**ANALYSIS OF EXTENSIBLE CURVED
PIPES CONVEYING FLUID**

by

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COMPONENTS TECHNOLOGY DIVISION

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ANALYSIS OF EXTENSIBLE CURVED PIPES
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ABSTRACT

A linear theory is presented to account for the motions of extensible curved pipes conveying fluid. Hamilton's principle is used to derive the equations of motions and the boundary conditions. Based on the theory, the flow-induced deformations are obtained in closed form. The natural frequency is also studied, and it is shown that the frequency decreases with increasing flow velocity. As the velocity approaches a certain value, the pipe loses stability by buckling. The critical flow velocity is calculated for two types of pipe end conditions.

1. INTRODUCTION

Vibration and stability of pipes conveying fluid are of concern in the design of oil pipelines, pump discharge lines, propellant lines, and, particularly, nuclear reactor system components. The dynamics of such coupled fluid/structural systems has been the subject of investigations since the early 1950's. However, most of these investigations concerned straight pipes containing a steady flow (see Stein [1], for example). Unny et al. [2] initiated the study of a curved pipe supported at both ends, and Chen [3-5] studied the vibration frequencies, instability mechanisms, and critical flow velocities for various end conditions. In both studies it was assumed that during linear strains the length of the central line remains invariant; i.e., there is no extension.

In this paper, the role of extension in the vibration and stability of curved pipes is studied. First, the equation of motion is derived from Hamilton's principle. Next, the prebuckling deformations and critical flow velocities are presented for fixed-fixed and hinged-hinged end conditions. Finally, the natural frequency is investigated, and the effect of flow velocity on natural frequencies is discussed.

II. EQUATIONS OF MOTION

Consider a pipe segment whose central line initially describes an arc of a plane circle as shown in Fig.1. The pipe has radius of curvature R , mass per unit length m , flexural rigidity EI , torsional rigidity GJ , extensional rigidity AE , and subtended angle α . The pipe conveys fluid of mass per unit length M , flowing with a constant velocity V . The following assumptions are made in deriving the equations of motion:

- (1) The rotatory inertia and shear deformation are negligible.
- (2) The effects of gravity and material damping are negligible.
- (3) The linear theory is valid.
- (4) The small-scale details of fluid motions are neglected.
- (5) The fluid is incompressible, and the total mass of fluid inside the pipe segment does not change during oscillations.

The equations are derived from Hamilton's principle. When the ends of a pipe are allowed to move, the fluid energy supply at the inlet is not necessarily equal to the fluid energy loss at the outlet. To account for the energy flux, Benjamin [6] has shown that Hamilton's principle in this case is

$$\delta \int_{t_1}^{t_2} \mathcal{L} dt - \int_{t_1}^{t_2} MV \left(\left. \frac{\partial \vec{R}}{\partial t} \right|_{\theta=\alpha} + V \vec{\tau} \right) \cdot \delta \vec{R} \Big|_{\theta=\alpha} dt = 0, \quad (1)$$

where $\vec{R}$ is the position vector of the central line of the pipe, $\vec{\tau}$ is a unit vector tangential to the pipe central line, and $\mathcal{L}$ is the Lagrangian given by

$$\mathcal{L} = T_p + T_f - V_p. \quad (2)$$

(b) STRESSED STATE

(a) UNSTRESSED STATE

Fig. 1. Deformation of a Uniformly Curved Pipe Conveying Fluid.

Here, T_p and T_f are the kinetic energies associated with the pipe and fluid, respectively, and V_p is the potential energy of the pipe.

