1
UNIVERSITY OF CALIFORNIA
Radiation Laboratory, Livermore Site
Livermore, California

Contract No. W-7405-eng-48

A METHOD OF SOLVING THE TWO-END-POINT PROBLEM
FOR SECOND-ORDER DIFFERENTIAL EQUATIONS

James E. Faulkner

October 1956

Printed for the U.S. Atomic Energy Commission

940 001

DISCLAIMER

This report was prepared as an account of work sponsored by an agency of the United States Government. Neither the United States Government nor any agency Thereof, nor any of their employees, makes any warranty, express or implied, or assumes any legal liability or responsibility for the accuracy, completeness, or usefulness of any information, apparatus, product, or process disclosed, or represents that its use would not infringe privately owned rights. Reference herein to any specific commercial product, process, or service by trade name, trademark, manufacturer, or otherwise does not necessarily constitute or imply its endorsement, recommendation, or favoring by the United States Government or any agency thereof. The views and opinions of authors expressed herein do not necessarily state or reflect those of the United States Government or any agency thereof.

DISCLAIMER

Portions of this document may be illegible in electronic image products. Images are produced from the best available original document.

A Method of Solving the Two-End-Point Problem
for Second-Order Differential Equations*

James E. Faulkner

University of California Radiation Laboratory
Livermore, California

Consider the following differential equation

$$(1) \quad d^2y/dx^2 - \alpha^2 y = f(x)$$

subject to the boundary conditions that $y(0)$ and $y(X)$ are given. The values of x are assumed to be in the range

$$(2) \quad 0 \leq x \leq X$$

Let this range be divided into N equal intervals of length Δx and let

$$(3a) \quad y(n \Delta x) = y_n$$

$$(3b) \quad f(n \Delta x) = f_n$$

The differential equation (1) goes over into the following difference equation.

$$(4) \quad \Delta x^{-2} (y_{n+1} - 2y_n + y_{n-1}) - \alpha^2 y_n = f_n$$

subject to the boundary conditions that y_0 and y_N are given.

Equation (4) may be solved by letting ^[1]

*Work performed under the auspices of U.S. Atomic Energy Commission.

$$(5) \quad y = Au + w$$

where w is an arbitrary solution of equation (4) subject only to the condition that $w_0 = y_0$ and u is a solution of the homogeneous equation

$$(6) \quad \Delta x^{-2}(u_{n+1} - 2u_n + u_{n-1}) - a^2 u_n = 0$$

The constant A is determined from the equation

$$(7) \quad y_N = Au_N + w_N$$

One problem which may arise can be seen by examining the differential form of equation (6)

$$(8) \quad d^2u/dx^2 - a^2 u = 0$$

The solution of equation (8) subject to the boundary condition that $u(0) = 0$ is

$$(9) \quad u = B \sinh ax$$

where B is an arbitrary constant. Thus the difference, $y - w$, can grow exponentially which implies a loss of accuracy in the final solution. In other words, both u and w can be large compared to y so that y is found by taking the difference of two numbers close to each other resulting in a loss of significant figures.

This loss of accuracy may be remedied in the following manner:
Suppose that

$$(10a) \quad \left| w_n^i - f_n \right| \leq \beta \quad n < j$$

$$(10b) \quad \left| w_n^i - f_n \right| > \beta \quad n = j$$

where w^i is a solution of equation (4) subject to the condition that $w_0^i = y_0$ and β is a predetermined constant depending on the accuracy desired. Define the constant A^{i+1} from the equation

$$(11) \quad f_j = A^{i+1} u_j + w_j^i$$

A new solution, w^{i+1} , to equation (4) may now be found.

$$(12a) \quad w_n^{i+1} = A^{i+1} u_n + w_n^i \quad n \leq j$$

$$(12b) \quad w_n^{i+1} = (2 + \alpha^2 \Delta x^2) w_{n-1}^{i+1} - w_{n-2}^{i+1} + \Delta x^2 f_{n-1} \quad n > j$$

w^{i+1} is generated recursively according to equation (12b) until condition (10b) is satisfied for some larger value of j or until $n=N$ at which time w^{i+2} is obtained from w^{i+1} in the same manner in which w^{i+1} was obtained from w^i .

The point of the method is to keep w the same order of magnitude as y . As outlined here, it assumes that y and f are the same order of magnitude.

The method has been tested on an IBM 650 with the aid of Robert Pexton. In the cases examined, $N=30$ and $\beta=10, 10^2, 10^3$, and 10^4 . y and f were of order of magnitude unity. Eight significant figures were carried. The results were satisfactory although four figures were lost for the largest value of β .

Reference

¹ W.E. Milne, "Numerical solution of differential equations." John Wiley and Sons, New York, 1953. p-100 ff.

1-4

THIS PAGE
WAS INTENTIONALLY
LEFT BLANK