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ORNL
CENTRAL FILES NUMBER

57-11-111

DATE: November 25, 1957
SUBJECT: Stress Analysis of
Flat Circular Plates
TO: Listed Distribution
FROM: F. J. Stanek

COPY NO. 51

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NOMENCLATURE AND SIGN CONVENTION FOR FLAT CIRCULAR PLATES

Fig. 1

- r - radial coordinate of plate
- a - radius of central hole in plate
- b - outer radius of plate
- $x = r/b$ - dimensionless coordinate of plate
- h - thickness of plate (uniform)
- w - lateral displacement of plate, positive downward
- u - radial displacement, positive outward
- p - uniform pressure acting on plate, positive downward
- M - bending moment in plate, positive when tensile stress in top surface
- Q - vertical shearing force in plate
- N - normal membrane force in plate, tension positive
- Subscripts a and b - inner and outer edge of plate, respectively
- Subscripts r and θ - radial and circumferential directions, respectively
- $(\sigma)_B, (\sigma)_m$ - bending and membrane internal unit stress in plate, respectively, tensile positive
- E - Young's modulus of Elasticity
- $D = Eh^3/10.92$ - flexural rigidity of the plate with Poisson's ratio = 0.3
- $\rho = pb^4/4D$ - load parameter due to uniform pressure p
- $K = (ab^2/4D)(Q_a - ap/2) = (b^3/4D)(Q_b - bp/2)$, load parameter due to edge shear forces and uniform pressure
- C_1, C_2, C_3, A_1 , and A_2 - constants of integration

Flat Circular Plates

Introduction

The present memorandum presents in a convenient and useable form the principal relations for the stress analysis of flat circular plates according to the classical theory. Most any flat circular plate of uniform thickness, with or without central holes, loaded with any combination of the axisymmetrical loads shown in Fig. 1, may be analysed by the procedure presented in this report. Poisson's ratio for the material of the plate should be approximately equal to 0.3, and the maximum lateral displacement should be less than approximately 0.6 of the thickness of the plate.

A dimensionless coordinate, x , is defined as the ratio of the radial coordinate r of the plate to the outer radius, b , of the plate; i.e., $x = r/b$. Hence, x will always fall in the range $0 \leq x \leq 1$. The plate formulae are expressed and listed in terms of x , and tables of values of the necessary functions of x are provided.

The application of the theory as presented here requires the solution of a set (three at most) of simultaneous linear equations which yield the constants of integration as obtained from the differential equation for the deflection of the plate. The procedure for the application of the stress and displacement analysis theory is demonstrated in three illustrative examples. The derivations of the formulae listed are shown in the appendix.

Plate Formulae

Formulae for the Bending Stress Analysis of Flat Circular Plates

$$w = K(x^2 \ln x) + \rho(x^4/16) + C_1(x^2/4) + C_2(\ln x) + C_3 \quad 1$$

$$\frac{dw}{dr} = (1/b) \left[K(2x \ln x + x) + \rho(x^3/4) + C_1(x/2) + C_2(1/x) \right] \quad 2$$

$$M_r = (D/b^2) \left[K(2.6 \ln x + 3.3) + \rho(3.3 x^2/4) + 0.65 c_1 - c_2(0.7/x^2) \right] \quad 3$$

$$M_\theta = (D/b^2) \left[K(2.6 \ln x + 1.9) + \rho(1.9 x^2/4) + 0.65 c_1 + c_2(0.7/x^2) \right] \quad 4$$

$$(\sigma_r)_B = \frac{\pm 6 M_r}{h^2} \quad 5$$

$$(\sigma_\theta)_B = \frac{\pm 6 M_\theta}{h^2} \quad \left(\begin{array}{l} + \text{ top surface} \\ - \text{ bottom surface} \end{array} \right) \quad 6$$