With reference to Fig. 1, the pipe deformations consist of radial displacement u along the N-axis, transverse displacement v along the z-axis, axial displacement w along the T-axis, and the twist angle ϕ . The potential and kinetic energies of the pipe are, respectively,

$$V_p = \int_0^\alpha \left(\frac{M_N^2}{2EI} + \frac{M_z^2}{2EI} + \frac{M_T^2}{2GJ} + \frac{N_T^2}{2EA} \right) R d\theta$$

(3)

and

$$T_p = \int_0^\alpha \frac{m}{2} \left[\left(\frac{\partial u}{\partial t} \right)^2 + \left(\frac{\partial v}{\partial t} \right)^2 + \left(\frac{\partial w}{\partial t} \right)^2 + \frac{J}{A} \left(\frac{\partial \phi}{\partial t} \right)^2 \right] R d\theta ,$$

where

$$\begin{aligned} M_N &= \frac{EI}{R^2} \left(R\phi - \frac{\partial^2 v}{\partial \theta^2} \right) , \\ M_z &= \frac{EI}{R^2} \left(\frac{\partial^2 u}{\partial \theta^2} + \frac{\partial w}{\partial \theta} \right) , \\ M_T &= \frac{GJ}{R^2} \left(R \frac{\partial \phi}{\partial \theta} + \frac{\partial v}{\partial \theta} \right) , \end{aligned}$$

(4)

and

$$N_T = \frac{AE}{R} \left(\frac{\partial w}{\partial \theta} - u \right) .$$

Next, consider fluid flow kinematics. The velocity at a point on the central line of the pipe is

$$\frac{\partial \vec{R}}{\partial t} = \frac{\partial u}{\partial t} \vec{e}_N + \frac{\partial v}{\partial t} \vec{e}_z + \frac{\partial w}{\partial t} \vec{e}_T \quad (5)$$

where $\vec{e}_N$, $\vec{e}_z$, and $\vec{e}_T$ are unit vectors along the N-axis, z-axis, and T-axis, respectively. The relative velocity of the fluid with respect to the pipe is: [7]

$$\vec{v}_r = \frac{V}{R} \left[\left(\frac{\partial u}{\partial \theta} + w \right) \vec{e}_N + \frac{\partial v}{\partial \theta} \vec{e}_z + \left(R - u + \frac{\partial w}{\partial \theta} \right) \vec{e}_T \right] . \quad (6)$$

For neglecting the effect of the twisting motion on the fluid, the absolute flow velocity is

$$\vec{V} = \left[\frac{\partial u}{\partial t} + \frac{V}{R} \left(\frac{\partial u}{\partial \theta} + w \right) \right] \vec{e}_N + \left(\frac{\partial v}{\partial t} + \frac{V}{R} \frac{\partial v}{\partial \theta} \right) \vec{e}_z + \left[\frac{\partial w}{\partial t} + \frac{V}{R} \left(R - u + \frac{\partial w}{\partial \theta} \right) \right] \vec{e}_T ; \quad (7)$$

therefore, the kinetic energy of the enclosed volume of fluid is

$$\begin{aligned} T_f = \frac{M}{2} \int_0^\alpha \left[\left(\frac{\partial u}{\partial t} \right)^2 + \left(\frac{\partial v}{\partial t} \right)^2 + \left(\frac{\partial w}{\partial t} \right)^2 + \frac{V^2}{R^2} \left(\frac{\partial u}{\partial \theta} + w \right)^2 \right. \\ \left. + 2 \frac{V}{R} \frac{\partial u}{\partial t} \left(\frac{\partial u}{\partial \theta} + w \right) + \frac{V^2}{R^2} \left(\frac{\partial v}{\partial \theta} \right)^2 + 2 \frac{V}{R} \frac{\partial v}{\partial t} \frac{\partial v}{\partial \theta} \right. \\ \left. + \frac{V^2}{R^2} \left(R + \frac{\partial w}{\partial \theta} - u \right)^2 + 2 \frac{V}{R} \frac{\partial w}{\partial t} \left(R + \frac{\partial w}{\partial \theta} - u \right) \right] R d\theta . \quad (8) \end{aligned}$$

