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NONLINEAR FINITE-ELEMENT
FORMULATION FOR TRANSIENT ANALYSIS
OF THREE-DIMENSIONAL THIN STRUCTURES

by

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NOMENCLATURE

A	Area of element	V	Volume of element
b_{iI}	Body coordinate unit vector of node I	(x, y, z)	Global coordinates
e_i	Element-coordinate unit vector	$(\hat{x}, \hat{y}, \hat{z})$	Element coordinates
E, E_p	Young's modulus and plastic modulus, respectively	$(\bar{x}, \bar{y}, \bar{z})$	Body coordinates
$[E^m], [E^f]$	Strain-nodal displacement matrix for membrane and flexural shape functions, respectively	α_i	Angular acceleration
f_{iI}	Nodal force at node I in i th direction	β	Fraction of critical damping
f_{IJ}	Tensile force along side joining nodes I and J	δ_{IJ}	Elongation of side connecting nodes I and J
$f_{iI}^{ext}, f_{iI}^{int}$	External and internal force at node I, respectively	$\{\hat{\epsilon}\}$	Strains in element coordinates
f_{IJ}^{vis}	Viscous force along side joining nodes I and J	$\hat{\theta}_{Ix}, \hat{\theta}_{Iy}$	Rotations relative to $(\hat{x}, \hat{y})$ plane
h	Thickness of element	$[\lambda]$	Body-global coordinate-transformation matrix
$I_{Ixx}, \text{etc.}$	Principal mass moments of inertia of node I	$[u]$	Element-global coordinate-transformation matrix
K	Effective stiffness for artificial viscosity	ν	Poisson's ratio
l_{IJ}	Length of side joining nodes I to J	ρ	Mass density
$\{\hat{m}\}$	Nodal moments	σ_y	Yield stress
M_I	Mass of node I	$\{\hat{\sigma}\}$	Stresses in element coordinates
t	Time	$[\varphi^f]$	Flexural shape functions
u_i	Displacement field	$[\varphi_x^m], [\varphi_y^m]$	Membrane shape functions
u_i^{def}, u_i^{rig}	Deformation and rigid-body displacements, respectively	ω_i	Angular velocity
v_i	Component of any vector	Superscript dots	Time derivatives
		Subscripts I, J, K	Node numbers or element numbers
		Subscripts i, j, k	Cartesian components

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ABSTRACT

This report describes a finite-element procedure for transient analysis of plates and shells in three-dimensional space, applicable to large displacements and nonlinear material properties. This procedure uses a convected coordinate formulation enabling the use of a simple strain-nodal displacement, as well as nodal force-stress relations. The plate/shell element can be subjected to linear in-plane displacements and cubic transverse displacements. The orientation of lumped masses is described by unit vectors, so that arbitrarily large rotations can be treated. The discretized equations of motion are integrated explicitly in time with a central difference formula. Membrane artificial viscosity is used to stabilize occasional oscillations.

The computational efficiency of the procedure is quite good: One element-time step takes 2 msec on an IBM 360/195 computer. Comparison of results with experimental data of impulsively loaded beams, plates, and shells shows good agreement. The program was applied to a hexagonal fuel subassembly loaded internally. Results are presented, and it is shown that, for this loading, two-dimensional cross-sectional models may be adequate.

I. INTRODUCTION

Concerns for the safety of reactor components have recently brought about the need for methods of analyzing structural components when subjected to large, transient loads. In these situations, the effects of large displacements and nonlinear material properties must be treated. Several general-purpose programs are available for solving problems of this type,^{1,2} but these employ implicit temporal integration and are quite time-consuming when applied to large-scale problems.

Here we investigate the applicability of a method with explicit temporal integration to three-dimensional plate and shell structures. The applicability

of such methods has recently been justified by two key papers: the numerical studies of Krieg and Key,³ who showed that the convergence of lumped-mass, explicit schemes are comparable to that of consistent-mass, implicit schemes; and the theoretical studies of Fujii,⁴ who rigorously proved the convergence of lumped-mass approximations in transient analysis.

Implicit integration schemes have generally been preferred in finite-element programs because of the absence of stability limitations on the time step. In elastic-plastic problems, this advantage is negated to a large extent by the need for a small time step in following the stress history of the path-dependent materials. Furthermore, when implicit schemes are applied to large nonlinear problems, the computational demands become overwhelming because of the need for solution of a system of linear algebraic equations at each time step. The computer time increases with the square of the bandwidth so that the size of problem that can be treated is limited.

