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SOME MECHANISMS OF DRY BEARING

by

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KNOLLS ATOMIC POWER LABORATORY

June 4, 1956

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ABSTRACT

Much study has been devoted to the motion of shafts in lubricated journal bearings. In contrast, the work done on the motion of shafts in unlubricated or dry bearings is negligible. The reason is obvious: in most applications, lubrication of the bearings is relatively simple. This is not the case in nuclear power plant application where bearings operate in low viscosity primary coolants of liquid metal or water without the benefit of coolant contaminating lubricants. For this reason, an analytical study was made to determine what motions would be excited in a shaft mounted in a dry bearing. The study splits naturally into two parts: (a) the shaft maintains continuous contact with the bearing, either rolling or sliding or a combination of the two; (b) the shaft dances (or chatters) about the bearing in a series of impacts. Typical motions in each of these cases (rolling oscillations, sliding oscillations, stick-slip oscillations, chatter) are calculated and plotted. Except for the stick-slip oscillation, none of the motions studied is self-sustaining; rolling and sliding oscillations will damp out due to frictional and other losses, and chatter degenerates into a rolling whirl of the shaft around the bearing.

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Nomenclature

- e - coefficient of restitution
- F - force
- F_0 - transverse applied force
- F_r - radial bearing reaction force
- F_θ - tangential bearing reaction force
- I - polar moment of inertia of shaft
- k - shaft radius of gyration
- k - spring constant
- M - mass of shaft
- o - subscript denoting value of variable at $t = 0$
- p - subscript denoting period
- r - radial coordinate of shaft axis
- R - shaft radius
- ΔR - radial clearance in bearing
- T - applied torque
- t - time
- τ - dimensionless time, $t \sqrt{k/M}$
- V - linear velocity
- x - linear coordinate function of t
- y - linear coordinate function of t
- α, α' - angle of incidence, rebound
- δ - parameter, $V/\Delta R \sqrt{k/M}$
- ϵ - parameter, $\Delta R/R$
- η - parameter, k/R
- ρ - dimensionless radius, $r/\Delta R$
- μ - coefficient of friction
- θ - angular coordinate of shaft axis

ω - angular velocity of shaft

τ - dimensionless time, $t \sqrt{F_0/MAR}$

$\circ \equiv d/d\tau$

$\cdot \equiv d/dt$

I. Introduction

The use of journal bearings in supporting or guiding rotating shafts is well known, and their simplicity has led to wide application. However, in certain applications, instabilities may arise which may lead to undesirable noise, wear, and even destruction of the bearing. In lubricated bearings there are instability phenomena such as "oil whip" occurring at angular velocities where hydrodynamic lubrication is effective. These have been widely studied and are now well understood. In this report, instabilities in unlubricated or "dry" journal bearings are examined. The bearing may be "dry" due to the absence of a lubricant or because the shaft speeds are too low to achieve hydrodynamic support of the transverse load from the low viscosity fluid present in the bearing. These instabilities, or "bearing chatter", exist in several forms: the shaft may remain in intimate contact with the bearing surface, and alternately roll and slide; or, it may dance about within the clearance of the bearing boundary, somewhat analogous to the action of a whirling drill bit in an oversize hole.

The advent of atomic power has revived and intensified interest in many problems such as bearing chatter which achieve new significance in this field. The use of low viscosity reactor coolants (i.e., water and liquid metals) and the need for simplicity, reliability, and freedom from contamination of these coolants has led to the use of simple bearings for pumps and control mechanisms, lubricated only by the coolant. The analysis presented here arose from the need to diagnose a bearing chatter problem which was encountered in the atomic energy field. However, the analysis and results may be of interest with respect to the problem of chatter during starting and low-speed operation of lubricated journal bearings and in applications such as electric circuit-breakers and other apparatus having journal bearings which operate intermittently with limited angular rotation.

Figure 1 illustrates the model which will be analyzed. The unlubricated bearing is mounted rigidly, while the stiff shaft may move within the confines of the bearing. Motion of the shaft center is described in polar coordinates (r, θ). For generality, a transverse force F_0 acts on the shaft. This may be a gravity or other applied load.

The purpose of this report is to investigate and analyze mechanisms which cause bearing chatter. The general equations of motion are derived, and solutions in terms of dimensionless parameters are obtained for particular initial conditions. The various types of motion studied are tabulated below, together with the pages in the report in which the analysis is covered in detail and a summary of results:

<u>Type of Motion</u>	<u>Results</u>	<u>pp</u>
1. Sliding and/or Rolling of Shaft in Continuous Contact with Bearing Surface		
a. No transverse force	After constant torque is applied to shaft and maintained, the shaft first slips (in	13

<u>Type of Motion</u>	<u>Results</u>	<u>pp</u>
	general) and then "spins up" to a rolling whirl, after which its angular velocity continues to increase linearly with time. Figure 3 shows angular velocity vs. time when the coefficient of friction is 0.25.	
b. Non-zero transverse force	After constant torque is applied to shaft initially at rest and maintained, the shaft may: (a) roll dead (and oscillate thereafter); (b) slip immediately; or (c) roll to some maximum angle less than 90° and then slip or fall. Figure 4 (a), 4 (b), and 4 (c) show this behavior for several values of system constants.	16
c. Oscillatory rolling	Figure 5 shows the time behavior of the angular amplitude of a shaft as a particular case of condition (a) above.	19
d. Continuous sliding	If the applied torque is kept large enough, continuous sliding (without oscillation or loss of contact with bearing surface) is possible if coefficient of friction > 0.707 . Figure 6 shows this result.	21
e. Oscillatory sliding	Figure 7 shows the time behavior of the angular amplitude of the shaft for $\mu = 0.25$, $T > 0.75$, after it has been started from rest.	24
f. Stick-slip oscillation	A particular case of periodic motion involving both rolling and slipping of the shaft is studied. For oscillations with large enough amplitudes, the motion is self-sustaining. Figures 8 and 10 show the time behavior of the shaft center amplitude and torque. Figures 9 and 11 show the period of oscillation and work done for cycle vs. shaft angular velocity.	24

<u>Type of Motion</u>	<u>Results</u>	<u>pp</u>
2. Impact of Shaft on Bearing Surface		
a. Impact on plane surface	Equations of impact of rotating shaft on flat plane are reviewed. It is concluded that if circular bearing can be represented by infinitely many-sided hexagon, any sequence of impacts will degenerate into a whirl.	31
b. Impact on curved surface	Equations of impact on a cylindrical surface are derived, to determine under what conditions energy would be fed into radial component of velocity after impact. Such an energy transfer was found that only when the shaft angular velocity decreased.	33

FIGURE 1 SHAFT AND BEARING SYSTEM FOR
CHATTER ANALYSIS

II. In-Plane Motion. Shaft in Continuous Contact with Bearing

Figure 1 (a) is redrawn as Figure 2 to permit an easy check on the derivation of the equations of motion given below:

Figure 2 Shaft Free Body Diagram
Figure for Derivation of In-Plane Equations of Motion

$$\left. \begin{aligned} F_r - F_0 \cos \theta &= M A R \ddot{\theta}^2 \\ F_\theta - F_0 \sin \theta &= M A R \ddot{\theta} \\ T - F_\theta R &= I \dot{\omega} \end{aligned} \right\} \dots \dots \dots (1)$$

F_r and F_θ are radial and tangential bearing forces, and F_0 and T are the force and torque applied to the shaft; θ is the angular coordinate locating the center of the shaft, and ω is the angular velocity of the shaft; M and I are the mass and moment of inertia of the shaft; and R and $2AR$ are the radius of the shaft and the diametral bearing clearance. Dots denote differentiation with respect to time.

Consider the case of a rotating shaft brought with negligible radial velocity into contact with its bearing surface; depending on the initial tangential velocity of the shaft, it will either roll along the bearing surface or be accelerated or decelerated by a friction force $F_\theta = \frac{1}{2} \mu F_r$, the shaft center describing a circular path of radius AR (half the bearing clearance). If the shaft slips initially, acceleration of its center of mass will continue until the relative motion at the point of contact with the bearing vanishes and rolling begins. Several examples of rolling and sliding motion will be examined. We consider the simplest case when the transverse force (F_0 , per Fig. 2) is zero. Such a condition could arise if a vertical shaft were perfectly centered in a horizontally mounted bearing:

A. $F_0 = 0$

Equations (1) become

$$\left. \begin{aligned} F_r &= M A R \ddot{\theta}^2 \\ F_\theta &= M A R \ddot{\theta} \\ T &= I \dot{\omega} + F_\theta R \end{aligned} \right\} \dots \dots \dots (2)$$

We consider pure rolling as the first special case. Then $R\omega = R\dot{\theta}$ at all times and Equations (2) may be rewritten in the form

$$\left. \begin{aligned} F_f &= M\Delta R\ddot{\theta}^2 \\ F_\theta &= M\Delta R\ddot{\theta} \\ \ddot{\theta} &= \frac{T}{MR\Delta R \left(1 + \frac{I}{MR^2}\right)} \end{aligned} \right\} \dots \dots \dots (3)$$

When the applied torque is constant (T_0 , say), the solution for θ is

$$\left. \begin{aligned} \theta &= \theta_0 + \dot{\theta}_0 t + \frac{T_0 t^2}{2MR\Delta R \left(1 + \frac{I}{MR^2}\right)} \\ \dot{\theta} &= \frac{R\omega}{\Delta R} = \dot{\theta}_0 + \frac{T_0 t}{MR\Delta R \left(1 + \frac{I}{MR^2}\right)} \end{aligned} \right\} \dots \dots \dots (4)$$

if the initial conditions are $\theta(0) = \theta_0$, $\dot{\theta}(0) = \dot{\theta}_0$.

