QUANTUM DYNAMICS OF A MASSLESS RELATIVISTIC STRING II

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ABSTRACT

The massless relativistic free string is studied in the gauge $x_0 = \tau$. It is found that the classical solutions include transverse and longitudinal vibrations. The problem is treated both in the Lagrangian and Hamiltonian formalism. Different ways of quantizing the system are investigated. The path integral quantization leads to a Poincaré invariant quantum theory in any number of dimensions.

1. Introduction

The massless relativistic string is an interesting system, to which much attention has been devoted lately⁽¹⁾. The most complete treatment has been given by Goddard, Goldstone, Rebbi and Thorn⁽²⁾ (GGRT). They found that a relativistically invariant quantum mechanical description of the string can be given only in 26 dimensions (1 time) and when the lowest lying excited state is a tachion. They also found that there exist no non-trivial motions of a string in two dimensions.

These conclusions followed from the fact that the action of the string being proportional to the area swept by the string in space-time, it is invariant under a reparametrization of the surface

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GGRT used the freedom of parametrizing the surface to choose τ , one of the two parameters necessary to describe the surface, proportional to $x_+ = \frac{1}{\sqrt{2}} (x_0 + x_3)$.

In the present paper we show that such a choice eliminates possible motions of the string. In particular we find that there exist non-trivial motions of the string in two dimensions, that is purely longitudinal vibrations in a space of higher dimensions. The absence of these motions is the reason for the breakdown of Lorentz invariance found by GGRT.

The impossibility of choosing $\tau \propto x_+$ leads to complications when we attempt quantization. We are faced with quantizing a system having an infinite number of degrees of freedom and constraints. Thus we can attempt either the manifestly covariant quantization or Feynman's path integral approach. We discuss both methods and point out difficulties associated with the covariant quantization. The path integral formalism allows the determination of the energy spectrum. We find a spectrum typical of a system of harmonic oscillators: equally spaced energy levels, starting at some value α (which we do not compute). Our treatment of the path integral can be used in the case of an arbitrary number of dimensions and when interactions of strings are included. It also has the virtue of making transparent Lorentz invariance at the quantum level.

The organization of the paper is as follows: in Section 2 we review the classical string and show that it is impossible to choose $\tau \propto x_+$. Section 3 contains the Lagrangian and Hamiltonian formalism in the gauge $x_0 = \tau$. The latter would be useful for canonical quantization. Classical solutions in two dimensions are thoroughly analyzed. Section 4 contains a discussion of different methods of quantization of the string. The Lorentz invariance of the path integral formalism is investigated in Section 5. There we comment also on the possibility of using such techniques to study interacting strings.

2. Review of classical mechanics of the string.

The dynamics of the string is contained in its action which is taken proportional to the area swept out by the string:

$$S = - \frac{1}{2\pi\alpha'hc^2} \int_{\tau_i}^{\tau_f} d\tau \int_0^\pi d\sigma \left\{ \left(\frac{\partial x}{\partial \sigma} \cdot \frac{\partial x}{\partial \tau} \right)^2 - \left(\frac{\partial x}{\partial \sigma} \right)^2 \left(\frac{\partial x}{\partial \tau} \right)^2 \right\}^{\frac{1}{2}}$$

$$= \int_{\tau_i}^{\tau_f} d\tau \int_0^\pi d\tau \quad (2.1)$$

In the above expression $X^\mu(\sigma, \tau)$ are the coordinate vectors in physical space of points on the surface; σ and τ are parameters used to label points on the surface; α' , $\hbar$ and c^2 are dimensional constants which we shall choose to be 1. The string being a physical system, the surface cannot be space-like, so that:

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$$\left(\frac{\partial x}{\partial \sigma} \cdot \frac{\partial x}{\partial \tau} \right)^2 - \left(\frac{\partial x}{\partial \sigma} \right)^2 \left(\frac{\partial x}{\partial \tau} \right)^2 \geq 0 \quad (2.2)$$

(our metric is $-g^{00} = g^{ii} = 1$). Also since we consider only free (non-interacting) strings, the surface must be simply connected.

The principle of least action applied to (2.1) yields the following equation of motion and boundary conditions

$$\frac{\partial}{\partial \tau} \frac{\partial \mathcal{L}}{\partial \dot{x}^\mu} + \frac{\partial}{\partial \sigma} \frac{\partial \mathcal{L}}{\partial x'^\mu} = 0 \quad (2.3a)$$

$$\frac{\partial \mathcal{L}}{\partial x'^\mu} (0, \tau) = \frac{\partial \mathcal{L}}{\partial x'^\mu} (\pi, \tau) = 0 \quad (2.3b)$$

where $\dot{x}^\mu \equiv \frac{\partial x^\mu}{\partial \tau}$ and $x'^\mu \equiv \frac{\partial x^\mu}{\partial \sigma}$.

The action being invariant under the Poincaré group, its generators P^μ and $M^{\mu\nu}$ are constants of motion. The generators are given by

$$P^\mu = \int_0^\pi d\sigma \mathcal{P}^\mu \quad (2.4a)$$

$$M^{\mu\nu} = \int_0^\pi d\sigma (x'^\mu x'^\nu - x'^\nu x'^\mu) \quad (2.4b)$$

where $\mathcal{P}^\mu \equiv \frac{\partial \mathcal{L}}{\partial \dot{x}^\mu}$.

The action is also invariant under an arbitrary non-singular reparametrization of the surface which leaves the lines x^μ ($\sigma = 0, \tau$) and x^μ ($\sigma = \pi, \tau$) unchanged (gauge freedom). One can show that one can always use this freedom of parametrization to choose σ and τ such that

$$\dot{x} \cdot x' = 0 \quad (2.5a)$$

$$\dot{x}^2 + x'^2 = 0 \quad (2.5b)$$

Such a gauge will be called orthonormal. The equations of motion become

$$\left(\frac{\partial^2}{\partial \tau^2} - \frac{\partial^2}{\partial \sigma^2} \right) x^\mu = 0 \quad (2.6a)$$

$$x'^\mu(0, \tau) = x'^\mu(\pi, \tau) = 0 \quad (2.6b)$$

The most general solution of (2.6a) and (2.6b) is

$$x^\mu(\sigma, \tau) = q^\mu + \sqrt{2} [a_0^\mu \tau + f^\mu(\tau + \sigma) + f^\mu(\tau - \sigma)] \quad (2.7)$$

where $f^\mu(u)$ is periodic

$$f^\mu(u) = f^\mu(u + 2\pi),$$

obeys the gauge conditions (5a) and (5b)

$$[a_0 + 2\dot{f}(u)]^2 = 0 \quad (2.8)$$

and is otherwise arbitrary. Therefore $f^\mu(u)$ can be expanded in a Fourier series:

$$x^\mu(\sigma, \tau) = q^\mu + \sqrt{2} \left[a_0^\mu \tau + i \sum_{\substack{n=-\infty \\ n \neq 0}}^{+\infty} \frac{a_n^\mu}{n} \cos n\sigma e^{-in\tau} \right] \quad (2.9)$$

The gauge condition (2.8) gives

$$\sum_{-\infty}^{\infty} a_{\ell+n} a_{-\ell} = 0 \quad (2.10)$$

for all n .

