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MICROINSTABILITY THEORY OF
TWO-ENERGY-COMPONENT TOROIDAL
SYSTEMS

BY

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PLASMA PHYSICS
LABORATORY

MASTER

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MICROINSTABILITY THEORY OF TWO-ENERGY-COMPONENT
TOROIDAL SYSTEMS

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ABSTRACT

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The microinstability theory of the two-energy-component toroidal system is developed and applied to systems typical of those in proposed fusion breakeven experiments. The energetic ion beam distributions $F_b(v_{\parallel}, v_{\perp}, t)$ obtained from approximate solutions of the Fokker-Planck equation are studied for high frequency microinstabilities arising from $\partial F_b / \partial v_{\parallel}^2 > 0$ or $\partial F_b / \partial v_{\perp}^2 > 0$ in both transient and collisional equilibrium states. For typical beam-plasma parameters the system is shown to be stable to high frequency modes during the parallel injection of a continuous, monoenergetic source.

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I. INTRODUCTION

Present ideas¹⁻³ for advanced CTR devices depend rather strongly on the use of energetic neutral beams for plasma heating and the production of significant fusion reaction power in systems with $n\tau_E$ less than 10^{14} sec/cm³. In some cases, particularly in so-called two component systems where thermonuclear reactions take place primarily between slowing down beam particles and background plasma ions, the energy density of the hot beam component may be comparable to the background plasma although the system's beta, $\beta = 8\pi p/B^2$, is small for both components. In the pure two component system one normally considers a deuteron beam injected into a tritium thermal plasma which is the system we assume in this work. The results obtained here, however, are easily modified to apply to other injection schemes. The thermonuclear power density is optimized³⁻⁴ when a deuteron beam is injected into a pure tritium background. Typically, the deuteron injection energy is 100-200 KeV while the background temperature for tritium and electrons 5-10 KeV.

Since the deuterons are injected at high energy there is a tendency for an anisotropic distribution function⁵⁻⁸ to

develop in steady state and pulsed injection. The anisotropy is then a possible source for microinstability^{3,5,6} that can lead to a more rapid beam energy loss rate than the Coulomb loss rate. As the Q (the ratio of power output to power input) of a TCT system is often not much larger than unity and the deuterons need to be energetic to undergo fusion, the presence of microinstability could severely reduce the amplification of a TCT device.

In this work we study the shape of the injected distribution function F_b to determine the structure of its free energy reservoir, and we survey plasma waves that depend on local plasma parameters to determine whether they tap the free energy reservoir of the hot component. The free energy reservoir arises from

$$(1) \quad \frac{\partial F_b}{\partial v_{\parallel}^2} > 0, \quad (2) \quad \frac{\partial F_b}{\partial v_{\perp}^2} > 0, \quad (3) \quad \frac{\partial F_b}{\partial x} \neq 0, \quad (4) \quad \frac{\partial F_b}{\partial v_{\perp}^2} \neq \frac{\partial F_b}{\partial v_{\parallel}^2}$$

where $v_{\parallel}$ and $v_{\perp}$ refer to velocity components parallel and perpendicular to the magnetic field. We primarily study beams injected parallel to the magnetic field although off-angle injection is also studied in less detail. In this

case there is some tendency for $\partial F_b / \partial v_{\parallel}^2 > 0$ to arise near the injection energy as well as at lower energy. However, instability from this source can be avoided for practical parameters. For example, although reservoir (1) introduces a destabilizing tendency to the ion acoustic wave, the background ion Landau damping is an overwhelmingly stabilizing mechanism. Alfven waves are less strongly Landau damped by the background plasma, and they can be destabilized by reservoir (1). However, coupling with the Alfven waves can be avoided if the injection velocity is less than the Alfven speed. This is generally satisfied for TCT, e.g., for $B = 50$ kg, and a tritium density of 10^{14} cm^{-3} , one requires the injection energy to be less than 0.5 MeV to avoid exciting Alfven waves.

Reservoir (2) can arise even from parallel injection under some circumstances. During steady state injection, this can arise when high Z impurities are present in the plasma. In this case the deuterons scatter more rapidly in angle than they slow down in speed in some regions of velocity space, and as a result they produce reservoir (2). However, for instability we require a weighted average of $\partial F_b / \partial v_{\parallel}^2$ be positive, and in

numerical evaluations we have been unable to find a net destabilizing deuteron contribution except for very high Z plasmas.

Hence it appears that a steady state distribution is not destabilized by reservoir (2) during parallel injection. At worst one may find very narrow bands of instability, which are stabilized by a very mild relaxation of the distribution function.

For transient pulsed and clamped operation of TCT reservoir (2) is more likely to arise from the Coulomb scattering. This reservoir can then be tapped by modes that propagate nearly perpendicular to the magnetic field and near harmonics of the beam cyclotron frequency, e.g., the lower hybrid wave, low frequency electron plasma waves and Bernstein modes.

Reservoir (3) which is due to the density gradient is stabilizing to electron drift waves and destabilizing to ion drift waves. This energy reservoir causes radial diffusion of the beam. However, as the wavelengths of the ion drift wave are short, the diffusion time is much longer than the slowing down time of the beam.

Because the beam is energetic, the magnetic drift frequency of the beam is larger than the diamagnetic drift

frequency of the background plasma. As a result one can obtain resonance between the deuteron magnetic drift frequency and the frequency of infinite medium modes. One mode of concern is the shear Alfvén wave for which we show that a simplified plasma model yields instability from reservoir (3) for reasonable TCT parameters. In Appendix C we also examine the turbulent diffusion of the deuterons due to trapped particle modes, and we show that the beam diffusion time is appreciably longer than the electron diffusion time.

Reservoir (4) is always present in the distribution function, but it is difficult to tap. For example, we show that this reservoir can excite whistler waves if the deuteron beam velocity is greater than $3.0 v_A$ where v_A is the Alfvén speed in a tritium plasma.

The structure of the paper is as follows. In Section II where the formal linear theory is presented, we show the general structure of the energy reservoir and discuss the equations for the plasma waves. In Section III we solve the Fokker-Planck equation for the deuteron beam and discuss the results in regards to the stability calculations. In Sections IV and V we discuss waves that tap reservoir (1) and (2), respectively, while in Section VI we discuss waves that tap reservoirs (3) and (4).

II. PLASMA WAVES AND BEAM-PLASMA WAVE ENERGY TRANSFER RATES

For our analysis of the waves of the system and the interaction of the energetic ion component with the waves, we use a slab model of the plasma where the magnetic field is along z and the density and magnetic field gradients are in the x direction. Strictly speaking, in this slab model only the grad-B drift proportional to the particle's perpendicular energy occurs; however, we shall add to this drift the curvature drift arising in tokamak geometry. From the orbit integration of the linearized Vlasov equation this is clearly the correct generalization of the slab model, and the principal oversimplification occurs in treating the combined drift term as constant rather than including its oscillatory dependence on position along the magnetic field line. Despite this shortcoming the constant drift approximation is informative for estimating the contributions from the gradient and curvature drifts.

In most of the mode analysis it is adequate to treat the energetic component as a perturbation to the waves of the background plasma as the beam density n_b is less than the

background plasma density n_0 . The MHD finite medium modes which depend on the sum of the component pressures are an exception and are not considered in the present work.

In the Vlasov equation the equilibrium is given by

$F = F_j(E, \mu, X)$ where $E = (v_{\parallel}^2 + v_{\perp}^2)/2$, $\mu = (v_{\perp}^2 - v_D^2)/2B(X)$, $X = x + v_Y/\omega_{cj}$ and $v_D = \hat{y}(v_{\perp}^2/\omega_{cj})(d \ln B/dx)$. The linearized Vlasov equation for the response $f_j \exp[i(\underline{k} \cdot \underline{x} - \omega t)]$ to an electric field of the form

$$\underline{e}(\underline{x}, t) = \underline{e} \exp[i(\underline{k} \cdot \underline{x} - \omega t)] \quad (\text{II-1})$$

is

$$\frac{df_j}{dt} = -\frac{e_j}{m_j} \left[\underline{e} \left(1 - \frac{\underline{k} \cdot \underline{v}}{\omega} \right) - \underline{k} \frac{\underline{e} \cdot \underline{v}}{\omega} \right] \left[\underline{v} \frac{\partial F_j}{\partial E} + \frac{(\underline{v} - \underline{v}_D)}{B} \frac{\partial F_j}{\partial \mu} \right.$$

$$\left. + \frac{\hat{y}}{\omega_{cj}} \frac{\partial F_j}{\partial X} \right] + O \left(\left| \rho_j \frac{d \ln B}{dx} \right|^2 \right)$$

$$= -\frac{e_j}{m_j} \left(\frac{i}{\omega \omega_{cj}} \frac{\partial F_j}{\partial X} \frac{d \underline{e}_Y}{dt} - \frac{i}{\omega B(X)} \frac{\partial F_j}{\partial \mu} \frac{d}{dt} [\underline{v}_{\parallel} \underline{e}_{\parallel} + \underline{v}_D \underline{e}_Y] \right.$$

(II-2)

$$\left. + \underline{v} \cdot \underline{e} \left\{ \frac{\partial F_j}{\partial E} + \left[1 - \frac{(\underline{k}_{\parallel} \underline{v}_{\parallel} + \underline{k}_Y \underline{v}_D)}{\omega} \right] \frac{1}{B(X)} \frac{\partial F_j}{\partial \mu} + \frac{\underline{k}_Y}{\omega \omega_{cj}} \frac{\partial F_j}{\partial X} \right\} \right).$$

Integrating Eq. (II-2) over the cyclotron orbits with the guiding center drifts, we obtain

$$\begin{aligned}
 f_j = & -\frac{ie_j}{m_j} \left\{ \frac{\epsilon_y}{\omega \omega_{cj}} \frac{\partial F_j}{\partial X} + \frac{1}{\omega B} \frac{\partial F_j}{\partial \mu} (\underline{v}_\perp - \underline{v}_D) \cdot \underline{\epsilon} + \sum_{\ell, m} J_m \left(\frac{k_\perp (2\mu B)^{1/2}}{\omega_{cj}} \right) \right. \\
 & \times \left(\frac{\partial F_j}{\partial E} + \frac{\ell \omega_{ci}}{\omega} \frac{\partial F_j}{\partial \mu} + \frac{k_y}{\omega \omega_{cj}} \frac{\partial F_j}{\partial X} \right) \frac{\exp[i(m-\ell)(\phi-\theta)]}{\omega - \ell \omega_{cj} - \omega_{Dj} - k_\parallel v_\parallel} \left[(\underline{v}_\parallel \underline{\epsilon}_\parallel + \underline{v}_{Dj} \underline{\epsilon}_y) J_\ell \right. \\
 & \left. \left. + \frac{1}{2} (2\mu B)^{1/2} \left(e^{-i\theta} \underline{\epsilon}_+ J_{\ell-1} + e^{i\theta} \underline{\epsilon}_- J_{\ell+1} \right) \right] \right\} \quad (II-3)
 \end{aligned}$$

where $\omega_{Dj} = k_y v_{Dj}$, $\underline{k}_\perp = k_\perp (\hat{x} \cos \theta + \hat{y} \sin \theta)$,

$\underline{v}_\perp = \underline{v}_{Dj} + (2\mu B)^{1/2} (\hat{x} \cos \phi + \hat{y} \sin \phi)$ and $\underline{\epsilon}_\pm = \underline{\epsilon}_x \pm i \underline{\epsilon}_y$.

From the f_j in Eq. (II-3) we calculate the perturbed current density $\underline{j}$ and substitute into Maxwell's equations to obtain

$$\nabla \times (\nabla \times \underline{\epsilon}) + \frac{1}{c^2} \frac{\partial^2 \underline{\epsilon}}{\partial t^2} + \frac{4\pi}{c^2} \frac{\partial \underline{j}}{\partial t} = -\frac{\omega}{c^2} \underline{D} \cdot \underline{\epsilon} = 0 \quad (II-4)$$

which defines the response function $\underline{D}$. We shall decompose

$\underline{D}$ as

$$\underline{D} = \underline{D}_0 - i\delta\underline{D}$$

where $\underline{D}_0$ is the nondissipative part of the background electron and ion response while $\delta\underline{D}$ is the dissipative response of all three species. (The nondissipative contribution from the energetic component is considered negligible). The perturbation to the mode frequency ω_k^0 then follows from

$$\delta\omega_k = \frac{i \int dx \underline{e}_0^* \cdot \delta\underline{D} \cdot \underline{e}_0}{\int dx \underline{e}_0^* \cdot \frac{\partial \underline{D}_0}{\partial \omega} \cdot \underline{e}_0}, \quad (\text{II-5})$$

and we define L as the average particle-wave energy transfer rate where

$$L = \int dx \underline{e}_0^* \cdot \delta\underline{D} \cdot \underline{e}_0 / \int dx$$

$$= \pi (\int dx)^{-1} \sum_j \omega_{pj}^2 \int dx \sum_\ell \int d\underline{u} \delta(\omega - k_{\parallel} v_{\parallel} - \omega_{Dj} - \ell \omega_{cj})$$

$$\times \left(\frac{\partial F_j}{\partial E} + \frac{\ell \omega_{cj}}{\omega B(X)} \frac{\partial F_j}{\partial \mu} + \frac{k_y}{\omega \omega_{cj}} \frac{\partial F_j}{\partial X} \right) \left| (v_{\parallel} \epsilon_{\parallel} + v_{Dj} \epsilon_y) J_{\ell} \right. \\ \left. + \frac{1}{2} (2\mu B)^{1/2} \left(e^{-i\theta} \epsilon_{+} J_{\ell-1} + e^{i\theta} \epsilon_{-} J_{\ell+1} \right) \right|^2. \quad (\text{II-6})$$

where $\underline{u} = \underline{v} - \underline{v}_D$.

We may also write that

$$\frac{\partial F_j}{\partial E} + \frac{\ell \omega_{cj}}{\omega B} \frac{\partial F_j}{\partial \mu} = \frac{2(\omega - \ell \omega_{cj})}{\omega} \frac{\partial F_j}{\partial v_{\parallel}^2} + \frac{2\ell \omega_{cj}}{\omega} \frac{\partial F_j}{\partial v_{\perp}^2} \quad (\text{II-7})$$

and

$$\frac{1}{2} \left(e^{-i\theta} J_{\ell-1} \epsilon_{+} + e^{i\theta} J_{\ell+1} \epsilon_{-} \right) = \frac{\ell \omega_{cj}}{k_{\perp} v_{\perp}} J_{\ell} \epsilon_k + i J'_{\ell} \epsilon_{\perp}$$

where

$$\hat{b} \times (\underline{\epsilon} \times \hat{b}) = \hat{k}_{\perp} \epsilon_k + \hat{b} \times \hat{k} \epsilon_{\perp}.$$

It is also convenient to introduce here a dimensionless beam-wave power transfer function $\epsilon_b^{\ell}(k_{\perp}, c_{\ell})$ used by Mai and Horton⁹ which is proportional to L_b for infinite medium electromagnetic modes with negligible $\epsilon_{\perp}$. It is defined as

$$\epsilon_b^{\ell}(k_{\perp}, c_{\ell}) = 4\pi^2 \tilde{v}_b^3 \int_0^{\infty} dv_{\perp} v_{\perp} J_{\ell}^2 \left(\frac{k_{\perp} v_{\perp}}{\omega_{cb}} \right) \left[\frac{\partial F_b}{\partial v_{\parallel}^2} \left(1 - \frac{\ell \omega_{cb}}{\omega} \right) + \frac{\ell \omega_{cb}}{\omega} \frac{\partial F_b}{\partial v_{\perp}^2} \right] \Bigg|_{v_{\parallel} = c_{\ell}}$$

where $\tilde{v}_b$ is a mean beam particle speed.

For infinite medium modes of a Maxwellian plasma we have that

$$\int dx \, \underline{\epsilon}_0^* \cdot \frac{\partial \underline{D}}{\partial \omega} \cdot \underline{\epsilon}_0 = |\underline{\epsilon}_0|^2 + \frac{c^2}{\omega^2} |\underline{k} \times \underline{\epsilon}_0|^2 + \int dx \sum_j \frac{2\omega_j^2}{v_j^2} \sum_{\ell} \int dv_{\parallel} \frac{F_j}{(\omega - k_{\parallel} v_{\parallel} - \omega_{Dj} - \ell \omega_{cj})^2} \quad (\text{II-8})$$

$$\times \left| v_{\parallel} \epsilon_{\parallel} J_{\ell} + \frac{1}{2} v_{\perp} \left(e^{-i\theta} J_{\ell-1} \epsilon_{+} + e^{i\theta} J_{\ell+1} \epsilon_{-} \right) \right|^2$$

which is intrinsically positive. For this case instability can arise only if $L_b > 0$.

We now review some aspects of the infinite, uniform plasma modes required for the stability analysis of the two-energy component system. Clearly the energetic ion component modifies the low frequency waves that depend on the ion dynamics.

Thus we consider $\omega \leq \omega_{pi} \ll \omega_{pe}, \omega_{ce}$, and similarly we

have $k_{\perp} \rho_e = k_{\perp} v_e / \omega_{ce} \ll 1$.