The computation time for explicit lumped methods depends only linearly on the number of degrees of freedom; it is independent of the bandwidth. Thus, these methods appear promising for large-scale problems. The only drawback is the need to use a time step below the stability limits, which confines its usefulness to loads of relatively short duration. Since the loads treated in reactor-safety analysis are often brief, lumped explicit methods appear appropriate for these problems.

Explicit lumped finite-element schemes for problems nonlinear in geometry and material behavior have been described by Wu and Witmer,⁵ Hartzmann and Hutchinson,⁶ Heifitz and Costantino,⁷ Belytschko and Hsieh,^{8,9} and Belytschko et al.¹⁰ In Refs. 6 and 7 the method was applied to two-dimensional continuum problems using linear displacement elements. In Refs. 8-10, the method was applied to flexural and continuum elements in two-dimensional space. A convected-coordinate technique was used which simplified the governing equations; convected-coordinate formulations were previously used in static problems by Argyris et al.¹¹ and Murray and Wilson.¹² The present report will develop a similar formulation for flat-plate elements in three-dimensional space. The analysis is limited to small strains but arbitrarily large displacements. The orientation of the nodes is described by unit vectors so that arbitrarily large rotations in three-dimensional space can be treated.

II. FORMULATION OF METHOD

A. Coordinate Systems

The discrete model of the structure consists of an assemblage of flat, triangular plane elements interconnected by nodes. Each triangular element

Fig. 1. Coordinate Systems.
ANL Neg. No. 900-3469.

is associated with a convected coordinate system $(\hat{x}, \hat{y}, \hat{z})$, originating at node 1 of the element, which follows the element so that the $(\hat{x}, \hat{y})$ plane always passes through the three nodes of the element and the $\hat{x}$ -axis always bisects the current angle between the sides connecting nodes 1 and 2 and 1 and 3 (see Fig. 1). Thus, the $(\hat{x}, \hat{y}, \hat{z})$ coordinate system approximates the rigid-body motion of the element. In addition, each node is associated with a set of body coordinates $(\bar{x}, \bar{y}, \bar{z})$ which rotate with the node and coincide with the principal axes of the mass moment of inertia of the node. The fixed global coordinate system is designated by (x, y, z) .

The unit vectors of the body and element coordinates are denoted by $\bar{b}_i$ and $\bar{e}_i$, respectively. The components of these unit vectors define the transformations between body and global components as well as between element and global components; that is, any vector $\bar{v}$ transforms as follows:

$$\begin{Bmatrix} v_x \\ v_y \\ v_z \end{Bmatrix} = \begin{bmatrix} b_{1x} & b_{2x} & b_{3x} \\ b_{1y} & b_{2y} & b_{3y} \\ b_{1z} & b_{2z} & b_{3z} \end{bmatrix} \begin{Bmatrix} \bar{v}_x \\ \bar{v}_y \\ \bar{v}_z \end{Bmatrix} = [\lambda] \{\bar{v}\}; \quad (1)$$

$$\begin{Bmatrix} v_x \\ v_y \\ v_z \end{Bmatrix} = \begin{bmatrix} e_{1x} & e_{2x} & e_{3x} \\ e_{1y} & e_{2y} & e_{3y} \\ e_{1z} & e_{2z} & e_{3z} \end{bmatrix} \begin{Bmatrix} \hat{v}_x \\ \hat{v}_y \\ \hat{v}_z \end{Bmatrix} = [\mu] \{\hat{v}\}. \quad (2)$$

The transformation matrices of Eqs. 1 and 2 are derived in Appendixes D and E, respectively.

B. Displacement Field

The displacement of each element is decomposed into rigid-body motion and deformation

$$u_i = u_i^{\text{rig}} + u_i^{\text{def}}. \quad (3)$$

The rigid-body motion is defined by the motion of the convected coordinate system $(\hat{x}, \hat{y}, \hat{z})$. If the state of strain is homogeneous, this corresponds to exact rigid-body motion. Otherwise, this decomposition is only approximate, but as long as variations in rotation within an element and the strains are relatively small, the errors introduced by this approximate decomposition are negligible. The deformation displacements are the displacements relative to the convected coordinates. Thus, they may be obtained by subtracting the rigid-body displacements from the total displacements or by characterizing them by modes that are independent of the rigid-body motion.