Notice that in an orthonormal gauge τ , x^μ and P^μ obey the following relation:

$$\int_0^\pi d\sigma x^\mu = \pi (q^\mu + 2 P^\mu \tau) \quad (2.11)$$

We observe that requiring a gauge to be orthonormal does not determine it uniquely. The most general reparametrization which preserves orthonormality is given by:

$$\begin{aligned}\tilde{\tau} &= \tau_0 + \tau + g(\tau + \sigma) + g(\tau - \sigma) \\ \tilde{\sigma} &= \sigma + g(\tau + \sigma) - g(\tau - \sigma)\end{aligned}\quad (2.12)$$

where $g(u)$ is periodic

$$g(u) = g(u + 2\pi),$$

leads to non-singular reparametrizations $\frac{\partial(\tilde{\tau}, \tilde{\sigma})}{\partial(\tau, \sigma)} \neq 0$

$$1 + 2\dot{g}(u) \neq 0 \quad (2.13)$$

and is otherwise arbitrary. Expanding $g(u)$ in a Fourier series gives:

$$\begin{aligned}\tilde{\tau} &= \tau_0 + \tau + \sum_{n \neq 0} A_n \cos n\sigma e^{-in\tau} \\ \tilde{\sigma} &= \sigma - i \sum_{n \neq 0} A_n \sin n\sigma e^{-in\tau}\end{aligned}\quad (2.14)$$

In their paper G. G. R. T. use this gauge freedom to choose τ such that

$$x^+ \equiv \frac{1}{\sqrt{2}} (x^0 + x^3) = 2P^+ \tau \quad (2.15)$$

That is, suppose we are in some orthonormal gauge (τ, σ) where $x^+(\tau, \sigma) \neq 2P^+ \tau$. Then, they say, from (2.7) and (2.12) it is apparent that we can always go to another orthonormal gauge $(\tilde{\tau}, \tilde{\sigma})$ where (2.15) holds true. In fact, if we are willing to look at the set of all possible motions of the string, the statement is false as we will show now.

It is obviously possible to describe the evolution of the string by taking pictures of it at a succession of times. That is to say, we can certainly choose

$$x^0(\tau, \sigma) = 2 P^0 \tau. \quad (2.16)$$

Then from (2.11) follows that if we want an orthonormal gauge, σ must be chosen such that

$$P^0 \sigma = \pi \int_{\text{fixed } \tau}^{\sigma} d\sigma' \dot{x}^0 \quad (2.17)$$

In this gauge the orthonormality condition (2.5b) requires the ends of the string to move with the speed of light.

The question is whether we can use the remaining gauge freedom to choose an orthonormal gauge where (2.15) is true. The answer is no because in general this would imply performing a singular change of variables. Indeed, the Jacobian of a transformation which made x^+ proportional to $\tilde{\tau}$ is proportional to

$$\frac{\partial(\tilde{\tau}, \tilde{\sigma})}{\partial(\tau, \sigma)} \propto \begin{bmatrix} a_0^+ + 2\dot{f}^+(\tau+\sigma) \\ a_0^+ + 2\dot{f}^+(\tau-\sigma) \end{bmatrix}$$

Without violating the gauge condition (2.8), one can have a motion of the string such that

$$a_0^+ + 2\dot{f}^+(u) = 0 \quad (2.18)$$

for some u . In fact this condition means only that at some time, the end of the string, which is always moving with the speed of light, has zero velocity in the x^1 and x^2 - directions (recall $x^+ = \frac{1}{\sqrt{2}}(x^0 + x^3)$).

Actually having the Jacobian vanish at some discrete value of u , does not prevent choosing x^+ proportional to $\tilde{\tau}$. The reason is that in such a case, the measurement of the area swept by the string is not viciated by this choice. However if (2.18) remains valid for an interval of finite length in u (see Fig. 1), then the transformation which makes x^+ proportional to $\tilde{\tau}$ is truly singular.

It is important to notice that in two dimensions the choice $x^+ = 2P^+ \tau$ implies that there are no non-trivial motions. In fact we will show later that the string in two dimensions has an infinite number of non-trivial solutions (Sect. 3). Clearly a string in two dimensions is like a string in any number of dimensions having a purely longitudinal motion. We conclude then that at the classical level a string in D dimensions $D \geq 3$ has both transverse and longitudinal excitations.

Before ending the discussion of the classical mechanics of the string, we should clarify a technical point, which might worry the reader. The question is this: the troublesome motions which do not allow the choice of x^+ proportional to τ , correspond to functions $f(u)$ which are non-analytic. If one allows such functions, are then the Euler-Lagrange equations of motion (2.3a) and (2.3b) still equivalent to the action principle

Firstly let us note that since the Lagrangian contains only first order derivatives of x^μ , we would not expect any trouble from functions for which the first and second derivatives exist.

For $D > 2$ one can have (2.18) hold true for an interval of finite length in u , even though the first and second derivatives of $f^+(u)$ are finite. Thus in a space of large enough dimensionality ($D > 2$), there can be no doubt that the choice $x^+ = 2P^+\tau$ is illegitimate and does not allow the determination of all classical motions.

Secondly let us investigate the case $D = 2$ and see if the type of functions which we allow as solutions of the Euler-Lagrange equations of motion (2.3a) and (2.3b), do not make the derivation of these equations from the action principle incorrect. As we shall see in Sect. 3c, the solutions are described by functions $f(u)$, which have a finite first derivative at all points u ; the second derivative is finite at all points except a number of discrete points at which it exists only as a left and right derivative.