Substituting for $\mathcal{L}$, $\frac{\partial \vec{R}}{\partial t}$, and $\vec{\tau}$ in Eq. 1, and integrating by parts yields the following Euler-Lagrange equations:

$$\left. \begin{aligned} \frac{EI}{R^3} \left(\frac{\partial^4 u}{\partial \theta^4} + \frac{\partial^3 w}{\partial \theta^3} \right) + \frac{AE}{R} \left(u - \frac{\partial w}{\partial \theta} \right) + \frac{MV^2}{R} \left(\frac{\partial^2 u}{\partial \theta^2} + 2 \frac{\partial w}{\partial \theta} - u \right) \\ + 2MV \left(\frac{\partial^2 u}{\partial \theta \partial t} + \frac{\partial w}{\partial t} \right) + R(m+M) \frac{\partial^2 u}{\partial t^2} + MV^2 = 0 , \\ \frac{AE}{R} \left(\frac{\partial u}{\partial \theta} - \frac{\partial^2 w}{\partial \theta^2} \right) - \frac{EI}{R^3} \left(\frac{\partial^3 u}{\partial \theta^3} + \frac{\partial^2 w}{\partial \theta^2} \right) + \frac{MV^2}{R} \left(\frac{\partial^2 w}{\partial \theta^2} - 2 \frac{\partial u}{\partial \theta} - w \right) \\ + 2MV \left(\frac{\partial^2 w}{\partial \theta \partial t} - \frac{\partial u}{\partial t} \right) + R(m+M) \frac{\partial^2 w}{\partial t^2} = 0 , \end{aligned} \right\} \quad (9)$$

$$\left. \begin{aligned} \frac{EI}{R^3} \left(R \frac{\partial^2 \phi}{\partial \theta^2} - \frac{\partial^4 v}{\partial \theta^4} \right) + \frac{GJ}{R^3} \left(R \frac{\partial^2 \phi}{\partial \theta^2} + \frac{\partial^2 v}{\partial \theta^2} \right) - \frac{MV^2}{R} \frac{\partial^2 v}{\partial \theta^2} \\ - 2MV \frac{\partial^2 v}{\partial \theta \partial t} - R(m+M) \frac{\partial^2 v}{\partial t^2} = 0 , \end{aligned} \right\} \quad (10)$$

and

$$\frac{GJ}{R^2} \left(R \frac{\partial^2 \phi}{\partial \theta^2} + \frac{\partial^2 v}{\partial \theta^2} \right) - \frac{EI}{R^2} \left(R\phi - \frac{\partial^2 v}{\partial \theta^2} \right) - R \left(\frac{J}{A} \right) \frac{\partial^2 \phi}{\partial t^2} = 0 .$$

The associated boundary conditions are:

$$\left. \begin{aligned}
& \left[\frac{EI}{R^3} \left(\frac{\partial^3 u}{\partial \theta^3} + \frac{\partial^2 w}{\partial \theta^2} \right) + \frac{MV^2}{R} \left(\frac{\partial u}{\partial \theta} + w \right) + MV \frac{\partial u}{\partial t} \right] \delta u \Big|_0^\alpha \\
& - \left[\frac{MV^2}{R} \left(\frac{\partial u}{\partial \theta} + w \right) + MV \frac{\partial u}{\partial t} \right] \delta u \Big|_{\theta=\alpha} = 0 , \\
& \left[\frac{EI}{R^3} \left(\frac{\partial^2 u}{\partial \theta^2} + \frac{\partial w}{\partial \theta} \right) \right] \delta \left(\frac{\partial u}{\partial \theta} \right) \Big|_0^\alpha = 0 , \\
& \left[\frac{EI}{R^3} \left(\frac{\partial^2 u}{\partial \theta^2} + \frac{\partial w}{\partial \theta} \right) + \frac{AE}{R} \left(\frac{\partial w}{\partial \theta} - u \right) - \frac{MV^2}{R} \left(R + \frac{\partial w}{\partial \theta} - u \right) \right. \\
& \quad \left. - MV \frac{\partial w}{\partial t} \right] \delta w \Big|_0^\alpha + \left[\frac{MV^2}{R} \left(R + \frac{\partial w}{\partial \theta} - u \right) + MV \frac{\partial w}{\partial t} \right] \delta w \Big|_{\theta=\alpha} = 0 , \\
& \left[\frac{EI}{R^3} \left(R \frac{\partial \phi}{\partial \theta} - \frac{\partial^3 v}{\partial \theta^3} \right) + \frac{GJ}{R^3} \left(R \frac{\partial \phi}{\partial \theta} + \frac{\partial v}{\partial \theta} \right) - \frac{MV^2}{R} \frac{\partial v}{\partial \theta} - MV \frac{\partial v}{\partial t} \right] \delta v \Big|_0^\alpha \\
& \quad + \left(\frac{MV^2}{R} \frac{\partial v}{\partial \theta} + MV \frac{\partial v}{\partial t} \right) \delta v \Big|_{\theta=\alpha} = 0 , \\
& \frac{EI}{R^3} \left(R \phi - \frac{\partial^2 v}{\partial \theta^2} \right) \delta \left(\frac{\partial v}{\partial \theta} \right) \Big|_0^\alpha = 0 , \\
& \frac{GJ}{R^2} \left(R \frac{\partial \phi}{\partial \theta} + \frac{\partial v}{\partial \theta} \right) \delta \phi \Big|_0^\alpha = 0 .
\end{aligned} \right\} \quad (11)$$

$$\left. \begin{aligned}
& \text{and} \\
& \frac{GJ}{R^2} \left(R \frac{\partial \phi}{\partial \theta} + \frac{\partial v}{\partial \theta} \right) \delta \phi \Big|_0^\alpha = 0 .
\end{aligned} \right\} \quad (12)$$

As in the inextensional theory [5], the in-plane motion and out-of-plane motion are uncoupled. Since the out-of-plane motion is exactly the same as the inextensional theory studied previously [5], only the in-plane motion is considered in this paper.

III. FLOW-INDUCED DEFORMATIONS AND CRITICAL FLOW VELOCITIES

With the nondimensionalization

$$\begin{aligned}\xi &= u/R, & \beta &= \frac{M}{m+M}, \\ \eta &= w/R, & \tau &= \left(\frac{EI}{m+M} \right)^{1/2} t/R^2, \\ \mu &= \left(\frac{M}{EI} \right)^{1/2} RV, & r &= \frac{AR^2}{I}.\end{aligned}\quad (13)$$

Eqs. 9 become

$$\begin{aligned}\frac{\partial^4 \xi}{\partial \theta^4} + \frac{\partial^3 \eta}{\partial \theta^3} + r \left(\xi - \frac{\partial \eta}{\partial \theta} \right) + \mu^2 \left(\frac{\partial^2 \xi}{\partial \theta^2} + 2 \frac{\partial \eta}{\partial \theta} - \xi \right) \\ + 2\beta^{1/2} \mu \left(\frac{\partial^2 \xi}{\partial \theta \partial \tau} + \frac{\partial \eta}{\partial \tau} \right) + \frac{\partial^2 \xi}{\partial \tau^2} + \mu^2 = 0,\end{aligned}\quad (14)$$

and

$$\begin{aligned}r \left(\frac{\partial \xi}{\partial \theta} - \frac{\partial^2 \eta}{\partial \theta^2} \right) - \frac{\partial^3 \xi}{\partial \theta^3} - \frac{\partial^2 \eta}{\partial \theta^2} + \mu^2 \left(\frac{\partial^2 \eta}{\partial \theta^2} - 2 \frac{\partial \xi}{\partial \theta} - \eta \right) \\ + 2\beta^{1/2} \mu \left(\frac{\partial^2 \eta}{\partial \theta \partial \tau} - \frac{\partial \xi}{\partial \tau} \right) + \frac{\partial^2 \eta}{\partial \tau^2} = 0.\end{aligned}$$

