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CAMBRIDGE ELECTRON ACCELERATOR

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RADIOFREQUENCY ACCELERATION II

Abstract

A radiofrequency system consisting of 16 cavities, strongly coupled together by transmission line, is analyzed. This system can be regarded as a single cavity operating in a particular mode, and is found to be a practical system for the accelerator. Also methods of coupling R.F. power into the system are considered.

The R.F. system with beam loading is analyzed for stability, and a relationship determined for the range of the cavity tuning over which the system is stable.

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Radiofrequency Acceleration II

In supplying the necessary radiofrequency voltage per turn to maintain the electrons in a phase stable position, it is desirable to use the largest number of radiofrequency cavities that is convenient in our packaging system. This reduces the power dissipated in the cavities to a minimum value, and since the cost of R.F. power in our accelerator seems to be much larger than the cost of cavities, this represents the most economical way of supplying the required voltage per turn. Also, the large number of cavities reduces the field strength in each cavity and the problems associated with sparking and multipactor breakdown. In our packaging system, it is convenient to use 16 symmetrically placed cavities.

There are two alternative methods of supplying the necessary R.F. power to each cavity. The first method would be to have an individual R.F. power unit for each cavity, located in close proximity to the cavity in the synchrotron tunnel, with the low level R.F. driving power distributed from a master oscillator to each unit. The alternative method is to use a small number of R.F. power units, located probably in the power building, and to distribute the high power from these units to the individual cavities by means of some kind of R.F. transmission lines. With separate R.F. units for each cavity it would be necessary to be able to remove and replace individual R.F. units in the synchrotron tunnel, so as to be able to operate the accelerator when one unit is being repaired. The failure of one unit of the 16 individual units would then cause little or no reduction of the operating energy of the accelerator, whereas the failure of one of the large R.F. power units would probably necessitate a larger reduction in

maximum energy of the accelerator. If the R.F. power system of a 6 Bev accelerator consisted of two units, it should operate at an energy of somewhat more than 5 Bev with only one unit. Some advantages of having large, centrally located R.F. power units are the accessibility of the units, permitting adjustment while the accelerator is operating. For units located in the tunnel it would probably be necessary to have several remote reading instruments and controls for each unit. Another advantage is the economy of obtaining the same R.F. power in a few large units as compared with 16 smaller units. With tubes now being developed, it will probably be possible to obtain a reduction by a factor of two in the total cost of the R.F. power units as compared to 16 smaller units.

With an individual R.F. power unit for each cavity it would be necessary to maintain the tuning of each cavity very accurately. Taking the Q of the R.F. cavity as 25,000, in order to maintain the phase of the cavity correct to 10° , it is necessary to maintain the tuning to about 4 parts per million. This accuracy would require a servo system which monitored the phase of the cavity voltage and automatically adjusted the tuning of each cavity so as to maintain the correct phase relationship.

With the system using central R.F. power units, the R.F. transmission lines which distribute the power to the cavities may also serve to stabilize the relative phase and amplitude of the R.F. fields in the individual cavities. This is done by coupling the individual cavities strongly together with transmission lines, such that the power flowing in each direction in the lines is large compared to the cavity dissipation. The natural form of this coupling system is a loop of R.F. transmission line in the synchrotron tunnel which is strongly coupled to each cavity. At the proposed operating frequency range of 400 to 500 m.c., this transmission line may be either rigid coaxial line or waveguide. Waveguide has the advantage of lower attenuation and higher power rating; also waveguide is

probably more convenient for making couplings to cavities with vacuum windows.

The strongly coupled system is then represented by a closed loop of transmission line coupled to equally spaced resonant cavities. Near resonance the cavities may be closely represented by parallel resonant circuit.

The current generators represent beam loading current or currents due to R.F. power sources.

The equations for the length of transmission line may be written:

$$i_1 = \frac{v_1}{Z_0} \coth \gamma - \frac{v_2}{Z_0} \operatorname{csch} \gamma$$

$$i_2 = -\frac{v_1}{Z_0} \operatorname{csch} \gamma + \frac{v_2}{Z_0} \coth \gamma$$

where $\gamma = \alpha + j\beta$ is the complex phase angle.