To show that rolling continues once it has been initiated, we compare μF_f with F_θ . Slipping will occur if $\mu F_f < F_\theta$, so the condition for rolling is $\mu F_f > F_\theta$ or by Equations (3) and (4).

$$\mu \left[\frac{T_0 t}{MR\Delta R \left(1 + \frac{I}{MR^2}\right)} + \dot{\theta}_0 \right]^2 - \frac{T_0}{MR\Delta R \left(1 + \frac{I}{MR^2}\right)} > 0 \dots \dots \dots (5)$$

Thus, if at $t = 0$, $\dot{\theta}_0$ is large enough to satisfy the inequality expressed above, the inequality will be satisfied for all $t > 0$, and rolling will continue. However, if $\dot{\theta}_0$ is not large enough to satisfy the inequality at $t = 0$, then the shaft will slip initially and will finally "spin up" to rolling velocity. This "spin up" constitutes the second special case and is defined by the condition that $R\omega(0) > \Delta R\dot{\theta}(0)$ and $F_\theta = \mu F_f$, μ being the coefficient of friction. Equations (2) then reduce to

$$\ddot{\theta} - \mu \dot{\theta}^2 = 0, \dots \dots \dots (6)$$

which has as its solution

$$\left. \begin{aligned} \theta &= \theta_0 - \frac{1}{\mu} \log(1 - \mu \dot{\theta}_0 t) \\ \dot{\theta} &= \frac{\dot{\theta}_0}{1 - \mu \dot{\theta}_0 t} \end{aligned} \right\} \dots \dots \dots (7)$$

when $\theta(0) = \theta_0$ and $\dot{\theta}(0) = \dot{\theta}_0$ (μ is assumed to be independent of slip velocity.) Equations (7) are valid until $\dot{\theta} = R/\Delta R \omega$. If the shaft angular velocity is constant (ω_0 , say), then the behavior of $\dot{\theta}$ as a function of $\mu \dot{\theta}_0 t$ is shown in Figure 3. It is seen that $\dot{\theta}_0$ increases rapidly until $R/\Delta R \omega_0$ is reached, after which it is synchronized with the shaft angular velocity. Similar reasoning can be used to show that if $R\omega(0) < \Delta R\dot{\theta}(0)$ initially, "spin down" to synchronous whirling speed will result.

Figure 3. Time Behavior of Shaft Center Angular Velocity during In-Plane Motion with No Transverse Force

B. $F_0 \neq 0$

Equations (1) are transformed to dimensionless form for convenience:

$$\begin{aligned}\bar{F}_r - \cos \theta &= \dot{\theta}^2 \\ \bar{F}_\theta - \sin \theta &= \dot{\theta}^3 \\ \bar{T} - \bar{F}_\theta &= \eta^2 \dot{\omega}^2\end{aligned} \quad (8)$$

where $\dot{\theta} \equiv d\theta/d\tau$, $\tau \equiv t\sqrt{F_0/MAR}$, dimensionless time

$\bar{F}_r \equiv F_r/F_0$, $\bar{F}_\theta \equiv F_\theta/F_0$, dimensionless forces

$\bar{T} \equiv T/F_0R$, dimensionless torque

$\dot{\omega} \equiv \omega/\sqrt{F_0/MAR}$, dimensionless angular velocity

$$\gamma^2 \equiv I/MR^2$$

$$\epsilon \equiv \Delta R/R$$

Assume first that the shaft rolls on the bearing without slipping; then $R\omega = \Delta R\dot{\theta}$, and Equations (8) can be rewritten in the following form by eliminating $\ddot{\omega}$:

$$\left. \begin{aligned} \ddot{\theta} (1 + \gamma^2) + \sin \theta &= \bar{T} \\ \bar{F}_r &= \cos \theta + \ddot{\theta}^2 \\ \bar{F}_\theta &= \sin \theta + \ddot{\theta} \end{aligned} \right\} \dots \dots \dots (9)$$

Thus, if $\bar{T}$ is given as a function of time, it is formally possible to solve for $\theta(\tau)$ from the first of Equations (9). However, the solution must be inspected to determine the existence of any of the following conditions:

- (1) Oscillation: If the solution shows that $\ddot{\theta}$ (or $\ddot{\theta}^2$) = 0, in the first quadrant, the shaft may roll up, stop, and roll back along the bearing surface in some periodic fashion (provided $\bar{T}$ is a sufficiently well behaved function of time).
- (2) Contact between shaft and bearing: If $\bar{F}_r$ as given by the second of Equations (9) vanishes, the shaft no longer touches the bearing. Examination of this equation shows that positive values of $\bar{F}_r$ are assured in the first quadrant if $\ddot{\theta}^2 > 0$; the converse is true in the second quadrant. If $\bar{F}_r = 0$ at some point in the second quadrant, the shaft, upon reaching that point, would obey the equations of motion of a freely falling body. Contact of the shaft with the bearing would occur after a time in free flight determined by its initial velocity and position at the time of losing contact with the bearing surface and $\bar{F}_\theta$.
- (3) Relative motion (slip) between shaft and bearing at their common point of contact: Sliding will result if $\bar{F}_r < \bar{F}_\theta$, and the equations of motion must be altered accordingly.

C. Rolling under Constant Torque $\bar{T}$

We consider now the special case when $\bar{T}$ is constant. The solution to the first of Equations (9) may then be written in the form*

$$\left. \begin{aligned} \frac{1}{2} (1 + \gamma^2) (\dot{\theta}^2 - \dot{\theta}_0^2) + (\cos \theta_0 - \cos \theta) - \bar{T} (\theta - \theta_0) &= 0 \\ \tau = \int_{\theta_0}^{\theta} \frac{d\theta'}{\dot{\theta}'} \end{aligned} \right\} (10)$$

when $\theta(0) = \theta_0$ and $\dot{\theta}(0) = \dot{\theta}_0$. (θ' and $\dot{\theta}'$ are dummy variables referring to θ and $\dot{\theta}$, respectively.) Thus, for given values of θ_0 , $\dot{\theta}_0$, γ^2 and $\bar{T}$, $\theta(\tau)$ may be found by quadratures, using the second of Equations (10).

* Reference 1, Page 546, Equation 6.19.

Before calculating any $\theta(t)$ curves, we raise the question: is it possible to roll the shaft completely around the bearing by the application of a constant torque $\bar{T}$ applied to the shaft initially at rest ($\theta_0 = \dot{\theta}_0 = 0$)? The answer can be obtained by determining if conditions (a) through (c) of Section II B can be met. Note that if the shaft once rolls up to the top of the bearing, it will thereafter maintain contact without slipping. These conditions can be rewritten with the aid of Equation (10) as follows:

$$(a) \text{ No oscillatory rolling: } \ddot{\theta}^2 > 0 (0 \leq \theta \leq \pi/2)$$

$$\text{Thus, } \bar{T} \theta - (1 - \cos \theta) > 0$$

$$(b) \text{ No loss of contact with bearing: } F_r > 0 (\pi/2 \leq \theta \leq \pi)$$

$$\text{Thus, } 2 \bar{T} \theta - 2 + (3 - \eta^2) \cos \theta > 0$$

$$(c) \text{ No slipping: } \mu F_r \geq F_\theta (0 \leq \theta \leq \pi)$$

$$\text{Thus, } \mu = \frac{\bar{T} + \eta^2 \sin \theta}{2 \bar{T} \theta - 2 + (3 - \eta^2) \cos \theta} > 0 \quad \dots (11)$$

For given values of η^2 and μ , Equation (11) prescribes the value of $\bar{T}$ necessary to assure continuous rolling of the shaft without oscillation, slipping, or falling. Setting each of the left-hand sides equal to zero defines curves of $\bar{T}$ vs θ which divide the $\bar{T}$ - θ plane into regions of continuous rolling, oscillatory rolling, slipping, and falling.

Typical curves are presented in Figure 4 (a), (b), (c) for $\eta^2 = 0.5, 1.0$, and 2.0 and $\mu = 0.25, 0.50, 0.75, 1.00$; and an "intermediate" value described in Appendix A. The lowest curve represents the boundary between continuous rolling and oscillatory rolling ($0 \leq \theta < \pi/2$), and falling ($\pi/2 < \theta < \pi$). It is seen that rolling to the top without oscillation or falling (neglecting slip for the time being) requires a $\bar{T}$ which exceeds one. This occurs about 30° from the top of the bearing and increases with η^2 . The remaining curves represent the boundaries between continuous rolling and slipping for various values of μ . It is seen that the shaft will always slip before losing contact with the bearing, and hence the range $\pi/2 < \theta < \pi$ is of no practical importance.