Deleting the time-dependent terms from Eqs. 14 yields

$$\begin{aligned}\frac{d^4 \xi}{d\theta^4} + \frac{d^3 \eta}{d\theta^3} + r \left(\xi - \frac{d\eta}{d\theta} \right) + \mu^2 \left(\frac{d^2 \xi}{d\theta^2} + 2 \frac{d\eta}{d\theta} - \xi \right) = -\mu^2, \\ \text{and} \\ r \left(\frac{\partial \xi}{\partial \theta} - \frac{\partial^2 \eta}{\partial \theta^2} \right) - \frac{\partial^3 \xi}{\partial \theta^3} - \frac{\partial^2 \eta}{\partial \theta^2} + \mu^2 \left(\frac{\partial^2 \eta}{\partial \theta^2} - 2 \frac{\partial \xi}{\partial \theta} - \eta \right) = 0.\end{aligned}\quad (15)$$

Equations 15 are easily solved, and the solutions are:

$$\begin{aligned}\xi &= B_1(p \cos \theta - \theta \sin \theta) + B_2(p \sin \theta + \theta \cos \theta) - B_3 \sin \theta + B_4 \cos \theta \\ &\quad - B_5(q \sin \mu \theta) + B_6(q \cos \mu \theta) + \frac{\mu^2}{\mu^2 - r},\end{aligned}\quad (16)$$

and

$$\eta = B_1 \theta \cos \theta + B_2 \theta \sin \theta + B_3 \cos \theta + B_4 \sin \theta + B_5 \cos \mu \theta + B_6 \sin \mu \theta,$$

where

$$p = \frac{r-1}{r+1-2\mu^2},$$

and

$$q = \frac{(r+1-2\mu^2)\mu}{r-\mu^2}. \quad (17)$$

The coefficients B_n are determined from the equations obtained by substituting Eqs. 16 into the boundary conditions; these equations can be written:

$$\{a_{jn}\} \{B_n\} = \{c_n\} ; \quad j, n = 1, 2, 3, 4, 5, 6 \quad (18)$$

where

$$c_1 = c_3 = c_4 = c_6 = 0,$$

and

$$c_2 = c_5 = \frac{\mu^2}{r-\mu^2}. \quad (19)$$

For clamped-clamped pipes:

$$\{a_{jn}\} = \begin{bmatrix} 0 & 0 & 1 & 0 & 1 & 0 \\ p & 0 & 0 & 1 & 0 & q \\ 0 & p+1 & -1 & 0 & -q\mu & 0 \\ a \cos \alpha & a \sin \alpha & \cos \alpha & \sin \alpha & \cos \mu \alpha & \sin \mu \alpha \\ p \cos \alpha & p \sin \alpha & -\sin \alpha & \cos \alpha & -q \sin \mu \alpha & q \cos \mu \alpha \\ -a \sin \alpha & +a \cos \alpha & \cos \alpha & \sin \alpha & -q \sin \mu \alpha & q \cos \mu \alpha \\ -a \cos \alpha & -a \sin \alpha & -\cos \alpha & \sin \alpha & -q \cos \mu \alpha & -q \sin \mu \alpha \\ -(p+1) \sin \alpha & +(p+1) \cos \alpha & -\cos \alpha & \sin \alpha & -q \cos \mu \alpha & -q \sin \mu \alpha \end{bmatrix}. \quad (20)$$