It is convenient to define normal modes for the system, equal to the number of resonant cavities, each mode being distinguished by the phase shift between adjacent cavities such that:

$$v_{p,n+1} = v_{p,n} e^{j\theta_p}$$

$$\theta_p = \frac{2\pi p}{N}$$

$$i_{p,n+1} = i_{p,n} e^{j\theta_p}$$

where N is number of resonant cavities.

Only one mode will be effective in accelerating electrons; the other undesired modes will only increase the power required.

A characteristic admittance may then be defined for each mode.

$$Y_p = \frac{i_{p,n}}{v_{p,n}}$$

From the representation of the coupled system

$$i_n = v_n \left[G + j\omega C + \frac{1}{j\omega L} \right] + v_n \frac{2 \coth \gamma}{Z_0} - (v_{n-1} + v_{n+1}) \frac{\operatorname{csch} \gamma}{Z_0}$$

then

$$Y_p = G + i\omega C + \frac{1}{i\omega L} + \frac{2\coth r}{Z_0} - (e^{-i\theta p} + e^{i\theta p}) \frac{\cosh r}{Z_0}$$

$$i\omega C + \frac{1}{i\omega L} \approx i 2 \frac{\Delta\omega}{\omega_0} QG$$

$$Y_p = G + i 2GQ \frac{\Delta\omega}{\omega_0} + \frac{2}{Z_0} \frac{\cosh r - \cos\theta p}{\sinh r}$$

It seems desirable to choose the mode of operation to be a non-generate mode with $\cos\theta p = 1$ or -1 , so as not to have an alternate mode with equal admittance.

Taking the fundamental mode to be $\theta p = 0$ $\cos\theta p = 1$

$$Y_0 = G + i 2GQ \frac{\Delta\omega}{\omega_0} + \frac{2}{Z_0} \frac{\cosh r - 1}{\sinh r}$$

$$\frac{\cosh r - 1}{\sinh r} = \tanh \frac{r}{2} = \frac{\sinh \alpha + i \sin \beta}{\cosh \alpha + \cos \beta}$$

taking $\alpha \ll 1$

$$Y_0 = G + \frac{2\alpha}{Z_0(1 + \cos \beta)} + i 2GQ \frac{\Delta\omega}{\omega_0} + i \frac{2 \sin \beta}{Z_0(1 + \cos \beta)}$$

The admittance of the other modes which are not useful in accelerating particle are then:

$$Y_p = Y_0 + \frac{1 - \cos\theta p}{\sinh r} = Y_0 + \frac{2}{Z_0} \frac{2 \sin^2 \frac{\theta p}{2}}{\sinh \alpha \cos \beta + i \cosh \alpha \sin \beta}$$

taking $\alpha \ll 1$

$$Y_p = Y_0 + \frac{4 \sin^2 \frac{\theta p}{2}}{Z_0(\alpha \cos \beta + i \sin \beta)}$$

$$Y_p = Y_0 + \frac{4 \sin^2 \frac{\theta p}{2}}{Z_0} \frac{\alpha \cos \beta}{\alpha^2 \cos^2 \beta + \sin^2 \beta} - i \frac{4 \sin^2 \frac{\theta p}{2}}{Z_0} \frac{\sin \beta}{\alpha^2 \cos^2 \beta + \sin^2 \beta}$$

$$Y_p = Y_0 + \frac{4 \sin^2 \frac{\theta p}{2}}{Z_0} \frac{\alpha \cos \beta}{\alpha^2 + \sin^2 \beta} - i \frac{4 \sin^2 \frac{\theta p}{2}}{Z_0} \frac{\sin \beta}{\alpha^2 + \sin^2 \beta}$$

It is desirable to make Y_p as large as possible in order to reduce the voltage amplitude and power in the spurious modes. Then Z_0 and $\sin \beta$ should be made small, in order to make Y_p large.

for $\sin \beta$ small

$$Y_0 \cong G + \frac{\alpha}{Z_0} + i \left(2GQ \frac{\Delta \omega}{\omega_0} + \frac{\sin \beta}{Z_0} \right)$$