Formulae for the Membrane Stress Analysis of Flat Circular Plates

$$u = A_1 x + A_2(1/x) \quad 7$$

$$N_r = (Eh/0.91 b) \left[1.3 A_1 - A_2(0.7/x^2) \right] \quad 8$$

$$N_\theta = (Eh/0.91 b) \left[1.3 A_1 + A_2(0.7/x^2) \right] \quad 9$$

$$(\sigma_r)_m = \frac{N_r}{h} \quad 10$$

$$(\sigma_\theta)_m = \frac{N_\theta}{h} \quad 11$$

Principal (total) stresses

$$\sigma_r = (\sigma_r)_B + (\sigma_r)_m \quad 12$$

$$\sigma_\theta = (\sigma_\theta)_B + (\sigma_\theta)_m \quad 13$$

Procedure

Each of the functions of x appearing in the plate formulae have been evaluated for x values from 0.05 to 1.00 in increments of 0.05 and are tabulated in Tables I, II, III, and IV. All of the various functions of x appear in the first four formulae. The functions appearing in Formula 1 are tabulated in Table I, and so on to Table IV. Each of the first four formulae are written at the top of their corresponding table. The functions $1/x$ and $0.7/x^2$ appearing in the membrane analysis are tabulated in Tables II and III (also IV), respectively.

The range of x values which will be used for the analysis of specific plates will be from the value of the a/b ratio of the plate to 1.0. The exact value of the ratio a/b may not be one of the x values appearing in the tables. Here, for most practical cases, one selects the x value appearing in the tables which is nearest to the exact value of a/b . This has the physical affect of changing a and/or b slightly so as to make a/b agree with the selected value of x . In general, a change in the value of " a " will introduce a smaller error in the results than a change in the value of " b ". The selected adjusted values for a and/or b should be used throughout the analysis so that the appropriate physical interpretation may be given to the results. For more accurate results the necessary functions of x should be evaluated for the exact value of $x_a = a/b$.

The term K appearing in the first four formulae is defined in the nomenclature. It is a load parameter of the inner or outer shear force and/or the uniform pressure acting on the plate. It should be noted that of the three external loads Q_a , Q_b , and p , only two may be arbitrarily designated. The third must maintain equilibrium; that is, the summation of forces parallel to the axis of the plate must be zero.

To apply the plate formulae to the analysis of a solid plate (one without a central hole), the constants of integration C_2 and A_2 must be zero.

The constant of integration C_3 is usually determined by specifying w at either edge of the plate. The constants of integration C_1 and C_2 are usually determined by specifying two boundary conditions among the three quantities $(w, dw/dr, M_r)$ defined by the first three formulae.

In the small deflection theory, ($w_{\max} < 0.6h$), the membrane analysis is independent of the bending analysis, and hence may be treated separately. The constants of integration A_1 and A_2 are usually determined by specifying two boundary conditions among the two quantities (u, N_r) defined by formulae 7 and 8. The principal or total stresses in the plate may then be obtained by adding the membrane and bending stresses as stated by formulae 12 and 13.

Once the constants of integration are determined, their values are substituted back into the appropriate formula to obtain the desired results.

Some of the above points and others on the analysis of circular plates are illustrated in three illustrative examples following Table IV.

Table I

Values* of functions appearing in the formula for the displacement w (Eq 1)