At oblique angles of propagation the dispersion relation $D_{\perp} \cdot \underline{\epsilon} = 0$ decouples approximately into the compressional mode with $\underline{\epsilon}_{\perp} = \underline{\epsilon} \cdot \hat{b} \times \hat{k}_{\perp} \gg \epsilon_k, \epsilon_{\parallel}$, from which one obtains at low frequency, $\omega < \omega_{ci}$, the compressional Alfvén wave

$$\omega^2 = k^2 c^2 \omega_{ci}^2 / \omega_{pi}^2 \equiv k^2 v_A^2 \quad (\text{II-9})$$

and the coupled electrostatic-

electromagnetic modes with $\underline{\epsilon} = \epsilon_k \hat{k}_{\perp} + \epsilon_{\parallel} \hat{b} \gg \epsilon_{\perp}$ for which

$$\left[k_{\perp}^2 c^2 - \omega^2 + \frac{\omega^2 \omega_{pe}^2}{k_{\parallel}^2 v_e^2} Z' \left(\frac{\omega}{k_{\parallel} v_e} \right) + \frac{\omega^2 \omega_{pi}^2}{k_{\parallel}^2 v_i^2} Z' \left(\frac{\omega}{k_{\parallel} v_i} \right) \right] \epsilon_{\parallel} = c^2 k_{\perp} k_{\parallel} \epsilon_k \quad (\text{II-10})$$

$$\left[k_{\parallel}^2 c^2 - \omega^2 \left(1 + \frac{\omega_{pe}^2}{\omega_{ce}^2} \right) - \frac{2\omega \omega_{pi}^2}{k_{\parallel} v_i} \sum_{\ell} \frac{\ell^2 \omega_{ci}^2}{k_{\perp}^2 v_i^2} e^{-b} I_{\ell}(b) Z \left(\frac{\omega - \ell \omega_{ci}}{k_{\parallel} v_i} \right) \right] \epsilon_k = c^2 k_{\perp} k_{\parallel} \epsilon_{\parallel} \quad (\text{II-11})$$

where $b = k_{\perp}^2 \rho_i^2 / 2$, $\rho_i = v_i / \omega_{ci}$, $v_j = (2T_j / m_j)^{1/2}$ and $Z(\zeta) = \pi^{1/2} \int_{-\infty}^{+\infty} dx e^{-x^2} / (x - \zeta)$ for $k_{\parallel} \text{Im} \zeta / |k_{\parallel}| > 0$. In expression (II-10) we

have neglected finite Larmor radius terms in the parallel ion current and in the off diagonal current response of the ions in Eqs. (II-10) and (II-11) which are small when either the Larmor radius or $k_{\parallel}/k_{\perp}$ approach zero. As a result Eq. (II-10) is accurate when the parallel ion current response is negligible compared to the parallel electron current and perpendicular ion current or when $k_{\perp}^2 \rho_i^2 < 1$. At times (particularly for $b > 1$ and $k_{\parallel} v_i \ll \omega$) it is convenient to rewrite Eq. (II-11) using the following identity:

$$\sum_{\ell=-\infty}^{\infty} \frac{\ell^2 \omega_{ci}^2}{v_i^2} I_{\ell} e^{-b} Z \left(\frac{\omega - \ell \omega_{ci}}{k_{\parallel} v_i} \right) \frac{\omega}{k_{\parallel} v_i} = \frac{\omega^2}{v_i^2}$$

$$- \frac{\omega}{\pi^{1/2} v_i^3} \sum_{\ell=-\infty}^{\infty} \int \frac{dv_{\parallel} (\omega - k_{\parallel} v_{\parallel})^2 \exp(-v_{\parallel}^2/v_i^2) I_{\ell} e^{-b}}{(\omega - k_{\parallel} v_{\parallel} - \ell \omega_{ci})}$$

(II-12)

$$\cong \frac{\omega^2}{v_i^2} \left[1 + \frac{\omega}{k_{\parallel} v_i} \sum_{\ell} Z_{\ell} \left(\frac{\omega - \ell \omega_{ci}}{k_{\parallel} v_i} \right) I_{\ell} e^{-b} + O \left(\frac{k_{\parallel}^2 v_i^2}{\omega^2} \right) \right]$$

The dispersion relation obtained from Eqs. (II-10) and (II-11) can be written as

$$\frac{c^2 k_{\parallel}^2 \sum_j \frac{\omega_j^2 p_j}{k_{\parallel}^2 v_j^2} z' \left(\frac{\omega}{k_{\parallel} v_j} \right)}{\left[c^2 k_{\perp}^2 + \sum_j \frac{\omega_j^2 \omega^2 p_j}{k_{\parallel}^2 v_j^2} z' \left(\frac{\omega}{k_{\parallel} v_j} \right) \right]} = 1 + \frac{\omega_{pe}^2}{\omega_{ce}^2} \quad (\text{II-13})$$

$$+ \frac{2\omega_{pi}^2}{\omega k_{\parallel} v_i} \sum_i \frac{\ell^2 \omega_{ci}^2}{k_{\perp}^2 v_i^2} e^{-b_i} I_{\ell} z \left(\frac{\omega - \ell \omega_{ci}}{k_{\parallel} v_i} \right)$$

and from Eq. (II-13) it is apparent that modes with

$$k_{\perp}^2 c^2 > \left| \frac{\omega^2 \omega_{pe}^2}{k_{\parallel}^2 v_e^2} z' \left(\frac{\omega}{k_{\parallel} v_e} \right) \right| \quad (\text{II-14})$$

essentially yield the usual electrostatic dispersion relation.

For nearly parallel propagation, $k_{\perp}^2 \ll k_{\parallel}^2 \left(1 + \left| \frac{\omega_{ci}}{\omega} \right|^{-1} \right)$. Eqs. (II-10) and (II-11) fail to describe the electromagnetic modes. Then circularly polarized waves and the longitudinal plasma oscillations are the natural solutions. The dispersion relation in the cold plasma approximation for the circularly polarized waves $\epsilon_{\pm} \gg \epsilon_{\parallel}$ is

$$k_{\parallel}^2 c^2 + \sum_j \frac{\omega \omega_j^2 p_j}{\omega \mp \omega_{cj}} = 0 \quad (\text{II-15})$$

and the plasma oscillations have $\epsilon_{\parallel} \gg \epsilon_{\pm}$ with the dispersion relation given by Eq. (II-10) with $k_{\perp} = 0$.

III. TCT COLLISIONAL DISTRIBUTION

The hot ion species in a TCT is characterized by a speed v_b that satisfies $v_i \ll v_b \ll v_e$ where v_i and v_e are the background plasma ion and electron thermal velocities, respectively. Furthermore, self-collisions of the hot ions may be ignored as the hot density is low. Hence, one can construct a simplified Fokker-Planck equation⁵⁻⁸ for the hot species that neglects the self-interaction and makes use of the velocity inequalities. In addition one can include the effects of energy pumping that arise from changing the radius of the tokamak. One then finds that the beam distribution F_b satisfies the equation

$$\frac{\partial F_b}{\partial \tau} + C \left[\frac{u(1+\mu^2)}{2} \frac{\partial F_b}{\partial u} + \frac{(1-\mu^2)}{2} \mu \frac{\partial F_b}{\partial \mu} \right] \quad (III-1)$$

$$- \frac{1}{u^2} \frac{\partial}{\partial u} [(1+u^3) F_b] - \frac{A}{2u^3} \frac{\partial}{\partial \mu} (1-\mu^2) \frac{\partial F_b}{\partial \mu} = S$$

where $u = v/v_I$, $v_I^3 = (3\pi^{1/2}/4) (m_e/m_b) v_e^3 \tilde{Z}$, $\mu = v_{\parallel}/v$, $\tau = vt$ with $v = 4\pi n_o e^4 \tilde{Z} \ln \Lambda / m_b^2 v_I^3$, $A = Z_{eff}/\tilde{Z}$, $Z_{eff} = \sum_j Z_j^2 n_j/n_o$, $\tilde{Z} = (m_b/n_o) \sum_j Z_j^2 n_j/m_j$, j refers to ion species, b refers to

the high energy component, $n_o = \sum_j Z_j n_j$ is the electron density, $T_e = \frac{1}{2} m_e v_e^2$, $C = -dR/R \dot{v} dt$ is the dimensionless compression rate where R is the major radius of the tokamak and $S(u, \mu, \tau)$ is the external beam source. Note that if the background ion density is dominated by species i (tritium for TCT) then $\tilde{Z} \approx Z_i m_b/m_i$ while Z_{eff} can be greater than $\tilde{Z}$ from a high Z_j impurity. Hence, for such cases $A \sim Z_{eff} m_b/m_i$. We shall refer to v_I and $\frac{1}{2} m_b v_I^2$ as the transition speed and transition energy, respectively. The energy pumping terms follow readily from the constancy of the magnetic moment, $\mu = v_{\perp}^2/2B \propto v_{\perp}^2 R = \text{const}$, and the longitudinal invariant, $\oint v_{\parallel} dl \propto v_{\parallel} R = \text{const}$, during the compression process.

In the collision term we have as dominant processes the electron and background ion drag and the angular scattering due to collisions with slow ions. For $A \sim 1$ the scattering term is small compared to the electron drag when $u = v/v_I > 1$ and competitive with the ion drag for $u < 1$. For $A > 1$ the scattering term dominates the drag for $u < A^{1/3}$.

From these observations one can draw the following qualitative conclusions about the relaxation of F_b in the

absence of energy pumping. If one injects at energies above the transition energy the beam degrades in energy with little angular scattering. Below the transition energy the beam isotropizes in angle and degrades in energy at comparable rates if $A \approx 1$. With energy pumping it is possible to clamp the beam energy; however, angular scattering continues to spread the beam in the energy shell. A qualitatively similar effect occurs when $A > 1$ and $v < v_I A^{1/3}$ as then angular scattering is faster than the energy degradation.

One can obtain an exact solution for the angle-averaged beam distribution $\bar{F}_b = \frac{1}{2} \int_{-1}^{+1} F_b d\mu$ in the absence of energy pumping. The angular scattering term integrates to zero, and $\bar{F}_b$ satisfies the equation

$$\frac{\partial \bar{F}_b}{\partial \tau} - \frac{1}{u^2} \frac{\partial}{\partial u} [(1 + u^3) \bar{F}_b] = \bar{S} \quad . \quad (\text{III-2})$$

In this case we have for the characteristics the equation for the drag on the particle of speed u

$$\frac{du}{d\tau} = - \frac{1 + u^3}{u^2} \quad (\text{III-3})$$

with the solution

$$u(u_0, \tau) = \left[(1 + u_0^3) e^{-3\tau} - 1 \right]^{1/3} \quad (\text{III-4})$$

where $u = u_0$ at $\tau = 0$. Without a source, $\bar{S} = 0$ in Eq. (III-2), the solution of Eq. (III-2) with the initial value $\bar{F}_b(u, 0)$ is

$$\begin{aligned} \bar{F}_b(u, \tau) &= \frac{1 + u_0^3(u, \tau)}{1 + u^3} \bar{F}_b(u_0(u, \tau), 0) \\ &= e^{3\tau} \bar{F}_b\left(\left[(1 + u^3) e^{3\tau} - 1\right]^{1/3}, 0\right) \end{aligned} \quad (\text{III-5})$$

where $u_0(u, \tau)$ is the inversion of $u(u_0, \tau)$ given in Eq. (III-4). For a monoenergetic source of constant strength in time, $\bar{S} = S_0 \Theta(\tau) \delta(u - u_b)$, the solution of Eq. (III-2) is

$$\bar{F}_b(u, \tau) = \frac{u_b^2 S_0 \left[\Theta(u - u_b(u_b, \tau)) - \Theta(u - u_b) \right]}{1 + u^3} \quad (\text{III-6})$$

where $u(u_b, \tau)$ is given by Eq. (III-4) for $\tau \leq \frac{1}{3} \ln(1 + u_b^3) \equiv \tau_f$ and vanishes thereafter.

A general solution for $F_b(u, \mu, \tau)$ can be obtained as an expansion of F_b in Legendre polynomials⁷⁻⁸ as shown in

Appendix A. However, the structure of F_b with such solutions is not transparent nor is it convenient for the stability theory calculations. Here we present approximate, analytic solutions which are valid for $v^2 \gg v_{\perp}^2$. We consider that either an initial beam distribution is given of the form

$$F_b(u, \mu, 0) = \bar{F}_b(u, 0) 2 \alpha_0 \exp[-\alpha_0(1 - \mu)] \quad (\text{III-7})$$

or that a constant strength, mononergetic source

$$S = S_0 \Theta(\tau) \delta(u - u_b) 2 \alpha_0 \exp[-\alpha_0(1 - \mu)] \quad (\text{III-8})$$

is given where $\alpha_0 \sim v_b^2/v_{b\perp}^2 \gg 1$. For such beam distributions peaked in the forward direction the angular scattering term reduces to

$$\frac{A}{u^3} \frac{\partial}{\partial \delta\mu} \left(\delta\mu \frac{\partial F_b}{\partial \delta\mu} \right)$$

where $\delta\mu = 1 - \mu \approx v_{\perp}^2/2v^2$. One now seeks solutions of the form

$$F_b(u, \mu, \tau) = \bar{F}_b(u, \tau) 2\alpha(u, u_0(u, \tau)) \exp[-\alpha(u, u_0(u, \tau)) \delta\mu]. \quad (\text{III-9A})$$

for the initial value problem with $\bar{F}_b(u, \tau)$ given in Eq. (III-5) and

$$F_b(u, \mu, \tau) = \bar{F}_b(u, \tau) 2\alpha(u, u_b) \exp[-\alpha(u, u_b) \delta\mu] \quad (\text{III-9B})$$

for the steady, monoenergetic source with $\bar{F}_b(u, \tau)$ given in Eq. (III-6).

Substituting Eq. (III-9A) into Eq. (III-1) using the simplified form of the angle scattering term one finds that for the initial value problem $\alpha(u, \tau)$ is given by

$$\alpha^{-1} = \alpha_o^{-1} + \frac{A}{3} \ln \left(\frac{1 + u^{-3}}{1 + u_o^{-3}(u, \tau)} \right) \quad (III-10)$$

$$= \alpha_o^{-1} + \frac{A}{3} \ln [1 + u^{-3} (1 - e^{-3\tau})] ,$$

and $\bar{F}_b(u, \tau)$ is given by Eq. (III-5). For the steady, monoenergetic source problem $\alpha(u)$ is given by

$$\alpha^{-1} = \alpha_o^{-1} + \frac{A}{3} \ln \left(\frac{1 + u^{-3}}{1 + u_b^{-3}} \right) , \quad (III-11)$$

and $\bar{F}_b(u)$ in Eq. (III-9B) is given by Eq. (III-6). These solutions are valid for $\alpha > 1$. In the region $u = v/v_I < 1$ the steady state solution (III-11) has $\alpha < 1$; however, for $v/v_I \ll 1$ the angle scattering term is important and the distribution tends to be isotropic. This is well satisfied for $A > 1$, and even for $A \approx 1$ the results of the numerical calculations of Callen and Rome⁷ for counter injection show

this isotropy. Hence, in this region $u < 1$, F_b is given approximately by

$$F_b \cong \bar{F}_b = \frac{u_b^2 S_o \Theta(u_b - u)}{1 + u^3}.$$

The steady state beam energy density can be computed from Eq. (III-6) when $\tau \geq \tau_f$, and one finds that

$$n_b E_b = \frac{m_b}{2} \int d^3v v^2 F_b \cong \frac{n_b m_b \tilde{v}_b^2}{2} \cong \frac{n_b m_b v_b^2}{4 \tau_f} \quad (\text{III-12})$$

where the approximation is valid for $u_b = v_b/v_I > 1$ and $n_b = 4\pi u_b^2 S_o \tau_f$ is the local steady state beam density.

In the case of energy pumping, $C \neq 0$, the exact solution to $\bar{F}_b$ is not readily obtainable, because the clamping forces are a function of pitch angle, and therefore the evolution of $\bar{F}_b$ depends on the detailed shape of F_b . However, here we shall solve for F_b for distributions initially peaked along the field lines for times less than the angular scattering time. If we transform to adiabatic invariant quantities $\tilde{u}$ and $\delta\tilde{\mu}$ defined by

$$\begin{aligned}\tilde{u} &= \left[u_{\perp}^2 e^{-C\tau} + u_{\parallel}^2 e^{-2C\tau} \right]^{1/2} \\ &= u e^{-C\tau} + \mathcal{O}(u_{\perp}^2/u^2) \\ \delta\tilde{\mu} &= \frac{u_{\perp}^2 e^{-C\tau}}{2\tilde{u}^2} = \frac{u_{\perp}^2}{2u^2} e^{C\tau} + \mathcal{O}(u_{\perp}^4/u^4)\end{aligned}$$

where $e^{-C\tau} = R(\tau)/R(0)$, we find that Eq. (III-1) is approximately given by

$$\frac{\partial F_b}{\partial \tau} + \frac{1}{\tilde{u}^2} \frac{\partial}{\partial \tilde{u}} \left(e^{-3C\tau} + \tilde{u}^3 \right) F_b + \frac{A}{\tilde{u}^3} e^{-2C\tau} \frac{\partial}{\partial \delta\tilde{\mu}} \frac{\delta\tilde{\mu} \partial F_b}{\partial \delta\tilde{\mu}} = 0 \quad (\text{III-13})$$

We shall consider the solution of an initial value problem (note that a steady state solution does not exist with clamping as $R(\tau) = R(0) e^{-C\tau}$ changes with time). For $C\tau$ and $\delta\tilde{\mu} \ll 1$, Eq. (III-13), to lowest order is the same as Eq. (III-1). Hence, for $C\tau$, $\alpha^{-1} \ll 1$, the initial distribution, $\alpha_0 e^{-\alpha_0 \delta\tilde{\mu}} F_b(\tilde{u})$ evolves the same as Eq. (III-9) in the variables $\tilde{u}$ and $\delta\tilde{\mu}$. In terms of the original speed variable, one finds for $\tau < 1$, that the characteristics for the speed u are

$$u(u_0, \tau) = u_0 - u_0 [1 + u_0^{-3} - C]\tau \quad (\text{III-14})$$

For $C < 1$ this solution is similar to Eq. (III-4) in that $u < u_0$ for $\tau > 0$. For $C > 1$ we observe that $u = (C-1)^{-1/3}$ is a point of balance between the forces of drag and energy pumping. Particles injected with $u > (C-1)^{-1/3}$ are accelerated to higher energies according to Eq. (III-14) and for $u < (C-1)^{-1/3}$ the particles are decelerated. Likewise, one finds for $\tau < 1$ that $\alpha^{-1}(\tau)$ is given by

$$\alpha^{-1}(\tau) = \alpha_0^{-1} + A\tau/u_0^3. \quad (\text{III-15})$$

Hence, if $A > [1 + (1-C)u_0^3]$ angle scattering dominates the evaluation of the distribution while the speed is roughly constant.