For pinned-pinned pipes:

$$\{a_{jn}\} = \begin{bmatrix} 0 & 0 & 1 & 0 & 1 & 0 \\ p & 0 & 0 & 1 & 0 & q \\ -(1+p) & 0 & 0 & 0 & 0 & -\mu(q\mu-1) \\ a \cos \alpha & a \sin \alpha & \cos \alpha & \sin \alpha & \cos \mu \alpha & \sin \mu \alpha \\ p \cos \alpha & p \sin \alpha & -\sin \alpha & \cos \alpha & -q \sin \mu \alpha & q \cos \mu \alpha \\ -a \sin \alpha & +a \cos \alpha & \cos \alpha & \sin \alpha & -q \sin \mu \alpha & q \cos \mu \alpha \\ -(1+p) \cos \alpha & -(1+p) \sin \alpha & 0 & 0 & \mu(q\mu-1) \sin \mu \alpha & -\mu(q\mu-1) \cos \mu \alpha \end{bmatrix}. \quad (21)$$

Equations 18 are used to solve for the coefficients B_n ; then, from Eqs. 16, the flow-induced static displacements can be calculated. These displacements depend on the subtended angle α , flow velocity μ , and the parameter r .

For example, Fig. 2 shows the displacements at the center of fixed-fixed pipes for $\alpha = 90^\circ$ and 180° , and $r = 100$. It is seen that the axial displacement is zero and the radial displacement increases with increasing flow velocity. For $\alpha = 180^\circ$, as the flow velocity increases to 3.00, the pipe is subjected to instability in the first asymmetric mode. For $\alpha = 90^\circ$, as the flow velocity increases to 5.04, the radial displacement becomes infinite.

The critical flow velocity at which a pipe loses stability can be calculated by the equation

$$|a_{jn}| = 0 \quad . \quad (22)$$

The critical flow velocity is a function of the subtended angle α and the geometric parameter r . Computations have been made for several values of r , and the results for the lowest two critical flow velocities are given in Figs. 3 and 4 for fixed-fixed and pinned-pinned end conditions, respectively. The mode shapes at instability of the first two modes for $r = 100$, and $K = 0.25$ and 0.5 are shown in Fig. 5, where $K = \alpha/2\pi$.

The pipe may lose stability in symmetric and asymmetric modes. The lowest critical flow velocity may associate with either mode, depending on the values of α and r . For example, Fig. 5 shows that for $r = 100$ the pipe with $K = 0.25$ will lose stability initially in the first symmetric mode, and the pipe with $K = 0.5$ will buckle initially in the first asymmetric mode.

The critical flow velocity associated with the first asymmetric mode is almost independent of r . Moreover, in calculating the critical flow

Fig. 2. Flow-Induced Deformations of Fixed-Fixed Pipes for $r = 100$.

Fig. 3. Dimensionless Critical Flow Velocities for Fixed-Fixed End Conditions.

Fig. 4. Dimensionless Critical Flow Velocities for Pinned-Pinned End Conditions.

Fig. 5. Symmetric and Asymmetric Instability Mode Shapes for $r = 100$ and $K = 0.25$ and 0.5 .

velocity for asymmetric modes, the pipe may be considered as inextensible. This is because the energy associated with axial strain is much smaller than that associated with the bending strain. On the other hand, the critical flow velocity associated with the symmetric mode is strongly dependent on r . For small α , the pipe is subjected to instability in its first symmetric mode. But as α increases, the instability mode changes to the second symmetric mode (see Figs. 3 and 5). For large α , the pipe will never lose stability in its first symmetric mode.

From Figs. 3 and 4, it is seen that for small α , the pipe segment approaches a straight pipe, and the lowest critical flow velocity is associated with the first symmetric mode. As the subtended angle α increases, the pipe loses stability in its first asymmetric mode. Clearly, there is an angle at which the critical flow velocity crosses over from the one associated with symmetric mode to the one associated with asymmetric mode. This crossover phenomenon of critical flow velocity is similar to that of frequency of vibration [8, 9]. As shown in Fig. 6, which compares the crossover angles for critical flow velocity (α_v) and natural frequency (α_f) for two end conditions, α_v is always less than α_f for all values of r .

Fig. 6. Crossover Angles for Critical Flow Velocity and Natural Frequency.