$$w = K(x^2 \ln x) + \rho(x^4/16) + C_1(x^2/4) + C_2(\ln x) + C_3$$

x		$(x^2 \ln x)$		$(x^4/16)$		$(x^2/4)$		$(\ln x)$
0.05	-3	-7.4893308	-7	3.9062500	-4	6.2500000		-2.9957323
0.10	-2	-2.3025851	-6	6.2500000	-3	2.5000000		-2.3025851
0.15	-2	-4.2685200	-5	3.1640625	-3	5.6250000		-1.8971200
0.20	-2	-6.4377516	-4	1.0000000	-2	1.0000000		-1.6094379
0.25	-2	-8.6643400	-4	2.4414063	-2	1.5625000		-1.3862944
0.30	-1	-1.0835755	-4	5.0625000	-2	2.2500000		-1.2039728
0.35	-1	-1.2860321	-4	9.3789063	-2	3.0625000		-1.0498221
0.40	-1	-1.4660652	-3	1.6000000	-2	4.0000000	-1	-9.1629073
0.45	-1	-1.6169781	-3	2.5628906	-2	5.0625000	-1	-7.9850770
0.50	-1	-1.7328680	-3	3.9062500	-2	6.2500000	-1	-6.9314718
0.55	-1	-1.8084569	-3	5.7191406	-2	7.5625000	-1	-5.9783700
0.60	-1	-1.8389722	-3	8.1000000	-2	9.0000000	-1	-5.1082562
0.65	-1	-1.8200578	-2	1.1156641	-1	1.0562500	-1	-4.3078292
0.70	-1	-1.7477072	-2	1.5006250	-1	1.2250000	-1	-3.5667494
0.75	-1	-1.6182116	-2	1.9775306	-1	1.4062500	-1	-2.8768207
0.80	-1	-1.4281187	-2	2.5600000	-1	1.6000000	-1	-2.2314355
0.85	-1	-1.1741993	-2	3.2625390	-1	1.8062500	-1	-1.6251893
0.90	-2	-8.5342021	-2	4.1006250	-1	2.0250000	-1	-1.0536052
0.95	-2	-4.6292198	-2	5.0906641	-1	2.2562500	-2	-5.1293294
1.00		0	-2	6.2500000	-1	2.5000000		0

*The first single digit is the exponent of ten by which the eight digit number needs to be multiplied. For example: -3 -7.4893308 = -7.4893308 x 10⁻³ = -0.0074893308.

Table II

Values* of functions appearing in the formula for dw/dr (Eq 2)

$$\frac{dw}{dr} = (1/b) \left[K(2x \ln x + x) + \rho(x^3/4) + C_1(x/2) + C_2(1/x) \right]$$

x	(2x ln x + x)		(x ³ /4)		(x/2)		(1/x)	
0.05	-1	-2.4957323	-5	3.1250000	-2	2.5000000	1	2.0000000
0.10	-1	-3.6051702	-4	2.5000000	-2	5.0000000	1	1.0000000
0.15	-1	-4.1913600	-4	8.4375000	-2	7.5000000		6.6666667
0.20	-1	-4.4377516	-3	2.0000000	-1	1.0000000		5.0000000
0.25	-1	-4.4314720	-3	3.9062500	-1	1.2500000		4.0000000
0.30	-1	-4.2238368	-3	6.7500000	-1	1.5000000		3.3333333
0.35	-1	-3.8487547	-2	1.0718750	-1	1.7500000		2.8571428
0.40	-1	-3.3303258	-2	1.6000000	-1	2.0000000		2.5000000
0.45	-1	-2.6865693	-2	2.2781250	-1	2.2500000		2.2222222
0.50	-1	-1.9314718	-2	3.1250000	-1	2.5000000		2.0000000
0.55	-1	-1.0762070	-2	4.1593750	-1	2.7500000		1.8181818
0.60	-2	-1.2990740	-2	5.4000000	-1	3.0000000		1.6666667
0.65	-2	8.9983500	-2	6.8656250	-1	3.2500000		1.5384615
0.70	-1	2.0065508	-2	8.5750000	-1	3.5000000		1.4285714
0.75	-1	3.1847689	-1	1.0546875	-1	3.7500000		1.3333333
0.80	-1	4.4297192	-1	1.2800000	-1	4.0000000		1.2500000
0.85	-1	5.7371782	-1	1.5459375	-1	4.2500000		1.1764705
0.90	-1	7.1035124	-1	1.8225000	-1	4.5000000		1.1111111
0.95	-1	8.5254274	-1	2.1434375	-1	4.7500000		1.0526315
1.00		1.0000000	-1	2.5000000	-1	5.0000000		1.0000000

*The first single digit is the exponent of ten by which the eight digit number needs to be multiplied. For example: -1 -2.4957323 = -2.4957323 x 10⁻¹ = -0.24957323.