To summarize the character of the distributions we have computed we note that if the source is on for a time $\tau > \tau_f$, the steady state distribution is relatively smooth up to the cut-off at the injection energy. However, for $\tau < \tau_f$ there is a sharp front in the distribution corresponding to the energy to which the initially injected particles have fallen from friction with the background plasma. If the injector delivers the beam in a time $\tau \ll \tau_f$ then the distribution remains localized in speed. After injection the distribution relaxes with a mild distortion of its shape but

with an overall slowing down due to the drag. For $A > |1 + (1-c)u_b^3|$ the distribution develops from angular scattering into a crescent shape.

An overall assessment of whether the hot component tends to be destabilizing to noncompressional waves ($\epsilon_{\perp} = 0$) can be made by analyzing the function

$$\epsilon_b^{\ell}(\rho, u_{\parallel}) = 4\pi^2 \tilde{u}_b^3 \int_0^{\infty} du_{\perp} u_{\perp} J_{\ell}^2(\rho u_{\perp}) \left[\left(1 - \frac{\ell \omega_{cb}}{\omega}\right) \frac{\partial F_b(u_{\perp}^2, u_{\parallel}^2)}{\partial u_{\parallel}^2} + \frac{\ell \omega_{cb}}{\omega} \frac{\partial F_b}{\partial u_{\perp}^2} \right] \quad (\text{III-16})$$

where $\rho = k_{\perp} v_I / \omega_{cb}$, and when $\epsilon_b^{\ell} > 0$ the hot component introduces a destabilizing tendency. Let us choose $C = 0$, so that $F_b(u_{\perp}, u_{\parallel}, \tau)$ is given by Eq. (III-9), where the normalization of ϵ_b^{ℓ} is chosen so that

$$\int du_{\parallel} F_b(\tau \geq \tau_f) = 1, \quad \tilde{u}_b^2 = \int du_{\parallel} u_{\parallel}^2 F_b(\tau \geq \tau_f).$$

For $\omega = \ell \omega_{cb}$, we find

$$\frac{\epsilon_b^0}{4\pi^2 \tilde{u}_b^3} = \int_{u_{\perp \min}}^{(u_b^2 - u_{\parallel}^2)^{1/2}} du_{\perp} u_{\perp} \frac{\alpha F_b J_0^2(\rho u_{\perp})}{2u_{\perp}^2} \left\{ \frac{A \left(1 - \frac{\alpha u_{\perp}^2}{2u_{\parallel}^2}\right)}{1 + u_{\parallel}^3} \right\}$$

$$\begin{aligned}
 & + \frac{u_{\perp}^2}{u^2} - \frac{3u^3}{\alpha(1+u^3)} \} + \frac{1}{2} F_b[u_{\perp \min}, u_{\parallel}] J_0^2[\rho u_{\perp \min}] \Theta(u_{\perp \min} - 0^+) \\
 & - \frac{1}{2} F_b[(u_b^2 - u_{\parallel}^2)^{1/2}, u_{\parallel}] J_0^2[\rho (u_b^2 - u_{\parallel}^2)^{1/2}] \quad (III-17)
 \end{aligned}$$

$$\begin{aligned}
 \frac{\epsilon_b^{\ell}}{4\pi^2 u_b^3} = & \int_{u_{\perp \min}}^{(u_b^2 - u_{\parallel}^2)^{1/2}} du_{\perp} u_{\perp} \frac{\alpha F_b J_{\ell}^2(\rho u_{\perp})}{2u^2} \left\{ \frac{A \left(1 - \frac{\alpha u_{\perp}^2}{2u^2} \right)}{1 + u^3} \frac{u_{\parallel}}{u} \right. \\
 & \left. - \frac{3u^3}{\alpha(1+u^3)} \right\} + \frac{1}{2} F_b[u_{\perp \min}, u_{\parallel}] J_{\ell}^2[\rho u_{\perp \min}] \\
 & - \frac{1}{2} F_b[(u_b^2 - u_{\parallel}^2)^{1/2}, u_{\parallel}] J_{\ell}^2[\rho (u_b^2 - u_{\parallel}^2)^{1/2}] \quad (III-18)
 \end{aligned}$$

where $u_{\perp \min}^2 = u_o^2(\tau) - u_{\parallel}^2$ if $u_o(\tau) > u_{\parallel}$ and $u_{\perp \min} = 0$ if $u_o(\tau) < u_{\parallel}$. In Appendix A we derive the expression for ϵ_b^{ℓ} using the exact Legendre expansion for F_b .

From the expressions in Eqs. (III-17) and (III-18) we see that the propagation front of the beam at $u = u_o(\tau)$ introduces a positive destabilizing contribution. The stationary front at $u = u_b$ is always stabilizing. For $\tau > \tau_f$, $u_o(\tau) = 0$ and a steady state distribution is achieved. For a given $u_{\parallel}$, ϵ_b^{ℓ} achieves its steady state value after a time

$$\tau > \frac{1}{3} \ln \left(\frac{1 + u_b^3}{1 + u_{\parallel}^3} \right) .$$

The $\ell = 0$ contribution is proportional to a weighted moment of $\partial F_b / \partial u_{\parallel}^2$. We see from the terms in the bracket of Eq. (III-17) that $\partial F_b / \partial u_{\parallel}^2 > 0$ in several regions of phase space, e.g. for large u if $au_{\perp}^2 / u^2 > 3$. However, the overall sign of ϵ_b^0 depends on the details of the weighting. For $\ell \geq 1$, ϵ_b^{ℓ} is proportional to a weighted moment of $\partial F_b / \partial u_{\perp}^2$. In the steady state we find $\partial F_b / \partial u_{\perp}^2 > 0$ when $A/(1+u^3) > 1$ if $au_{\perp}^2 / 2u^2 \ll 1$. Again, a weighted moment determines the overall sign. In Figs. (1) through (5) we show several plots of ϵ_b^{ℓ} for $\ell = 0, 1$ at several values of $u_{\parallel}$ from which we see that the maximum destabilization occurs transiently. In the cases shown the maximum destabilizing values for both the $\partial F_b / \partial u_{\parallel}^2 > 0$ and $\partial F_b / \partial u_{\perp}^2 > 0$ contributions occur for $t/\tau_s \sim 0.3 - 0.5$ where the beam is at approximately 50 to 70% of its final steady state density n_b . In Fig. (1) we compare the matrix element for $\ell=1$ evaluated with the asymptotic distribution in Eq. (III-9B) which yields Eq. (III-18) and the more precise evaluation with the Legendre polynomial expansion of the beam distribution given in Appendix A. We note that during the transient response with $u_{\parallel} \leq u_0(\tau)$

the agreement is good with a relative error of order 10% whereas in the collisional equilibrium regions $u_{\parallel} > u_o(\tau)$ there is only a broad qualitative agreement.

An insight into the analytic structure of ϵ_b^{ℓ} can be obtained by examining the following model crescent shaped distribution of a transient response for $\tau \ll \tau_f$

$$F = 2 S_o \tau \frac{u_b \Theta(u^2 - (u_b - \Delta u)^2) \Theta(u_b^2 - u^2) \Theta(u_{\parallel} - u_1)}{(u_b - u_1) \Delta u}$$

where we assume $\Delta u_1 (u_b - u_1) \ll u \sim u_b$. This distribution models the transient Coulomb distribution given in Eqs. (III-9) and (III-1) for $\tau \ll \tau_f$ with the association $\alpha \sim u_b / (u_b - u_1)$ and $\Delta u = u_b (1 + u_b^{-3}) \tau$. We then find that ϵ_b^{ℓ} is given by

$$\frac{\epsilon_b^{\ell}}{4\pi^2 \tilde{u}_b^3} = \frac{\tau [J_{\ell}^2(Y_1) \Theta((u_b - \Delta u)^2 - u_{\parallel}^2) - J_{\ell}^2(Y_2) \Theta(u_b^2 - u_{\parallel}^2)]}{4\pi \tau_f \Delta u (u_b - u_1) u_b}$$

where

$$Y_1 = (k_{\perp} v_I / \omega_{cb}) [(u_b - \Delta u)^2 - u_{\parallel}^2]^{1/2}$$

$$Y_2 = (k_{\perp} v_I / \omega_{cb}) (u_b^2 - u_{\parallel}^2)^{1/2}$$

and $4\pi S_o u_b^2 \tau_f = 1.$

As $\Delta u \rightarrow 0$ we obtain

$$\lim_{\Delta u \rightarrow 0} \frac{\epsilon_b^\ell}{4\pi^2 \tilde{u}_b^3} = - \frac{\tau}{4\pi \tau_f} \frac{Y[J_\ell^2(y)]' \Theta(u_b^2 - u_\parallel^2)}{(u_b^2 - u_\parallel^2)(u_b - u_\parallel)}$$

where $y = (k_\perp v_I / \omega_{cb}) (u_b^2 - u_\parallel^2)^{1/2}$.

A rough estimate of ϵ_b^ℓ can be obtained by using the $\Delta u \rightarrow 0$ limit if $\lambda \equiv u_b \Delta u \ell^{2/3} / (u_b^2 - u_\parallel^2) < 1$ or by neglecting $J_\ell^2(y_2)$ in ϵ_b^ℓ if $\lambda > 1$. If we use $\text{Max}_\ell J_\ell^2(x) \approx 0.4/\ell^{2/3}$ and $\text{Max}_x \{x[J_\ell^2(x)]'\} \sim 0.7$ we have roughly by interpolation

$$\text{Max}_{k_\perp} \frac{\epsilon_b^\ell}{4\pi^2 \tilde{u}_b^3} \approx \frac{0.7 \tau}{4\pi \tau_f (u_b - u_\parallel)} \left[1 + \frac{1.75 u_b \Delta u \ell^{2/3}}{(u_b^2 - u_\parallel^2)} \right]^{-1}$$

We now consider the formation of the beam distribution on magnetic surfaces where the beam direction is at an angle $\theta_o \approx v_{b\perp}/v_b = (1 - u_o^2)^{1/2}$ to the magnetic field and note that for tangential injection the angle θ_o is of order the local inverse aspect ratio $\epsilon = r/R$.

This analysis also applies for perpendicular injection where $\theta_o = \pi/2$. We assume that the intrinsic

angular width of the source α_0^{-1} is less than θ_0^2 . We can then simplify Eq. (III-1) to (for $C = 0$)

$$\frac{\partial F_b}{\partial \tau} - \frac{1}{u^2} \frac{\partial}{\partial u} [(1+u^3) F_b] - \frac{A(1-\mu_0^2)}{2u^3} \frac{\partial^2 F_b}{\partial \delta \mu^2} = S \quad (\text{III-19})$$

Considering the source to be of the form

$$S = S_0 \Theta(\tau) \delta(u - u_b) \pi^{-1/2} \alpha_0 \exp[-\alpha_0^2 (\delta\mu - \delta\mu_0)^2] \quad (\text{III-20})$$

where $\delta\mu_0 = 1 - \mu_0$,

we seek solutions of Eq. (III-19) of the form

$\bar{F}_b(u) \alpha(u) \exp[-\alpha^2 (\delta\mu - \delta\mu_0)^2]$. We obtain for the source

problem

$$F_b(u, \mu, \tau) = \frac{u_b^2 S_0 \alpha(u) \exp[-\alpha^2 (\delta\mu - \delta\mu_0)^2] [\Theta(u - u(u_b, \tau)) - \Theta(u - u_b)]}{\pi^{1/2} (1 + u^3)}$$

(III-21)

where $u = u(u_b, \tau)$ is given in Eq. (III-4) and where α is now

given by

$$\alpha^{-2} = \alpha_0^{-2} + \frac{2A(1-\mu_0^2)}{3} \ln \left(\frac{1 + u^{-3}}{1 + u_b^{-3}} \right) \quad (\text{III-22})$$

$$\approx \alpha_0^{-2} + \frac{2A(1-\mu_0^2)(u_b - u)}{u_b(1 + u_b^3)} \quad \text{for } u \lesssim u_b$$

or, since $u_b - u \approx u_b (1 + u_b^{-3}) \tau$,

$$\alpha^{-2} \approx \alpha_o^{-2} + \frac{2A(1 - \mu_o^2) \tau}{u_b^3} \quad (\text{III-23})$$

From Eq. (III-23) we see then the distribution fills to

$\mu = 1$ region in the time τ_c

$$\tau_c = \frac{\delta \mu_o u_b^3}{4A} \quad (\text{III-24})$$

and the approximate solution (III-21) fails about this time.

For $\tau \gg \tau_c$ the solutions are qualitatively similar to those given above, e.g. (III-9), which are peaked about $\mu = 1$.

To obtain the distribution given by Eq. (19) it is necessary for the entire beam to be deposited in a time

$\tau_o < \tau_c$. The angular spread $\langle \delta \theta \rangle$ of the distribution is

then found from Eq. (III-23) to be approximately $\langle \delta \theta \rangle / \theta_o \approx$

$$(\tau_o / \tau_c)^{1/2}.$$

The off-angle distribution is characterized by $\partial F_b / \partial u_{\perp}^2 > 0$ for sufficiently small $u_{\perp}$. To aid in the stability analysis, we model the form of F_b , given by Eq. (III-21) to a simpler form,

$$F_b = \frac{2S_0 u_b^4 \theta[u_{\parallel} - u(u_b, \tau)] \theta(u_b - u_{\parallel}) \theta(u_{b\perp}^2 + \Delta u_{\perp}^2 - u_{\perp}^2) \theta(u_{\perp}^2 - u_{b\perp}^2 - \Delta u_{\perp}^2)}{\Delta u_{\perp}^2 (1 + u_b^3)} \quad (\text{III-25})$$

where $\Delta u_{\perp}^2$ is chosen to be comparable with the perpendicular velocity spread in Eq. (III-21), viz.,

$$\Delta u_{\perp}^2 = u_b^2 \left[\alpha_0^{-2} + 2A u_{b\perp} (u_b - u_{\parallel}) / u_b^3 (1 + u_b^3) \right] \quad (\text{III-26})$$

The sharp edges in $u_{\parallel}$ is quite reasonable as $u_{\parallel} = u - \Theta(u_{\perp}^2/u^2)$. The distribution in $u_{\perp}$ is really smooth and the sharp edges overestimate the destabilization somewhat. As a measure of the destabilizing tendency in $\partial F_b / \partial u_{\perp}^2 > 0$, we again compute ϵ_b^{ℓ} for $\ell \geq 1$ and with the normalization used in Eq. (III-16) we find

$$\frac{\epsilon_b^{\ell}}{4\pi^2 \tilde{u}_b^3} = \frac{u_b^2 \tilde{\epsilon}_b^{\ell} \theta(u_{\parallel} - u_b - u(u_b, \tau)) \theta(u_b - u_{\parallel})}{4\pi \tau_f u_{b\perp}^2 (1 + u_b^3)} \quad (\text{III-27})$$

with

$$\tilde{\epsilon}_b^\ell = \frac{u_{b\perp}^2}{\Delta u_\perp^2} \left[J_\ell^2(x^-) - J_\ell^2(x^+) \right] \quad (\text{III-28})$$

where $x^\pm = (k_\perp v_I / \omega_{cb}) (u_{b\perp}^2 \pm \Delta u_\perp^2)^{1/2}$. When $\Delta u_\perp^2 \rightarrow 0$, $\tilde{\epsilon}_b^\ell$ reduces to

$$\lim_{u_\perp^2 \rightarrow 0} \tilde{\epsilon}_b^\ell = -x \frac{d}{dx} J_\ell^2(x) \quad (\text{III-29})$$

For $x^- > \ell$, we observe that $J_\ell^2(x^-)$ tends to be larger than $J_\ell^2(x^+)$, and hence ϵ_b^ℓ tends to be positive and therefore destabilizing. There are bands where $\epsilon_b^\ell < 0$ due to the oscillatory structure of J_ℓ^2 . This oscillating structure is a consequence of the narrowness of F_b and the sharp boundaries we have taken. If there is a spread in the boundaries by an amount $x^- \delta v_\perp / v_{I0} > 1$ ($x^- > \ell$), we expect the oscillations to be negligible. If we neglect the oscillatory terms in Eq. (III-28), then ϵ_b^ℓ is positive for roughly $x^- \gg \ell$.