We consider here a plate element with linear in-plane displacements and cubic transverse displacements. The deformation displacements can then be described by

$$\begin{Bmatrix} \hat{u}_x^{\text{def}} \\ \hat{u}_y^{\text{def}} \end{Bmatrix} = \begin{bmatrix} \varphi_x^m \\ \varphi_y^m \end{bmatrix} \{\delta\} \quad (4)$$

and

$$\{\hat{u}_z^{\text{def}}\} = [\varphi^f] \{\hat{\theta}\}, \quad (5)$$

where

$$\{\hat{\theta}\}^T = \{\hat{\theta}_{1x} \hat{\theta}_{1y} \hat{\theta}_{2x} \hat{\theta}_{2y} \hat{\theta}_{3x} \hat{\theta}_{3y}\} \quad (6)$$

and

$$\{\delta\}^T = \{\delta_{12} \delta_{23} \delta_{31}\}. \quad (7)$$

Here δ_{IJ} is the elongation of the element side connecting nodes I and J; $\hat{\theta}_{1x}$ and $\hat{\theta}_{1y}$ are the rotations relative to the moving $(\hat{x}, \hat{y})$ plane. These variables constitute the deformation modes of the element. The matrices $[\varphi_x^m]$, $[\varphi_y^m]$, and $[\varphi^f]$ are the membrane and flexural shape functions, respectively, the latter of which is described in Appendix G. The deformation of the element is assumed to be followed by rigid-body translation and rotation, so that the actual configuration of the deformed element is achieved.

For any configuration, the deformation displacements can be computed from (a) the total displacements by the equations

$$\delta_{IJ} = [2(x_{JI}u_{JIx} + y_{JI}u_{JIy} + z_{JI}u_{JIZ}) + u_{JIx}^2 + u_{JIy}^2 + u_{JIZ}^2]/(\ell_{IJ}^0 + \ell_{IJ}), \quad (8)$$

where ℓ_{IJ} and ℓ_{IJ}^0 are the current and original sides of the element joining nodes I and J,

$$x_{JI} = x_J - x_I, \quad u_{JIx} = u_{Jx} - u_{Ix}, \text{ etc.}, \quad (9)$$

and (b) the rotations of the element

$$\hat{\theta}_{Ix} = -\hat{e}_{3y}^0, \quad \hat{\theta}_{Iy} = \hat{e}_{3x}^0, \quad (10)$$

where

$$\{\hat{e}_3^0\} = [u]^T[\lambda]\{\bar{e}_3^0\}. \quad (11)$$

Here $\bar{e}_3^0$ is the unit vector normal to the plane of the element in the undeformed configuration.

The equations for $\hat{\theta}_{Ix}$ and $\hat{\theta}_{Iy}$ can be derived by noting that the relative rotation at a node is given by the cross product of the current normal vector $\bar{e}_3$ and the original vector $\bar{e}_3^0$ which rotates with the node; i.e.,

$$\bar{\theta} \sin \theta = \bar{e}_3 \times \bar{e}_3^0. \quad (12)$$

From the assumption that the strains are small, the relative rotations must be small, so if Eq. 12 is expressed in components of the element coordinates, Eq. 9 must follow immediately.

The novel concept of defining the deformation of the constant strain element by the elongation of its sides was originated by Argyris et al.;¹¹ it is used here more or less in the form given in Ref. 11, except that Argyris used incremental relations for δ_{IJ} rather than the total expression of Eq. 8. For the flexural deformations, the nonconforming shape functions developed by Bazeley et al.¹³ are used here.

C. Strain-Displacement Equations

For problems with small strains but arbitrarily large displacements, the strains can be measured by

$$\hat{e}_{ij} = \frac{1}{2}(\partial \hat{u}_i^{\text{def}} / \partial \hat{x}_j + \partial \hat{u}_j^{\text{def}} / \partial \hat{x}_i), \quad (13)$$

as shown in Ref. 8. These strain components are measured in the element coordinates $(\hat{x}, \hat{y}, \hat{z})$. When the deformation (i.e., engineering strain) is small, these strains are equivalent to the difference between the stretch unit tensors.