The curves illustrate several points in the behavior of the shaft after the sudden application of a constant torque $\bar{T}$: For very small values of $\bar{T}$, and non-zero μ , the shaft "rolls dead" and drops back, producing oscillatory rolling; however, above a certain intermediate value of μ (defined below), a range of $\bar{T}$'s will accelerate the shaft to some angle in the first quadrant where slipping impends. $\bar{T}$'s above this range produce immediate slip. The slip angle for this range of $\bar{T}$'s is determined by the equation

$$\theta_{\text{slip}} = \frac{1}{2\mu} \mu_{\text{int}} \quad \dots (12)$$

which is valid for all values of the coefficient of friction μ which exceed the "intermediate" value defined by the simultaneous transcendental equations

$$1 - 2\mu_{\text{int}} \theta_{\text{max}} = 0 \quad \dots (13)$$

$$\mu_{\text{int}} (3 - \eta^2) \cos \theta_{\text{max}} - 2\mu_{\text{int}} - \eta^2 \sin \theta_{\text{max}} = 0$$

REGIONS OF ALLOWED $\bar{T}$ FOR CONTINUOUS ROLLING OF SHAFT IN BEARING

$$\bar{T} \equiv T/F_0 R$$

T - APPLIED TORQUE, CONSTANT

F_0 - TRANSVERSE FORCE

R - SHAFT RADIUS

μ - COEFFICIENT OF FRICTION

$\eta^2 = I/MR^2$, I AND M BEING SHAFT POLAR MOMENT OF INERTIA AND MASS, RESPECTIVELY.

FOR a, b, c

FIGURE 4 (a, b, c)

Subscripts "int" and "max" refer to intermediate (friction coefficient) and maximum (angle), respectively. Note that the maximum angle which can be reached before slip (or oscillation) occurs is associated with the intermediate coefficient of friction by Equation (13). It may be obtained from single transcendental equation by eliminating μ_{int} from Equation 13:

$$(3 + \gamma^2) \cos \theta_{max} - 2 (1 + \gamma^2 \theta_{max} \sin \theta_{max}) = 0 \dots \dots \dots (14)$$

For a given γ^2 , the band of intermediate values of $\bar{T}$ which will accelerate the shaft to θ_m is then found by the following equation.

$$\bar{T}_{min} = \frac{1 - \cos \theta_m}{\theta_m} \dots \dots \dots (15)$$

$$\bar{T}_{max} = (1 + \gamma^2) \mu_{int}$$

To prevent slippage once the shaft begins to roll, μ must also satisfy the condition that

$$\mu > \frac{\gamma}{\sqrt{2(1 + \gamma^2)}} \dots \dots \dots (16)$$

We note that μ_{int} does not always satisfy this condition (Fig. 4 (c)), and θ_{max} will not be reached. (Proofs of Equations (12) through (16) and statements made in the discussion are found in Appendix A).

Thus, we conclude that if a transversely loaded shaft initially at rest is subjected to a step torque which is maintained thereafter, it may begin to roll without slipping, but that it will either begin to slip or stop dead and roll back before leaving the first quadrant. An example of the latter motion, oscillatory rolling, will be studied next.

D. Oscillatory Rolling

Inspection of Figure 4(c) shows that in a bearing with a coefficient of friction of 0.5, oscillatory rolling of a shaft initially at rest will be excited by $\bar{T} \leq 0.27$. As an example, we take $\bar{T} = 0.25$ and $\gamma^2 = 2$ and plot the resulting rolling oscillation as $\theta(t)$ in Figure 5. Details of the calculation are outlined in Appendix B.

The time behavior of θ is seen to be nearly sinusoidal with a period of $\tau_p \approx 11.0$. The shaft oscillates about its rest position which is determined by setting $\dot{\theta} = 0$ in the first of Equations (9) to obtain

$$\theta_{st} = \sin^{-1} \bar{T} \dots \dots \dots (17)$$

It should be recognized that under actual operating conditions some damping would be present to cause the amplitude of oscillation to decay with time. The final angular displacement would thus be θ_{st} .

20

PLOT OF ROLLING OSCILLATION FOR CONDITIONS OF $\eta^2 = 2.0$
 $\mu = 0.5$
 $\bar{T} = 0.25$
 $\phi_0 = \phi_0' = 0$

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FIGURE 5

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L. Sliding

Figures 4(a) through (c) show that if, for a given μ , $\bar{T}$ exceeds a certain value, the shaft will slip initially. In this case, $\bar{F}_0 = \mu \bar{F}_r$ (only the case when $R\omega > \Delta R \dot{\theta}$ is considered) and Equations (8) become

$$\left. \begin{aligned} \ddot{\theta} - \mu \dot{\theta}^2 + \sin \theta - \mu \cos \theta &= 0 \\ \bar{F}_r &= \cos \theta + \dot{\theta}^2 \\ \bar{F}_0 &= \mu \bar{F}_r \\ \bar{T} &= \mu \bar{F}_r + \frac{2}{\epsilon} \dot{\theta}^2 \end{aligned} \right\} \dots \dots \dots (18)$$

The solution for θ is*

$$\left. \begin{aligned} \tau(\theta) &= \int_{\theta_0}^{\theta} \frac{d\theta}{f(\theta)}, \text{ where} \\ \dot{\theta}^2 &= e^{2\mu(\theta-\theta_0)} \left\{ \dot{\theta}_0^2 - \frac{2}{1+\mu^2} \left[(1-2\mu^2) \cos \theta_0 + 3\mu \sin \theta_0 \right] \right. \\ &\quad \left. + \frac{2}{1+\mu^2} \left[(1-2\mu^2) \cos \theta + 3\mu \sin \theta \right] \right\} \end{aligned} \right\} (19)$$

when the initial conditions are $\theta(0) = \theta_0$ and $\dot{\theta}(0) = \dot{\theta}_0$. Thus, once μ , θ_0 , and $\dot{\theta}_0$ are specified, θ , and hence $\bar{F}_r$ and $\bar{F}_0$, can be found.

The tendency is for the shaft to "spin up" to rolling speed, $\dot{\theta} = \frac{R}{\Delta R} \omega$ as in Section II A. If slip is to be maintained $\dot{\theta} < \frac{R}{\Delta R} \omega$. This condition plus two others necessary to maintain continuous slip are listed below.

- $\bar{\omega} > \epsilon \dot{\theta}$ (discussed above). The torque must be increased with time to satisfy this condition.
- $\dot{\theta}^2 > 0$ ($0 < \theta < \pi/2$) if the shaft is to slip continuously along the bearing without stopping.
- $\bar{F}_r > 0$ ($\pi/2 < \theta < \pi$) to assure that the shaft does not lose contact with the bearing surface in the second quadrant.

By manipulating Equations (17) and (18), conditions (a) through (c) can be written in equation form. We consider the special case of the application of a stop torque $\bar{T}$ of sufficient magnitude to cause slipping to a shaft initially at rest ($\theta_0 = \dot{\theta}_0 = 0$). Conditions (a) through (c) are satisfied if

* Reference 1.

(a) Continued slip: $\bar{\omega} > \varepsilon \dot{\theta}$ ($\theta > 0$), or

$$\frac{1}{\mu^2} \left\{ \bar{T} \mu \frac{2}{1+\mu^2} \int_0^{\theta} e^{2\mu\theta(r')} dr' - \frac{3\mu}{1+\mu^2} \int_0^{\theta} (\cos \theta + \mu \sin \theta) d\theta \right\} > \left\{ \frac{2}{1+\mu^2} \left[-(1-2\mu^2) e^{2\mu\theta} + (1-2\mu^2) \cos \theta + 3\mu \sin \theta \right] \right\}^{\frac{1}{2}}$$

(b) No oscillatory slipping: $\dot{\theta} > 0$ ($0 < \theta < \pi/2$), or

$$-(1-2\mu^2) e^{2\mu\theta} + (1-2\mu^2) \cos \theta + 3\mu \sin \theta > 0$$

(c) No loss of contact with bearing: $F_r > 0$ ($\pi/2 < \theta < \pi$), or

$$-2(1-2\mu^2) e^{2\mu\theta} + 3(\cos \theta + \mu \sin \theta) > 0$$

(20)

Let the first of conditions (20) be a requirement on the magnitude of $\bar{T}$; i.e., $\bar{T}$ must be large enough to satisfy the inequality over a specified arc length, say $0 < \theta < \pi$. It is recognized that for a constant value of $\bar{T}$, the shaft may eventually spin up and slip will cease and Equation (10) of rolling will apply. However, before rolling is considered, the second and third of conditions (19) must be inspected to determine if the shaft can possibly slide up to the top of the bearing and around before it begins oscillatory sliding or loses contact with the bearing surface.

Setting the last two of conditions (19) equal to zero provides a continuous equation for μ as a function of θ in the range $0 \leq \theta \leq \pi$. The resulting curves divide the μ - θ plane into regions of continuous slipping, oscillatory slipping, and falling. This plot is given as Figure 6. Details of the calculation appear in Appendix C.

It is seen that if $\mu \leq 0.6$, the shaft will never slide above the first quadrant, but will oscillate up and down the bearing. An example of this motion will be described in the next section. If $\mu > 0.707$, the shaft will slide continuously at least to the top of the bearing. However, if $0.6 \leq \mu < 0.707$, the shaft will lose contact with the bearing and fall, causing impact of the bearing when it strikes the surface.

We conclude that the application of a step torque to a transversely loaded shaft of sufficient magnitude to assure slip at all times may excite three different shaft motions:

- oscillatory slip within the first quadrant if $0 \leq \mu < 0.6$;
- continuous slip into the second quadrant with subsequent loss of contact with the bearing surface if $0.6 \leq \mu < 0.707$;
- continuous slip completely around the bearing if $\mu > 0.707$ (and $\bar{T}$ remains sufficiently large).