We note that ϵ_b^ℓ peaks at $x \sim \ell$. If $\ell(\Delta u_\perp/u_{b\perp})^3 < 1$, the maximum of ϵ_b^ℓ can be estimated from the maximum of Eq. (III-29) which we find to be ~ 0.7 . If $\ell(\Delta u_\perp/u_{b\perp})^3 > 1$, we can neglect $J_1^2(x^+)$ near the maximum of $J_1^2(x^-)$. Roughly, we find $\max_\ell J_\ell^2(x) \approx 0.4/\ell^{2/3}$. Hence, we use as an interpolation formula

$$\max \tilde{\epsilon}_b^\ell \approx 0.7 \left[1 + 1.75 \ell^{2/3} \Delta u_\perp^2 / u_{b\perp}^2 \right]^{-1} \quad (\text{III-30})$$

This evaluation of ϵ_b^ℓ will be used in Section V in the analysis of the stability of off angle distribution functions.

Finally, we note in passing that there is a third way $\partial F_b / \partial u_\perp^2 > 0$ can evolve. If a distribution is formed extremely rapidly, and then allowed to relax in time, the energy is well defined since the entire distribution feels the same drag. However, the angular scattering will cause the angular distribution to evolve into a crescent shape. For example, in the extreme limit where $\tau = 0$, $F_b = \delta(u - u_b) \alpha_0 \exp(-\alpha_0 \delta\mu)$, then at time τ we have, $F_b = \delta[u - u(\tau)] \alpha(\tau) \exp(-\alpha(\tau) \delta\mu)$ where $u(\tau)$ and $\alpha(\tau)$ are given in Eqs. (III-4) and (III-10).

IV. ENERGY RESERVOIR FROM PARALLEL VELOCITY GRADIENT

A. Ion Acoustic Waves

First of all let us investigate the ion acoustic waves at both frequencies above and below the ion cyclotron frequency and show that in either case the waves are dominated by ion Landau damping for regimes of practical interest. For sound waves we have $k_{\perp}^2 c^2 \gg \omega^2 \omega_{pe}^2 / k_{\parallel}^2 v_e^2$ and hence from Eq. (II-14) we have that the waves are electrostatic and have a frequency $\omega_{\underline{k}} = k_{\parallel} c_s$ below the ion cyclotron frequency and $\omega_{\underline{k}} \cong k c_s$ when above but not resonant with the ion cyclotron frequency or its harmonics.

Let us first consider the high frequency waves where the effects of the curvature drift and density gradient in the energetic component are negligible. The high frequency waves resonant with the beam particles when

$$\omega_{\underline{k}} = k c_s = k_{\parallel} v_{\parallel} + \ell \omega_{cb} \quad (\text{IV-1})$$

which implies $k_{\parallel}/k \ll 1$ when $v_b \gg c_s$. A destabilizing contribution from the beam arises when

$$\left(1 - \frac{\ell \omega_{cb}}{\omega}\right) \frac{\partial F_b}{\partial v_{\parallel}^2} + \frac{\ell \omega_{cb}}{\omega} \frac{\partial F_b}{\partial v_{\perp}^2} > 0$$

with $v_{\parallel} = (kc_s - l\omega_{cb})/k_{\parallel}$. Frequently, such as during the transient phase of injection and in the steady state when the beam is injected parallel to the magnetic field, one has $\partial F_b / \partial v_{\parallel}^2 > 0$ and $\partial F_b / \partial v_{\perp}^2 < 0$. Thus, the $l = 0$ beam resonance is destabilizing and the higher harmonics also have a destabilizing resonance as long as $l\omega_{cb} < k_{\parallel} v_{\parallel} \sim \omega_k = kc_s$. Comparing the terms in the above equation we have that for $l \leq L = kc_s \langle v_{\perp}^2 \rangle / 2\omega_{cb} v_b \Delta v_{b\parallel}$ the destabilizing harmonic terms are nearly the same as the $l = 0$ term. For $l > L$ the distributions with negative $\partial F_b / \partial v_{\perp}^2$ have stable resonances.

Making a pessimistic assumption or, in other words, placing an upper limit on the maximum destabilizing contribution by neglecting the terms with $l > L$, we have for the beam destabilization

$$L_b \approx \sum_{l=-L}^L 2\pi \omega_{pb}^2 \omega^2 \int dv J_l^2 \frac{\partial F_b}{\partial v_{\parallel}^2} \delta(\omega - l\omega_{cb} - k_{\parallel} v_{\parallel}) |\phi|^2 \quad (IV-2)$$

which for $k_{\perp} \rho_b > L$ with $J_l^2(x) \approx 1/\pi x$ and for $\langle v_{\perp}^2 \rangle \leq v_b \Delta v_{b\parallel}$ then L_b reduces to

$$L_b \approx \frac{2\omega_{pb}^2 L c_s \omega_{cb}}{v_{b\perp} \Delta v_{b\parallel} 2} |\phi|^2 \quad (IV-3)$$

where $v_{b\perp}$ and $\Delta v_{b\parallel}$ are average widths of the perpendicular and parallel velocity distributions. The evaluation of the electron contribution is straightforward and gives

$$L_e = \frac{-2\pi^{1/2} \omega_{pe}^2 k^2 c_s^2}{|k_{\parallel}| v_e^3} |\phi|^2 \quad (IV-4)$$

The background ion contribution involves many large ℓ terms since $\omega = kc_s \gg \omega_{ci}$. When these terms are summed the Landau damping response is essentially that of the zero magnetic field case which is given by

$$L_i = 2\pi \omega_{pi}^2 \omega^2 \int d\underline{v} \frac{\partial F_i}{\partial v^2} \delta(\omega - \underline{k} \cdot \underline{v}) |\phi|^2 \quad (IV-5)$$

$$= \frac{-2\pi^{1/2} \omega_{pi}^2 |k| c_s^2}{v_i^3} \exp\left(-\frac{T_e}{2T_i}\right) |\phi|^2$$

Comparing the electron and ion damping we have

$$\frac{L_i}{L_e} \cong \frac{c_s}{v_b} \left(\frac{m_i}{m_e}\right)^{1/2} \left(\frac{T_e}{T_i}\right)^{3/2} \exp\left(-\frac{T_e}{2T_i}\right) \quad (IV-6)$$

using $k_{\parallel}/k \cong c_s/v_b$. For $c_s \sim v_b/4$ we have $L_i/L_e > 1$ unless $T_e/T_i > 13$. Hence, ion Landau damping is the dominant stabilizing effect for reasonable plasma parameters.

Now comparing the energetic ion component's destabilizing contribution L_b with the background stabilizing contribution we have

$$\frac{L_b}{L_i} \cong \eta \frac{c_s^3 v_{b\perp}}{v_b \Delta v_{b\parallel}^3} \left(\frac{T_i}{T_e} \right)^{3/2} \exp \left(\frac{T_e}{2T_i} \right) \quad (\text{IV-7})$$

where $\eta = n_b/n_o$. Now the factor $(T_i/T_e)^{3/2} \exp(T_e/2T_i) \leq 1$ for $T_e/T_i \leq 5$ while $v_{b\perp} \sim \Delta v_{b\parallel}$ and $v_i/v_b \sim 0.2$, and hence $L_b/L_i < 1$ even for η of order unity. The system is stable due to ion Landau damping to the high frequency acoustic mode.

For the low frequency acoustic mode given in Eq. (II-20) the ion damping is just as strong as for the high frequency mode, and in addition the wave does not resonate with beam particles since $\omega/k_{\parallel} = c_s \ll v_b$.

B. Low Frequency Compressional Alfven Wave

Now let us examine the low frequency compressional Alfven wave $\omega = kv_A = k_{\parallel} v_{\parallel} \ll \omega_{ci}$. For $k_{\perp}/k_{\parallel} > (\omega/\omega_{ci})^{1/2}$ the electric field polarization is $\underline{\epsilon} = \epsilon_{\perp} \hat{b} \times \hat{k}$ with $\epsilon_{\parallel}, \epsilon_k \ll \epsilon_{\perp}$. For a beam distribution peaked at $v_{\parallel} \approx v_b$ with a width $\Delta v_{b\parallel}$ the destabilization is given by

$$L_b = 2\pi \omega_{pb}^2 \int d\underline{v} \delta(kv_A - k_{\parallel} v_{\parallel}) v_{\perp}^2 J_0'^2 \frac{\partial F_b}{\partial v_{\parallel}^2} |\epsilon_{\perp}|^2 \quad (IV-8)$$

which for $k_{\perp} \bar{v}_{\perp} / \omega_{cb} < 1$ reduces to

$$L_b \cong \frac{\pi \omega_{pb}^2 k_{\perp}^2 \langle v_{\perp}^4 \rangle}{2 kv_A \omega_{cb}^2 \Delta v_{b\parallel}^2} |\epsilon_{\perp}|^2 \quad (IV-9)$$

where $\langle v_{\perp}^4 \rangle = \int d\underline{v} F_b v_{\perp}^4 / \int d\underline{v} F_b$. For electron Landau damping we have

$$L_e = - \frac{2\pi^{1/2} \omega_{pe}^2}{|k_{\parallel}| v_e^5} \int_0^{\infty} dv_{\perp}^2 \exp(-v_{\perp}^2/v_e^2) \left| \epsilon_{\parallel} \frac{\omega}{k_{\parallel}} - i \epsilon_{\perp} \frac{k_{\perp} v_{\perp}^2}{2\omega_{ce}} \right|^2 \quad (IV-10)$$

Now from the component of Maxwell's equation parallel to the magnetic field we have from Eq. (II-10)

$$\epsilon_{\parallel} = - \frac{k_{\perp} k_{\parallel}^3 c^2 v_e^2}{2\omega_{pe}^2 k^2 v_A^2} \epsilon_{\perp},$$

and hence the $\epsilon_{\perp}$ contribution is larger than the $\epsilon_{\parallel}$ contribution in L_e by $(k^2/k_{\parallel}^2) (\omega_{ci}/\omega)$ compared to unity. Thus, for $k_{\perp}/k_{\parallel} > (\omega/\omega_{ci})^{1/2}$ we have for instability

$$\frac{n_b}{n_e} > \frac{m_e v_e v_b \Delta v_{b\parallel}^2}{m_b v_{b\perp}^4} \quad (\text{IV-11})$$

where $v_{b\perp}^4 = \langle v_{\perp}^4 \rangle$. The beam density threshold for the compressional Alfvén wave is low unless there are very few beam particles with finite $v_{\perp}$. In the case of an essentially isotropic peaked ($\Delta v_b/v_b < 0.3$) energy shell distribution the criterion (IV-11) reduces to

$$\frac{n_b}{n_e} > \frac{m_e v_e}{m_b v_b}$$

as given earlier by Korablev and Rudakov.¹⁰

It is interesting that the steady state Coulomb distribution from parallel injection can be unstable when $v_b > v_A$. In this case we find

$$L_b = \frac{2\pi \omega_{pb}^2 k_{\perp}^2 |\epsilon_{\perp}|^2}{\tau_f \omega_{cb}^2 |k_{\parallel}| v_{\parallel}} \frac{\partial}{\partial v_{\parallel}} \left(\frac{v_{\parallel}^6}{(v_{\parallel}^3 + v_I^3) \alpha^2} \right)$$

which is destabilizing when there is sufficient angular spread in the beam given by $\alpha_o^{-1} > (A/3)(u_{\parallel}^{-3} + u_b^{-3})$. However, at lower velocities L_b is negative. Hence for parallel injection with $v_b > v_A$ there is a small region in $v_{\parallel}$ of instability below the injection energy.

Now assuming $v_b > v_A \gg v_I$ we find that

$$L_b = \frac{6\pi}{\tau_f} \cdot \frac{\omega_{pb}^2 k_{\perp}^2 k v_A}{\omega_{cb}^2 k_{\parallel}^2 \alpha} [\alpha_o^{-1} - (A/3)(u_{\parallel}^{-3} + u_b^{-3})] |\epsilon_{\perp}|^2. \quad (IV-12)$$

Balancing L_b from Eq. (IV-13) with electron Landau damping

(IV-10) gives for instability with $k_{\parallel} \sim k$, $v_A \sim v_b$ and

$$\alpha_o^{-1} > (A/3)(u_{\parallel}^{-3} + u_b^{-3})$$

$$\frac{n_b}{n_e} > \frac{m_e v_e}{m_b v_b} \frac{\alpha_o^2 \tau_f}{6\pi^{1/2}} \quad (IV-13)$$

or in terms of beam to plasma energy density ratio $\Gamma = n_b E_b /$

$(3/2) n_e T_e (1 + T_i/T_e)$ with E_b defined in Eq. (III-12) we have

for $T_e = T_i$ that $\Gamma > \alpha_o^2 / (36 \pi^{1/2}) (v_b/v_e)$; a condition

readily violated if $v_b > v_A$. In the unstable regime the growth rate is quite rapid compared to a typical confinement time as we have

$$\gamma \approx \frac{3\pi \eta_h k_{\perp}^2 k v_A m_b}{\alpha_o^2 \tau_f k_{\parallel}^2 m_i} \quad . \quad (IV-14)$$

In Appendix B we study the quasilinear relaxation of the beam distribution and show that the plateau state $\partial F / \partial v_{\parallel} = 0$ is not energetically accessible from the initial state. Hence, we conclude $v_{\parallel} < v_A$ to have a net damping of the compressional Alfvén wave.

C. Low Frequency Shear Alfven Wave

Now let us consider the beam interaction with the shear Alfven wave where $\omega = k_{\parallel} v_A \gg k_y v_{Db}$ (the case $\omega \sim k_y v_{Db}$ is discussed in Section VI). The $l = 0$ Landau damping is given by

$$L_b = 2\pi \omega_{pb}^2 \int dv_{\parallel} \delta(\omega - k_{\parallel} v_{\parallel}) \left(\frac{\partial F_b}{\partial v_{\parallel}^2} + \frac{k_y}{\omega \omega_{cb}} \frac{\partial F_b}{\partial x} \right) \left| (v_{\parallel} \mathcal{E}_{\parallel} + v_D \mathcal{E}_y) J_0 + i v_{\perp} J_0' \mathcal{E}_{\perp} \right|^2 \quad (\text{IV-15})$$

where $v_D = v_{\parallel}^2 / \omega_{cb} R$ with R the major radius. We need to determine the relative magnitude of the three polarization terms in Eq. (IV-15). The principal polarization of the mode is $\mathcal{E}_{\perp} = \mathcal{E}_k \hat{k}_{\perp}$ while $\mathcal{E}_{\parallel}$ and $\mathcal{E}_{\perp}$ are found to be

$$\mathcal{E}_{\perp} = -i \frac{k_{\parallel}^3 v_A \mathcal{E}_k}{k^2 \omega_{ci}} \quad \text{and} \quad \mathcal{E}_{\parallel} = -\frac{T_e}{T_i} \frac{k_{\perp} k_{\parallel} v_i^2}{\omega_{ci}^2}$$

for $k_{\perp} > k_{\parallel}$. Now taking $v_{\parallel} \sim v_b \sim v_A$, $k_y \sim k_{\perp}$, $T_i = T_e$ and $k_{\perp} \rho_b < 1$ ($\rho_b = v_b / \omega_{cb}$) we find that the relative

magnitude of the three polarization terms are

$$|v_{\parallel} \epsilon_{\parallel}| : |v_D \epsilon_k| : \left| \frac{k_{\perp}^2 v_{\perp}^2}{2 \omega_{cb}} \epsilon_{\perp} \right|$$

$$= (k_{\parallel} R) (k_{\perp} \rho_i) \frac{v_i}{v_{\parallel}} : 1 : \frac{k_{\parallel} R}{2} \frac{k_{\parallel}^2}{k^2} k_{\perp} \rho_b \frac{v_{b\perp}^2}{v_b^2}$$

If $k_{\parallel} R$ is not too large we see that there is a large range of parameters where the $v_D \epsilon_k$ -term dominates. Hence the curvature drift can drive a shear Alfvén instability.

Now we find the instability threshold when

$$\partial F_b / \partial v_{\parallel}^2 \text{ is positive and greater than } (k_{\parallel} / \omega_{cb}) (\partial F_b / \partial x).$$

We then find

$$\frac{L_b}{L_e} \approx \eta \left(\frac{m_i}{m_e} \right)^{1/2} \left(\frac{m_b}{m_i} \right) \frac{v_b}{v_i} \left(\frac{v_b}{\Delta v_b} \right)^2 \frac{1}{(k_{\parallel} R)^2 (k_{\perp} \rho_b)^2}$$

and hence the critical beam density η_c and relative energy density are

$$\eta_c = \left(\frac{m_e}{m_i} \right)^{1/2} \left(\frac{m_i}{m_b} \right) \frac{v_i}{v_b} \left(\frac{\Delta v_b}{v_b} \right)^2 (k_{\perp} \rho_b)^2 (k_{\parallel} R)^2 \quad (\text{IV-16})$$

and using Eq. (III-12)

$$\Gamma_c = \frac{1}{6\tau_f} \left(\frac{m_e}{m_i} \right)^{1/2} \left(\frac{m_i}{m_b} \right) \left(\frac{\Delta v_b}{v_b} \right)^2 (k_{\parallel} R)^2 (k_{\perp} \rho_b)^2. \quad (\text{IV-17})$$

From Eqs. (IV-16) and (IV-17) the threshold values of η and Γ for instability are low, and we have a growth rate above threshold from Eq. (II-5) given by

$$\gamma \approx \eta \left(\frac{m_b}{m_i} \right) \left(\frac{v_b}{\Delta v_b} \right)^2 \frac{1}{k_{\parallel} R} \frac{v_b}{R}$$

which is short compared to the required plasma confinement time.