Table III

Values* of functions appearing in the formulae for M_r (Eq 3)

(Poisson's ratio taken = 0.3)

$$M_r = (D/b^2) \left[K(2.6 \ln x + 3.3) + \rho(3.3x^2/4) + 0.65 c_1 - c_2(0.7/x^2) \right]$$

x	(2.6 ln x + 3.3)	(3.3 x ² /4)	(0.7/x ²)
0.05	-4.4889040	-3 2.0625000	2 2.8000000
0.10	-2.6867213	-3 8.2500000	1 7.0000000
0.15	-1.6325120	-2 1.8562500	1 3.1111111
0.20	-1 -8.8453850	-2 3.3000000	1 1.7500000
0.25	-1 -3.0436540	-2 5.1562500	1 1.1200000
0.30	-1 1.6967070	-2 7.4250000	7.7777777
0.35	-1 5.7046250	-1 1.0106250	5.7142857
0.40	-1 9.1764670	-1 1.3200000	4.3750000
0.45	1.2238826	-1 1.6706250	3.4567901
0.50	1.4978199	-1 2.0625000	2.8000000
0.55	1.7456238	-1 2.4956250	2.3140495
0.60	1.9718560	-1 2.9700000	1.9444444
0.65	2.1799644	-1 3.4856250	1.6568047
0.70	2.3726452	-1 4.0425000	1.4285714
0.75	2.5520266	-1 4.6406250	1.2444444
0.80	2.7198268	-1 5.2800000	1.0937500
0.85	2.8774508	-1 5.9606250	-1 9.6885813
0.90	3.0260626	-1 6.6825000	-1 8.6419753
0.95	3.1666374	-1 7.4456250	-1 7.7562326
1.00	3.3000000	-1 8.2500000	-1 7.0000000

*The first single digit is the exponent of ten by which the eight digit number needs to be multiplied. For example: -3 2.0625000 = 2.0625000 x 10⁻³ = 0.0020625000.

Table IV

Values* of functions appearing in the formula for M_θ (Eq 4)

(Poisson's ratio taken = 0.3)

$$M_\theta = (D/b^2) \left[K(2.6 \ln x + 1.9) + \rho(1.9x^2/4) + 0.65 c_1 + c_2(0.7/x^2) \right]$$

x	(2.6 ln x + 1.9)		(1.9x ² /4)		(0.7/x ²)	
0.05		-5.8889040	-3	1.1875000	2	2.8000000
0.10		-4.0867213	-3	4.7500000	1	7.0000000
0.15		-3.0325120	-2	1.0687500	1	3.1111111
0.20		-2.2845385	-2	1.9000000	1	1.7500000
0.25		-1.7043654	-2	2.9687500	1	1.1200000
0.30		-1.2303293	-2	4.2750000		7.7777777
0.35	-1	-8.2953750	-2	5.8187500		5.7142857
0.40	-1	-4.8235590	-2	7.6000000		4.3750000
0.45	-1	-1.7612000	-2	9.6187500		3.4567901
0.50	-2	9.7817300	-1	1.1875000		2.8000000
0.55	-1	3.4562380	-1	1.4368750		2.3140495
0.60	-1	5.7185340	-1	1.7100000		1.9444444
0.65	-1	7.7996700	-1	2.0068750		1.6568047
0.70	-1	9.7264780	-1	2.3275000		1.4285714
0.75		1.1520292	-1	2.6718750		1.2444444
0.80		1.3198268	-1	3.0400000		1.0937500
0.85		1.4774508	-1	3.4318750	-1	9.6885813
0.90		1.6260626	-1	3.8475000	-1	8.6419753
0.95		1.7666374	-1	4.2868750	-1	7.7562326
1.00		1.9000000	-1	4.7500000	-1	7.0000000

*The first single digit is the exponent of ten by which the eight digit number needs to be multiplied. For example: -3 1.1875000 = 1.1875000 x 10⁻³ = 0.0011875000.