The power dissipation associated with the fundamental mode of operation is then:

$$P_0 = V_0^2 G_0 = V_0^2 \left(G + \frac{\alpha}{Z_0} \right)$$

The fractional increase in power dissipation required for this strong coupling scheme, due to power dissipated in the transmission lines is then:

$$\frac{\Delta P_0}{P_0} = \frac{\alpha}{G Z_0}$$

For commercially available waveguide in the 400-500 m c. range, the attenuation is about .05 dB per hundred feet. Taking the distance between cavities to be 50', this gives a value of $\alpha \cong .0029$. If the power dissipated in the waveguide is taken to be 5% of the power dissipated in the cavities, or about 3.5% of the total power.

$$\frac{1}{Z_0} = \frac{G}{\alpha} \frac{\Delta P_0}{P_0} = \frac{G}{.0029} \times .05 \cong 17G$$

To calculate the effect of changes in tuning of cavities, it is assumed that the admittances of spurious modes are large compared with the fundamental mode, such that an imperfection of the system, as the change of a cavity tuning, may be calculated as a first order effect. (This assumption will be investigated later)

A change in the tuning of an individual cavity will add a current of

$i V_0 \cdot 2 \frac{\Delta \omega'}{\omega_0} GQ$ at that point in the system. The fundamental mode component of this current will then be $i \frac{V_0}{N} \cdot 2 \frac{\Delta \omega'}{\omega_0} GQ$. The effect of this detuning on the admittance of the fundamental mode can be then eliminated by changing the tuning of all the cavities by an amount $-\frac{\Delta \omega'}{N}$, or by changing the frequency by $\frac{\Delta \omega'}{N}$. In this type of strongly coupled system then, it is not necessary to maintain the tuning of each individual cavity accurately, but just to maintain the tuning of the system as a whole. This can be accomplished by small changes in frequency within a range of about $\pm 10^{-4}$. For larger, long term drifts it would be necessary to control the tuning of the cavities, probably by controlling the temperature of the cavities, by means of the cooling water system.

The change in tuning of a cavity also produces a current in every spurious mode equal to $i \frac{V_0}{N} \cdot 2 \frac{\Delta \omega'}{\omega_0} GQ$. The voltage generated in a spurious mode by the change of cavity tuning is then:

$$V_p = \frac{V_0}{N} \cdot 2 \frac{\Delta \omega'}{\omega_0} \frac{GQ}{Y_p}$$

and the power in the spurious mode is

$$P_p = V_0^2 \left(\frac{1}{N} \cdot 2 \frac{\Delta \omega'}{\omega_0} Q \right)^2 \frac{G^2}{Y_p^2} G_p$$

with
$$Y_p = Y_0 + \frac{4}{Z_0} \sin^2 \frac{\theta_p}{2} \frac{\alpha \cos \beta}{\alpha^2 + \sin^2 \beta} = \frac{4}{Z_0} \sin^2 \frac{\theta_p}{2} \frac{\sin \beta}{\alpha^2 + \sin^2 \beta}$$

$$G_p = G_0 + \frac{4}{Z_0} \sin^2 \frac{\theta_p}{2} \frac{\alpha \cos \beta}{\alpha^2 + \sin^2 \beta}$$

$\sin \beta$ can probably be made smaller than 0.1. This would correspond to adjusting the electrical length of the transmission line to about 1 part in 1000. In the accelerator R.F. system using cavity resonators and

waveguides. $\sin\beta$ would not correspond exactly to the electrical length of the waveguide between cavities, due to characteristics of the cavity and junction to the waveguide. Rather $\sin\beta$ would be made small by adjusting the two independent parameters of cavity tuning and electrical length of waveguide to make the system resonant to excitation in the fundamental mode, and to have a maximum admittance to excitation in a spurious mode. Another viewpoint would be to adjust the system to obtain the largest frequency spacing between the fundamental mode and the resonant frequencies of the spurious modes.