In TCT this mode can be avoided by having $v_b < v_A$. However, MHD kink modes and other finite medium MHD modes have a similar field polarization as the shear Alfvén wave but can propagate with a smaller parallel phase velocity. The finite medium theory requires further work to determine the critical v_b/v_A .

V. ENERGY RESERVOIR FROM POSITIVE PERPENDICULAR VELOCITY GRADIENT

In the previous section we showed that the energy reservoir from $\partial F_b / \partial v_{\parallel}^2 > 0$ can persist in a stable plasma as it is difficult to find plasma waves that resonate with beam particles for $c_s \ll v_b < v_A$. However, for waves propagating nearly perpendicular to the magnetic field and at frequencies near the cyclotron harmonics beam particles in this velocity range can form a cyclotron resonance at $v_{b\parallel} = (\omega_k - \ell\omega_{cb})/k_{\parallel}$. Furthermore, an appreciable inversion in the perpendicular energy distribution can occur in pulsed systems and in the transient response of a steady source. In Figs. (1) and (2) we see that a steady state distribution does not yield a destabilizing contribution. For definiteness we confine our analysis to the transient response of a steady source where we can have an appreciable destabilizing effect if $u(u_b, \tau) \geq (\omega_k - \ell\omega_{cb})/k_{\parallel}$ with $u(u_b, \tau)$ given in Eq. (III-4).

For the noncompressional oblique plasma waves the beam-wave resonance in terms of the matrix element defined following Eq. (II-7) is

$$L_b \cong \frac{\omega_{pb}^2 \epsilon_b^{\ell}(k_{\perp}, c_b^{\ell})}{|k_{\parallel}| \tilde{v}_b^3} \left(\frac{\ell\omega_{cb}}{k_{\perp}} \right)^2 |\epsilon_k|^2 \quad (V-1)$$

where ω_{pb}^2 is defined with the final steady density established by the beam. The dissipation in the thermal plasma from electron Landau damping for these modes is given by

$$L_e = - \frac{2\pi^{1/2} \omega_{pe}^2 \exp(-\omega^2/k_{\parallel}^2 v_e^2)}{|k_{\parallel}| v_e^3} \left(\frac{\omega}{k_{\parallel}} \right)^2 |\mathcal{E}_{\parallel}|^2, \quad (V-2)$$

and the ion Landau damping is exponentially small in v_b^2/v_i^2 .

From the parallel component of the plasma wave equation (II-10) we obtain for the polarizations that the modes with $\omega/k_{\parallel} v_e < 1$ have

$$|\mathcal{E}_k/\mathcal{E}_{\parallel}|^2 = (k_{\perp}/k_{\parallel})^2 \left| 1 - \omega^2 \omega_{pi}^2/k_{\parallel}^2 k_{\perp}^2 c^2 c_s^2 \right|^2 \quad (V-3)$$

and the modes with $\omega/k_{\parallel} v_e > 1$ have

$$|\mathcal{E}_k/\mathcal{E}_{\parallel}|^2 = \left(k_{\perp}^2/k_{\parallel}^2 \right) \left| 1 + \omega_{pe}^2/k_{\perp}^2 c^2 \right| \quad (V-4)$$

with the conditions for electrostatic polarization clearly

following. From Eqs. (V-1), (V-2), (V-3) and (V-4) we readily compute the critical condition $L_b/L_e = 1$ for the various noncompressional, oblique plasma waves.

A. Ion Cyclotron Waves and Harmonics

For the cyclotron harmonic waves with $\omega \approx l\omega_{ci} < k_{\parallel} v_e$ and $|\omega - l\omega_{ci}| > k_{\parallel} v_i$ we obtain from Eq. (II-13)

$$\frac{\omega_o - l\omega_{ci}}{\omega_{ci}} = \frac{-l I_l(b) e^{-b}}{\frac{P_l T_i/T_e}{1 - P_l} - k_{\perp}^2 \lambda_{Di}^2 (1 + \omega_{pe}^2/\omega_{ce}^2) - (1 - I_l(b) e^{-b})} \quad (V-5)$$

where $P_l = k_{\perp}^2 k_{\parallel}^2 c^2 c_s^2 / l^2 \omega_{ci}^2 \omega_{pi}^2$. The beam resonance condition for these modes is $\omega_o - l'\omega_{cb} = k_{\parallel} v_b$ where l' is the nearest integer to $l\omega_{ci}/\omega_{cb}$, and this resonance condition determines $k_{\parallel}$ as a function of $k_{\perp}$. We note that for a deuterium beam in a tritium plasma $l = (3/2)l'$ and beam resonances with l' even coincide with harmonics of the background plasma. The critical condition for instability that follows from Eqs. (V-1), (V-2) and (V-3) is

$$\frac{L_b}{L_e} = \frac{n_b}{n_e} \frac{m_e v_e^3}{m_b v_b^3} \frac{\epsilon_b^{l'}(k_{\perp}, c_b^{l'})}{2\pi^{1/2}} |1 - 1/P_l|^2 \quad (V-6)$$

where we shall write $m_b \tilde{v}_b^3 / m_e v_e^3 = (3\pi^{1/2} \tilde{Z}/4) \tilde{u}_b^3$ recalling the definitions $\tilde{u}_b = v_b/v_I$ and $\tilde{Z} = (m_b/n_o) \sum_j n_j z_j^2/m_j$ from Section III, and in the following we shall define $\eta = n_b/n_e$.

First we consider modes near the background ion cyclotron frequency. The mode frequency for $\ell = 1$ from Eq. (V-5) is above ω_{ci} for $P_1 > 1$ and below ω_{ci} for $P_1 < 1$. Thus, for $P_1 < 1$ the deuteron beam in a tritium plasma does not produce a resonance. If instead we consider like charge-to-mass ratio beam-plasma systems a resonance occurs for

$$\frac{\omega_o - \omega_{cb,i}}{\omega_{cb,i}} = \frac{\omega_o - \omega_c}{\omega_c} \approx -\frac{1}{2} \frac{\omega_{pi}^2}{k_{\parallel}^2 c^2} \approx \frac{k_{\parallel} v_b}{\omega_c} \quad (V-7)$$

as obtained from Eq. (V-5) with $\ell = 1$ and for $b < P_1 < 1$. Using the resonant $k_{\parallel}$ from Eq. (V-7) in Eq. (V-6) we obtain for the critical beam density η_c

$$\eta_c = 0.31 \frac{\tilde{u}_b^3}{\tilde{Z}^{4/3}} \left(\frac{v_A}{v_b} \right)^{4/3} \left(\frac{m_e}{m_i} \right)^{2/3} \text{Min}_{k_{\perp}} \left[\frac{(k_{\perp} v_I / \omega_c)^4}{\epsilon_b^1(k_{\perp} v_I / \omega_c, \tilde{u}_b)} \right] \quad (V-8)$$

which gives a small η_c instability threshold for typical systems as seen from Fig. (1). In the above and the following calculations the critical beam density η_c taken in the collisional equilibrium ($\tau \geq \tau_f$) implies from Eq. (III-12) a critical relative beam energy density at $\tau \geq \tau_f$ given by

$$\Gamma_c = \frac{2 \eta_c (m_b/m_e)^{1/3} \tilde{u}_b^2}{3(1 + T_i/T_e)} \left(\frac{3\pi^{1/2} \tilde{z}}{4} \right)^{2/3}.$$

We now investigate the waves at higher harmonics of the cyclotron frequency which are probably only important when the frequency shift from the harmonic is not excessively small which requires $b^{1/2} \gg \ell$. For $b^{1/2} \gg \ell$ we use the approximation

$$I_\ell(b) e^{-b} \cong (2\pi b)^{-1/2} \exp(-\ell^2/2b)$$

and obtain for $P_\ell < 1$ the frequency from Eq. (V-5) and the resonance condition given by

$$\omega_o - \ell \omega_{ci} = \omega_o - \ell' \omega_{cb} = \omega_{ci} \left(\frac{\ell^2}{2\pi b} \right)^{1/2} \exp(-\ell^2/2b) = k_{\parallel} v_b.$$

(V-9)

Using the resonant $k_{\parallel}$ to calculate P_ℓ we find that the

resonant waves satisfy $P_\ell < 1$ if

$$P_\ell = \frac{1}{2\pi} \left(\frac{T_e}{T_i} \right) \left(\frac{v_A}{v_b} \right)^2 \exp(-\ell^2/b) < 1 \quad (V-10)$$

which tends to be satisfied for TCT parameters. For consistency with respect with respect to $\omega/k_\parallel v_e < 1$ we have from

$$\frac{k_\parallel v_e}{\omega_o} \approx \frac{v_e}{v_b} \exp \frac{(-\ell^2/2b)}{(2\pi b)^{1/2}} > 1$$

the requirement that $b^{1/2} < b_c^{1/2} = (v_e/v_b) (2\pi)^{-1/2}$. When these conditions ($b < b_c$ and $P_\ell < 1$) are satisfied we have from Eq. (V-6) the critical beam density

$$\eta_c = \frac{3}{8\pi} \tilde{u}_b^3 \left(\frac{T_e}{T_i} \right)^2 \left(\frac{v_A}{v_b} \right)^4 \text{Min}_{k_\perp} \left[\frac{\exp(-2\ell^2/b)}{e_b^{\ell'} (k_\perp v_I / \omega_{cb}, \tilde{u}_b)} \right] \quad (V-11)$$

Note that we have a low density threshold for $\ell^2 > b$; however, the modes at small b/ℓ^2 are highly resonant and hence are expected to be very sensitive to finite scale length effects.

For $b > b_c$ we assume that $\omega/k_\parallel v_e > 1$ and return to Eq. (II-13) to calculate the frequency shift

$$\frac{\omega_o - \ell \omega_{ci}}{\omega_{ci}} = \frac{\ell (T_e/T_i) I_\ell(b) e^{-b}}{1 + k_\perp^2 \lambda_{De}^2 (1 + \omega_{pe}^2/\omega_{ce}^2) + (T_e/T_i) (1 - I_0(b) e^{-b})} \quad (V-12)$$

$$\approx \frac{\ell I_\ell(b) e^{-b}}{1 + T_i/T_e}$$

for $b > 1$ and $k_\perp^2 \lambda_{De}^2 (1 + \omega_{pe}^2/\omega_{ce}^2) < 1$. Since the frequency shift differs from that in Eq. (V-9) by $(1 + T_e/T_i)^{-1}$, the consistency condition for $\omega/k_\parallel v_e > 1$ remains $b > b_c$.

The mode is electrostatic since $b > b_c$ leads to

$k_\perp c/\omega_{pe} > 1$. Including the contribution of ion Landau damping with $\ell \omega_{ci} \approx \ell' \omega_{cb}$ we arrive at the critical beam density

$$\eta_c = \frac{3\pi}{2} \min_{k_\perp} \left[\frac{\tilde{u}_b^3 \left[\exp(-c_e^2) + \left(\frac{m_i}{m_e}\right)^{1/2} \left(\frac{T_e}{T_i}\right)^{3/2} I_{\ell'}(b) \exp(-b - v_b^2/v_i^2) \right]}{\epsilon_b^{\ell'}(k_\perp v_i/\omega_{cb}, \tilde{u}_b)} \right]$$

where $c_e = (v_b/v_e) (1 + T_i/T_e) (2\pi b)^{1/2} \exp(\ell^2/2b)$.

B. Lower Hybrid and Electron Plasma Waves

Now we consider the higher frequency modes at the lower hybrid frequency and the electron plasma oscillations. These waves have $\omega \geq \omega_{pi} (1 + \omega_{pe}^2 / \omega_{ce}^2)^{-1/2}$ and $\omega > k_{\parallel} v_e$, and they require $\omega \sim \ell' \omega_{cb} \sim k_{\perp} v_{b\perp}$ for strong interaction with beam. The wave frequency and resonance conditions are

$$\begin{aligned} \omega_k &= \frac{(\omega_{pi}^2 + \omega_{pe}^2 k_{\parallel}^2 / k^2)^{1/2}}{(1 + \omega_{pe}^2 / \omega_{ce}^2)^{1/2}} \\ &= \ell' \omega_{cb} + k_{\parallel} v_b \end{aligned} \quad (V-14)$$

For significant beam resonance $k_{\perp} v_{b\perp} \sim \ell' \geq (\omega_{pi} / \omega_{ci}) (1 + \omega_{pe}^2 / \omega_{ce}^2)^{-1/2}$ which leads to $k_{\perp} c / \omega_{pe} > 1$ if $(m_i v_{b\perp}^2 / m_e c^2) (1 + \omega_{pe}^2 / \omega_{ce}^2) < 1$. We shall assume that $v_{b\perp}$ is sufficiently small to satisfy this inequality. The thermal dissipation for the modes is

$$\begin{aligned} L_e + L_i &= - \frac{2\pi^{1/2} \omega_{pe}^2 \omega_k^2}{|k_{\parallel}| v_e^3} \left[\exp \left(- \frac{\omega_k^2}{k_{\parallel}^2 v_e^2} \right) \right. \\ &\quad \left. + \left(\frac{m_i}{m_e} \right)^{1/2} \left(\frac{T_e}{T_i} \right)^{3/2} I_{\ell}(b) \exp \left(-b - v_b^2 / v_i^2 \right) \right] |\phi|^2 \end{aligned} \quad (V-15)$$

compared to the resonant beam contribution

$$L_b = \frac{\omega_{pb}^2 \ell'^2 \omega_{cb}^2 \epsilon_b^{\ell'} |\phi|^2}{|k_{\parallel}| \tilde{v}_b^3} \quad (V-16)$$

The critical beam density is

$$\eta_c = \frac{3\pi \tilde{u}_b^3}{2} \text{Min}_{k_{\perp}} \frac{1}{\epsilon_b^{\ell'}} \left[\exp \left(- \frac{\omega_k^2}{k_{\parallel}^2 v_e^2} \right) \right] \quad (V-17)$$

$$+ \left(\frac{m_i}{m_e} \right)^{1/2} \left(\frac{T_e}{T_i} \right)^{3/2} I_{\ell}(b) \exp(-b - v_b^2/v_i^2) \Big]$$

which (1) for $|k_{\parallel}/k_{\perp}| < (m_e/m_i)^{1/2} (v_{b\perp}/v_b) (T_i/T_e)^{1/2}$ is dominated by ion Landau damping and gives

$$\eta_c = \frac{3\pi \tilde{u}_b^3}{2} \left(\frac{T_e}{T_i} \right)^{3/2} \text{Min}_{k_{\perp}} \left[\frac{\exp(-\ell^2/2b)}{(2\pi b)^{1/2} \epsilon_b^{\ell'} (k_{\perp} v_i / \omega_{cb})} \right] \quad (V-18)$$

which is low threshold when $\epsilon_b^{\ell'}$ is positive and (2) for

$|k_{\parallel}/k_{\perp}| > (m_e/m_i)^{1/2} (v_{b\perp}/v_b) (T_i/T_e)^{1/2}$ is dominated by electron Landau damping and gives

$$\eta_c = \frac{3\pi \tilde{u}_b^3}{2} \exp \left(- \omega_{pe}^2 v_{b\perp}^2 / \ell'^2 \omega_{cb}^2 v_e^2 \right) \text{Min}[\epsilon_b^{\ell'} (k_{\perp} v_I / \omega_{cb})]^{-1}_{k_{\perp}} \quad (\text{V-19})$$

In deriving the critical conditions in Eqs. (V-18) and (V-19) we have implicitly assumed that $k_{\parallel} \Delta v_b < \omega_{cb}$ so that only one resonant harmonic in L_b need be considered. If we estimate $k_{\parallel} \Delta v_b$ for $k_{\parallel}/k_{\perp} > (m_e/m_i)^{1/2}$ and $k_{\perp} v_{b\perp}/\omega_{cb} \sim \omega/\omega_{cb}$, we find that

$$\begin{aligned} \frac{k_{\parallel} \Delta v_b}{\omega_{cb}} &\approx \frac{\omega k_{\perp} \Delta v_b}{\omega_{pi} \omega_{cb}} \left(\frac{m_e}{m_i} \right)^{1/2} (1 + \omega_{pe}^2 / \omega_{ce}^2)^{1/2} \\ &\approx \frac{\omega^2}{\omega_{pi} \omega_{cb}} \frac{v_{b\perp}}{v_b} \cdot \left(\frac{m_e}{m_i} \right)^{1/2} (1 + \omega_{pe}^2 / \omega_{ce}^2)^{1/2} \end{aligned}$$

where we have used $\Delta v_b \approx v_{b\perp}^2 / v_b$ which follows from the geometry of a crescent shaped distribution. Hence, single beam harmonics contribute for wave frequencies less than ω_1 that is

$$\omega < \omega_1 = \frac{(\omega_{pi} \omega_{cb})^{1/2}}{(1 + \omega_{pe}^2 / \omega_{ce}^2)^{1/4}} \left(\frac{v_b}{v_{b\perp}} \right)^{1/2} \left(\frac{m_i}{m_e} \right)^{1/4} \quad (\text{V-20})$$

For $\omega > \omega_1$ several cyclotron harmonics resonate with the beam, and it can be shown that the beam wave interaction can be evaluated by the straight-line orbit approximation. This approximation leads to the relation

$$L_b = \frac{2\pi^2 \omega_{pb}^2 \omega^2 |\epsilon_k|^2}{k_\perp^2} \int \left(\frac{\omega}{k_\perp}\right)^2 \frac{dv_\perp^2}{(k_\perp^2 v_\perp^2 - \omega^2)^{1/2}} \int_{-\infty}^{+\infty} \frac{\partial F_b}{\partial v_\perp^2} dv_\parallel, \quad (V-21)$$

and it can be shown that the previously discussed crescent shaped distribution which model the transient distribution in Section III lead to $L_b < 0$. Hence, instability should not arise for $\omega > \omega_1$ during parallel injection.