Further inspection of the pure rolling and pure sliding solutions (Equations (10) and (18), respectively) shows that proper combinations of sliding and rolling can move the shaft completely around the bearing without losing

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CURVE OF COEFFICIENT OF FRICTION NECESSARY
TO ASSURE CONTINUOUS SLIDING TO GIVEN HEIGHT
(AS MEASURED BY θ) ON BEARING

$\theta_0 = \theta_c = 0$
1/16 INCH FORCE
ENOUGH TO MAINTAIN
SLIP

CONTINUOUS SLIDING

FALLING

OSCILLATORY
SLIDING

θ_0

θ_c

θ_m

θ

FIGURE 6

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contact at values of $\mu < 0.707$. Furthermore, appropriate time behavior of $\bar{T}$ may produce the same results. Therefore, more specialized cases than those reported here must be investigated separately to determine whether motion with or without loss of contact will occur. If contact is maintained, "spin up" to large values of $\dot{\theta}$ (and accompanying high centrifugal forces against the bearing surface) may result; if contact is lost, impacting of the bearing surface which may excite bearing chatter (a self-excited impact vibration) is a possibility. In general, the value of the coefficient of friction is so small that the transversely loaded shaft remains at or near dead center. However, the mechanism of self-excitation in bearing chatter is investigated in later sections.

F. Oscillatory Sliding

By inspection of Figures 4 and 6, we see that oscillatory sliding will be produced at a $\mu = 0.25$ if $\bar{T}$ is large enough. Assuming that this requirement is satisfied, sliding oscillations given as $\theta(t)$ in Figure 7 will result when $\theta_0 = \dot{\theta}_0 = 0$. Equation (19) is used to obtain the curve, and details are given in Appendix D.

As in the rolling oscillation, the arc described by the center of the shaft varies almost sinusoidally with time. The steady-state value of $\theta (= \theta_{st})$ shown as the horizontal line in Figure 7 is obtained by setting both $\ddot{\theta}$ and $\dot{\theta}$ in the first of Equation (18) equal to zero:*

$$\theta_{st} = \tan^{-1} \mu \dots \dots \dots (21)$$

If there is any damping present in the system (which is inevitable), the oscillations will die down to this steady-state value. The shaft will then continue to slide continuously at this equilibrium point.

G. Stick-Slip Oscillation

One final oscillatory motion in which the shaft and bearing remain in contact combines both rolling and sliding motions. The shaft may roll part way up the bearing, slip and fall back, and catch and begin to roll up again. In this event of unequal static and dynamic coefficients of friction, the oscillations are self-excited and may build up until limited by fluid friction losses or altered by changing boundary conditions (i.e., if the arc of oscillation exceeds 90° , the shaft will lose contact with the bearing and fall). The equations applicable to this motion are found in Appendix E.

As an example, we consider a system with static and dynamic coefficients of friction $\mu_0 = 0.50$ and $\mu_1 = 0.25$, respectively. The applied torque is assumed variable so as to maintain a constant shaft angular velocity ω_0 . Such conditions might exist in a shaft with little torsional rigidity but with a large torsional moment of inertia, I . There would then be considerable windup in the shaft (causing small fluctuations in applied torque), but little change in angular velocity at the bearing end.

* Note the equivalence between $\bar{T}$ and μ in Equations (16) and (20); the fact that θ_{st} is equal in Figures 5 and 7 is due to a choice of $\bar{T} = \mu = 0.25$

PLOT OF OSCILLATORY SLIDING FOR CONDITIONS
 OF $\mu = 0.25$
 $\bar{T} > 0.75$
 $\phi_0 = \phi_0' = 0$

FIGURE 7

Shaft motion begins with the shaft rolling from rest ($\theta = \dot{\theta} = 0$). Ensuing oscillations ($\theta(\tau)$) are plotted in Figure 8 for given values of shaft angular velocity $\dot{\theta}_0$ (or $\dot{\theta}_0 = \omega_0$, which proves to be simpler in the derivation). It may be seen that for large values of the shaft angular velocity the shaft center rises quickly through a large arc before slip begins. During the period of slip, the over and undershoot of the shaft center are large before the relative velocity at the point of contact vanishes and rolling commences once more. For smaller values of the shaft angular velocity, the arc described by the shaft center decreases as well as the over and under-shoot during slip. Also, the ratio of dwell times during slip and stick reduces as the shaft angular velocity becomes smaller. Finally, the period of a complete stick slip cycle is plotted in Figure 9 as a function of $\dot{\theta}_0$ (which is proportional to shaft angular velocity). As is indicated in Figure 8, only very long-period oscillations are excited when the shaft angular velocity is small. As the angular velocity is increased, the period of oscillations decreases until it reaches a zone where the period is very insensitive to $\dot{\theta}_0$; then after a minimum value the period rises gradually. Note that $\dot{\theta}_0$ must not be so large that it causes rolling out of the first quadrant with resultant falling of the shaft.

The torque required to maintain a constant shaft angular velocity is plotted as Figure 10. For large values of shaft angular velocity, the torque peaks rapidly. However, as soon as slip begins, the applied torque reverses in sign to keep the angular velocity of the shaft from increasing. When rolling recommences, the applied torque immediately jumps to the value necessary to offset any drop in shaft angular velocity. The same pattern is carried on for decreasing values of $\dot{\theta}_0$. Since the work done/cycle on the system is

$$W = \int_0^{\tau_p} \bar{T}(\tau) \bar{\omega}(\tau) d\tau, \quad \dots \dots \dots (22)$$

we recognize that the area under $\bar{T}(\tau)$ curve in Figure 10 is proportional to W because $\bar{\omega}(\tau) = \dot{\theta}_0$, a constant. Therefore, if the negative area is larger than the positive area, there is work done by the system, rather than on it. In that hypothetical case, the point of application of the torque would have to provide a means of dissipating the energy. Such an oscillation is self-sustaining, and the amplitude would increase until the shaft lost contact with the bearing surface.

For the system studied here, the work/cycle is plotted versus $\dot{\theta}_0$ in Figure 11. It is seen that for small values of shaft angular velocity, work must be done on the system to produce the oscillations. Beyond a threshold value of $\dot{\theta}_0$ ($= 0.11$ in this case), the work is done by the system, so that the oscillation is self-excited.

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STICK-SLIP OSCILLATION OF SHAFT IN BEARING WITH FOLLOWING CONDITIONS:

- $\mu_s = 0.50$ (STATIC COEFFICIENT OF FRICTION)
 $\mu_d = 0.25$ (DYNAMIC COEFFICIENT OF FRICTION)
 $\phi = \frac{\mu_s}{e}$, $\phi_0 =$ SHAFT ANGULAR VELOCITY
 (CONSTANT)
 $e = \frac{L R}{R}$, RATIO OF RADIAL CLEARANCE
 TO SHAFT RADIUS
 $\phi_0 =$ ANGLE AT WHICH SLIP BEGINS

FIGURE 8

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MEMO 11111

PERIOD OF STICK-SLIP OSCILLATIONS VS. SHAFT ANGULAR VELOCITY

T_p - PERIOD
 ω_s - SHAFT ANGULAR VELOCITY
 E - AR/R , CLEARANCE RATIO

FIGURE 9

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TORQUE APPLIED TO SHAFT DURING STICK-SLIP MOTION NECESSARY TO MAINTAIN CONSTANT SHAFT ANGULAR VELOCITY $\dot{\omega} = \epsilon \dot{\theta}_0$

$$\dot{\theta}_0 = \frac{1}{\epsilon} \dot{\omega}_0; \dot{\omega}_0 \sim \text{SHAFT ANGULAR VELOCITY (CONSTANT)}$$

$$\epsilon = \frac{DR}{P}, \text{ RATIO OF RADIAL CLEARANCE TO SHAFT RADIUS}$$

$\theta_0 = \text{ANGLE AT WHICH SLIP BEGINS}$

MEMO 11-1-1

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MEMO 11-1

VARIABLE PROPORTIONAL TO WORK / CYCLE DONE ON SYSTEM TO
 MAINTAIN STICK-SLIP OSCILLATION VS. SHAFT CENTER
 INITIAL ANGULAR VELOCITY

W - WORK / CYCLE
 $\dot{\theta}_0$ - SHAFT ANGULAR VELOCITY
 E - GRIP, CLEARANCE RATIO

FIGURE 11

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III. In-Plane Motion - Impact

Up to this point, only examples of oscillation of the shaft center in continuous contact with the bearing were considered. Of these, only the stick-slip oscillation was found to be self-sustaining; i.e., capable of sustaining a periodic motion in the presence of external damping forces. In this section, motion in which the shaft skips about the bearing, spending most of its time in free flight between short duration impacts on the bearing surface, is investigated. It is believed that rapid impacting of this type may be responsible for chatter in unlubricated bearings.

A. Impact on a Plane Surface

It is recognized that in order to produce a sustained chatter by in-plane impacting, some rotational energy of the shaft must be transferred during impact to translational energy to compensate for damping losses. (The hydraulic effect of any fluid present tending to produce whip in the bearing is assumed to be negligible). By way of introduction to the problem, it is assumed that the curvature of the bearing may be neglected. The bearing surface is then represented by a flat rigid plane which is struck by the shaft at some arbitrary angle of incidence α (See Figure 12 below):

Figure 12 - Impact on a Plane

Appendix F or reference 2 may be consulted for details of the derivation of the equations of impact. The results are presented below:

Slip during impact ($R\omega > V \sin \alpha$):

$$\begin{aligned} V' \sin \alpha' &= V \sin \alpha - \frac{1}{1 + e} V \cos \alpha \\ V' \cos \alpha' &= e V \cos \alpha \quad \dots \dots \dots (23) \\ \omega' &= \omega - \frac{1}{1 + e} \frac{V \cos \alpha}{R} \end{aligned}$$

Stick practically throughout impact ($\frac{1}{2} F_r > F_0$):

$$\begin{aligned} V' \sin \alpha' &= \frac{1}{1 + \frac{1}{2} \frac{I}{R^2}} V \sin \alpha - \frac{\frac{1}{2} \frac{I}{R^2}}{1 + \frac{1}{2} \frac{I}{R^2}} R \omega \\ V' \cos \alpha' &= e V \cos \alpha \quad \dots \dots \dots (24) \end{aligned}$$

$$\omega = \frac{V^2}{1 + \frac{1}{e^2}} \omega + \frac{1}{1 + \frac{1}{e^2}} (V \sin \alpha)/R \quad \dots \dots \dots (24, \text{cont'd})$$

All symbols have been defined previously except for the coefficient of restitution, e . We note that replacing μ by $-\mu$ in Equation (22) covers the case when $R\omega < V \sin \alpha$. The applied force and moment F_0 and T are assumed to be negligible compared to the inertia forces and torques during impact.