Taking $\alpha \ll \sin\beta \ll 1$
 then $Y_p \approx -i \frac{4}{Z_0 \sin\beta} \sin^2 \frac{\theta_p}{2} \quad p \neq 0$
 $G_p \approx G + \frac{4\alpha}{Z_0 \sin^2\beta} \sin^2 \frac{\theta_p}{2} \approx \frac{4\alpha}{Z_0 \sin^2\beta} \sin^2 \frac{\theta_p}{2} \quad \text{for } \beta < 0.1$

The ratio of power dissipated in spurious modes to power dissipated in the fundamental mode for change in tuning of one cavity by $\Delta\omega'$ is then:

$$\frac{\Delta P}{P_0} = \sum_{p \neq 0} \left(\frac{1}{N} 2 \frac{\Delta\omega'}{\omega_0} Q \right)^2 \frac{G G_p}{Y_p^2}$$

For variations of resonant frequency of all cavities by an r.m.s.

value $\Delta\omega'$

$$\frac{\Delta P}{P_0} = \sum_{p \neq 0} \frac{1}{N} \left(2 \frac{\Delta\omega'}{\omega_0} Q \right)^2 \frac{G G_p}{Y_p^2} = \left(2 \frac{\Delta\omega'}{\omega_0} Q \right)^2 \frac{G Z_0 \alpha}{4N} \sum_{p \neq 0} \frac{1}{\sin^2 \frac{\theta_p}{2}}$$

for $N=16 \quad \theta_p = \frac{2\pi p}{16} \quad \sum_{p \neq 0} \frac{1}{\sin^2 \frac{\theta_p}{2}} = 87.2$

Taking

$$\frac{G}{Z_0} = 17 \quad \alpha = .0029 \quad Q = 25,000$$

$$\frac{\Delta P}{P_0} = 5.7 \times 10^5 \left(\frac{\Delta\omega'}{\omega_0} \right)^2$$

Then to make the power dissipated in the spurious modes equal 5% of the power in the fundamental mode.

$$\frac{\Delta\omega'}{\omega_0} = \left(\frac{.05}{5.7 \times 10^5} \right)^{1/2} = \underline{2.96 \times 10^{-4}}$$

This r.m.s. variation of the tuning of the individual cavities from the average value should easily be maintained over a long period of time, without adjustment of the individual cavities. This variation would correspond to an r.m.s. temperature variation of about 20°C .

Under these conditions, the voltage in the modes adjacent to the fundamental mode will be:

$$V_1 = V_0 \frac{1}{\sqrt{N}} \left(2 \frac{\Delta \omega'}{\omega_0} \right) \frac{QG}{Y_p} = \frac{1}{2} V_0 \left(\frac{\Delta \omega'}{\omega_0} \right) \frac{QG}{\sqrt{N}} \frac{Z_0 \sin \beta}{\sin^2 \frac{\theta_1}{2}}$$

$$\text{taking } \sin \beta = 0.1$$

$$\frac{V_1}{V_0} = \frac{2.9 \times 10^{-4} \times 2.5 \times 10^4 \times .1}{2 \times 4 \times 17 \times .038} = \underline{.14}$$

This mode has a difference in phase angle per revolution of 2π radians with respect to the fundamental mode and corresponds to a principal orbit removed by about 3 inches from the chosen principal orbit. Since the voltages in the spurious modes are small compared to the fundamental mode and their frequency as seen by electrons in phase stable positions on the fundamental mode is large compared with the synchronous frequency, these spurious modes should have little effect on the motions of the electrons.

This system of strongly coupled cavities is then regarded as a single cavity into which R.F. power will be introduced at a number of points, by transmission lines from a central power source. These transmission lines will probably be adjusted to be terminated by their characteristic impedance at the point of maximum power in the acceleration cycle.

Considering two transmission lines, feeding power from a central power source to diametrically opposite cavities. In this case, equal current will be introduced into every even mode, and zero current into every odd mode. There will also be a circulating beam in the accelerator which will introduce

a current into the fundamental mode only.

$$\text{Then } V_0 = \frac{I + I_L}{Y_0}$$

I represents the current from the R.F. power source.

$$I = V_0 Y_0 - I_L$$

I_L is the current of the beam

It is assumed the system is tuned to compensate for the reactive component of I_L at the point of maximum power, so as to make I in phase with V_0 . Also, taking the beam loading power equal to the dissipation at the point of maximum power, then at point of maximum power:

$$I = 2 V_0 G \quad P_0 = 2 V_0^2 G$$

$$P_p = \frac{I^2}{Y_p^2} G_p \quad p \text{ even}$$

Ratio of total power in spurious modes to power in fundamental mode:

$$\frac{\Delta P}{P_0} = 2 \sum_{\substack{p \text{ even} \\ p \neq 0}} \frac{G G_p}{Y_p^2} = \frac{G Z_0 \alpha}{2} \sum_{\substack{p \text{ even} \\ p \neq 0}} \frac{1}{\sin^2 \theta_p} = .0017 = .17 \%$$

At lower cavity voltages the relative power in the spurious modes would be somewhat higher, but this is unimportant in the operation of the R.F. power source, as maximum power is not required at these lower voltages.