When $\eta > \eta_c$ as given by either Eqs. (V-18) or (V-19) we have the growth rate given by

$$\frac{\gamma}{\omega_{cb}} = \frac{L_b}{2 \omega_{cb} k_\perp^2 |\phi|^2} = \frac{\ell'^2 \omega_{pb}^2 \omega_{cb} \epsilon_b^{\ell'}}{2 k_\perp^2 |k_\parallel| \tilde{v}_b^3} \quad (V-22)$$

when Eq. (V-22) predicts $\gamma > |k_\parallel| \Delta v_b$ or ω_{cb} then the formula (V-22) is inadequate and the evaluation of the growth rate is considerably more complicated requiring

numerical evaluation in general. If the right hand side of Eq. (V-20) is greater than $k_{\parallel} \Delta v_b / \omega_{cb}$ then the dispersion relation reduces to

$$\frac{(\omega - \omega_k)^2}{\omega_{cb}^2} = - \frac{\ell'^2 \omega_{pb}^2}{2 k_{\perp}^2} \int \frac{\partial F_b}{\partial v_{\perp}^2} J_{\ell}^2 dv_{\perp} \quad (V-23)$$

with $\omega_k = \ell' \omega_{cb}$ which for the model crescent distribution in Eq. (III-) reduces to

$$\begin{aligned} \frac{(\omega - \omega_k)^2}{\omega_{cb}^2} = & \frac{\ell'^2 \omega_{pb}^2}{2 k_{\perp}^2 v_b \Delta v (v_b - v_1)} \int_{v_1}^{v_b} \left[J_{\ell}^2(Y_1) \Theta[(v_b - \Delta v)^2 - v_{\parallel}^2] \right. \\ & \left. - J_{\ell}^2(Y_2) \right] dv_{\parallel} \end{aligned} \quad (V-24)$$

where $Y_1 = k_{\perp} [(v_b - \Delta v)^2 - v_{\parallel}^2]^{1/2} / \omega_{cb}$ and $Y_2 = k_{\perp} [v_b^2 - v_{\parallel}^2]^{1/2} / \omega_{cb}$.

If $J_{\ell}^2(Y)$ is replaced by $\pi^{-1} [Y^2 - \ell'^2]^{-1/2} \Theta(Y^2 - \ell'^2)$, it

can be shown that the right hand side of Eq. (V-24) is posi-

tive which shows stability for the $k_{\parallel}$ -modes where Eq. (V-23)

applies. We note that there can be a remnant instability in

Eq. (V-24). For example such modes have been explicitly shown

by Mai and Horton⁹ and Korablev¹⁰ for a monoenergetic distribution $F_b \propto \delta(v - v_b)$. However, this remnant of instability should disappear once the sharp edges in $v_\perp$ of the distribution function diffuse by an amount $\Delta v_\perp > \omega_{cb}/k_\perp$.

Finally, we note that we have shown $\gamma < \omega_{cb}$ for the parallel injected distributions since we have $\omega < \omega_\perp$ and $\gamma < k_\parallel \Delta v_b < \omega_{cb}$. We now consider briefly the growth rates from Eqs. (V-22) and (V-23) for the early time model off-angle distribution.

For a transient off-angle distribution $k_\parallel = 0$ instability modes can occur. For the model off-angle distribution in Eq. (III-25) we find that for $\gamma < k_\parallel \Delta v_b < \omega_{cb}$ the growth rate from Eq. (V-22) is

$$\frac{\gamma}{\omega_{cb}} = \frac{\pi \omega_{pb}^2 \ell'^2 v_b^2 [J_\ell^2(x^-) - J_\ell^2(x^+)]}{2 \tau_f k_\perp^2 k_\parallel \Delta v_\perp^2 (v_b^3 + v_I^3)} \quad (V-25)$$

where $x^\pm = k_\perp (v_{b\perp}^2 \pm \Delta v_\perp^2)^{1/2} / \omega_{cb}$. This calculated growth becomes large at small $k_\parallel$ and from Eq. (V-23) we obtain for $k_\parallel \Delta v_b < \gamma < \omega_{cb}$ that

$$\frac{\gamma^2}{\omega_{cb}^2} = \frac{\ell'^2 \omega_{pb}^2 (\tau/\tau_f)}{4 k_{\perp}^2 \Delta v_{\perp}^2} \left[J_{\ell}^2(x^-) - J_{\ell}^2(x^+) \right] \quad (V-26)$$

where $\tau < \tau_f$. If ω_{pb} is large enough that $\gamma > \omega_{cb}$ from from Eq. (V-26) then we sum the harmonic contributions by using the straight-line approximation in Eq. (V-21) to obtain

$$\frac{\gamma}{\omega_k} = \frac{\omega_k \omega_{pb}^2 (\tau/\tau_f)}{4 k_{\perp}^3 \Delta v_{\perp}^2} \left[\frac{1}{(v_b^2 - \Delta v_{\perp}^2)^{1/2}} - \frac{1}{(v_b^2 + \Delta v_{\perp}^2)^{1/2}} \right]$$

valid for $k_{\perp} > \omega_k/v_{b\perp} \gg \omega_{cb}/v_{b\perp}$.

In this section we have surveyed the waves that can be destabilized when there arises a positive velocity gradient of the beam distribution in the perpendicular direction. Although there are many waves that can tap this positive gradient, we note that the energy reservoir that excites these waves is quite feeble. Instability does not occur for steady state distributions, and even in the transient case it would appear that a minor readjustment of the distribution will destroy the energy reservoir.

VI. VELOCITY SPACE ANISOTROPY AND DENSITY GRADIENT ENERGY RESERVOIR

A. Anisotropy Instability in the Whistler Wave

A strong instability source can be tapped from the anisotropy of the hot component if a resonance for $\ell = -1$ can be established. In the case of nearly parallel propagation where $\theta \cong 0$ the Landau resonance for the energetic component given by Eq. (II-6) is

$$L_b = 2\pi \omega_{pb}^2 \int d\mathbf{v} \delta(\omega + \omega_{cb} - k_{\parallel} v_{\parallel}) \left(\frac{k_{\parallel} v_{\parallel}}{\omega} \frac{\partial F_b}{\partial v_{\parallel}^2} - \frac{\omega_{cb}}{\omega} \frac{\partial F_b}{\partial v_{\perp}^2} \right) \\ \times \left| \frac{1}{2} v_{\perp} J_2 e^{-i\theta} \epsilon_+ + \frac{1}{2} v_{\perp} J_0 e^{i\theta} \epsilon_- \right|^2$$

(VI-1)

For $k_{\perp} v_{b\perp} / \omega_{cb} < 1$ this reduces to

$$L_b = \frac{\pi \omega_{pb}^2}{2 |k_{\parallel}|} \left[\frac{\omega_{cb}}{\omega} - \frac{(\omega + \omega_{cb})}{\omega} \left\langle \frac{v_{\perp}^2}{v_{\parallel}^2} \right\rangle \right] |\epsilon_-|^2 \int d^2 v_{\perp} F_b$$

(VI-2)

where

$$\left\langle \frac{v_{\perp}^2}{v_{\parallel}^2} \right\rangle = \frac{-1}{\int d^2 v_{\perp} F_b} \int d^2 v_{\perp} v_{\perp}^2 \frac{\partial F_b}{\partial v_{\parallel}^2}$$

If $\langle v_{\perp}^2/v_{\parallel}^2 \rangle$ is sufficiently small then we have a strong destabilizing source.

A mode that can tap this source is the whistler whose dispersion relation is given by Eq. (II-15) with the lower sign, that is

$$k_{\parallel}^2 c^2 + \frac{\omega \omega_{pi}^2}{\omega + \omega_{ci}} - \frac{\omega \omega_{pi}^2}{\omega_{ci}} = 0 \quad (VI-3)$$

Setting $k_{\parallel} v_{\parallel} = \omega + \omega_{cb}$ for the beam resonance, we find that the resonant $v_{\parallel}$ is given by

$$\frac{v_{\parallel}}{v_A} = \frac{1}{2} y + \frac{1}{2} (4 + y^2)^{1/2} + \frac{1}{y} \frac{\omega_{cb}}{\omega_{ci}}$$

where $y = k_{\parallel} v_A / \omega_{ci}$. The minimum value of $v_{\parallel}$ for resonance of a deuterium injected into a tritium plasma is found to be

$$\text{Min } (v_{\parallel}/v_A) = 3.0 \quad (VI-4)$$

which occurs for $k_{\parallel} c / \omega_{pi} = 1.4$ and $\omega / \omega_{ci} = 2.6$. For a system with $\omega_{ci} = \omega_{cb}$ the minimum velocity is $v_{\parallel} / v_A = 2.6$ occurring for $k_{\parallel} c / \omega_{pi} = 1.15$ and $\omega / \omega_c = 2.0$. Hence, instability can arise from the anisotropy if $\langle v_{\perp}^2 / v_{\parallel}^2 \rangle < \omega_{cb} / (\omega + \omega_{cb}) \approx 1/3$ and $v_b > 3 v_A$. This threshold beam velocity is much higher than is contemplated for TCT corresponding to multi-Mev injection energies.

B. Beam Density Gradient Reservoir for the Shear Alfvén Mode

We now consider instabilities that arise when the energy reservoir from the spatial beam gradient term $(k_y / \omega \omega_{cb}) (\partial F_b / \partial x)$ dominates. It is interesting that this source can be tapped by infinite medium modes when their oscillation frequency is comparable or less than the curvature drift frequency $\omega_{Db} = k_y v_{\parallel}^2 / \omega_{cb} R$ of the beam particles.

For example, let us consider the shear Alfvén wave with $\omega = k_{\parallel} v_A$. The resonance condition is

$$\omega = k_{\parallel} v_A = k_{\parallel} v_{\parallel} + \omega_{Db} \ll \omega_{cb} . \quad (\text{VI-5})$$

Even if $v_b < v_A$ the resonance condition can be satisfied for modes with $k_{\parallel} \ll k_y$ with $k_{\parallel} v_A = \omega_{Db}$. We note that

for long wavelengths where $k_{\parallel} = p/qR$ with q the tokamak safety factor and p low integer the resonance condition is $k_y v_{\parallel} / \omega_{cb} = p v_A / q v_{\parallel}$; and, hence, there is a range of p where $k_y v_{\parallel} / \omega_{cb} < 1$ and $J_o^2(k_{\perp} v_{\perp} / \omega_{cb}) \approx 1$. When Eq. (VI-5) is satisfied we have

$$\frac{k_y}{\omega_{cb}} \frac{\partial F_b}{\partial x} \bigg/ 2 \frac{\partial F_b}{\partial v_{\parallel}^2} \approx \frac{R}{R_b} \frac{\Delta v_b}{v_b}$$

Hence, if the beam is sufficiently spread, $\Delta v_b / v_b > \epsilon = R_b / R$, then the spatial gradient is the most important driving source. In this case we find from Eq. (II-7) that

$$L_b = \frac{\pi k_y \omega_{pb}^2 |e_y|^2}{k_{\parallel} v_A \omega_{cb}} \int dv_{\parallel} v_{Db}^2 \frac{\partial F_b}{\partial x} \delta(k_{\parallel} v_A - k_y v_{Db}) \quad (VI-6)$$

We note that the destabilizing contributions can come only from resonating with particles drifting with bad curvature if $\partial F_b / \partial x < 0$.

To estimate Eq. (VI-6), we take $v_{Db} = v_{\parallel}^2 / \omega_{cb} R$ and $\partial F_b / \partial x = F_b / R_b$, and for F_b we use the steady state Coulomb distribution. We obtain

$$L_b = \frac{\pi}{2} \frac{\omega_{pb}^2 I_o(b_t) e^{-b_t} |\epsilon_Y|^2}{\tau_f |k_Y| R_b \omega_{cb}} \frac{v_{\parallel}^3}{v_{\parallel}^3 + v_I^3} \left| v_{\parallel} = \left(\frac{k_{\parallel}}{k_Y} \omega_{cb} R v_A \right)^{1/2} \right.$$

(VI-7)

where $b_t = (k_{\perp} v_{\parallel})^2 / \alpha \omega_{cb}^2$. For the stabilization from electron Landau damping we have Eq. (V-2) with $\omega/k_{\parallel} v_e \rightarrow v_A/v_e < 1$ and Eq. (V-) for $|\epsilon_k/\epsilon_{\parallel}|^2$ to obtain

$$\frac{L_b}{L_e} = \frac{\pi^{1/2} \eta}{4 \tau_f} \frac{(m_e v_e^3 / m_i) \omega_{ci}^3 I_o(b_t) e^{-b_t}}{R_b |k_{\parallel}| |k_Y|^3 v_A^2 c_s^4} \cdot \frac{v_{\parallel}^3}{v_{\parallel}^3 + v_I^3} \left| v_{\parallel} = \left(\frac{k_{\parallel}}{k_Y} \omega_{cb} R v_A \right)^{1/2} \right.$$

(VI-8)

The largest L_b/L_e occurs for $v_{\parallel} \approx v_b$ and the smallest $k_{\parallel}$ which we take to be $k_{\parallel} R = p/q$ with $p \sim 1$. For this mode $b_t = (p v_A)^2 / (2 \alpha q^2 v_A^2) \ll 1$, and we obtain for the critical beam density

$$\eta_c = \left(\frac{2}{\pi}\right)^{1/2} \left(\frac{m_e}{m_i}\right)^{1/2} \left(\frac{m_i}{m_b}\right)^3 \left(\frac{p}{q}\right)^4 \frac{\tau_f \epsilon c_s v_A^5}{v_b^6}$$

(VI-9)

and the critical Γ

$$\Gamma_c = \frac{1}{6} \left(\frac{2}{\pi}\right)^{1/2} \frac{(m_i/m_b)^2 (m_e/m_i)^{1/2}}{(1 + T_i/T_e)} \left(\frac{p}{q}\right)^4 \frac{\epsilon v_A^5}{c_s v_b^4}$$

(VI-10)

For typical TCT parameters we take $\tilde{Z} = 2/3$, $\epsilon = 0.25$,

$q = 3$, $T_e = T_i$, $m_b/m_i = 2/3$ $(m_i/m_e)^{1/2} = 74$, $\beta = 0.01$,

$E_b/T_i = 20$, $E_b = 100$ KeV, $B = 50$ kg and obtain $\Gamma_c \cong$

$0.07 p^4 / N^{5/2}$ where N is the density in units of 10^{14} cm^{-3} .

Above the threshold for instability we obtain using Eqs.

(VI-8) and (II-) the growth rate

$$\gamma = \frac{\pi}{4\tau_f} \frac{\eta \omega_{ci}}{|k_y R_b|} < \frac{\pi}{4\tau_f} \frac{\eta q v_b^2 (m_b/m_i)}{v_A R_b}$$

where the minimum k_y is determined by the resonance condition and $k_{\parallel} R = 1/q$. Thus, we see that the growth rate can be comparable to the transit rate v_b/qR of the beam around the torus.

Clearly, for an actual tokamak a more refined study of this mode is needed. For example, we need to consider the effects of the spatial $\cos \theta$ variation of the beam drift velocity, the effect of trapped electrons enhancing L_e and the effect of magnetic shear. This study is presented elsewhere¹¹ and indicates that the mode including these effects is considerably more stable than the present slab model calculation indicates.

In a similar way, low frequency sound waves with $\omega = k_{\parallel} c_s \ll \omega_{cb}$ can be shown to resonate with the beam drift frequency ω_{Db} . However, for this mode the background ion Landau damping is an overwhelming stabilizing influence, and we will not present the analysis here.

C. Beam Density Gradient Reservoir for Drift Waves

For electron drift waves, $\omega \approx \omega_{*e}$, we note that the beam drift term is normally a stabilizing influence if $\partial F_b / \partial x \quad \partial n_e / \partial x > 0$. If the geometrical properties of the injection system create an inverted beam density profile, the destabilizing influence of the beam would tend to cause an inward beam diffusion which would actually aid in containment. We will not discuss this case further here.

Electron drift wave turbulence is believed to be responsible for the energy diffusion in tokamaks. An important question to investigate is whether this turbulence can cause energy loss of the beam at a rate comparable or faster than the electron loss. In the TCT regime the drift waves will occur as the trapped electron and trapped ion modes as $\omega_{*e} < \omega_{be}$ where ω_{be} is the electron transit frequency. In Appendix C we show that the beam turbulent lifetime is longer than the background plasma under the most pessimistic conditions. Hence, this factor is not a problem in the operation of TCT.

The density gradient energy reservoir has a destabilizing tendency for ion diamagnetic drift waves ($\omega = \omega_{*i} = k_y v_i^2 / 2\omega_{ci} R_p$) which arise when $\omega / k_{||} v_e \gg 1$. For these waves, the mode frequency is less than the curvature drift frequency, $\omega_{Db} = k_y v_{||}^2 / \omega_{cb} R$, so that the resonance condition is approximately $\ell\omega_{cb} \cong \omega_{Db}$ with $\ell \geq 1$.