Consider the case when the shaft is already rotating at a constant angular velocity ω_0 when the shaft center acquires a linear velocity V due to dynamic unbalance, the application of F_0 , or some other mechanism. Assuming that $R\omega_0 > V \sin \alpha$, the impact involves slip, so that Equations (22) apply. It is seen that the tangential component of linear velocity $V' \sin \alpha'$ after impact is greater than that before. The radial component after impact, $V' \cos \alpha'$, never exceeds the value before impact since the coefficient of restitution does not exceed 1. The angular velocity is reduced by the impact, but we assume that the shaft can speed up to ω_0 again before the next impact. Also, the angle of rebound α' exceeds the angle of incidence because of the gain in the tangential component of velocity during impact.

Since it is assumed that the bearing surface can be adequately represented by a series of tangential planes, the shaft center will move freely through the bearing clearance before striking the (plane) surface again. In the absence of F_0 and hydrodynamic effects, the shaft center in free flight moves along a chord of the bearing circle and impacts the surface with a new angle of incidence equal to the angle of rebound preceding the free flight.

Figure 13 illustrates this motion:

Figure 13 - Successive Impacts on Plane Surface

Figure 13 gives an indication of what to expect from successive impacts. If some damping exists during free flight, the effect of the impacts is to offset this and increase $V \sin \alpha \rightarrow R\omega$. However, although V may increase, α increases as well, and the radial component of velocity, $V \cos \alpha$, will actually decrease since $e < 1$ for any practical system. Therefore, the effect of impact with slip is to cause the shaft center "chatter" motion to degenerate into a whirl. As the shaft comes up to "whirling speed", $R\omega = V \sin \alpha = R\omega_0$, it tends to stabilize in this orbit. If, before some impact, $V \sin \alpha > R\omega_0$ (before $V \cos \alpha$ has vanished entirely) friction forces during impact act to return the linear velocity of the shaft to the stable condition, and the shaft will eventually whirl about the bearing after $V \cos \alpha$ vanishes entirely. Even when the shaft is very nearly at whirling speed and Equation (23) may apply, the effect is seen to be the same: the tangential component of velocity increases while the radial component decreases slightly.

B. Impact on Curved Surface

Since the plane surface approximation does not reveal any mechanism for sustained chatter, the effect of a curved surface will be examined. The equations of motion (Equation (1)) are rewritten to include radial as well as tangential motion:

Figure 14 - Impact on a Curved Surface

$$\left. \begin{aligned} F_r - F_0 \cos \theta &= M (r \ddot{\theta}^2 - \dot{\theta}^2) \\ F_\theta - F_0 \sin \theta &= M (r \ddot{\theta} + 2 \dot{r} \dot{\theta}) \\ T - F_\theta R &= I \dot{\omega} \end{aligned} \right\} \dots \dots \dots (25)$$

As shown in Figure 1 (a), r is the radial distance of the shaft center from the bearing center. For impact, $r \geq \Delta R$.

If F_0 and T are specified, then there are three equations and five unknowns. The condition of slip ($F_\theta = \mu F_r$) or stick ($R\dot{\omega} = r\dot{\theta}$) provides a fourth equation, but another relationship is necessary to take care of the additional unknown, the radial motion. Since there is radial deformation of the shaft at the point of contact with the bearing (which is assumed rigid), there will exist some relationship between the radial force and the resultant deformation. For small deformations it will be assumed that the relationship is a linear one:

$$F_r = K_r (r - \Delta R) = K (r - \Delta R), \dots \dots \dots (26)$$

where K is the compressional spring constant of the shaft surface.

Taking the applied loads to be negligible compared to the inertia loads, we arrive at the following (dimensionless) set of equations (applied forces and torques have been removed):

Slip during impact (condition: $\frac{1}{E} \ddot{w} > \dot{\theta}$):

$$\left. \begin{aligned} \rho'' - \rho \dot{\theta}^2 + \rho - 1 &= 0 \\ \rho \dot{\theta} + 2 \rho' \dot{\theta} - \mu (\rho - 1) &= 0 \\ \frac{1}{E} \ddot{w} + \mu (\rho - 1) &= 0 \end{aligned} \right\} \dots \dots \dots (27)$$

Stick during impact (condition: $\mu(\rho-1) > \rho \ddot{\theta} + 2 \dot{\rho} \dot{\theta}$):

$$\left. \begin{aligned} \ddot{\rho} - \rho \dot{\theta}^2 + \rho - 1 &= 0 \\ (1 + \gamma^2) \rho \ddot{\theta} + (2 + \gamma^2) \dot{\rho} \dot{\theta} &= 0 \\ \frac{1}{L} \bar{W} &= \rho \dot{\theta} \end{aligned} \right\} \dots \dots \dots (28)$$

where $\rho \equiv r/\Delta R$
 $\dot{\theta} \equiv d\theta/dt$ $t \equiv t\sqrt{R/M}$
 $\bar{W} \equiv W\sqrt{R/M}$

Other terms have been defined previously. The deviation of Equations (27) and (28) (28) are given in App. F.

Equations (27) and (28) are both nonlinear, although (28) may be integrated directly. No simple method has been found to solve Equation (27). Therefore, solutions for slip during impact have been obtained by utilizing the following observations:

(a) The initial conditions at the beginning of impact for $\dot{\rho}$ and $\rho \dot{\theta}$ are
 $\dot{\rho}_0 = \frac{V}{\Delta R \sqrt{R/M}} \cos \alpha$ and $(\rho \dot{\theta})_0 = \dot{\theta}_0 = \frac{V}{\Delta R \sqrt{R/M}} \sin \alpha$ (see Fig. 12)
 from the definitions of $\dot{\rho}$ and $\rho \dot{\theta}$.

(b) The parameter $\delta \equiv \frac{V}{\Delta R \sqrt{R/M}}$ is considered small because the compressional spring constant k is large enough to offset the smallness of ΔR in the denominator.

A solution in the form of a series in powers of δ with time dependent coefficients may therefore be assumed:

$$\left. \begin{aligned} \rho(t) &= 1 + \delta x_1(t) + \delta^2 x_2(t) + \dots \\ \dot{\theta}(t) &= \delta y_1(t) + \delta^2 y_2(t) + \dots \end{aligned} \right\} \dots \dots \dots (29)$$

Following the method outlined in App. F, the functions $x_1(t)$, $x_2(t)$, $y_1(t)$, $y_2(t)$, which satisfy Equation (27) and the initial conditions of $x_1(0) = x_2(0) = y_1(0) = y_2(0) = 0$ and $\dot{x}_1(0) = y_1(0) = 1$ and $\dot{x}_2(0) = \dot{y}_2(0) = 0$ are

$$\left. \begin{aligned} x_1 &= \cos \alpha \sin t \\ x_2 &= \left[(\sin \alpha + \mu \cos \alpha)^2 + \frac{1}{2} \mu^2 \cos^2 \alpha \right] - \left[\frac{1}{3} \mu^2 \cos^2 \alpha + (\sin \alpha + \mu \cos \alpha)^2 \right] \cos t - \mu \cos \alpha (\sin \alpha + \mu \cos \alpha) t \sin t \\ &\quad - \frac{1}{6} \mu^2 \cos^2 \alpha \cos 2t \end{aligned} \right\} \dots (30)$$

* The value of θ at which impact occurs is immaterial and may be taken to zero.