IV. FREE VIBRATION

Based on the linear theory, the effect of static deformations on free vibration is not considered; i.e., the nonhomogeneous term in Eqs. 14 is not included in the analysis of free vibration. Let

$$\xi(\theta, \tau) = \Phi(\theta) \exp(i\Omega\tau),$$

and

$$\eta(\theta, \tau) = \Psi(\theta) \exp(i\Omega\tau),$$

where $i = \sqrt{-1}$, and Ω is a dimensionless frequency related to the circular frequency of motion ω by

$$\Omega = [(m + M)/EI]^{1/2} R^2 \omega. \quad (24)$$

Substitution of Eqs. 23 into Eqs. 14 yields

$$\begin{aligned} \frac{d^4 \Phi}{d\theta^4} + \frac{d^3 \Psi}{d\theta^3} + r \left(\Phi - \frac{d\Psi}{d\theta} \right) + \mu^2 \left(\frac{d^2 \Phi}{d\theta^2} + 2 \frac{d\Psi}{d\theta} - \Phi \right) \\ + 12\beta^{1/2} \mu \Omega \left(\frac{d\Phi}{d\theta} + \Psi \right) - \Omega^2 \Phi = 0 \\ r \left(\frac{d\Phi}{d\theta} - \frac{d^2 \Psi}{d\theta^2} \right) - \frac{d^3 \Phi}{d\theta^3} - \frac{d^2 \Psi}{d\theta^2} + \mu^2 \left(\frac{d^2 \Psi}{d\theta^2} - 2 \frac{d\Phi}{d\theta} - \Psi \right) \\ + 12\beta^{1/2} \mu \Omega \left(\frac{d\Psi}{d\theta} - \Phi \right) - \Omega^2 \Psi = 0. \end{aligned} \quad (25)$$

The solutions of Eqs. 25 are

$$\Phi = \sum_{n=1}^6 D_n \exp(i\lambda_n \theta),$$

and

$$\Psi = i \sum_{n=1}^6 D_n \delta_n \exp(i\lambda_n \theta),$$

where the D_n are some constants to be determined using the boundary conditions, and the λ_n are the six roots of the equation

$$\begin{aligned}
& (r - \mu^2)\lambda^6 - 2\beta^{1/2}\mu\Omega\lambda^5 + [\mu^4 + (2 - r)\mu^2 - 2r - \Omega^2]\lambda^4 \\
& + 2\beta^{1/2}\mu\Omega(2\mu^2 - r + 1)\lambda^3 - [2\mu^4 - (4\beta\Omega^2 + 2r + 2\Omega^2 - 1)\mu^2 \\
& + (\Omega^2 + r\Omega^2 - r)]\lambda^2 + 2\beta^{1/2}\mu\Omega(r - 2\mu^2 + 2\Omega^2)\lambda \\
& + (\Omega^2 + \mu^2)(\Omega^2 + \mu^2 - r) - 4\beta\mu^2\Omega^2 = 0,
\end{aligned}$$

and

(27)

$$\delta_n = \frac{\lambda_n^3 + (r - 2\mu^2)\lambda_n - 2\beta^{1/2}\mu\Omega}{(\mu^2 - r - 1)\lambda_n^2 + 2\beta^{1/2}\mu\Omega\lambda_n + \Omega^2 + \mu^2}.$$

Therefore, the complete solutions are given by

$$\xi(\theta, \tau) = \operatorname{Re} \left[\sum_{n=1}^6 D_n \exp(i\lambda_n \theta) \right]$$

and

(28)

$$\eta(\theta, \tau) = \operatorname{Re} \left[i \sum_{n=1}^6 D_n \delta_n \exp(i\lambda_n \theta) \right].$$

The equations to determine D_n can be written

$$\{b_{jn}\}\{D_n\} = \{0\}; \quad j, n = 1, 2, 3, 4, 5, 6. \quad (29)$$

The elements b_{jn} are given in Table 1 for two types of end conditions.