To obtain the equations of motion from the action principle one has to choose the number of points n where the function is allowed to have only a left and right second derivative⁽³⁾. In the τ - σ plane these points lie on some curves (see Fig. 2)

$$\sigma = c_i(\tau) \quad i = 1, 2, \dots, n. \quad (2.19)$$

The functions $c_i(\tau)$ are unknown and will be determined by the action principle. (Only their number is given.) The domain of integration in τ and σ is split by the n lines into a number of subdomains, in each of which the functions we consider are twice differentiable. Therefore we can integrate by parts in each of

these subdomains. The action principle yields the old equations (2.3a) and (2.3b), together with a new set of equations provided by the vanishing of the surface terms along the lines $c_i(\tau)$. They are:

$$-\frac{\partial \mathcal{L}}{\partial \dot{x}^\mu} + \frac{\partial \mathcal{L}}{\partial x'^\mu} \frac{dc_i}{d\tau} \bigg|_{\text{left}}^{\text{right}} = 0 \quad (2.20)$$

These equations determine $c_i(\tau)$. They are trivially satisfied if x^μ is twice differentiable.

3. Lagrangian and Hamiltonian formalism; string in two-dimensions.

3a) Lagrangian Formalism

It is convenient for later purposes to derive the equations of motion from the Lagrangian using Lagrange multipliers to introduce the constraints which fix the gauge. (In Sect. 2 we imposed the gauge fixing constraints directly in the equations of motion, not in the action.) We will consider two ways of fixing the gauge:

i) We require

$$\dot{x} \cdot x' = 0 \quad (3.1a)$$

$$\dot{x}^2 + x'^2 = 0 \quad (3.1b)$$

The action becomes:

$$I = \int_{\tau_i}^{\tau_f} d\tau \int_0^\pi d\sigma \left[\mathcal{L} + \lambda (\dot{x}x') + \frac{\mu}{2} (\dot{x}^2 + x'^2) \right] \quad (3.2)$$

The principle of least action yields

$$\dot{x} x' = 0 \quad (3.3a)$$

$$\dot{x}^2 + x'^2 = 0 \quad (3.3b)$$

$$\ddot{x}^\nu + \frac{\partial}{\partial \tau} [\lambda x'^\nu + \mu \dot{x}^\nu] - x''^\nu + \frac{\partial}{\partial \sigma} [\lambda \dot{x}^\nu + \mu x'^\nu] = 0 \quad (3.3c)$$

$$-x'^\nu + \lambda \dot{x}^\nu + \mu x'^\nu = 0 \quad \text{at} \quad \sigma = 0, \pi \quad (3.3d)$$

Multiplying (3.3c) by $\dot{x}^\nu$ and x'^ν respectively and using (3.3a) and (3.3b) we get

$$\dot{x}^\nu \frac{\partial}{\partial \tau} [\lambda x'^\nu + \mu \dot{x}^\nu] + \dot{x}^\nu \frac{\partial}{\partial \sigma} [\lambda \dot{x}^\nu + \mu x'^\nu] = 0 \quad (3.4a)$$

$$x'^\nu \frac{\partial}{\partial \tau} [\lambda x'^\nu + \mu \dot{x}^\nu] + x'^\nu \frac{\partial}{\partial \sigma} [\lambda \dot{x}^\nu + \mu x'^\nu] = 0 \quad (3.4b)$$

This can be rewritten with the help of (3.3a) and (3.3b) as:

$$\frac{\partial}{\partial u} [(\lambda + \mu) \dot{x}^2] = 0 \quad (3.5a)$$

$$\frac{\partial}{\partial v} [(-\lambda + \mu) \dot{x}^2] = 0 \quad (3.5b)$$

where $u = \tau + \sigma$ and $v = \tau - \sigma$. These equations imply:

$$\lambda \dot{x}^2 = g(v) - f(u) \quad (3.6a)$$

$$\mu \dot{x}^2 = g(v) + f(u) \quad (3.6b)$$

From (3.3d) after multiplication by $\dot{x}^\nu$ we have

$$\lambda \dot{x}^2 = 0 \quad \text{at} \quad \sigma = 0, \pi \quad (3.7)$$

This implies

$$g(u) = f(u) = f(u + 2\pi) \quad (3.8)$$

Finally multiplying (3.3d) by x'^ν we get

$$(-1 + \mu) \dot{x}^2 = 0 \quad \text{at} \quad \sigma = 0, \pi \quad (3.9)$$

This implies

$$2f(\tau) = \dot{x}^2(\tau; \sigma = 0) \quad (3.10)$$

From (3.6a) and (3.6b) we observe that if our solutions do not violate causality, we must have $\dot{x}^2(\tau; \sigma = 0) = 0$. Therefore $\lambda = \mu = 0$. The equations of motion (3.3c) and (3.3d) become:

$$\ddot{x}^\nu - x^{\nu\prime\prime} = 0 \quad (3.11a)$$

$$x^{\nu\prime} = 0 \quad \text{at} \quad \sigma = 0, \pi \quad (3.11b)$$

Notice that we obtained the same result as in Sect. 2 (Eq. (2.6a) and (2.6b)).

ii) We require

$$x^0 = \tau \quad (3.12a)$$

$$(\dot{x} + x^1)^2 = 0 \quad (3.12b)$$

The action becomes:

$$I = \int_{\tau_i}^{\tau_f} d\tau \int_0^\pi d\sigma \left[-1 + \lambda (x^0 - \tau) + \frac{\mu}{2} (\dot{x} + x^1)^2 \right] \quad (3.13)$$

The principle of least action yields:

$$x^0 = \tau \quad (3.14a)$$

$$(\dot{x} + x^1)^2 = 0 \quad (3.14b)$$

$$\partial_u \mu + \lambda = 0 \quad (3.14c)$$

$$\ddot{x}_1 - x_1^{\prime\prime} + \partial_u [\mu \partial_u x_1] = 0 \quad (3.14d)$$

$$x_1^{\prime} + \mu (\dot{x}_1 + x_1^1) = 0 \quad \text{at} \quad \sigma = 0, \pi \quad (3.14e)$$

From (3.14b) we have

$$1 - (\dot{x}_1 + x_1')^2 = 0 \quad (3.15a)$$

$$(\ddot{x}_1 - x_1'')(\dot{x}_1 + x_1') = 0 \quad (3.15b)$$

These equations together with (3.14d) and (3.14e) imply

$$\mu \partial_u x_1(\tau, u) = -x_1'(\tau, \sigma, 0)$$

To preserve causality we must have $\mu = \lambda = 0$. The equations obeyed by x_1 are

$$\ddot{x}_1 - x_1'' = 0 \quad (3.16a)$$

$$x_1' = 0 \quad \text{at} \quad \sigma = 0, \pi \quad (3.16b)$$

3b) Hamiltonian Formalism

For quantization purposes, it is desirable to derive the classical Hamiltonian formalism of the problem.