If we impose the Kirchhoff assumptions that normals to the midplane remain straight and normal, we obtain

$$\left. \begin{aligned} \hat{e}_{xx} &= (\partial \hat{u}_x^{\text{def}} / \partial \hat{x}) - \hat{z}(\partial^2 \hat{u}_z^{\text{def}} / \partial \hat{x}^2); \\ \hat{e}_{yy} &= (\partial \hat{u}_y^{\text{def}} / \partial \hat{y}) - \hat{z}(\partial^2 \hat{u}_z^{\text{def}} / \partial \hat{y}^2); \\ 2\hat{e}_{xy} &= (\partial \hat{u}_y^{\text{def}} / \partial \hat{x}) + (\partial \hat{u}_x^{\text{def}} / \partial \hat{y}) - 2\hat{z}(\partial^2 \hat{u}_z^{\text{def}} / \partial \hat{x} \partial \hat{y}). \end{aligned} \right\} \quad (14)$$

The decomposition into deformation and rigid-body displacements is necessary to describe adequately the bending strains unless they are defined in terms of curvatures for flexural elements. When the element undergoes large rotations, the transverse displacement cannot be clearly defined, so that the terms $\partial^2 u_z / \partial x^2$, etc., are not indicative of the curvature. If the total displacement transverse to the original configuration of the element is used in the Lagrangian counterpart of Eq. 14, bending strains are induced by the in-plane displacements. The inclusion of higher-order von Karman terms in the strain-displacement equations does not eliminate this difficulty.

Inserting Eqs. 4 and 5 into Eq. 14, we obtain

$$\{\hat{e}\} = [E^m]\{\delta\} - \hat{z}[E^f]\{\hat{\theta}\}, \quad (15)$$

where

$$\{\hat{e}\}^T = \{\hat{e}_{xx} \ \hat{e}_{yy} \ \hat{e}_{xy}\} \quad (16)$$

and

$$\left. \begin{aligned} [E^m] &= \begin{bmatrix} \partial \varphi_{xI}^m / \partial \hat{x} \\ \partial \varphi_{yI}^m / \partial \hat{y} \\ (\partial \varphi_{xI}^m / \partial \hat{y}) + (\partial \varphi_{yI}^m / \partial \hat{x}) \end{bmatrix}; \\ [E^f] &= \begin{bmatrix} \partial^2 \varphi_I^f / \partial \hat{x}^2 \\ \partial^2 \varphi_I^f / \partial \hat{y}^2 \\ 2\partial^2 \varphi_I^f / \partial \hat{x} \partial \hat{y} \end{bmatrix}. \end{aligned} \right\} \quad (17)$$

See Appendixes F and G for the derivation of Eqs. 17.

D. Nodal Forces

The nodal forces, following the derivations in Ref. 8, are given by

$$\begin{Bmatrix} f_{12} \\ f_{23} \\ f_{31} \end{Bmatrix} = \int_{V_I} [E^m]^T \{\hat{\sigma}\}, \quad (18)$$

where $\{\hat{\sigma}\}$ are the stresses measured in the convected coordinates and are defined as

$$\{\hat{\sigma}\}^T = \{\hat{\sigma}_{xx} \ \hat{\sigma}_{yy} \ \hat{\sigma}_{xy}\}. \quad (19)$$

The local moments are given by

$$\{\hat{m}\} = - \int_{V_I} \hat{z} [E^f]^T \{\hat{\sigma}\}, \quad (20)$$

where

$$\{\hat{m}\}^T = \{\hat{m}_{1x} \ \hat{m}_{1y} \ \hat{m}_{2x} \ \hat{m}_{2y} \ \hat{m}_{3x} \ \hat{m}_{3y}\}. \quad (21)$$

The integrations of Eqs. 18 and 20 are performed using a five-point trapezoidal integration over $\hat{z}$ (see Appendix H) and a three-point integration given in Ref. 14 over the $(\hat{x}, \hat{y})$ plane.