Illustrative Examples

Illustrative Example 1

A flat circular plate with inner radius of 5 inches, and outer radius of 12 inches, is loaded with a shear force of Q_a pounds per inch along the inner edge of the plate. The thickness of the plate is h inches and the outer edge is simply supported. Determine the circumferential bending stress and the lateral deflection at the inner edge of the plate.

Solution: With $a = 5$ and $b = 12$; $x_a = 5/12 = 0.4167$. Since the nearest value to 0.4167 appearing in the tables for x is 0.40 we select $x_a = 0.40$. The values (or expressions) for K and ρ for this problem are

$$K = ab^2 Q_a / 4D \quad \text{and} \quad \rho = 0$$

The next step is to state the boundary conditions or edge conditions which will determine the constants of integration C_1 , C_2 and C_3 . For this problem the boundary conditions are;

1. $w = 0$ at $x = 1.0$
2. $M_r = 0$ at $x = 0.4$
3. $M_r = 0$ at $x = 1.0$

With formulae 1 and 3 and the above boundary conditions, we write the necessary set of simultaneous equations.

$$\text{By condition 1: } 0.2500 C_1 + C_3 = 0$$

$$\text{by condition 2: } 0.9176 K + 0.65 C_1 - 4.375 C_2 = 0$$

$$\text{by condition 3: } 3.300 K + 0.65 C_1 - 0.7000 C_2 = 0$$

*For convenience and saving of space, all numerical values in the illustrative examples are rounded off to four significant figures.

Solving this set of equations yields

$$\begin{aligned} C_1 &= -5.775 K \\ C_2 &= -0.6482 K \\ C_3 &= 1.444 K \end{aligned}$$

Substituting these values back into formula 1 with $x = 0.40$ we find the lateral deflection at the inner edge of the plate. This yields

$$w_a = 1.660 K = 0.4150 ab^2 Q_a / D = 4.532 ab^2 Q_a / Eh^3$$

Similarly, by using formula 4, we obtain

$$b^2 M_\theta / D = -7.072 K,$$

or

$$M_\theta = -1.768 a Q_a.$$

And by Eq (6),

$$(\sigma_\theta)_B \text{ at inner edge} = -10.61 a Q_a / h^2.$$

The negative sign denotes compressive stresses in the top surface of the plate. The relative influence of a and b upon the results is self evident. Hence, the change in a and/or b so as to make $a/b = 0.40$ is left up to the judgement of the reader.

Illustrative Example 2

A flat circular plate of radius b inches is loaded with a concentrated load of P pounds in the center. The thickness is h inches and the outer edge is rigidly fixed. Determine the lateral deflection w_0 at the center of the plate and the radial bending stress at the outer edge.

Solution: For a plate without a central hole, $C_2(A_2$ for the membrane analysis) must be taken equal to zero. The second definition of K is used with $Q_b = P/2\pi b$.

Hence,

$$K = b^3 Q_b / 4D = Pb^2 / 8\pi D \quad \text{and} \quad \rho = 0.$$

The boundary conditions for this example are

1. $w = 0$ at $x = 1.0$
2. $dw/dr = 0$ at $x = 1.0$

which yield, by formulae 1 and 2,

$$\text{by condition 1: } 0.2500 C_1 + C_3 = 0$$

$$\text{by condition 2: } K + 0.5000 C_1 = 0.$$

Therefore,

$$C_1 = -2K \quad \text{and} \quad C_3 = 0.5000 K.$$

Then by formula 1, and since the limit of $x^2 \ln x$, as x goes to zero, is zero,

$$w_0 = C_3 = 0.5000 K = Pb^2 / 16 D.$$

For a solid plate with a concentrated load at the center, it should be noted that the bending moments (formulae 3 and 4) tend to infinity near the center of the plate. Hence, one must exercise caution in evaluating the stresses near concentrated loads as the classical theory fails. However, satisfactory evaluation of the stresses away from the concentrated load may be obtained. Hence, by formula 3

$$b^2 M_r / D \text{ (at } x = 1.0) = 2 K$$

or

$$M_r = P/4\pi,$$

and $(\sigma_r)_B \text{ (at } x = 1.0) = 1.5P/\pi b^2$, tensile stresses in top surface.