Setting the determinant of the coefficients in Eq. 29 equal to zero gives the frequency equation. The frequency depends on the parameters μ , r , α , and β ; therefore, the frequency equation may be written

$$F(\Omega, r, \alpha, \mu, \beta) = 0. \quad (30)$$

Figures 7 and 8 show the frequencies of a fixed-fixed pipe with $\alpha = 45^\circ$ and $r = 10^3$ and 10^6 , respectively. (The general characteristics of the frequency of a pinned-pinned pipe are similar.) With no flow, the pipe behaves as a ring segment [9]. As the flow velocity increases, the natural frequency always decreases. When the flow velocity reaches a certain value (critical flow velocity), the first frequency becomes zero, and the pipe is subjected to buckling. With further increase in

TABLE 1. BOUNDARY CONDITIONS AND ELEMENTS b_{jn}

End Conditions		Fixed-Fixed	Pinned-Pinned
Boundary Conditions	$\theta = 0$	$\phi = 0$	$\phi = 0$
		$\psi = 0$	$\psi = 0$
		$\frac{\partial \phi}{\partial \theta} = 0$	$\frac{\partial^2 \phi}{\partial \theta^2} + \frac{\partial \psi}{\partial \theta} = 0$
	$\theta = \alpha$	$\phi = 0$	$\phi = 0$
		$\psi = 0$	$\psi = 0$
		$\frac{\partial \phi}{\partial \theta} = 0$	$\frac{\partial^2 \phi}{\partial \theta^2} + \frac{\partial \psi}{\partial \theta} = 0$
Elements b_{jn}	b_{1n}	1	1
	b_{2n}	δ_n	δ_n
	b_{3n}	λ_n	$\lambda_n(\lambda_n + \delta_n)$
	b_{4n}	$\exp(i\lambda_n \alpha)$	$\exp(i\lambda_n \alpha)$
	b_{5n}	$\delta_n \exp(i\lambda_n \alpha)$	$\delta_n \exp(i\lambda_n \alpha)$
	b_{6n}	$\lambda_n \exp(i\lambda_n \alpha)$	$\lambda_n(\lambda_n + \delta_n) \exp(i\lambda_n \alpha)$

Fig. 7. Natural Frequency of a Fixed-Fixed Pipe as a Function of Flow Velocity for $\alpha = 45^\circ$ and $r = 10^3$.

Fig. 8. Natural Frequency of a Fixed-Fixed Pipe as a Function of Flow Velocity for $\alpha = 45^\circ$ and $r = 10^6$.

flow velocity, other frequencies also become zero. It is noted that the frequency depends on the parameter β ; however, the critical flow velocity is independent of β . This characteristic is the same as that of inextensional theory [3].

In the inextensional theory, the lowest frequency is always an asymmetric mode; however, in the extensional theory, it may be either a symmetric or asymmetric mode. As shown in Figs. 7 and 8, for $r = 10^3$ it is a symmetric mode, and for $r = 10^6$ it is an asymmetric mode.

V. SUMMARY

A general theory is presented to account for the motions of extensible, curved pipes conveying fluid. The equations of motion are derived by Hamilton's principle; both in-plane and out-of-plane equations and the associated boundary conditions are obtained. Based on this theory, the flow-induced deformations of pipes can be calculated easily. In contrast to the inextensional theory [3, 4], for a pipe supported at both ends, the prebuckling state of the pipe is generally not a circular arc. The natural frequencies of fixed-fixed and pinned-pinned pipes are also studied. It is shown that the frequency decreases with increasing flow velocity, and depends on the mass ratio β . As the flow velocity reaches a certain value, the frequency becomes zero and the pipe buckles. The critical flow velocity has been calculated for fixed-fixed and pinned-pinned end conditions. The results reveal that the pipe may lose stability in symmetric and asymmetric modes; the lowest critical flow velocity is associated with the first symmetric mode for $\alpha < \alpha_v$, and with the first asymmetric mode for $\alpha > \alpha_v$.

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