The canonical coordinates are $x^\mu(\tau, \sigma)$ and the canonical momenta are $\mathcal{P}^\mu(\tau, \sigma) = \frac{\partial \mathcal{L}}{\partial \dot{x}^\mu}$.

We observe that by definition, the momenta $\mathcal{P}^\mu$ obey the following constraints:

$$x' \cdot \mathcal{P} = 0 \quad (3.17a)$$

$$x'^2 + \mathcal{P}^2 = 0 \quad (3.17b)$$

We postulate the fundamental P. B.

$$\{x^\mu(\tau, \sigma), \mathcal{P}^\nu(\tau, \sigma')\} = g^{\mu\nu} \delta(\sigma - \sigma') \quad (3.18)$$

From the coordinates and the momenta we can construct the generators of the Poincare' group as in (2.4a) and (2.4b). If we choose $x_0 = \tau$, the momentum $\mathcal{P}_0$ becomes illdefined and so do the generators P_0 and $M^{\mu 0}$.

We can remedy this situation using a method devised by Dirac⁽⁴⁾. Namely, we observe that the P.B. of (3.17b) with the generators of the Poincare' group is vanishing. Then we try to find functions λ_μ and $\lambda_{\mu\nu} = -\lambda_{\nu\mu}$ such that

$$\tilde{\mathcal{P}}_\mu = \mathcal{P}_\mu + \lambda_\mu [x_1'^2 + \mathcal{P}_1^2] \quad (3.19)$$

$$\tilde{M}_{\mu\nu} = x_\mu \mathcal{P}_\nu - x_\nu \mathcal{P}_\mu + \lambda_{\mu\nu} [x_1'^2 + \mathcal{P}_1^2]$$

do not contain $\mathcal{P}_0$. From the above remark follows that the integrals of $\tilde{\mathcal{P}}_\mu$ and $\tilde{M}_{\mu\nu}$ obey the Poincare' algebra. The λ 's have the values:

$$\begin{aligned} \lambda_1 &= 0 & \lambda_0 &= - \left(\mathcal{P}_0 + \sqrt{\mathcal{P}_1^2 + x_1'^2} \right)^{-1} \\ \lambda_{10} &= - x_1 \left(\mathcal{P}_0 + \sqrt{\mathcal{P}_1^2 + x_1'^2} \right)^{-1} \end{aligned} \quad (3.20)$$

Therefore the new Hamiltonian is

$$\tilde{P}_0 = \int_0^\pi d\sigma \quad \tilde{\mathcal{P}}_0 = \int_0^\pi d\sigma \quad \sqrt{x_1'^2 + \mathcal{P}_1^2} \quad (3.21)$$

The Hamiltonian equations of motion are

$$\{x_1, \tilde{P}_0\} = \frac{\mathcal{P}_1}{\sqrt{x_1'^2 + \mathcal{P}_1^2}} = \dot{x}_1 \quad (3.22a)$$

$$\{\mathcal{P}_1, \tilde{P}_0\} = \frac{\partial}{\partial \sigma} \frac{x_1'}{\sqrt{x_1'^2 + \mathcal{P}_1^2}} = \dot{\mathcal{P}}_1 \quad (3.22b)$$

These equations can be rewritten as

$$\frac{\partial}{\partial \tau} \dot{x}_1 \frac{x_1'}{\sqrt{1-\dot{x}^2}} = \frac{\partial}{\partial \sigma} x_1' \frac{\sqrt{1-\dot{x}^2}}{x_1'} \quad (3.23a)$$

$$\mathcal{P}_1 = \frac{\dot{x}_1 x_1'}{\sqrt{1-\dot{x}^2}} \quad (3.23b)$$

Notice that we have not fixed the gauge completely since we have required only $x_0 = \tau$. Thus we could choose σ such that

$$x_1' = \sqrt{1-\dot{x}^2} \quad (3.24)$$

This is equivalent to asking $\dot{x}^2 + x_1'^2 = 0$. Then the equations of motion become

$$\ddot{x}_1 - x_1'' = 0 \quad (3.25a)$$

$$\mathcal{P}_1 = \dot{x}_1 \quad (3.25b)$$

Notice though that (3.24) implies

$$\tilde{P}_0 = \int_0^\pi d\sigma \sqrt{x_1'^2 + \mathcal{P}_1^2} = \pi \quad (3.26)$$

Thus we still have a subsidiary condition, (similar to (2.17)), which will lead to difficulties when we will attempt quantization (see Sect. 4).

3c) Classical Solutions

We will describe here the classical solutions in two dimensions in an orthonormal gauge. We choose

$$x^0 = \tau \quad (3.27)$$

Then x_1 must obey

$$\left. \begin{aligned} \dot{x}_1 x_1' &= 0 \\ 1 - \dot{x}_1^2 + x_1'^2 &= 0 \\ \ddot{x}_1 - x_1'' &= 0 \\ x_1' &= 0 \quad \text{at } \sigma = 0, \pi \end{aligned} \right\} \quad (3.28)$$

The most general solution to (3.28) is

$$x_1(\tau, \sigma) = a + \frac{1}{2} f(\tau + \sigma) + \frac{1}{2} f(\tau - \sigma) \quad (3.29)$$

where $f(u)$ is arbitrary subject to

$$\dot{f}(u) = \pm 1 \quad (3.30a)$$

$$\dot{f}(u) = \dot{f}(u + 2\pi) \quad (3.30b)$$

We observe that the discontinuities in the first derivative of $x_1(\tau, \sigma)$ occur along lines of slope $c_1' = \pm 1$. Thus these solutions obey (2.20). Examples of classical motions are given in Fig. 3 and Fig. 4. Notice that the motion in Fig. 4 is that of a string which starts out folded once. The fold disappears at $x_0 = \frac{\pi}{6}$, reappears at $x_0 = \frac{\pi}{3}$ and so on. The number of folds is arbitrary and is determined by the number of discontinuities of $f(u)$.