The relative voltage in the adjacent even mode is given by:

$$\frac{V_2}{V_0} = \frac{I}{Y_2 V_0} = \frac{Y_0}{Y_2} - \frac{I_L}{Y_2 V_0}$$

At low cavity voltages the second term due to beam loading will become dominant. With the conditions of beam loading assumed above:

$$I_L V_0 \cos \phi_s = V_{0 \max}^2 G \quad \phi_s \text{ is synchronous phase angle at maximum power}$$

then for small cavity voltages

$$\frac{V_2}{V_0} = \frac{V_{0 \max} G}{Y_2 V_0 \cos \phi_s} = \frac{Z_0 G \sin \beta}{4 \cos \phi_s \sin^2 \frac{\theta_2}{2}} \cdot \frac{V_{0 \max}}{V_0}$$

$$\text{taking } \sin \beta = 0.1 \quad \cos \phi_s = \frac{1}{\sqrt{2}} \quad \sin^2 \frac{\theta_2}{2} = .383$$

$$\frac{V_2}{V_0} = .014 \frac{V_{\max}}{V_0}$$

The R.F. voltage at injection will probably not be less than $1/20$ of the peak voltage. Then $V_2 = .26$ at injection. This will then have little effect on the electron motion as V_2 corresponds to a principal orbit about 6" from the center of the vacuum chamber, and the frequency of V_2 as seen by electrons in phase with V_0 will be twice the orbital frequency.

The klystron R.F. power source and strongly coupled system may be represented by:

where i_k represents the klystron beam current and i_L is the beam load current. One possibility is to make l an integral number of half wave lengths, then klystron and synchrotron cavity system are tightly coupled and may be considered as one cavity.

The characteristic of a klystron at the coupling loop in the output cavity is roughly of the form (for constant beam current and R.F. drive).

For voltages less than the optimum voltage, the klystron is almost a constant current generator and G_{kr} is small compared with G_C .

i_L is constant in magnitude and at a phase angle of 45° with respect to the cavity voltage. The cavity will be tuned so as to compensate for the reactive component of i_L at the time of maximum R.F. power.

$$\text{Then } i_{kmax} - V_{max} G_C - \frac{i_L}{\sqrt{2}} = 0$$

Taking beam power equal to cavity dissipation at point of maximum power

$$\text{then } V_{max} G_C = \frac{i_L}{\sqrt{2}}$$

$$i_{kmax} = \sqrt{2} i_L$$

At the beginning of acceleration cycle v is very small.

$$\text{then } i_{kmin} \cong i_L$$

$$\text{and } \frac{i_{kmin}}{i_{kmax}} = \frac{1}{\sqrt{2}}$$

Then for the klystron cavity coupled tightly to the synchrotron cavity, the klystron beam current during the first part of the acceleration cycle may approach $\frac{1}{\sqrt{2}}$ of the peak beam current. This would necessitate a much larger average power dissipation in the klystron collector, than indicated by the average power required to operate the R.F. system.

A more attractive possibility is to make the coupling line an odd quarter wavelength.

For characteristic impedance of line to be matched to load at peak R.F. power

$$Z_0 = \frac{1}{2G_c}$$

From the equations for a quarter wave line at the beginning of the acceleration cycle

$$i_{k \min} - v_{k \min} G_k - \frac{v_{c \min}}{Z_0} = 0$$

$$\text{Since } v_{c \min} \text{ is small } v_{k \min} \cong i_L Z_0$$

$$\text{then: } i_{k \min} - \frac{i_L}{2G_c} G_k - v_{c \min} \cdot 2G_c = 0$$

$$i_{k \min} = i_L \frac{G_k}{2G_c} + 2 v_{c \min} G_c$$

$$\begin{aligned} \text{and } \frac{i_{k \min}}{i_{k \max}} &= \frac{G_k}{2\sqrt{2} G_c} + \frac{2 v_{c \min} G_c}{\sqrt{2} i_L} \\ &= \frac{G_k}{2\sqrt{2} G_c} + \frac{v_{c \min}}{v_{c \max}} \end{aligned}$$

Then since $G_K \ll G_C$ and $v_{cmin} \ll v_{cmax}$ the beam current and collector dissipation at the beginning of the acceleration cycle will be much less than the peak values.