$$y_1 = k \sin \alpha + \mu \cos \alpha (k - \sin k)$$

$$y_2 = \frac{1}{2} \mu [(\sin \alpha + \mu \cos \alpha)^2 + \frac{1}{2} \mu^2 \cos^2 \alpha + \frac{1}{2} \cos^2 \alpha] k^2 + \frac{1}{2} \mu^2 \cos^2 \alpha (\sin \alpha + \mu \cos \alpha) + 2 \cos \alpha (\sin \alpha + \mu \cos \alpha) \dots (30 \text{ Cont})$$

$$+ \frac{1}{3} \mu^2 \cos^2 \alpha + (\sin \alpha + \mu \cos \alpha)^2 \frac{1}{3} \cos k - \frac{1}{4} \mu \cos^2 \alpha (3/2 - 1/6 \mu^2) \cos 2k + \mu^2 \cos \alpha (\sin \alpha + \mu \cos \alpha) \times \cos \alpha + k \sin k - \frac{1}{2} \sin \alpha \cos \alpha (1 + 2 \mu^2) + \mu \cos 2\alpha$$

$$(5/8 + 27/8 \mu^2) + \mu$$

To determine what conditions are necessary to transfer energy to the radial motion, during impact the radial velocity at the end of impact, $\dot{\rho}(t_p)$, must be inspected. The impact duration t_p is determined by the condition $\rho(t_p) = 1$. Substituting this condition into Equation (28), we obtain

$$x_1(t_p) + \delta x_2(t_p) + \dots = 0 \dots \dots \dots (31)$$

an equation containing an infinite number of transcendental terms. If $\delta \ll 1$, we can get a good approximation for t_p by neglecting all terms in Equation (31) except the first. Then, from Equation (29),

$$t_p \approx \pi \dots \dots \dots (32)$$

Velocities at the end of impact are then

$$\dot{\rho}(t_p) = \delta x_1'(t_p) + \delta^2 x_2'(t_p) +$$

$$\dot{\theta}(t_p) = \delta y_1'(t_p) + \delta^2 y_2'(t_p) +$$

where $x_1'(t_p) \approx -\cos \alpha$

$$x_2'(t_p) \approx \mu \cos \alpha (\sin \alpha + \mu \cos \alpha)$$

$$y_1'(t_p) \approx \sin \alpha + 2 \mu \cos \alpha$$

$$y_2'(t_p) \approx \pi \mu [1 - 1/2 \cos^2 \alpha (1 - \mu^2) + \mu \sin \alpha \cos \alpha]$$

..... (33)

Comparison of Equation (33) with (23) shows that after dimensionalizing and setting $\Delta R \dot{\theta} = V \sin \alpha$ and $\dot{r} = V \cos \alpha$, the first terms in the series for $\dot{\rho}(t_p)$ and $\dot{\theta}(t_p)$ are identical with the rebound velocities from a plane surface if the coefficient of restitution $e = 1$. Higher terms in the series may therefore be interpreted as corrections to the solution for impact on a plane surface which account for the curvature of the surface. The second term in the series for $\dot{\rho}(t_p)$ shows that as long as $\frac{1}{2} \bar{w} > \rho \theta$, ("positive" slip), the absolute value of the radial velocity at rebound is less than that at impact and consequently energy is not fed into radial motion under these conditions. (Higher terms in the series were not examined since their contributions would be very small). However, when $\frac{1}{2} \bar{w} < \rho \theta$ ("negative" slip, when the shaft rotation is being slowed down deliberately, for example), the μ appearing in Equations (27), (30), and (33) is replaced by $-\mu$. Then from Equation (33) it can be seen that $x_2(t_p)$ will add to $x_1(t_p)$ if

$$\tan \alpha > \mu \dots \dots \dots (34)$$

However, inspection of Equation (33) shows that if $\frac{1}{k} \dot{\omega} < \rho \dot{\theta}$, each impact will act to correct the inequality; i.e., reduce $\dot{\theta}$ (if $\dot{\omega}$ is being continuously slowed down). This increase in radial velocity and decrease in tangential velocity produces a decrease in the angle of incidence until Equation (34) is no longer satisfied. Therefore, transfer of energy to radial motion during an impact with slip is at best a transient affair which occurs only when $\frac{1}{k} \dot{\omega} < \rho \dot{\theta}$. This would happen only when the shaft's rotation is reduced by the application of torque T.

In the event that the shaft rolls during impact, solutions to Equation (28) apply. We note first that the condition for rolling which heads Equation (28), $\mu(\rho - 1) > \rho \dot{\theta} + 2 \dot{\rho} \dot{\theta}$, cannot be satisfied at the initial instant of contact. This may be easily proved by substituting for $\rho \dot{\theta}$ from the second of Equation (28). Then the condition for rolling becomes

$$\mu(\rho - 1) > \frac{1}{1 + \eta^2} \dot{\rho} \dot{\theta} \quad \dots \dots \dots (35)$$

At the instant of contact $\rho = 1$, while $\dot{\rho}$ and $\dot{\theta}$ are both positive. Hence, slip must take place initially, but to determine the effect of impact with stick, we assume that rolling takes place practically throughout the period of contact. The solution then is written (see App. F) as:

$$\begin{aligned} \dot{\rho} &= \frac{1}{k} \sqrt{\dot{\rho}_0^2 + \frac{\theta_0^2}{1} [1 - \rho^{-2(k-1)}]} \cdot \sqrt{1 + \eta^2} \\ \dot{\theta} &= \dot{\theta}_0 / \rho k \end{aligned} \quad \dots \dots \dots (36)$$

where the initial conditions are

$$\begin{aligned} \rho(0) &= 1 \\ \dot{\theta}(0) &= \dot{\theta}_0 \end{aligned}$$

$$\text{and } \eta = 2 + \dot{\rho}_0^2 / 1 + \dot{\theta}_0^2.$$

Equation (36) show that at the end of contact when $\rho = 1$ once again, both tangential and radial velocities are unchanged in magnitude, as would be expected. Hence, no transfer of energy is produced during impact with rolling.

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APPENDIX A

Examination of Separation Curve Equations for Figure 4.

Rewriting Equation (11) for the limit curves $\bar{T}(\theta)$, we obtain

$$\left. \begin{aligned} \bar{T} &= \frac{1 - \cos \theta}{\theta}, \quad 0 \leq \theta \leq \pi/2 \\ \bar{T} &= \frac{2 - (3 + \mu^2) \cos \theta}{2\theta}, \quad \pi/2 \leq \theta \leq \pi \\ \bar{T} &= \frac{(3 + \mu^2)\mu \cos \theta - 2\mu - \mu^2 \sin \theta}{1 - 2\mu\theta}, \quad 0 \leq \theta \leq \pi \end{aligned} \right\} \dots \dots \dots (A-1)$$

These curves separate the region of continuous rolling from the regions of oscillation, falling, and slipping, respectively. Comparison of the first two of Equation (A-1) with the first two inequalities of Equation (11) shows that continuous rolling is produced by values of $\bar{T}$ greater than those calculated from the equations above. For example, any value of $\bar{T}$ which intersects the curve given by the first of Equation (A-1) is not sufficient to roll the shaft further and rolling oscillations result. If a given $\bar{T}$ intersects the curve given by the second of Equation (A-1), the shaft loses contact with the bearing and falls because the magnitude of $\bar{T}$ is insufficient to accelerate the shaft to a whirling velocity adequate to maintain contact with the bearing throughout the second quadrant of travel.

Closer inspection of the third of Equation (A-1) shows that even if $\bar{T}$ is large enough to assure contact between the shaft and the bearing in the second quadrant, the shaft begins to slip back before it ever enters that region. This result may be shown most simply by rewriting the last of Equation (11) in the form

$$f(\mu, \bar{T}, \mu^2, \theta) = \mu - \frac{\bar{T} \mu^2 \sin \theta}{2\bar{T}\theta - 2\mu(3 + \mu^2) \cos \theta} \dots \dots \dots (A-2)$$

and asking what conditions are necessary for $f > 0$ (to assure rolling without slipping). The limit curve is given by $f = 0$, for which $\bar{T}$ is defined in terms of θ , μ , and μ^2 by the last of Equation (A-1). We now ask: What is the effect on f if we fix θ , μ , and μ^2 , but vary $\bar{T}$ by a small amount $\Delta \bar{T}$? Is $(\partial f / \partial \bar{T}) \Delta \bar{T} \geq 0$? If it is positive, then the condition of rolling without slipping is assured; if it is negative, the shaft will slip. Carrying out the partial differentiation, we obtain:

$$\frac{\partial f}{\partial \bar{T}} \Delta \bar{T} = \frac{[(3 + \mu^2) \cos \theta - 2(1 + \mu^2 \theta \sin \theta)] \Delta \bar{T}}{[2\bar{T}\theta - 2\mu(3 + \mu^2) \cos \theta]^2} \dots \dots \dots (A-3)$$

Rolling without slipping will occur if

$$[(3 + \mu^2) \cos \theta - 2(1 + \mu^2 \theta \sin \theta)] \Delta \bar{T} > 0, \dots \dots \dots (A-4)$$

Since the denominator of Equation (A-3) is never negative. Thus, if

$$(3 + \mu^2) \cos \theta > 2(1 + \mu^2 \theta \sin \theta), \Delta \bar{T} < 0, \dots \dots \dots (A-5)$$

The region of pure rolling lies below the curve defined by the last of Equation (A-1). However, if

$$(3 \mu^2) \cos \theta < 2 (1 \mu^2 \sin \theta) \Delta T > 0, \dots \dots \dots (A-6)$$

the region of pure rolling lies above the curve. It may be seen that for $\theta \rightarrow 0$, Equation (A-5) will be satisfied, and consequently the region of pure rolling must lie initially below the curves separating the regions of slipping and rolling. However, as θ increases, a point is reached where

$$(3 \mu^2) \cos \theta = 2 (1 \mu^2 \sin \theta) \dots \dots \dots (A-7)$$

and for values of θ greater than those obtained in Equation (A-7), the region of pure rolling lies above the curve. Because of the boundary condition that $\bar{T}$ is applied as a step and held constant, it may be seen that the value of θ defined in Equation (A-7) represents an upper limit to the arc through which the shaft will roll. A second point of interest is that for small values of the coefficient of friction μ and moderate torques $\bar{T}$, the shaft will begin to slip as soon as it starts to move. This is due to the fact that $\bar{T}(\theta)$ as given by the last of Equation (A-1) either increases or decreases monotonically with θ depending on the value of μ . The limit is obtained by setting $(\partial \bar{T} / \partial \theta) \theta = 0$ to zero. The value of μ is then found to be:

$$\mu_{\min} = \frac{2}{\sqrt{1 + \gamma^2}} \dots \dots \dots (A-8)$$

Values of μ necessary to prevent initial slippage at maximum initial $\bar{T}$ are given in Figure 4 (a), (b), and (c).