Both the end points and folds move with the speed of light, while the other points on the string are at rest in the c.m. The moving points have variable mass. Their mass is determined by the length of the σ -interval over which $x_1' = 0$. (These characteristics of the two dimensional string make it resemble a system of yo-yos.

We are grateful to T. Regge for suggesting the analogy.)

Let us compute the area swept by a string which starts out at τ_i as $x(\tau_i, \sigma)$ and ends up at τ_f as $x(\tau_f, \sigma)$. It is given by:

$$A [x(\tau_f, \sigma); x(\tau_i, \sigma)] = \frac{1}{4} \int_{\tau_i}^{\tau_f} d\tau \int_0^\pi d\sigma [1 - \dot{f}(u) \dot{f}(v)] \quad (3.31)$$

Notice that because $\dot{f}(u)$ is periodic (3.30b), if

$$\tau_f - \tau_i = 2n\pi \quad n = 1, 2, \dots \quad (3.32)$$

then

$$x[\tau_i + 2n\pi, \sigma] = x[\tau_i, \sigma] + n\Delta. \quad (3.33)$$

Here Δ is some constant determined by the c.m. momentum.

It follows that area swept by the string in one period $\tau_f - \tau_i = 2\pi$ is independent of its initial configuration $x(\tau_i, \sigma)$ and depends only on Δ . Indeed from (3.31) we have

$$A [x(\tau_i + 2\pi, \sigma); x(\tau_i, \sigma)] = \frac{1}{4} \int_{\tau_i}^{\tau_i + 2\pi} d\tau \int_0^\pi d\sigma [1 - \dot{f}(u) \dot{f}(v)] \quad (3.34)$$

Making use of the periodicity of $\dot{f}(u)$ we can change the region of integration as in Fig. 5 and obtain:

$$A [x(\tau_i + 2\pi, \sigma); x(\tau_i, \sigma)] = \left(\int_{\tau_i}^{\tau_i + \pi} du \int_{\tau_i - \pi}^{\tau_i + \pi} dv + \int_{\tau_i + \pi}^{\tau_i + 2\pi} du \int_{\tau_i}^{\tau_i + 2\pi} dv \right) \cdot \frac{1}{4} [1 - \dot{f}(u) \dot{f}(v)] \quad (3.35)$$

Thus we find

$$A [x(\tau_i + 2\pi, \sigma); x(\tau_i, \sigma)] = \pi^2 - \frac{1}{4} \Delta^2 \quad (3.36)$$

In particular for a string whose c.m. is moving with the speed of light ($\Delta = 2\pi$) the area is zero.

4. Quantum Mechanics of the String.

The quantization of a system with an infinite number of degrees of freedom and constraints is a difficult problem, and no standard prescription exists. We will discuss the non-covariant and covariant quantization proposed by GGRT, together with the quantization via the path integral formalism.

4a) Non-covariant Quantization

In this method of quantization, one must first choose the independent degrees of freedom, then compute their Poisson brackets. Quantization is achieved by transforming P. B. into commutators.

In Sect. 3c we wrote the fundamental P. B. in a gauge where $x^0 = \tau$. Unfortunately, the choice $x^0 = \tau$ does not specify the gauge uniquely. We managed to complete this specification, but at the price of introducing a constraint (Eq. (3.24)) on the independent degrees of freedom $x_1(\tau, \sigma)$ and $\mathcal{P}_1(\tau, \sigma)$.

Alternatively one could choose the orthonormal gauge where $x^0 = 2P^0\tau$ and regard the Fourier coefficients a_n^μ in (2.9) as the dynamical variables. First we notice that $a_n^0 = 0$ for $n \neq 0$.

The orthonormality conditions (2.10) become:

$$\begin{aligned} \sum_{l=-\infty}^{\infty} a_{l+n}^1 a_{-l}^1 &= 0 \quad \text{all } n \neq 0 \\ \sum_{l=-\infty}^{\infty} a_l^1 a_{-l}^1 &= (a_0^0)^2 \end{aligned} \quad (4.1)$$

The author has not been able to "solve" these equations, in the sense of expressing everything in terms of some unconstrained dynamical variables. Thus so far it has not been possible to attempt a non-covariant quantization of the string⁽⁵⁾.

4b) Covariant Quantization

This method of quantization retains all degrees of freedom as independent and enforces the constraints on the physical states. In our case the operators are q , a_0 and a_n ; they obey the following commutations relations:

$$\begin{aligned} [a_n^\mu, a_m^\mu] &= g^{\mu\nu} \delta_{n,m} \\ [q^\mu, a_0^\nu] &= i\sqrt{2} g^{\mu\nu} \end{aligned} \quad (4.2)$$

The gauge conditions (2.10) require all physical states in our theory to obey

$$L_N |\psi\rangle = 0 \quad N > 0 \quad (4.3a)$$

$$L_0 |\psi\rangle = \alpha_0 |\psi\rangle \quad (4.3b)$$

where

$$L_N = \frac{1}{2} \sum_{l=-\infty}^{\infty} : a_{N+l} \cdot a_{-l} : \quad (4.4)$$

The solutions to (4.3a) and (4.3b) have been studied⁽⁶⁾. The results are that for $\alpha_0 = 1$ and $1 \leq D \leq 26$ and for $\alpha_0 < 1$ and $1 \leq D \leq 25$ there are no negative norm states solutions to (4.3a) and (4.3b) (D is the dimensionality of spacetime). However, in general the solutions include null states (states of zero norm orthogonal to all the other solutions of (4.3a) and (4.3b)), which are undesirable, as they introduce an arbitrariness in the definition of physical states. Thus if $|p\rangle$ has positive norm and $|n\rangle$ is null, then

$$|P'\rangle = |P\rangle + \lambda |n\rangle$$

has the same norm and couplings as $|p\rangle$. The stipulation made by GGRT was that the existence of null states reflects the arbitrariness in choosing some orthonormal gauge from the many possible ones. Indeed, in enforcing (4.3a) and (4.3b) we require only that the gauge be orthonormal, but do not specify it otherwise.

A natural way of fixing the gauge would be to transform (2.16) into a matrix statement:

$$\langle \psi_1 | x^0(\tau, \sigma) | \psi_2 \rangle = 2\tau \langle \psi_1 | P^0 | \psi_2 \rangle \quad (4.5)$$

with $|\psi_1\rangle$ and $|\psi_2\rangle$ any states which obey (4.3a) and (4.3b).