Another possibility is to insert a non-reciprocal isolating device between the klystron and the cavity, which absorbs all reflected power.

At the beginning of the acceleration cycle when V_c is very low, the current flowing in the end of the line is i_L , and the power flowing in each direction will be:

$$P_{min} = \frac{1}{4} i_L^2 Z_0$$

From previously derived relations

$$Z_0 = \frac{1}{2G_C} \quad V_{max} G_C = \frac{i_L}{\sqrt{2}} \quad P_{max} = 2 V_{max}^2 G_C$$

$$\text{then } \frac{P_{min}}{P_{max}} = \frac{1}{8}$$

Then during the portion of the acceleration cycle, when the R.F. voltage is small, the klystron must produce 1/8 of the peak power, almost all of which will be dissipated in the isolator. For a peak power of 1.4 MW, the power dissipated in the isolator at the beginning of the acceleration cycle would be 175 KW, and the average power dissipated in the isolator would be about 1/3 of this or about 60 KW.

In order to obtain the dynamic behavior of R.F. system with beam loading, the system is represented by:

With the resonant circuit representing the strongly coupled system and the impedance of the R.F. source at the point of operation, and i_L is the beam load current.

The response of the resonant circuit to a unit step of current is:

$$V(t) = I_0 Z + [V(0) - I_0 Z] e^{-\alpha t} e^{i(\beta - \omega_0)t}$$

with α the natural rate of response of the tuned circuit $= \frac{\omega_0}{2Q}$

$$\beta = \sqrt{\omega_1^2 - \alpha^2} \approx \omega_1, \text{ the natural frequency of the resonant circuit}$$

and V and I are complex variables representing amplitude and phase of R.F.

quantities

$$\text{At } t=0$$

$$\begin{aligned} \frac{dV(t)}{dt} &= -(V - I_0 Z)(\alpha - i\omega) = -(V - I_0 Z) \alpha (1 - i \frac{2\Delta\omega}{\omega_0} Q) \\ &= -\alpha (V - I_0 Z) \frac{R}{Z} = I_0 \alpha - V \alpha R Y \\ &= I_0 \alpha - V \alpha - i V \alpha \tan \phi_Y \end{aligned}$$

where ϕ_Y is the phase angle of the admittance of the resonant circuit.

The following relationships may be derived from a vector diagram representation of the system:

$$I_v = -(I_L + I_b)$$

$$\frac{d\Delta V}{dt} = -\alpha V_L \cos \phi_s \Delta \phi_s - \alpha V_L \frac{\sin \phi_s}{I_L} \Delta I - \alpha \Delta V + \alpha V \tan \phi_v \Delta \phi_v$$

$$\frac{d\phi_v}{dt} = -\alpha \frac{V_L}{V} \sin \phi_s \Delta \phi_s + \alpha \frac{V_L}{V} \cos \phi_s \frac{\Delta I}{I_L} - \alpha \Delta \phi_v - \alpha \tan \phi_v \frac{\Delta V}{V}$$

ΔV , $\Delta \phi_v$, ΔI , $\Delta \phi_s$ being small changes from the equilibrium values of the amplitude and phase of the cavity voltage and beam current. $V_L = I_L R$ is the voltage the beam would induce in the cavities in the absence of R.F. power, when tuned to resonance.

Equations for the motion of the electron beam may be written:

$$\frac{d\Delta E}{dt} = f_0 V \cos \phi_s \Delta \phi_s - f_0 V \cos \phi_s \Delta \phi_v + f_0 \sin \phi_s \Delta V$$

$$\text{and } \frac{d\Delta \phi_s}{dt} = - \frac{2\pi f_0 k \alpha'}{E} \Delta E$$

where E is the electron energy
 k is the harmonic order
 f_0 is the orbital frequency
 α' is the reduced compaction factor

In this analysis small damping terms due to change in radiation loss with energy are neglected.

$$\text{Also } \Delta I = -I_L \frac{\Delta \Gamma}{R_0} = -I_L \alpha' \frac{\Delta E}{E}$$

In the strong focusing accelerator ΔI will be very small and will be neglected.