We conclude, therefore, that rolling of the transversely loaded shaft completely around the bearing is impossible if a step torque is applied to the shaft when it is initially at rest. The maximum angle to which it can roll (before slipping or rolling dead) is found from Equation (A-7) by setting $\mu^2 = 0$:

$$\theta_{\max} \approx 48^\circ (\mu^2 \rightarrow 0) \dots \dots \dots (A-9)$$

It is seen that the so-called "intermediate" value of coefficient of friction (see page 9) is associated with $\theta_{\max}$. This results from the fact that the "discrete" value of $\bar{T}$ occurs when the numerator and denominator of the last of Equation (A-1) both vanish identically. Then

$$\left. \begin{aligned} (3 \mu^2) \cos \theta - 2 \mu - \mu^2 \sin \theta &= 0 \\ 1 - 2 \mu \theta &= 0 \end{aligned} \right\} \dots \dots \dots (A-10)$$

By substituting for μ into the first of Equation (A-10), we obtain Equation (A-7). Solving for $\theta_{\max}$, we may then find μ_{int} by substituting into the second of Equation (A-10).

The range of $\bar{T}$'s which will accelerate the shaft to $\theta_{\max}$ is bounded on the lower side by a value which is just large enough to produce slip rather than stick and on the upper side by a value which is not quite large enough to cause initial rolling when the coefficient of friction is μ_{int} . These limiting values are given by the equations below:

$$\left. \begin{aligned} \bar{T}_{i_{min}} &= \frac{1 - \cos \theta_{max}}{\theta_{max}} \\ \bar{T}_{i_{max}} &= (1 / \eta^2) \bar{\tau}_{int} \end{aligned} \right\} \dots \dots \dots (A-11)$$

Note that $\bar{\tau}_{int}$ must satisfy the condition given by Equation (10-8).

APPENDIX BCalculations for Oscillatory Rolling

When a step torque $\bar{T}$ is applied to a transversely loaded shaft initially at rest, the rolling solution may be written from Equation (10) by setting $\theta_0 = \dot{\theta}_0 = 0^*$:

$$\tau(\theta) = \int_0^\theta \frac{d\theta'}{\frac{1}{\theta'}} \quad , \quad \text{where} \quad \left. \begin{aligned} \theta &= \sqrt{\frac{2}{1-f^2}} \sqrt{\bar{T}\theta - (1 - \cos \theta)} \end{aligned} \right\} \dots \dots \dots (B-1)$$

If the integrand $1/\theta$ is well behaved, $\tau(\theta)$ can be found easily by quadratures. However, for oscillatory rolling, $\theta = 0$ at $\theta = 0$ and $\theta = \theta_m$ (determined by setting the second of Equation (B-1) to zero). At these points the integrand in the first of Equation (B-1) becomes infinite, thereby making numerical or graphical integration impossible.

By subdividing the range of integration, this difficulty can be avoided, as will be shown. Consider the equation for the half-period which contains both units of infinite integrands in it:

$$\tau(\theta) = \tau_p = \left[\int_0^{\Delta\theta_1} \frac{d\theta'}{\frac{1}{\theta'}} + \int_0^{\theta_m - \Delta\theta_2} \frac{d\theta'}{\frac{1}{\theta'}} + \int_{\theta_m - \Delta\theta_2}^{\theta_m} \frac{d\theta'}{\frac{1}{\theta'}} \right] \dots (B-2)$$

$$\text{where } \Delta\theta_1 \rightarrow 0$$

$$\Delta\theta_2 \rightarrow 0$$

Using the mean value theorem, we write the first and last terms on the right hand side of equation (B-2) as

$$\int_0^{\Delta\theta_1} \frac{d\theta'}{\frac{1}{\theta'}} \approx \sqrt{\frac{1-f^2}{2}} \frac{1}{\sqrt{\bar{T}}} \int_0^{\Delta\theta_1} \frac{d\theta'}{\sqrt{\theta'}} = \sqrt{\frac{1-f^2}{2}} \cdot 2 \sqrt{\frac{\Delta\theta_1}{\bar{T}}}$$

$$\int_{\theta_m - \Delta\theta_2}^{\theta_m} \frac{d\theta'}{\frac{1}{\theta'}} \approx \sqrt{\frac{1-f^2}{2}} \frac{1}{\sqrt{\sin \theta_m - \bar{T}}} \int_0^{\Delta\theta_2} \frac{d\theta'}{\sqrt{\theta'}} = \sqrt{\frac{1-f^2}{2}} \cdot 2 \sqrt{\frac{\Delta\theta_2}{\sin \theta_m - \bar{T}}}$$

* The $\pm$ sign associated with θ indicates that the solution admits of oscillatory rolling; i.e., for a given angle, the shaft can be either rolling up or down.

Thus, the numerical or graphical integration may be carried out to any degree of accuracy by making $\Delta\theta_1$ and $\Delta\theta_2$ sufficiently small. It was found that taking $\Delta\theta_1$ or $\Delta\theta_2$ less than 0.05 did not improve three-place accuracy in the results and this value was used for all integrations.

APPENDIX CInvestigation of Conditions of Slip.

Figure 6 is divided into regions of continuous slip, oscillatory slip, and falling by curves defined by the equations below (taken from conditions (19)):

$$\left. \begin{aligned} -(1 - 2\mu^2)e^{2\mu\theta} + (1 - 2\mu^2)\cos\theta + 3\mu\sin\theta &= 0 \quad (0 < \theta < \pi/2) \\ -2(1 - 2\mu^2)e^{2\mu\theta} + 3(\cos\theta + \mu\sin\theta) &= 0 \quad (\pi/2 < \theta < \pi) \end{aligned} \right\} \dots (C-1)$$

To determine the location of the three regions with relation to the curves, we define the function $g(\mu, \theta)$ to be equal to the left hand sides of Equation (C-1) in the proper range of θ . We now ask: for a small vertical excursion from the $\mu(\theta)$ curve defined by setting $g(\mu, \theta) = 0$, is $g(\mu + \Delta\mu, \theta) \geq 0$? If $g(\mu + \Delta\mu, \theta)$ is positive, condition (19) is satisfied, and we are in the region of continuous slip; conversely if $g(\mu + \Delta\mu, \theta)$ is negative we are either in the range of oscillatory slip ($0 < \theta < \pi/2$) or falling ($\pi/2 < \theta < \pi$).

$g(\mu + \Delta\mu, \theta)$ can be expanded about the point $g(\mu, \theta)$:

$$\left. \begin{aligned} g(\mu + \Delta\mu, \theta) &= g(\mu, \theta) + \frac{\partial g}{\partial \mu} \Delta\mu + \dots \\ &\approx \frac{\partial g}{\partial \mu} \Delta\mu \end{aligned} \right\} \dots (C-2)$$

(θ is held fixed). Carrying out the differentiation, we obtain

$$\frac{\partial g}{\partial \mu} = \left\{ \begin{aligned} -\{[2(1 - 2\mu^2) - 4\mu]e^{2\mu\theta} + 4\mu\cos\theta + 3\sin\theta\}\Delta\mu, \quad (0 < \theta < \pi/2) \\ \{[-4(1 - 2\mu^2)\theta + 8\mu]e^{2\mu\theta} + 3\sin\theta\}\Delta\mu \quad (\pi/2 < \theta < \pi) \end{aligned} \right\} \dots (C-3)$$

Substitution of calculated values for $\mu(\theta)$ at various values of θ for Figure 6 shows that the coefficients of $\Delta\mu$ in Equation (C-3) are always positive. Thus, if $\Delta\mu$ is positive (that is, we explore the region above the $\mu(\theta)$ curve), continuous slipping is produced. The regions are so indicated in the figure.

APPENDIX D

Calculations for Oscillatory Sliding.

If a step torque of sufficient magnitude to cause slip is applied to a shaft initially at rest in a bearing, it will excite oscillatory sliding if $0 \leq \mu \leq 0.6$. The motion can be calculated from the following equations (taken from Equation (18) by setting $\theta_0 = \dot{\theta}_0 = 0$):

$$\tau(\theta) = \int_0^\theta \frac{d\theta'}{\frac{d\theta'}{dt}}, \text{ where} \quad \left. \begin{aligned} \theta^2 = \frac{2}{1 + 4\mu^2} \left[-(1 - 2\mu^2)e^{2\mu\theta} + (1 - 2\mu^2) \cos \theta + 3\mu \sin \theta \right] \end{aligned} \right\} \dots (D-1)$$

The integration may be carried out by quadratures. As in App. B, it should be recognized that the integrand becomes infinite at the lower and upper limits of travel ($\theta = 0$ and $\theta = \Theta$, defined by setting the second of Equation (D-1) equal to zero). By limiting the range of numerical or graphical integration to $\Delta\theta_1 < \theta' < \Theta - \Delta\theta_2$ as described in App. B, we can evaluate the integral within the small intervals $\Delta\theta_1$ and $\Delta\theta_2$ by use of the mean value theorem. The contributions to the calculation of τ are shown in the relation for the half period $\tau_p/2$:

$$\tau(\Theta) = \frac{\tau_p}{2} = \sqrt{\frac{1 - 4\mu^2}{2}} \left[\int_0^{\Delta\theta_1} \frac{d\theta'}{\frac{d\theta'}{dt}} + \int_{\Delta\theta_1}^{\Theta - \Delta\theta_2} \frac{d\theta'}{\frac{d\theta'}{dt}} + \int_{\Theta - \Delta\theta_2}^{\Theta} \frac{d\theta'}{\frac{d\theta'}{dt}} \right] \quad \dots (D-2)$$

where $\int_0^{\Delta\theta_1} \frac{d\theta'}{\frac{d\theta'}{dt}} = \frac{2\sqrt{\Delta\theta_1}}{\sqrt{\mu(1 + 4\mu^2)}}$

$$\int_{\Theta - \Delta\theta_2}^{\Theta} \frac{d\theta'}{\frac{d\theta'}{dt}} = \frac{2\sqrt{\Delta\theta_2}}{\sqrt{(1 - 2\mu^2)(2\mu e^{2\mu\Theta} + \sin \Theta) - 3\mu \cos \Theta}}$$