Unfortunately, we will show now that (4.5) is satisfied by any states $|\psi\rangle$ solution of (4.3a) and (4.3b), so that it does not eliminate null states if there are any.

The proof is trivial once one observes that from (4.2) it follows that

$$[a_n^\nu, L_0] = -n a_n^\nu \quad (4.6)$$

This equation is the cause of all the problems of a covariant quantization. As we already said, it prevents us from using (4.5) to eliminate null states. Its more unpleasant consequence is that it leads to a quantum mechanical Hilbert space on which not all possible classical motions can be realized. Indeed it should be possible to find a state $|\psi\rangle$ solution of (4.3a) and (4.3b) such that $\hat{x}^\nu(\tau, \sigma)$, defined by

$$\hat{x}^\nu(\tau, \sigma) = \langle \psi | \hat{x}^\nu(\tau, \sigma) | \psi \rangle \quad (4.7)$$

represents any classical motion we please. But from (4.6) it follows that for any state $|\psi\rangle$ which obeys the gauge conditions, we have:

$$\langle \psi | \hat{x}^\nu(\tau, \sigma) | \psi \rangle = \langle \psi | (q^\nu + \sqrt{2} \alpha_0^\nu \tau) | \psi \rangle \quad (4.8)$$

This fact leaves the covariant quantization an open problem.

We should point out that most likely, these difficulties have to do with the careless use of unbounded operators. Indeed a similar situation can be created in the ordinary quantum mechanics of a non-relativistic point particle in 1 dimension. For instance let x and p be the usual position and momentum operators, and $|\psi\rangle$ be such that

$$(p^2 - p_0^2) |\psi\rangle = 0$$

Then from $[x, p^2] = 2ip$, we would conclude that $\langle \psi | p | \psi \rangle = 0$, whereas in fact we know that $\langle \psi | p | \psi \rangle = \pm p_0$.

Returning now to the problem of the null states, we notice that irrespective of the previous difficulties, for a free string the

null states can be eliminated by hand in a Poincare' invariant manner. This is a corollary to the following theorem: if $|p\rangle$ and $|n\rangle$ satisfy (4.3a) and (4.3b) and if 0 is a gauge invariant operator ($[0, L_N] = 0$ for all N), then $\langle p|0|n\rangle = 0$. The proof is based on the observation that 0 being gauge invariant, $0|p\rangle$ also satisfies (4.3a) and (4.3b). Then $|n\rangle$ being a null state, the theorem follows.

The corollary is that if 0 is gauge invariant and $|n\rangle$ a null state, then $0|n\rangle$ is also null. The generators of the Poincare' group are gauge invariant, hence the proof to our claim.

4c) Path Integral Quantization in Two Dimensions

This method of quantization, devised by Feynman⁽⁷⁾, computes first the kernel K in terms of the action. For the case of problems involving local gauge invariance the correct procedure has been established by Fadeev and Popov⁽⁸⁾. It gives

$$K[x_f(\sigma); x_i(\sigma); \tau_f - \tau_i] = N \int \exp[iS] \prod_{\tau, \sigma} \delta[x_0 - \tau] \delta[(\dot{x} + x^1)^2] \Delta[x] dx^\mu \quad (4.9)$$

Here N is a normalization factor, the two δ -functions represent the gauge fixing conditions and $\Delta[x]$ is defined by

$$\Delta[x] \int \prod_{\tau, \sigma} d\Omega \delta[x_0^\Omega - \tau] \delta[(\dot{x}^\Omega + x^{1\Omega})^2] = 1 \quad (4.10)$$

where the integration is to be performed over the group of gauge transformations which leave the action invariant. In the present case

the dynamical quantity is the area of a given surface. The two parameters τ and σ label points on the surface and thus can be used to define a functional

$$A[\tau, \sigma(\tau)] = \int d\tau' \int_{\sigma(\tau')}^{\sigma(\tau)} d\sigma' \{(\dot{x} \cdot x')^2 - \dot{x}^2 x'^2\}^{\frac{1}{2}} \quad (4.11)$$

The group of invariance in our problem is the group of relabelings of points on the surface; they leave the area invariant. Thus for an infinitesimal reparametrization

$$\begin{aligned} \tau &\rightarrow \tau + e_1(\tau, \sigma) \\ \sigma &\rightarrow \sigma + e_2(\tau, \sigma) \end{aligned} \quad (4.12)$$

we have

$$A[\tau, \sigma(\tau)] \rightarrow A[\tau, \sigma(\tau)] + \mathcal{L} e_1 e_2 \quad (4.13)$$

We conclude that the measure of integration in (4.10) is $\mathcal{L} de_1 de_2^{(9)}$.

Because of the δ -functions in (4.9), we receive contribution in (4.10) only around the identity transformation. For infinitesimal transformations (4.12) we have:

$$\begin{aligned} x_0 &\rightarrow x_0 + e_1 \\ (\dot{x} + x')^2 &\rightarrow (\dot{x} + x')^2 + 4 \mathcal{L} \partial_u (e_1 - e_2) \end{aligned} \quad (4.14)$$

Therefore

$$\Delta[x] = 4 \det \left(\frac{\partial}{\partial \tau} + \frac{\partial}{\partial \sigma} \right) \quad (4.15)$$

Including these factors into the normalization, we have the basic formula for K :

$$K[x_f(\sigma); x_i(\sigma); \tau_f - \tau_i] = N' \int_{\tau, \sigma} \exp[iS] \prod \delta[x_0 - \tau] \delta[(\dot{x} + x')^2] dx_\nu \quad (4.16)$$

It is convenient to express the δ -functions as

$$\begin{aligned}\delta[x_0 - \tau] &\propto \int \exp[i\lambda(x_0 - \tau)] \prod_{\tau, \sigma} d\lambda \\ \delta[(\dot{x} + \dot{x}')^2] &\propto \int \exp\left\{i \frac{\mu}{2} (\dot{x} + \dot{x}')^2\right\} \prod_{\tau, \sigma} d\mu\end{aligned}\quad (4.17)$$

Then K contains the integral

$$\int \exp\left\{i \int_{\tau_i}^{\tau_f} d\tau \int_0^\pi d\sigma \left[\frac{1}{2}(\dot{x}^2 - x'^2) + \lambda(x_0 - \tau) + \frac{\mu}{2}(\dot{x} + \dot{x}')^2\right]\right\} \prod_{\tau, \sigma} d\lambda d\mu dx_\nu \quad (4.18)$$

