by a solution f_1 of norm β and $N(K)$.

Picard's theorem gives criterion for the existence of a solution of (1) and gives the form for a solution.

V. Examples

$$(Kf)(x) = \int_0^1 (x^2 + y)f(y)dy = \frac{x^2}{3} + \frac{1}{4} \text{ has a} \quad (4)$$

$$\text{solution } f_1(y) = y^2 \text{ of norm } \left(\int_0^1 y^4 dy\right)^{\frac{1}{2}} = \frac{1}{\sqrt{5}}.$$

$$u(y) = \sqrt{\frac{15}{17}}(-12y^2 + 10y - 1) \in N(K) \text{ and is of norm } 1.$$

$$\langle f_1, u \rangle = \sqrt{\frac{15}{17}} \int_0^1 y^2 (-12y^2 + 10y - 1) dy = -\frac{7}{30} \sqrt{\frac{15}{17}}$$

so

$$f_u = y^2 - 2 \cdot \left(-\frac{7}{30} \sqrt{\frac{15}{17}}\right) \left(\sqrt{\frac{15}{17}}(-12y^2 + 10y - 1)\right)$$

$$= y^2 + \frac{7}{17}(-12y^2 + 10y - 1)$$

$$= \frac{-67y^2 + 70y - 7}{17} \text{ is a solution of norm } \frac{1}{\sqrt{5}}.$$

Now for the same example consider $\beta^2 < \frac{1}{5} = \|f_2\|^2$ where $f_2(y) = y^2$ and

$u \in N(K)$ above. Suppose $f_1 = f_2 + \epsilon u$

$$\beta = \int_0^1 f_1^2(y) dy$$

$$= \int_0^1 (f_2(y) + \epsilon(-12y^2 + 10y - 1) \sqrt{\frac{15}{17}})^2 dy$$

$$= \int_0^1 f_2^2(y) dy + 2\epsilon \sqrt{\frac{15}{17}} \int_0^1 (-12y^4 - 10y^3 - y^2) dy + \epsilon^2 \frac{15}{17} \int_0^1 (-12y^2 + 10y - 1)^2 dy$$

$$= \frac{1}{5} - \frac{7}{15} \sqrt{\frac{15}{17}} \epsilon + \epsilon^2$$

so

$$\epsilon^2 - \frac{7}{15} \sqrt{\frac{15}{17}} \epsilon + \frac{1}{5} - \beta^2 = 0$$

$$\epsilon = \frac{-\frac{7}{15} \sqrt{\frac{15}{17}} \pm \sqrt{\frac{49}{155} - 4\left(\frac{1}{5} - \beta^2\right)}}{2}$$

$$= \frac{-\frac{7}{15} \sqrt{\frac{15}{17}} \pm \sqrt{-\frac{75 + 620\beta^2}{155}}}{2}.$$

Now choose

$$-75+620\beta^2 = 0$$

$$\beta^2 = \frac{75}{620} = \frac{15}{124} < \frac{1}{5}$$

so

$$\epsilon = -\frac{7}{30}\sqrt{\frac{15}{17}}$$

Take

$$f_1 = y^2 - \frac{7}{30}\sqrt{\frac{15}{17}}(-12y^2+10y-1)\sqrt{\frac{15}{17}}$$

$$= y^2 - \frac{7}{34}(-12y^2+10y-1)$$

$$= \frac{118^2 - 70y + 7}{34}$$

is a solution of (4) of norm less than that of f_2 such that $u \perp f_1$.

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