Then the equations for the system may be written:

$$\frac{1}{E} \frac{d\Delta E}{dt} = \frac{f_0 V \cos \phi_s}{E} \Delta \phi_I - \frac{f_0 V \cos \phi_s}{E} \Delta \phi_V + \frac{f_0 V \sin \phi_s}{E} \frac{\Delta V}{V}$$

$$\frac{d\Delta \phi_I}{dt} = -K \cdot 2\pi f_0 \alpha' \frac{\Delta E}{E}$$

$$\frac{d\Delta \phi_V}{dt} = -\alpha \frac{V_L}{V} \sin \phi_s \cdot \Delta \phi_I - \alpha \Delta \phi_V - \alpha \tan \phi_V \frac{\Delta V}{V}$$

$$\frac{1}{V} \frac{d\Delta V}{dt} = -\alpha \frac{V_L}{V} \cos \phi_s + \alpha \tan \phi_V \Delta \phi_V - \alpha \frac{\Delta V}{V}$$

The natural frequencies of the system are then found by solving the secular equation.

$$\begin{vmatrix} -\omega & \frac{f_0 V \cos \phi_s}{E} & -\frac{f_0 V \cos \phi_s}{E} & \frac{f_0 V \sin \phi_s}{E} \\ -2\pi f_0 k \alpha' & -\omega & 0 & 0 \\ 0 & -\alpha \frac{V_L}{V} \sin \phi_s & -\alpha - \omega & -\alpha \tan \phi_V \\ 0 & -\alpha \frac{V_L}{V} \cos \phi_s & \alpha \tan \phi_V & -\alpha - \omega \end{vmatrix} = 0$$

By reducing the determinant an equation for ω is obtained.

$$\omega^4 + \omega^3 \cdot 2\alpha + \omega^2 [\omega_s^2 + \alpha^2 + \alpha^2 \tan^2 \phi_Y] + \dots$$

$$+ \omega \cdot 2\alpha \omega_s^2 + [\alpha^2 + \alpha^2 \tan^2 \phi_Y - \alpha^2 \frac{V_L}{V} \frac{\tan \phi_Y}{\cos \phi_s}] \omega_s^2 = 0$$

with $\omega_s = \left[\frac{2\pi f_0^2 \alpha^1 V \cos \phi_s}{E} \right]^{1/2}$, the synchronous frequency of the individual electrons in the accelerating voltage.

By means of Routh's criterion the equation is tested for roots with positive real parts. The condition for no roots to have positive real parts, which is necessary for a stable solution is found to be:

$$0 < \frac{\tan \phi_Y}{\cos \phi_s} < \frac{V}{V_L} \sec^2 \phi_Y$$

$$\text{or } 0 < \sin 2\phi_Y < 2 \frac{V}{V_L} \cos \phi_s$$

Taking the beam load at maximum voltage equal to the cavity dissipation, with a synchronous phase angle of 45° , then the voltage induced by the beam on the cavities unconnected to the R.F. generator is $\frac{V_{max}}{\sqrt{2}}$, taking the conductance of the R.F. generator equal to twice that of the cavities, then

$$V_L = \frac{V_{max}}{3\sqrt{2}}$$

Shortly after injection the synchronous phase angle will probably be small then:

$$0 < \sin 2\phi_Y < 6\sqrt{2} \frac{V}{V_{max}}$$

Then the region of tuning over which the system is stable becomes smaller as the cavity voltage is reduced and the relative beam loading increases. Taking

$$\frac{V}{V_{max}} = \frac{1}{20}$$

near injection

then $0 < \phi_v < 12.5^\circ$

In order for the system to present a resistive impedance to the generator at the maximum voltage under the above conditions $\phi_v = 45^\circ$. Then the tuning of the cavity system should be changed during the acceleration cycle, by slightly varying the R.F. frequency.

Possible modifications of the R.F. system would be an automatic system for maintaining the desired cavity voltage throughout the acceleration cycle, compensating for changes in beam loading. This would probably use a sensing device to measure the cavity voltage, and control the klystron current by means of an error signal derived from the cavity voltage.