Making the substitution

$$\begin{aligned}x_\nu &= \bar{x}_\nu + y_\nu \\ \lambda &= \bar{\lambda} + \ell \\ \mu &= \bar{\mu} + m\end{aligned}\quad (4.19)$$

where $\bar{x}_\nu$, $\bar{\lambda}$ and $\bar{\mu}$ are the solutions to the classical problem (Sect. 3a)

$$\delta \int_{\tau_i}^{\tau_f} d\tau \int_0^\pi d\sigma \left[\mathcal{L} + \lambda(x_0 - \tau) + \frac{\mu}{2}(\dot{x} + \dot{x}')^2 \right] = 0$$

we obtain

$$\begin{aligned}K &\propto e^{iS_{cl}}[x_f(\sigma); x_i(\sigma); \tau_f - \tau_i] \\ &\cdot \int \exp\left\{i \int_{\tau_i}^{\tau_f} d\tau \int_0^\pi d\sigma \left[\frac{1}{2}(\dot{y}^2 - y'^2) + \ell y_0 + m(\dot{\bar{x}} + \dot{\bar{x}}') \cdot (\dot{y} + y') + \frac{m}{2}(\dot{y} + y')^2\right]\right\} \prod_{\tau, \sigma} d\ell dm dy_\nu\end{aligned}\quad (4.20)$$

Notice that $y_\nu(\tau, \sigma)$ are restricted by

$$y_\nu(\tau_i, \sigma) = y_\nu(\tau_f, \sigma) = 0 \quad (4.21)$$

Doing the x and m integrations we get back δ -functions. We use one δ -function to do the y_0 -integration and get

$$K[x_f(\sigma); x_i(\sigma); \tau_f - \tau_i] = N^i e^{iS_{cl}} \int_{\tau_i}^{\tau_f} d\tau \int_0^\pi d\sigma (\partial_u y_1)(\partial_v y_1) \prod_{\tau, \sigma} \delta [2(\partial_u \bar{x}_1)(\partial_u y_1) + (\partial_u y_1)^2] dy_1 \quad (4.22)$$

In (4.22) the integral depends only on $\tau_f - \tau_i$ and is independent of $x_i(\sigma)$ and $x_f(\sigma)$. The reason is that it depends on $x_i(\sigma)$ and $x_f(\sigma)$ only through $|\partial_u \bar{x}_1|$. But $\bar{x}_1(u, v)$ being a classical solution, from (3.29) and (3.30a) we have $|\partial_u \bar{x}_1| = \frac{1}{2}$. To see that indeed only $|\partial_u \bar{x}_1|$ is important in (4.22), consider an approximation to the integral obtained by covering the domain of integration with a finite number of squares (see Fig. 6) and replacing derivatives by ratios of differences. We distinguish two types of squares depending upon the fact that they touch the lines $\tau = \tau_i$ or $\tau = \tau_f$ or not. In squares touching τ_i or τ_f , from (4.21) we know that $\partial_u y_1 = \partial_v y_1$ so that only $|\partial_u x|$ matters. In all the other squares $\partial_u y_1 = 0$ or $\partial_u y_1 = -\frac{1}{2} \partial_u \bar{x}_1 = \pm \frac{1}{4}$ but $\partial_v y_1$ is unrestricted. Thus the contribution of such squares depends again only on $|\partial_u \bar{x}_1|$. Therefore the kernel has the form:

$$K[x_f(\sigma); x_i(\sigma); \tau_f - \tau_i] = e^{iS_{cl}[x_f(\sigma); x_i(\sigma); \tau_f - \tau_i]} F(\tau_f - \tau_i) \quad (4.23)$$

The computation of the integral giving $F(\tau_f - \tau_i)$ is difficult. However we can use the closure property of the kernel to find some characteristics of $F(\tau_f - \tau_i)$. The closure can be expressed symbolically

as

$$K[x_f; x_i; s+t] = \int dy K[x_f; y; t] K[y; x_i; s] \quad (4.24)$$

We observe firstly that F must become infinite when $\tau_f - \tau_i \rightarrow 0$.

Secondly taking $s = 2\pi$ and recalling that all classical motions have period 2π in their c.m., it follows that

$$F(t+2\pi) = F(t) r(2\pi) \quad (4.25)$$

where $r(2\pi)$ is finite and equal to

$$r(2\pi) = \frac{F(2\pi)}{F(0)}.$$

(The harmonic oscillator has the same properties.)

The knowledge of the kernel allows the determination of the energy spectrum. Indeed we can write K as

$$K[x_f(\sigma); x_i(\sigma); \tau_f - \tau_i] = \sum_n \Phi_n^* [x_f(\sigma)] e^{-iE_n(\tau_f - \tau_i)} \Phi_n [x_i(\sigma)] \quad (4.26)$$

where $\Phi_n[x(\sigma)]$ are energy eigenstates. Taking $x_f(\sigma) = x_i(\sigma)$ and integrating over all configurations $x(\sigma)$ we obtain

$$I(\tau_f - \tau_i) \equiv \int dx(\sigma) K[x(\sigma); x(\sigma); x(\sigma); \tau_f - \tau_i] = \sum_n e^{-iE_n(\tau_f - \tau_i)} \quad (4.27)$$

The kernel is given by (4.23). In Sect. 3a we showed that S_{cl} is just the area of the surface generated by the classical motion ($\bar{\lambda} = \bar{\mu} = 0$ in (4.19)). We also know from Sect. 3c that the area of the surface described by any classical solution over one period is independent of the initial configuration (Eq. (3.31)). These facts

together with the property (4.25) of $F(\tau_f - \tau_i)$ imply that as a function of $\tau_f - \tau_i$ the l.h.s of (4.27) obeys

$$I(\tau_f - \tau_i + 2\pi) = e^{-i\pi/2} r(2\pi) I(\tau_f - \tau_i) \quad (4.28)$$

This shows that $I(\tau_f - \tau_i)$ is of the form:

$$I(\tau_f - \tau_i) = e^{-i\alpha(\tau_f - \tau_i)} \sum_{n=1}^{\infty} f_n e^{-in(\tau_f - \tau_i)} \quad (4.29)$$

The constant α is determined by $F(\tau_f - \tau_i)$. Comparing (4.29) with (4.27) we conclude that there are an infinite number of equally spaced energy levels, the lowest one being at α . That n is positive in (4.29) is guaranteed by the convergence of the integral giving $F(\tau_f - \tau_i)$. (We have not proved this, but it is reasonable to assume it.) Indeed the integral exists only if it is possible to make a Wick rotation, so that F cannot contain both positive and negative frequencies.