For $k_{||} = 0$ the growth rate γ due to the hot component is found to be

$$\gamma = \pi \eta \omega_{*i}^2 \frac{n}{\partial n / \partial x} \int d\mathbf{v} \frac{\partial F_b}{\partial x} J_\ell^2 \left(\frac{k_\perp v_\perp}{\omega_{cb}} \right) \delta(\omega - \ell\omega_{cb} - \omega_{Db})$$

$$\gamma \approx \frac{\eta}{8} \left(\frac{m_i}{m_b} \right)^2 \cdot \frac{R}{R_p R_b} \cdot \frac{v_i^4}{v_b \Delta v_{b\parallel} v_{b\perp}} \quad (\text{VI-13})$$

where $\Delta v_{\parallel}$ is the parallel velocity width of the beam,

$$v_{b\perp}^{-1} = \pi k_{\perp} / \omega_{cb} \int J_{\ell}^2 F_b d^2 v, \quad \ell = 1; \quad \text{and we take}$$

$$d \ln F_b / d \ln n = 1.$$

As $k_{\perp}$ is quite large for these modes, the turbulent diffusion time τ_b can be expected to be much longer than the containment time for the thermal plasma in TCT. For if we assume that the diffusion coefficient is $\gamma/k_{\perp}^2$ then

$$\begin{aligned} \tau_b &\approx \frac{k^2 R_p^2}{\gamma} = \omega_{cb}^4 R^2 R_p^2 / \gamma v_b^4 \\ &= \frac{8}{\eta c} \left(\frac{R_p}{\rho_b} \right)^3 \left(\frac{R_p}{\rho_i} \right)^2 \frac{\Delta v_{\parallel} v_{b\perp}}{v_i^2 \omega_{ci}}. \end{aligned} \quad (\text{VI-14})$$

For the nominal parameters, $T_i = 5$ keV, $E_b = 100$ keV,

$c = 0.25$, $B = 50$ kg, and $R_p = 40$ cm, we obtain

$$\tau_b \approx \frac{0.3}{\eta} \frac{\Delta v_{\parallel} v_{b\perp}}{v_i^2} \text{ sec}$$

which is much longer than we need to confine the plasma.

Additional studies of the ion diamagnetic drift waves have been made by Coppi, Sleeper and Bhadra¹² who investigate a variant of this mode when $\omega < \omega_{be}$ where ω_{be} is the electron bounce frequency.

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APPENDIX A: LEGENDRE POLYNOMIAL EXPANSION OF THE BEAM DISTRIBUTION FUNCTION

In the numerical evaluations of the resonant beam-wave matrix elements ϵ_b^ℓ we have used both the asymptotic beam distributions given in Sec. III and exact solutions obtained from the Legendre polynomial expansion of the distribution function. For the asymptotic beam distributions the matrix elements are computed by performing numerically the integrals in Eqs. (III-17) and (III-18). A more precise evaluation of ϵ_b^ℓ is made by using the expansion

$$F_b(u, \mu, \tau) = \sum_{\ell=0}^{\infty} F_\ell(u, \tau) P_\ell(\mu) \quad (A-1)$$

Substituting Eq. (A-1) into Eq. (III-1) with a steady mono-energetic source given by

$$S = \delta(u - u_b) \sum_{\ell=0}^{\infty} S_\ell P_\ell(\mu)$$

we obtain the solution

$$F_b(u, \mu, \tau) = F_f(u, \mu) \{ \Theta[u - u(u_b, \tau)] - \Theta(u - u_b) \}$$

(A-2)

with $u(u_b, \tau)$ given by Eq. (III-4) and with F_f given by

$$F_f = \sum_{\ell=0}^{\infty} \frac{u_b^2 S_{\ell} P_{\ell}(\mu)}{1 + u^3} \left(\frac{1 + 1/u_b^3}{1 + 1/u^3} \right)^{A\ell(\ell+1)/6} \quad (A-3)$$

In our applications of Eq. (A-3) we have found it adequate to consider the highly peaked source with $S = S_0 \delta(u - u_b) \delta(\mu - \mu_0)$ for which $S_{\ell} = S_0 (\ell + 1/2) P_{\ell}(\mu_0)$.

Recognizing that the final steady state beam density n_b is given by $n_b = 4\pi u_b^2 S_0 \tau_f$ where $\tau_f = 1/3 \ln(1 + u_b^3)$, we then write that

$$F_f(u, \mu) = \frac{n_b}{2\pi \tau_f (1 + u^3)} \sum_{\ell=0}^{\infty} (\ell + 1/2) P_{\ell}(\mu) P_{\ell}(\mu_0) \left(\frac{1 + 1/u_b^3}{1 + 1/u^3} \right)^{A\ell(\ell+1)/6} \quad (A-4)$$

We compute the matrix element separately for the $\ell = 0$ and $\omega/\omega_c \approx \ell \neq 0$ modes and neglect the $(1 - \ell\omega_c/\omega) \partial F_b / \partial v_{\parallel}^2$ - term in the latter case. For the $\ell \neq 0$ matrix element we integrate by parts to obtain

$$\epsilon_b^l(\rho_I u_{\parallel}) = \frac{\pi^2 \tilde{u}_b^3 \rho_I^2}{2l} \int_{u_{\perp \min}^2}^{u_b^2 - u_{\parallel}^2} du_{\perp}^2 F_f(u_{\perp}, u_{\parallel}) \left[J_{l+1}^2(\rho_I u_{\perp}) - J_{l-1}^2(\rho_I u_{\perp}) \right] \quad (A-5)$$

where $\rho_I = k_{\perp} v_I / \omega_{cb}$, $\tilde{u}_b = \tilde{v}_b / v_I$, $u_{\parallel} = (\omega_k - l\omega_{cb}) / k_{\parallel} v_I$ and $u_{\perp \min}$ is given at Eq. (III-18). For the $l = 0$ matrix element we have

$$\epsilon_b^0(\rho_I, u_{\parallel}) = 2\pi^2 \tilde{u}_b^3 \frac{\partial}{\partial u_{\parallel}^2} \int_{u_{\perp \min}^2}^{u_b^2 - u_{\parallel}^2} du_{\perp}^2 F_f(u_{\perp}, u_{\parallel}) J_0^2(\rho_I u_{\perp}). \quad (A-6)$$

The calculations of ϵ_b^l from Eqs. (A-4), (A-5) and (A-6) are compared to the calculations made from Eqs. (III-17) and (III-18) using the asymptotic beam distribution. The results of the calculations are shown in Figs. (1-2).

APPENDIX B: QUASILINEAR THEORY FOR COMPRESSIONAL ALFVEN MODE

In Sec. III we showed that the low frequency compressional Alfvén mode has a low beam density threshold for beams with $v_b > v_A$ and finite $\langle v_\perp^4 \rangle$. In fact we find that for $v_b > v_A$ the collisional steady state distribution is unstable if Eq. (IV-7) is greater than unity. Here we consider the quasilinear evaluation from the steady state collisional distribution and show that the plateau solution with nonzero $|\epsilon_{\underline{k}}|^2$ is not consistent with energy conservation.

The quasilinear equation for F_b is

$$\frac{\partial F_b}{\partial t} = \frac{\pi e^2 v_\perp^2}{m_b^2} \int \frac{d\underline{k}}{(2\pi)^3} |\epsilon_{\underline{k}}|^2 J_0'^2 \left(\frac{k_\perp v_\perp}{\omega_{cb}} \right) \frac{\partial}{\partial v_\parallel} \left(\frac{\delta(kv_A - k_\parallel v_\parallel)}{v_\parallel^2} \frac{\partial F_b}{\partial v_\parallel} \right) \quad (B-1)$$

where we have assumed that the turbulence, is cylindrically symmetric about the magnetic field. We can construct an H-theorem from this expression by multiplying by F_b and integrating over $v_\parallel$ to obtain

$$\frac{1}{2} \frac{d}{dt} \int_{-\infty}^{+\infty} F_b^2 dv_\parallel = \frac{-\pi e^2 v_\perp^2}{m_b^2} \int \frac{d\underline{k}}{(2\pi)^3} |\epsilon_{\underline{k}}|^2 J_0'^2 \left(\frac{k_\perp v_\perp}{\omega_{cb}} \right) \int_{-\infty}^{+\infty} \frac{dv_\parallel}{\omega_{cb}} \delta(kv_A - k_\parallel v_\parallel) \left(\frac{1}{v_\parallel} \frac{\partial F_b}{\partial v_\parallel} \right)^2. \quad (B-2)$$

From this expression one concludes that a stationary state can

be achieved only if $\partial F_b / \partial v_{\parallel} = 0$ for $v_{\parallel} > v_A$ (plateau solution) or $|\epsilon_{\perp k}|^2 = 0$. The latter condition implies that the beam distribution evolves to a stable state where the waves subsequently damp away. In fact, the steady state Coulomb distribution with $x = v_b/v_A > 1$ must evolve to such a stable state as we now show. If we suppose to the contrary that a plateau solution is formed then the final distribution is uniquely specified in the region $v_{\parallel} > v_A$ by particle conservation

$$\int_{v_A}^{v_b} dv_{\parallel} F_b^f(v_{\parallel}, v_{\perp}) = \int_{v_A}^{v_b} dv_{\parallel} F_b^i(v_{\parallel}, v_{\perp}) \quad (B-3)$$

where the lower limit is the smallest velocity that can resonate with the waves. The upper limit follows since the wave spectrum is initially arbitrarily small beyond the beam velocity and any possible distortion of the distribution to larger velocities leads to negative $v_{\parallel}$ slope distributions beyond v_b which continue to damp the waves. Hence, for $\omega/k_{\parallel} > v_b$ $|\epsilon_{\perp k}|^2 \approx 0$.

For the initial state we use

$$F_b^i = \frac{\eta \alpha_o \exp(-\alpha_o v_{\perp}^2 / 2v_{\parallel}^2)}{2\pi \tau_f v_{\parallel}^3} \quad (B-4)$$

for $v_{\parallel} \geq v_A \gg v_I$. For a plateau solution which is independent of $v_{\parallel}$ for $v_{\parallel} > v_A$ we must have for the final state from Eq. (B-3) that

$$2\pi F_b^f = \frac{\eta \alpha_o}{\tau_f (v_b - v_A)} \int_{v_A}^{v_b} \frac{dv_{\parallel}}{v_{\parallel}^3} \exp(-\alpha_o v_{\perp}^2 / 2v_{\parallel}^2)$$

$$= \frac{\eta [\exp(-\alpha_o v_{\perp}^2 / 2v_b^2) - \exp(-\alpha_o v_{\perp}^2 / 2v_A^2)]}{\tau_f v_{\perp}^2 (v_b - v_A)} \quad (B-5)$$

We first observe that the waves do not scatter the perpendicular particle velocities, and hence it is only the distribution of parallel kinetic energy that is changed by the wave spectrum. The initial parallel kinetic energy is

$$T_{b\parallel}^i = 2\pi \int_0^{\infty} dv_{\perp} v_{\perp} \int_{v_A}^{v_b} dv_{\parallel} \frac{1}{2} m_b v_{\parallel}^2 F_b^i$$

$$= \frac{m_b \eta (v_b^2 - v_A^2)}{4\tau_f} \quad (B-6)$$

The parallel kinetic energy of the plateau state is

$$T_{b\parallel}^f = \frac{m_b \eta \alpha_o}{\tau_f (v_b - v_A)} \int_0^\infty \frac{dv_\perp}{2v_\perp} \left[\exp \left(- \frac{\alpha_o v_\perp^2}{2 v_b^2} \right) - \exp \left(- \frac{\alpha_o v_\perp^2}{2 v_A^2} \right) \right] \int_{v_A}^{v_b} dv_\parallel v_\parallel^2$$

$$= \frac{m_b}{12 \tau_f} \eta (v_b^3 + v_b v_A + v_A^3) \ln (v_b^3 / v_A^3)$$

Now, from Eqs. (B-6) and (B-7) we have for the plateau

$$\frac{T_{b\parallel}^f}{T_{b\parallel}^i} = \frac{2(x^2 + x + 1)}{3(x^2 - 1)} \ln x \approx \begin{cases} \frac{2}{3} \ln x, & x \gg 1 \\ 1 + (x-1)^2, & 0 < x - \ll 1 \end{cases}$$

where $x = v_b / v_A$. It is simple to show from Eq. (B-8) that $T_{b\parallel}^f / T_{b\parallel}^i > 1$ for all $x > 1$. Hence, the plateau solution is not energetically accessible and the solution must evolve to a damped state where $|\mathcal{E}_{\underline{k}}|^2 = 0$.

APPENDIX C: HOT COMPONENT ENERGY LOSS RATE DUE TO TRAPPED PARTICLES MODES

For optimum fusion energy output in TCT it is necessary that the energetic component have an energy containment time τ_b comparable to that of the background plasma.³ The high energy component lifetime is governed by the Coulomb slowing down time while it is expected that the background plasma lifetime will be governed by turbulence generated by trapped particle instabilities.¹³ This turbulence also interacts with the hot component and in this section we show that the turbulent lifetime that results is a great deal longer than that of the turbulent lifetime of the background plasma. Hence, trapped particle instabilities are not expected to affect the hot component lifetime in TCT.

We first calculate the thermal flux of electrons generated by a trapped particle instability. The perturbed electron distribution f_e is given by

$$f_e = \frac{2n_o}{m_e v_e^2} \frac{(\omega - \omega_{*e}^T)}{\omega + i \nu_{eff}} \phi_k F_m \Theta(2\epsilon v_\perp^2 - v_\parallel^2) \quad (C-1)$$

where

$$\Theta(x) = \begin{cases} 1 & x > 0 \\ 0 & x < 0 \end{cases} \quad \text{and} \quad F_m = (\pi v_e^2)^{-3/2} \exp(-v^2/v_e^2),$$

$$v_{\text{eff}} = v_o v_e^3 / \epsilon v^3 \quad \text{where} \quad v_o \quad \text{the } 90^\circ \text{ electron}$$

$$\text{scattering rate, } \epsilon = r/R, \quad \omega_{*e}^T = \omega_{*e} [1 - \eta(3/2 - v^2/v_e^2)],$$

$$\omega_{*e} = k_y v_e^2 (2\omega_{ce} n_o)^{-1} \partial n_o / \partial x, \quad \text{and} \quad \eta = \partial \ln T / \partial \ln n. \quad \text{The real part of the frequency } \omega \text{ for trapped ion modes is}$$

$$\omega_{oi} = (2\epsilon)^{1/2} \omega_{*e} / (1 + T_e/T_i) \quad \text{where it is necessary that}$$

$$\omega_o < \omega_{bi} \equiv (\epsilon/2)^{1/2} v_i / qR \quad \text{and} \quad \omega_o < v_{\text{eff}} \quad \text{where } q \text{ is the}$$

$$\text{tokamak safety factor } (q = rB_\zeta / RB_\theta). \quad \text{For trapped electron}$$

$$\text{modes we have } \omega_{oe} \approx \omega_{*e} \quad \text{when} \quad \omega_{bi} < \omega_o < \omega_{be}. \quad \text{We shall}$$

$$\text{restrict ourselves to } \omega_o < v_{\text{eff}} \quad \text{in this discussion. In}$$

$$\text{tokamak geometry all linear perturbations are proportional to}$$

$$\exp(i\ell\zeta) \quad \text{where } \zeta \text{ is the toroidal angle and } \ell \text{ is an}$$

$$\text{integer with which we can associate } k_y \equiv \ell q / r. \quad \text{The growth}$$

$$\text{rate } \gamma_{Tp} \text{ is found to be}$$

$$\gamma_{Tp} = \beta \omega_o \int dv \frac{F_m \Theta(2\epsilon v_\perp^2 - v_\parallel^2)}{v_{\text{eff}}^2 + \omega_o^2} v_{\text{eff}} (\omega_{*e}^T - \omega_o)$$

$$\approx \frac{2\beta \omega_o (2\epsilon)^{3/2}}{(1 + 2\epsilon)^{1/2} \pi^{1/2}} \left[\frac{\omega_{*e} - \omega_o + 3/2 \omega_{*e} \eta}{v_o} \right], \quad v_o \gg \omega_o \epsilon$$

where $\beta = (1 + T_e/T_i)^{-1}$ for the trapped ion mode in $\beta = 1$ for the trapped electron mode.

The electron energy flux Fl_e that leaves the plasma due to this instability is

$$\begin{aligned}
 Fl_e &= - \frac{c}{B} \sum_{\underline{k}} k_y \phi_{-k} \int dv \frac{m_e v^2}{2} \text{Im } f_e \\
 &= \frac{en_o c}{v_e^2 B} \sum_{\underline{k}} k_y |\phi_k|^2 \int \frac{dv v^2 (\omega_o - \omega_{*e}^T) F_m}{\omega_o^2 + v_{eff}^2} \Theta(2\epsilon v_{\perp}^2 - v_{\parallel}^2) v_{eff} \\
 &\cong - 15 \left[\frac{2\epsilon}{\pi(1+2\epsilon)} \right]^{1/2} m_e v_e^2 \frac{\partial n_o}{\partial x} \frac{\epsilon}{v_o} \sum_{\ell} \\
 &\quad \times \left| \frac{\ell q c \phi_{\ell}}{r_D} \right|^2 \left[\eta - \frac{2}{5} \left(\frac{\omega_o}{\omega_{*e}} - 1 \right) \right] \quad (C-3)
 \end{aligned}$$

where we set $k_y = \ell q/r$ as this quantity is the toroidal geometry equivalent to k_y .