APPENDIX E

Equations of Motion for Stick-Slip Oscillation

As adapted from Equations (9), (17), and (18), the necessary solutions to the equations of motion for stick-slip oscillation when $\omega = \omega_0$, a constant, are:

$$\left. \begin{aligned} \dot{\theta}_0 &= \frac{1}{k} \bar{\omega}_0 \\ \theta &= \theta_0 + \dot{\theta}_0 \tau \\ \bar{T} &= \sin \theta \end{aligned} \right\} \dots \dots \dots (E-1)$$

during rolling, and

$$\begin{aligned} \tau - \tau_0 &= \int_{\theta_0}^{\theta} \frac{d\theta}{\dot{\theta}}; \text{ where} \\ \dot{\theta} &= \pm \left[e^{2\mu_1 (\theta - \theta_0)} \left\{ \dot{\theta}_0^2 - \frac{2}{1 + 4\mu_1^2} \times \left[(1 - 2\mu_1^2) \cos \theta_0 + 3\mu_1 \sin \theta_0 \right] \right. \right. \\ &\quad \left. \left. + \frac{3}{1 + 4\mu_1^2} \left[\cos \theta + \mu_1 \sin \theta \right] \right\} \right]^{\frac{1}{2}} \\ \bar{T} &= \mu_1 e^{2\mu_1 (\theta - \theta_0)} \left\{ \dot{\theta}_0^2 - \frac{2}{1 + 4\mu_1^2} \left[(1 - 2\mu_1^2) \cos \theta_0 + \right. \right. \\ &\quad \left. \left. 3\mu_1 \sin \theta_0 \right] \right\} + \frac{3}{1 + 4\mu_1^2} \left[\cos \theta + \mu_1 \sin \theta \right] \end{aligned} \quad (E-2)$$

during sliding. μ_0 and μ_1 are the static and dynamic coefficients of friction. The first of Equation (E-2) may be solved by quadratures as described in App. B.

Rolling (Equation E-2) prevails until the shaft center reaches θ_0 when $F_{\theta} = \mu_0 F_r$, or by Equation (9),

$$\sin \theta_0 = \mu_0 (\dot{\theta}_0^2 + \cos \theta_0) \dots \dots \dots (E-3)$$

and sliding continues until

$$\dot{\theta} = \pm \bar{\omega}_0 = \pm \dot{\theta}_0 \dots \dots \dots (E-4)$$

The plus sign is included in Equation (E-4) to indicate that upon slipping, the shaft first falls back with a negative shaft center angular velocity and under-shoots before the angular velocity becomes positive. After the second pass when $\theta > 0$, rolling will begin again.

APPENDIX F

Impact Equations

A. Impact of Shaft on Plane

From reference (2), p. 472, the equations of impact of a shaft (or a disk) on a plane surface may be written as follows:

Figure F-1 - Impact on a Plane

Assuming that during impact, F_0 and T are negligible compared to the impact forces and torques F_x , F_y , and $F_x R$, we write the impact equations as

$$\left. \begin{aligned} \int_0^{t_p} F_x dt &= M(V' \sin \alpha' - V \sin \alpha) \\ \int_0^{t_p} F_y dt &= M(V' \cos \alpha' - V \cos \alpha) \\ \int_0^{t_p} F_x R dt &= -I(\omega' - \omega) \end{aligned} \right\} \dots \dots \dots (F-1)$$

where t_p is the duration of impact and α and α' the angles of incidence and rebound.

The magnitude of the radial component of velocity is unchanged except for a coefficient of restitution correction:

$$V' \cos \alpha' = e V \cos \alpha \dots \dots \dots (F-2)$$

A. Impact of Shaft on Plane (Cont'd)

The two conditions which may exist during impact are sliding and rolling. For each of these conditions, the following relations must hold:

$$\left. \begin{aligned}
 &\text{Sliding } (R\omega > V \sin \alpha \text{ and } R\omega > V' \sin \alpha) \\
 &\quad F_x = \mu F_y \\
 &\text{Rolling } (\mu F_y > F_x)^* \\
 &\quad R\omega' = V' \sin \alpha
 \end{aligned} \right\} \dots \dots \dots (F-3)$$

Equations (F-1) through (F-3) may then be combined to obtain Equations (22) and (23).

B. Impact of Shaft on Cylindrical Surface

Reference is made to Figure 14 and Equations (25) and (26). If sliding occurs during impact, $F_\theta = \mu F_r = \mu k(r - \Delta R)$, $(R\omega > r\dot{\theta})$. If rolling occurs, $R\omega = r\dot{\theta}$ ($\mu F_r > F_\theta$). Equations (24) and (25) may be combined with these conditions to yield:

$$\left. \begin{aligned}
 &\text{Sliding, } R\omega > r\dot{\theta}: \\
 &\quad k(r - \Delta R) - F_0 \cos \theta = M(r\ddot{\theta} - \ddot{r}) \\
 &\quad \mu k(r - \Delta R) - F_0 \sin \theta = M(r\ddot{\theta} / 2 + \ddot{r} \dot{\theta}) \\
 &\quad T - \mu R k(r - \Delta R) = I \ddot{\omega} \\
 &\text{Rolling, } \mu F_r > F_\theta: \\
 &\quad k(r - \Delta R) - F_0 \cos \theta = M(r\ddot{\theta} - \ddot{r}) \\
 &\quad F_\theta - F_0 \sin \theta = M(r\ddot{\theta} / 2 + \ddot{r} \dot{\theta}) \\
 &\quad T - F_\theta R = I (r\ddot{\theta} / r\dot{\theta})
 \end{aligned} \right\} \dots \dots \dots (F-4)$$

Equation (F-4) are non-dimensionalized by dividing by $k\Delta R$ and $k\Delta RR$. Since $F_0 \ll k\Delta R$ and $T \ll k\Delta RR$, they are neglected. The results are found as Equations (27) and (28).

A solution to Equation (27) is obtained by substituting the solutions

* Reference 2 uses the condition $\int_0^{t_p} F_y dt > \int_0^{t_p} F_x dt$ which is less stringent and realistic, since it admits some slip during impact. That it must is demonstrated by the fact that $R\omega \neq V \sin \alpha$ initially.

B. Impact of Shaft on Cylindrical Surface (Cont'd)

$$\left. \begin{aligned} \rho(t) &= 1 + \delta x_1(t) + \delta^2 x_2(t) + \dots \\ \dot{\theta}(t) &= \delta y_1(t) + \delta^2 y_2(t) + \dots \end{aligned} \right\} \dots \dots \dots (F-5)$$

(where $\delta = V/AR \sqrt{k/M}$) into Equation (27). The result is

$$\left. \begin{aligned} \delta(x_1'' + x_1) + \delta^2(x_2'' + x_2 - \dot{y}_1^2) + \dots &= 0 \\ \delta(y_1'' + \mu x_1) + \delta^2(y_1'' + x_1 \dot{y}_1 + 2 \dot{x}_1 \dot{y}_1 - \mu x_2) + \dots &= 0 \end{aligned} \right\} \dots \dots \dots (F-6)$$

Since δ may take on arbitrary values, its coefficients in each of Equation (F-6) must vanish independently. The resulting differential equations

$$\left. \begin{aligned} x_1'' + x_1 &= 0 \\ y_1'' &= -\mu x_1 \\ x_2'' + x_2 &= \dot{y}_1^2 \\ y_2 &= -(x_1 \dot{y}_1 + 2 \dot{x}_1 \dot{y}_1 - \mu x_2) \end{aligned} \right\} \dots \dots \dots (F-7)$$

may be solved stepwise by straightforward, if increasingly lengthy, integration. The results appear in Equation (30).

Solutions to Equation (28) are relatively simple because the variables are separable in the second of these equations:

$$\frac{\rho'}{\rho} = - \frac{1 + \mu^2}{2 + \mu^2} \frac{\theta'}{\theta} \dots \dots \dots (F-8)$$

Equation (F-8) is integrated directly and is integrated directly and appears as the second of Equation (36). This solution is then used to substitute for θ in the first of Equation (27):

$$\rho'' - \frac{\mu^2}{2} \rho^{-2k+1} + \rho - 1 = 0 \dots \dots \dots (F-9)$$

This may be integrated by multiplying each term by ρ' and noticing that the product may then be written as

$$\frac{d}{dt} \left[\frac{1}{2} \rho'^2 + \frac{\mu^2}{2(k-1)} \rho^{-2(k-1)} + \frac{1}{2} \rho^2 - \rho \right] = 0 \dots \dots (F-10)$$

With the initial conditions that $\rho(0) = 1$ and $\rho'(0) = \rho'_0$, the solution appears as the first of Equation (36).

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