In principle one could go on and determine the density of eigenstates f_n and the eigenstates $\phi_n[x(\sigma)]$.

5. Lorentz Invariance at the Quantum Level; Final Remarks.

In Section 2 we showed that the choice $x^+ \propto \tau$ leads to the elimination of valid classical motions of the string. We will demonstrate now that this choice was also the reason for the breakdown of Lorentz invariance found by GGRT. On the contrary, the gauge $x_0 = \tau$ leads to a Poincaré-invariant quantum mechanics of the string in any number of dimensions and probably without having the ground state a tachion.

This could have been seen already from the covariant quantization of the string, together with our comments about null states. However, for the reasons explained in Section 4b, the covariant quantization has some problems which make it untrustworthy; hence we will disregard it and concentrate on the path integral formalism.

Firstly we observe that in two-dimensions, the kernel is given by a Poincaré-invariant expression (Eq. (4.23)). Indeed $x^\mu(\tau, \sigma)$ enters that formula only through $S_{cl}[x(\tau, \sigma)]$, which is the area swept by the string.

Secondly let us consider the case $D > 2$. We will prove now that formula (4.23) for the kernel remains valid for $D > 2$. The following facts will be needed for this purpose:

- a) The Lagrangian formalism developed in Section 3a is unchanged by the addition of extra space-like dimensions. In the case when the gauge is fixed by

$$x^0 = \tau \quad (5.1a)$$

$$(\dot{x} + x')^2 = 0 \quad (5.1b)$$

the spatial components of x^μ obey the following equations and constraints;

$$\ddot{x}_i - x_i'' = 0 \quad (5.2a)$$

$$x_i' = 0 \quad \text{at} \quad \sigma = 0, \pi \quad (5.2b)$$

$$(\partial_u x_i) (\partial_u x_i) = 1 \quad (5.2c)$$

b) The most general solution to (5.2) is

$$x_i(\tau, \sigma) = a_i + \frac{1}{2} f_i(u) + \frac{1}{2} f_i(v) \quad (5.3)$$

where $f_i(u)$ are arbitrary subject to

$$\dot{f}_i(u) = \dot{f}_i(u + 2\pi) \quad (5.4a)$$

$$\dot{f}_i \dot{f}_i = 1 \quad (5.4b)$$

c) The area swept by a string in one period $\tau_f - \tau_i = 2\pi$ is independent of its initial configuration $x^\mu(\tau_i, \sigma)$ and depends only on its c.m. momentum. Thus formula (3.36) remains valid

$$A[x^\mu(\tau_i + 2\pi, \sigma); x^\mu(\tau_i, \sigma)] = \pi^2 - \frac{1}{4} \Delta_i \Delta_i \quad (5.5)$$

These facts are straightforward generalization of things already proved and we leave their verification to the reader.

Using (a) and (b) it is easy to realize that the kernel retains its form (4.23) in any number of dimensions

$$K[x_f^\mu(\sigma); x_i^\mu(\sigma); \tau_f - \tau_i] = e^{iS_{cl}[x_f^\mu(\sigma); x_i^\mu(\sigma); \tau_f - \tau_i]} F(\tau_f - \tau_i) \quad (5.6)$$

This proves that the quantum mechanical treatment is Poincaré invariant in any number of dimensions.

Finally from (c) we observe that $F(\tau_f - \tau_i)$ has the same properties as in 2-dimensions, so that the energy spectrum remains unchanged: equally spaced levels starting at some value α . This constant α is determined by $F(\tau_f - \tau_i)$, an integral which we do not compute. Presumably the integral would be divergent since there is an infinite number of oscillators; thus α could be obtained only after some sort of renormalization had been performed. There is no reason to suspect that α would come out imaginary, so that there would be a tachion in the spectrum⁽¹⁰⁾.

In spite of its merits in proving the Poincaré invariance of the theory, a word of caution about the path integral formalism is necessary. We have not proved the existence and closure of the functional integral, thus we cannot be sure that the resulting theory is unitary. In fact the covariant quantization would suggest that only for certain values of D and α_0 there are no ghosts.

In closing we would like to point out that although we have ignored the problem of interactions of strings, our way of treating the path integral could be used in that case also. Indeed in the approach of Gervais and Sakita⁽⁹⁾ and Mandelstam⁽¹¹⁾, the interaction is represented by a term quadratic in x^μ . Having the

effective action quadratic in the fields was the crucial feature used by us. Of course, it may be that that type of interaction will lead to the Veneziano model, with its tachion. Then one would have to look for a different interaction, which might be more than quadratic in x^μ , in which case a new approach would be needed.

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FOOTNOTES AND REFERENCES

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which we criticized in Section 2. Also their Fadeev-Popov determinant contains one extra factor $\prod_{\tau, \sigma} \mathcal{L}[x(\tau, \sigma)]$. This factor comes from their measure of integration over the group, which is taken to be $de_1 de_2$. That is they take the action of the group to be (4.12), whereas we take it to be (4.13) since the area is the dynamical quantity. A similar situation arises in the case of a free relativistic particle, where the relevant quantity is the length of the world line.

10. In the Veneziano model α_0 must be chosen 1 or else the vertex is not conformally invariant. Thus it is reasonable to suspect that interactions of strings might constrain α_0 , but that there are no restrictions at the free string level.
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FIGURE CAPTIONS

- Fig. 1 World line of one end of the string; the end moves with the speed of light in the x^3 -direction for a finite time interval.
- Fig. 2 Curves in the τ - σ plane along which $x^\mu(\tau, \sigma)$ has only a left and right 2-nd derivative.
- Fig. 3 Motion of the string in 2-dimensions; there are no folds.
- Fig. 4 Motion of the string in 2-dimension; there is one fold (heavy line). The cross-hatched regions give a double contribution to the area because the surface is folded once.
- Fig. 5 The region of integration can be changed because of the periodicity of $\dot{f}(u)$. Triangles bearing the same label give equal contribution.
- Fig. 6 Grid used in approximating the functional integral with an integral over a finite number of variables.

FIG. 1

FIG. 2

FIG. 3

FIG. 4

FIG.5

FIG. 6