Now considering the beam particles we have that the radial thermal flux Fl_b of the passing particles is given by

$$Fl_b = i \sum_k \int \frac{d\theta}{2\pi} \frac{c}{B} \underline{E}_{-k} \times \underline{b} \cdot \hat{r} \int d^3v \frac{m_b v^2}{2} \text{Im } f_k \quad (C-4)$$

where $\underline{E}_{-k} \times \underline{b} = i \phi_{-\ell} \underline{k} \times \underline{b} \approx (i \ell q/r) \phi_{-\ell} \hat{r}$ and θ is the minor toroidal angle of a circular tokamak.

Now, $\text{Im } f_{-k}$ is given by

$$\begin{aligned} \text{Im } f_k &= \frac{|e|}{m_b} \text{Re} \int_{-\infty}^0 dt' \exp(-i\omega t') \phi_{\ell}(r(t')) \\ &\times \left[\omega \frac{\partial F}{\partial E} + \frac{\ell q}{r\omega_{cb}} \frac{\partial F}{\partial x} \right] \end{aligned} \quad (C-5)$$

Since $\omega \sim \omega_{*e}$, we have

$$\omega \frac{\partial F}{\partial E} \bigg/ \frac{k_y}{\omega_{cb}} \frac{\partial F}{\partial x} \approx \frac{T_e}{E_b} \ll 1$$

and we may neglect the $\partial F/\partial E$ term.

Now for $\phi(\underline{r})$ we take

$$\phi_{\ell}(\underline{r}) = \phi_{\ell} \exp \left[i \ell \zeta + i p \int \frac{d\theta' r}{R'} \frac{B'_{\zeta}}{B'_{\theta}} \right] \quad (C-6)$$

and we treat ζ independent of r and θ . Note that we have the quantization condition

$$p \oint_{\theta} \frac{d\theta'}{2\pi} \frac{r}{R} \frac{B_{\zeta}}{B_{\theta}} \equiv p \bar{q} = s \quad (C-7)$$

where s is an integer and $\bar{q}$ is the mean safety factor.

Now to evaluate the time integral in Eq. (5) we assume that for the hot component $\mu B \ll E \cong m_b v_{\parallel}^2/2$ so that the equations of motion for ζ and θ are

$$\frac{d\theta}{dt} = \frac{v_{\parallel} B_{\theta}}{rB} - \frac{v_{\parallel}^2}{\omega_{cb} R r} \frac{B_{\zeta}}{B} \cos \theta \quad (C-8)$$

$$\frac{d\zeta}{dt} = \frac{v_{\parallel}}{R} \frac{B_{\zeta}}{B} + \frac{v_{\parallel}^2}{\omega_{cb} R^2} \frac{B_{\theta}}{B} \cos \theta \quad (C-9)$$

where to order $\epsilon = r/R$, $B_{\zeta} = B_{\zeta 0} (1 - \epsilon \cos \theta)$, $B_{\theta} = B_{\theta 0} (1 - \epsilon \Lambda \cos \theta)$, $R = R_0 (1 + \epsilon \cos \theta)$. We also take $B_{\zeta}/B \cong 1$ as $B_{\theta}/B \ll 1$. We now solve by iteration in the small parameters ϵ , $v_{\parallel}/\omega_{cb} R$ and find

$$\theta = \theta_0(t) + \delta\theta(t)$$

where

$$\frac{d\theta_0}{dt} = \omega_B \left[1 + \epsilon (1 - \Lambda) \cos (\theta_1 + \omega_B t) \right] \quad (C-10)$$

$$\begin{aligned} \frac{d\theta}{dt} &= -\omega_D \cos \theta_o(t) \\ &\cong -\omega_D \left[\cos(\theta_1 + \omega_B t) - \frac{\epsilon}{2} (1 - \Lambda) \right] \end{aligned} \quad (C-11)$$

where $\omega_B = v_{\parallel}/qR$, $\omega_D = v_{\parallel}^2/\omega_{cb} Rr$ and θ_1 is the initial angle. We then integrate to obtain

$$\begin{aligned} \theta(t) - \theta_1 &= \left[\omega_b + \frac{\epsilon}{2} (1 - \Lambda) \omega_D \right] t \\ &+ \left[\frac{q v_{\parallel}}{\omega_{cb} r} + \epsilon(1 - \Lambda) \right] \left[\sin(\theta_1 + \omega_B t) - \sin \theta_1 \right] \end{aligned} \quad (C-12)$$

Similarly, we can integrate Eq. (9) using (C-12).

In this case we can neglect the curvature drift term as it is $\mathcal{O}(B_{\theta}/B)$. We find that

$$\zeta(t) - \zeta_1 = \frac{v_{\parallel}}{R} t - \epsilon q \left[\sin(\theta_1 + \omega_B t) - \sin \theta_1 \right] \quad (C-13)$$

We now need to consider the integral

$$I = \int_{-\infty}^0 dt \exp \left[-i\omega t + i\ell(\zeta(t) - \zeta) + ip \int_0^t dt' \frac{r}{R} \frac{B_{\theta}}{B} \frac{d\theta}{dt'} \right] \quad (C-14)$$

which is obtained from Eqs. (5) and (6). Using Eqs. (10)-(13) we find

$$I = \int_{-\infty}^0 dt \exp\left\{-i\omega t + i(\ell+p)\left[\frac{v_{\parallel}}{R_0} t - \epsilon q(\sin(\theta + \omega_B t) - \sin \theta)\right.\right. \\ \left.\left.+ ip\left[\frac{v_{\parallel}}{\omega_{cb} r} q^2(\sin(\theta + \omega_B t) - \sin \theta) + \frac{\epsilon \omega_D}{2} q(3-2\Lambda)t\right]\right]\right\}. \quad (C-15)$$

Now, integrating the time dependence of Eq. (15) we obtain

$$I = i \sum_{r,s} J_r(\alpha) J_s(\alpha) \exp[i(r-s)\theta] / \Omega_{\ell p s} \quad (C-16)$$

$$\overline{I} = \int \frac{d\theta}{2\pi} \text{Re } I = \pi \sum_s J_s^2(\alpha) \delta(\Omega_{\ell p s}) + \gamma_{Tp} \sum_s J_s^2(\alpha) / \Omega_{\ell p s}^2 \quad (C-17)$$

where $\alpha = pq^2 v_{\parallel} / r\omega_{cb} - (\ell+p)\epsilon q$ and $\Omega_{\ell p s} = \omega_0 - (\ell+p)v_{\parallel}/R_0 + p(\epsilon\omega_D/2)q(3-2\Lambda) - s\omega_B$. The first term gives rise to diffusion due to the resonance interaction while the second

term gives rise to nonresonant diffusion that is due to the turbulent background of waves.

The hot component energy flux is then found from Eqs. (4), (5), (6), (14) and (16) to be

$$Fl_b = - \frac{m_b}{2} \sum_l \left| \frac{c l q}{r B} \phi_l \right|^2 \int d\underline{v} v^2 \frac{\partial F_b}{\partial x} \bar{I} . \quad (C-18)$$

For F_b we choose

$$F_b = \frac{n_b(x) \alpha(v) \exp [-\alpha(v) v_{\perp}^2 / 2 v^2]}{2 \pi \tau_f (v^3 + v_I^3)} , \quad (C-19)$$

and we shall assume $\alpha(v) \gg 1$ so that we have $v_{\parallel} \cong v$.

The lifetime of the beam and the electrons can be obtained from the energy continuity relation

$$n_j E_j \frac{\pi r^2}{\tau_j} = 2 \pi r Fl_j \quad (C-20)$$

or $\tau_j^{-1} = 2 Fl_j / (r n_j E_j)$ where $E_b \approx m_b v_b^2 / 4 \tau_f$, and $E_j = \frac{3}{4} m_j v_j^2$.

We are interested in the ratio τ_e / τ_b and we wish to show that

this quantity is large. In general, both τ_e and τ_b consist of the sum of several ℓ -numbers, but for simplicity we consider each ℓ -number separately.

For small ℓ -numbers we consider the trapped ion mode for which

$$\omega_{oi} \approx \left(\frac{\epsilon}{2}\right)^{1/2} \frac{\ell q v_i^2}{2r^2 \omega_{ci}} < \left(\frac{c}{2}\right)^{1/2} \frac{v_i}{qR}, \quad \frac{v_o}{\epsilon} \quad (C-21)$$

The inequalities follow from the constraint that ω_{oi} is less than the trapped ion bounce frequency and trapped electron collision frequency respectively. These constraints limit the ℓ -number to be

$$\ell < \begin{cases} \ell_1 \equiv 2\epsilon r/q^2 \rho_i \approx 12/W^{1/2} \\ \ell_2 \equiv \frac{r^2 v_o}{q \rho_i^2 \omega_{ci}} \left(\frac{2}{\epsilon}\right)^{3/2} \approx 20 N/W^{5/2} \end{cases} \quad (C-22)$$

where $T_e = T_i$, $r = 50$ cm, $B = 50$ kG, $\epsilon = 1/4$, $q = 2.5$ and W is temperature in units of 5 keV and N is density in units of 10^{14} cm^{-3} . With these constraints on ℓ it is

unlikely that the hot component can resonate with the wave.

To demonstrate this we estimate the magnitude of each of the terms in $\Omega_{\ell, p, s}$, and we find that

$$\omega : (\ell + p)v_{\parallel}/R_0 : p\epsilon\omega_D q(3 - 2\Lambda)/2 : s\omega_B$$

$$\approx \left(\frac{\epsilon}{2}\right)^{1/2} \frac{v_i^2 \ell q}{2\omega_{ci} r^2} : (\ell + p) \frac{v_b}{R} : \frac{\epsilon \ell v_b^2 q}{\omega_{cb} R r} (3 - 2\Lambda)/2 : \frac{sv_b}{qR}$$

(C-23)

Now, typically $m_b v_b^2 > m_i v_i^2 / \epsilon^{3/2}$, so that $\omega < \epsilon\omega_D q(3 - 2\Lambda)/2$, and the first term on the right hand side can be ignored. In the term, $(\ell + p)v_{\parallel}/R_0$, we are interested in the case $p \approx -\ell$. However, p is limited by the quantization condition $p\bar{q} =$ integer. Hence on averaging between rational surfaces the lower limit on $\ell + p$ will be of order $1/2q$. Hence on setting $\Omega_{\ell, p, s} = 0$, we find

$$|\ell| = \frac{(1+2s)\omega_{cb} r}{(3-2\Lambda)\epsilon q^2 v_b} \quad (C-24)$$

and $\min |\ell| \sim \omega_{cb} r / q^2 \epsilon v_b \equiv \ell_r \sim 25$ where we used the parameters given after Eq. (22) and $v_b \sim 3 \times 10^8$ cm/sec. Comparing Eq. (22) with Eq. (24), we see that ℓ_1 and ℓ_2 are less than $\min |\ell|$ given by Eq. (24). Hence only the nonresonant contribution to $\bar{I}$ is important in Eq. (18). The argument of the Bessel functions tends to be less than unity, so that the nonresonant contribution can be obtained from just the $s = 0$ term in $\bar{I}$. We then find

$$F_{b,\ell}^1 \approx -2m_b \gamma_{Tp} \left| \frac{c \ell q \phi}{rB} \right|^2 \frac{\partial n_b}{\partial x} q^2 R^2 \quad (C-25)$$

where

$$\gamma_{Tp} \approx \frac{2\epsilon^2}{\pi^{1/2}} \frac{\omega^2}{v_o} \frac{*e}{v_o} (1 + 3\eta/2) .$$

Using Eqs. (3), (20), and (25) we find that τ_e/τ_b is given by

$$\frac{\tau_b}{\tau_e} \approx \frac{5\sqrt{2} \epsilon^{3/2} (\eta + 2/5) (v_b/v_i)^2}{\tau_f \ell^2 q^4 \left(\frac{\partial}{\partial x} \ln n_b \right) \left(\frac{\partial}{\partial x} \ln n_o \right) \rho_i^2 (1 + 3\eta/2)}$$

$$\ll \frac{v_b^2}{\tau_f \epsilon^{1/2} v_i^3} \quad (C-26)$$

where $\max |l|$ is determined from $\omega_{oi} < (\epsilon/2)^{1/2} v_i/qR$ and $\partial/\partial x \ln n_o \approx \partial/\partial x \ln n_b \sim 1/r$. Hence, for the trapped ion mode the hot component confinement time is a good deal longer than the electron confinement time.

Now let us consider the trapped electron mode where $\omega \sim \omega_{*e} < v_o/\epsilon$. This restricts us to $l < l_3 \equiv (2v_o r^2)/(\epsilon q \rho_i^2 \omega_{ci}) \approx 8N/W^{5/2}$. Here the resonant interaction of the hot component is unlikely since we would require l_3 greater than the minimum value of Eq. (24). Assuming $s = 0$ is the main contribution to the ion term, we now find that τ_b/τ_e is given by

$$\frac{\tau_b}{\tau_e} = \frac{10 \epsilon^2 v_b^2}{\tau_f l^2 q^4 v_i^2 \rho_i^2 \frac{\partial}{\partial x} \ln n_b \frac{\partial}{\partial x} \ln n_o} < \frac{5}{6} \frac{\epsilon^4}{(v_o r)^2 q^2} \frac{v_b^2}{\tau_f}$$

$$\sim \frac{60}{\tau_f} \frac{W^3}{N^2} \quad (C-27)$$

At high density and low background temperature [i.e., $l_3 > l_r$ or $v_o > v_i^2/(v_b q r)$] a resonant interaction of the beam with the trapped electron mode is possible. For this case when we assume $\omega_{*e} < (\epsilon/2) \omega_D (3 - 2\Lambda)$ (which is marginally satisfied for TCT) and the hot component flux becomes

$$Fl_{b,l} \approx - \frac{\pi m_b}{2 \partial x} \left| \frac{c l q \phi_l}{r B} \right|^2 \sum_s \int \frac{dv_{\parallel} v_{\parallel}^4 J_s^2(\alpha)}{(v_{\parallel}^3 + v_I^3) \tau_f} \cdot \delta \left(\frac{v_{\parallel} (1+2s)}{2 q R} - \frac{\epsilon l v_{\parallel}^2 q (3-2\Lambda)}{2 \omega_{cb} R r} \right) \quad (C-28)$$

where we set $l + p = 1/2q$. At resonance $\alpha \cong (1+2s)/\epsilon(3-2\Lambda) > 1$, so that

$$J_s^2(\alpha) \approx \frac{\epsilon}{\pi} \left| \frac{3-2\Lambda}{1+2s} \right|.$$

Adding the contributions from $s = 0, -1$ and setting $\Lambda = 1$, we find

$$Fl_{b,l} \approx - \frac{2m_b}{\tau_f} \frac{\partial n_b}{\partial x} q R v_b \epsilon \left| \frac{c l q \phi_l}{r B} \right|^2$$

Hence, we find for τ_b/τ_e

$$\tau_b/\tau_e \approx \frac{5}{4} \left(\frac{2}{\pi} \right)^{1/2} \eta \frac{v_b \epsilon^{1/2} \frac{\partial}{\partial x} \ln n}{v_o q R \frac{\partial}{\partial x} \ln n_b} \approx 13 \eta w^{3/2}/N. \quad (C-29)$$

Observe that the nonresonant turbulence confinement time ratios given by Eqs. (26) and (27) are quite large, and the resonant contribution given by Eq. (29) appears large although smaller than the nonresonant contributions. Hence, the hot component diffusion time τ_b is appreciably greater than the background electron diffusion time τ_e .

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Fig. 1. Beam-wave matrix element for noncompressional oblique modes near the beam cyclotron frequency evaluated with the asymptotic beam distribution (dashed curve) in Eq. (III-18) and with the Legendre polynomial expansion of the beam distribution (solid curve) in Eq. (A-5) for the collisional transient of a single parallel injected monoenergetic source with $v_b/v_T = 2.0$ in a background plasma with $A(\approx Z_{\text{eff}}) = 5$.

Fig. 2. Beam-wave matrix element for noncompressional oblique modes near the beam cyclotron frequency evaluated with the Legendre polynomial expansion of the beam distribution for the collisional transient of a single parallel injected monoenergetic source with $v_b/v_T = 2.0$ in a background plasma with $A(\approx Z_{\text{eff}}) = 1$.

Fig. 3. Beam-wave matrix element for low frequency ($\omega < \omega_{cb}$) noncompressional oblique modes evaluated with the Legendre polynomial expansion of $F_b(v, \mu, t)$ for the collisional transient of a single parallel injected monoenergetic source with $v_b/v_I = 2.0$ in a background plasma with $A(\approx Z_{eff}) = 5$.

Fig. 4. Beam-wave matrix element for low frequency ($\omega < \omega_{cb}$) noncompressional modes evaluated with the Legendre polynomial expansion of $F_b(v, \mu, t)$ for the collisional transient of a single parallel injected monoenergetic source with $v_b/v_I = 2.0$ in a background plasma with $A(\approx Z_{eff}) = 1$.

Fig. 5. Beam-wave matrix element for low frequency compressional oblique modes and circularly polarized parallel modes evaluated with the Legendre polynomial expansion of $F_b(v, \mu, t)$ for the collisional transient of a single parallel injected monoenergetic source with $v_b/v_I = 2.0$ in a background plasma with $A(\approx Z_{\text{eff}}) = 5$.

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