Illustrative Example 3

A flat circular plate of thickness h inches has an outer radius of b inches and inner radius of $a = 0.5b$ inches. The plate is loaded with a uniform pressure of p pounds per square inch between the radii of $c = 0.75b$ and b inches. The inner portion of the plate between the radii of $a = 0.5b$ and $c = 0.75b$ is free of external loads and supports. The outer edge is simply supported. Determine the lateral deflection at the inner edge of the plate and the bending stresses at $r = c$.

Solution: To analyze a plate loaded in this manner it is helpful to designate the two portions of the plate as

$$\text{portion 1: } 0.5 \leq x \leq 0.75$$

$$\text{portion 2: } 0.75 \leq x \leq 1.0$$

For portion 1, $Q_a = Q_b = p = 0$ and hence $K_1 = \rho_1 = 0$. For portion 2, $(Q_b)_2 = p(b^2 - c^2)/2b$ and therefore $K_2 = -pb^2c^2/4D = -0.28125\rho_2$, where, by definition, $\rho_2 = pb^4/4D$.

Since there are two portions to this plate, six boundary conditions are necessary to solve for the six constants of integration. These may be listed as follows:

1. $(M_r)_1 = 0$ at $x = 0.5$
2. $w_1 = w_2$ at $x = 0.75$
3. $\left(\frac{dw}{dr}\right)_1 = \left(\frac{dw}{dr}\right)_2$ at $x = 0.75$
4. $(M_r)_1 = (M_r)_2$ at $x = 0.75$
5. $w_2 = 0$ at $x = 1.0$
6. $(M_r)_2 = 0$ at $x = 1.0$

Conditions 1, 5 and 6 were obtained from the conditions at the two edges of the plate. Conditions 2, 3 and 4 were obtained by equating the deflection (w), slope (dw/dr) and radial bending moment (M_r) at the common radius of the two portions.

Using the prefixes 1 and 2 before the subscripts on the constants of integration to designate portion 1 and 2, respectively, the above boundary conditions yield the following set of six simultaneous equations.

$$1. \quad 0.65 c_{11} - 2.800 c_{12} = 0$$

$$2. \quad 0.1406 c_{11} - 0.2877 c_{12} + c_{13} = 0.06528 \rho_2 + 0.1406 c_{21} \\ - 0.2877 c_{22} + c_{23}$$

$$3. \quad 0.3750 c_{11} + 1.333 c_{12} = 0.0159 \rho_2 + 0.3750 c_{21} + 1.333 c_{22}$$

$$4. \quad 0.65 c_{11} - 1.244 c_{12} = -0.2536 \rho_2 + 0.65 c_{21} - 1.244 c_{22}$$

$$5. \quad 0.06250 \rho_2 + 0.2500 c_{21} + c_{23} = 0$$

$$6. \quad -0.1031 \rho_2 + 0.65 c_{21} - 0.7000 c_{22} = 0$$

Solving the above set of equations, yields

$$c_{11} = -0.2204 \rho_2$$

$$c_{21} = 0.0183 \rho_2$$

$$c_{12} = -0.05117 \rho_2$$

$$c_{22} = -0.1303 \rho_2$$

$$c_{13} = 0.05454 \rho_2$$

$$c_{23} = -0.06707 \rho_2$$

Then, by formula 1, with $x = 0.5$

$$w_a = 0.0625 c_{11} - 0.6931 c_{12} + c_{13}$$

$$= 0.07624 \rho_2 = 0.2081 \text{ pb}^4/\text{E}$$

By formula 3, with $x = 0.75$

$$(b^2 M_r / D)_c = 0.65 c_{11} - 1.244 c_{12} = -0.0797 p_2$$

or

$$(M_r)_c = -0.01993 pb^2$$

and

$$\left[(\sigma_r)_B \right]_c = -0.1196 pb^2/h^2 \text{ (compressive stress in top surface) .}$$

Similarly, by formula 4 with $x = 0.75$

$$(b^2 M_\theta / D)_c = 0.65 c_{11} + 1.244 c_{12} = -0.2070 p$$

or

$$(M_\theta)_c = -0.05175 pb^2$$

and

$$\left[(\sigma_\theta)_B \right]_c = -0.3105 pb^2/h^2 \text{ (compressive stress in top surface) .}$$

The reader may readily verify that the same results are obtained by using the formulae for portion 2 at $x = 0.75$.

A procedure similar to the one illustrated in this example may be used for the analysis of a plate loaded with a circular line load between the inner and outer radii.

APPENDIXDerivations of the Plate FormulaeBending Analysis

The differential equation for the lateral displacement w of a circular plate loaded axisymmetrically is¹

$$\frac{d}{dr} \left[\frac{1}{r} \frac{d}{dr} \left(\frac{rdw}{dr} \right) \right] = Q/D \quad 1A$$

With the transformation $x = r/b$,

$$\frac{dw}{dr} = \frac{1}{b} \frac{dw}{dx} ,$$

and equation 1A transforms to

$$\frac{d}{dx} \left[\frac{1}{x} \frac{d}{dx} \left(\frac{xdw}{dx} \right) \right] = Qb^3/D \quad 2A$$

Before Eq (2A) can be integrated the vertical shearing force Q must be expressed in terms of x . This is readily done by summing forces along the axis of the plate for an arbitrary radius r . Hence, for the general loading shown in Figure 1,

$$Q = a(Q_a - pa/2)/bx + pbx/2 \quad 3A$$

For the plate to remain in equilibrium, the following relationship must always hold true,

$$Q_a = bQ_b/a - p(b^2 - a^2)/2a \quad 4A$$

1. See Chapter III, "Theory of Plates and Shells", S. Timoshenko, McGraw-Hill, 1940.

Hence, by substituting Eq (4A) for Q_a into Eq (3A), Q may be written in the following alternate form,

$$Q = (Q_b - pb/2)/x + pbx/2 \quad . \quad 3aA$$

For convenience, define

$$K = ab^2(Q_a - ap/2)/4D = b^3(Q_b - bp/2)/4D$$

and

$$\rho = pb^4/4D \quad .$$

Substituting Eq (3A), or (3aA), for Q in Eq (2A) and introducing the above definitions of K and ρ , yields

$$\frac{d}{dx} \left[\frac{1}{x} \frac{d}{dx} \left(\frac{xdw}{dx} \right) \right] = 4K/x + 2\rho x \quad 2aA$$

Integrating Eq (2aA) once and multiplying through by x yields

$$\frac{d}{dx} \left(\frac{xdw}{dx} \right) = 4Kx \ln x + \rho x^3 + C_{oo}x \quad ,$$

where C_{oo} is a constant of integration.

Integrating again,

$$\frac{xdw}{dx} = K(2x^2 \ln x - x^2) + \rho x^2/4 + C_{oo}x^2/2 + C_2 \quad ,$$

which may be written in the form

$$\frac{xdw}{dx} = 2Kx^2 \ln x + \rho x^2/4 + C_o x^2/2 + C_2 \quad ,$$

where $C_o = C_{oo} - 2K$ (this merely re-defines the original unknown constant of integration C_{oo}), and C_2 is another constant of integration. Dividing by x and integrating again, yields formula 1. That is

$$w = K(x^2 \ln x) + \rho x^4/16 + C_1 x^2/4 + C_2 \ln x + C_3, \quad 1.$$

where $C_1 = C_0 - 2K$ and C_3 is the third constant of integration.

With the transformation relationship,

$$\frac{dw}{dr} = \frac{1}{b} \frac{dw}{dx},$$

formula (2) is readily obtained.

The expressions for the bending moments M_r and M_θ in terms of the displacement w are²

$$M_r = D \left(\frac{d^2 w}{dr^2} + \frac{\mu}{r} \frac{dw}{dr} \right) = \frac{D}{b^2} \left(\frac{d^2 w}{dx^2} + \frac{\mu}{x} \frac{dw}{dx} \right) \quad 5A$$

$$M_\theta = D \left(\frac{1}{r} \frac{dw}{dr} + \mu \frac{d^2 w}{dr^2} \right) = \frac{D}{b^2} \left(\frac{1}{x} \frac{dw}{dx} + \mu \frac{d^2 w}{dx^2} \right), \quad 6A$$

where μ is Poisson's ratio.

Substituting Eq (1) for w in Eqs (5A) and (6A) yields M_r and M_θ , respectively

$$M_r = (D/b^2) \left\{ K \left[2(1+\mu) \ln x + 3 + \mu \right] + (3+\mu) \rho x^2/4 + (1+\mu) C_1/2 - (1-\mu) C_2/x^2 \right\}$$

$$M_\theta = (D/b^2) \left\{ K \left[2(1+\mu) \ln x + 1 + 3\mu \right] + (1+3\mu) \rho x^2/4 + (1+\mu) C_1/2 + (1-\mu) C_2/x^2 \right\}$$

By taking $\mu = 0.3$, formulae (3) and (4) immediately follow.

If the plate is loaded with a variable pressure (p = a function of r), the two terms containing p in Eq (3A) are replaced with

$$\frac{1}{r} \int_a^r r' p(r') dr' = \frac{b}{x} \int_{x_a}^x x' p(x') dx'$$

where $p(r)$ is a function of r and $p(x)$ is the corresponding function in x .

2. See Chapter III, "Theory of Plates and Shells", by S. Timoshenko, McGraw-Hill, 1940. The positive directions for M_r and M_θ are changed herein.

Membrane Analysis

The formulae for the membrane analysis of circular plates originate from the following set of equations³,

$$\frac{dN_r}{dr} + \frac{N_r - N_\theta}{r} = 0 \quad 7A$$

$$N_r = \frac{Eh}{1 - \mu^2} \left(\frac{du}{dr} + \mu \frac{u}{r} \right) \quad 8A$$

and

$$N_\theta = \frac{Eh}{1 - \mu^2} \left(\frac{u}{r} + \mu \frac{du}{dr} \right) \quad 9A$$

Transforming the above equations to the coordinate x yields

$$\frac{dN_r}{dx} + \frac{N_r - N_\theta}{x} = 0 \quad 7aA$$

$$N_r = \frac{Eh}{b(1 - \mu^2)} \left(\frac{du}{dx} + \mu \frac{u}{x} \right) \quad 8aA$$

$$N_\theta = \frac{Eh}{b(1 - \mu^2)} \left(\frac{u}{x} + \mu \frac{du}{dx} \right) \quad 9aA$$

Substituting Eqs (8aA) and (9aA) for N_r and N_θ , respectively, in Eq (7aA) yields the following differential equation

$$\frac{d^2u}{dx^2} + \frac{1}{x} \frac{du}{dx} - \frac{u}{x^2} = 0,$$

3. See page 58 for Eq (7A); and page 65 for Eqs (8A) and (9A) - "Theory of Elasticity", S. Timoshenko and J. N. Goodier, 2nd edition, McGraw-Hill, 1951.

which may be written

$$\frac{d}{dx} \left(\frac{1}{x} \frac{d(ux)}{dx} \right) = 0 \quad 10A$$

Solving Eq (10A) yields formula (7)

$$u = A_1 x + A_2 (1/x) \quad (7)$$

Substituting Eq (7) for u in Eqs (8aA) and (9aA) yields for N_r and N_θ , respectively

$$N_r = \frac{Eh}{b(1 - \mu^2)} \left[(1 + \mu)A_1 - (1 - \mu)A_2/x^2 \right]$$

$$N_\theta = \frac{Eh}{b(1 - \mu^2)} \left[(1 + \mu)A_1 + (1 - \mu)A_2/x^2 \right]$$

Taking $\mu = 0.3$, yields formulae (8) and (9).

ACKNOWLEDGEMENT

The author is indebted to M. Sanders for her help in evaluating and tabulating the numerical values of the various functions of x included in this report.