The transverse nodal forces are obtained from the requirement that the nodal forces of an element must be self-equilibrated. Moment equilibrium yields

$$\begin{Bmatrix} \hat{f}_{2z} \\ \hat{f}_{3z} \end{Bmatrix} = \frac{1}{2A} \begin{bmatrix} \hat{x}_3 & \hat{y}_3 \\ -\hat{x}_2 & -\hat{y}_2 \end{bmatrix} \begin{Bmatrix} \hat{m}_{1x} + \hat{m}_{2x} + \hat{m}_{3x} \\ \hat{m}_{1y} + \hat{m}_{2y} + \hat{m}_{3y} \end{Bmatrix}, \quad (22)$$

whereas transverse equilibrium yields

$$\hat{f}_{1z} = -\hat{f}_{2z} - \hat{f}_{3z}, \quad (23)$$

where A is the element area.

The nodal forces $\{f_{IJ}\}$ are transformed to global coordinates by

$$\begin{Bmatrix} f_{Ix} \\ f_{Iy} \\ f_{Iz} \end{Bmatrix} = \frac{f_{IJ}}{l_{IJ}} \begin{Bmatrix} x_{IJ} + u_{IJx} \\ y_{IJ} + u_{IJy} \\ z_{IJ} + u_{IJz} \end{Bmatrix} + \frac{f_{IK}}{l_{IK}} \begin{Bmatrix} x_{IK} + u_{IKx} \\ y_{IK} + u_{IKy} \\ z_{IK} + u_{IKz} \end{Bmatrix} + \hat{f}_{Iz} \begin{Bmatrix} e_{3x} \\ e_{3y} \\ e_{3z} \end{Bmatrix}. \quad (24)$$

The moments are transformed to body coordinates of the associated nodes by

$$\begin{Bmatrix} \bar{m}_{Ix} \\ \bar{m}_{Iy} \\ \bar{m}_{Iz} \end{Bmatrix} = [\lambda]^T [\mu] \begin{Bmatrix} \hat{m}_{Ix} \\ \hat{m}_{Iy} \\ \hat{m}_{Iz} \end{Bmatrix}, \quad (25)$$

where $[\lambda]$ and $[\mu]$ are defined by Eqs. 1 and 2.

E. Equations of Motion

The equations of motion are

$$\ddot{u}_{Ji} = (f_{Ji}^{\text{ext}} - f_{Ji}^{\text{int}})/M_J \quad (26)$$

and

$$\begin{cases} \bar{\alpha}_{Jx} = [\bar{m}_{Jx}^{\text{ext}} - \bar{m}_{Jx}^{\text{int}} + \bar{w}_{Jy} \bar{w}_{Jz} (\bar{I}_{Jyy} - \bar{I}_{Jzz})]/\bar{I}_{Jxx}; \\ \bar{\alpha}_{Jy} = [\bar{m}_{Jy}^{\text{ext}} - \bar{m}_{Jy}^{\text{int}} + \bar{w}_{Jz} \bar{w}_{Jx} (\bar{I}_{Jzz} - \bar{I}_{Jxx})]/\bar{I}_{Jyy}; \\ \bar{\alpha}_{Jz} = [\bar{m}_{Jz}^{\text{ext}} - \bar{m}_{Jz}^{\text{int}} + \bar{w}_{Jx} \bar{w}_{Jy} (\bar{I}_{Jxx} - \bar{I}_{Jyy})]/\bar{I}_{Jzz}. \end{cases} \quad (27)$$

Equation 26 represents the translational equations of motion in global coordinates, whereas Eqs. 27 gives the Euler equations of motion in the moving nodal coordinates. The internal forces are obtained by a topologically appropriate summation of all the element nodal forces given by Eqs. 24 and 25.

The lumped masses and rotational inertias of the nodes are obtained as follows. Each element is subdivided into three equal parts, each associated with one of the nodes of the elements. The translational mass and moments of inertia about the node of each of these parts are then computed, and the contributions of all nodes adjacent to an element are summed. The principal moments of inertia and the associated directions are then found for all nodes; the latter constitute the initial body coordinates $\bar{b}_i$ (see Appendix A).

The equations of motion are integrated by the Newmark- β method¹⁵ (with $\beta = 0$), which gives

$$u_{iI}(t + \Delta t) = u_{iI}(t) + \Delta t \dot{u}_{iI}(t) + \frac{1}{2} \Delta t^2 \ddot{u}_{iI}(t); \quad (28)$$

$$\dot{u}_{iI}(t + \Delta t) = \dot{u}_{iI}(t) + \frac{1}{2} \Delta t [\ddot{u}_{iI}(t) + \ddot{u}_{iI}(t + \Delta t)]. \quad (29)$$

The method for updating the body unit vectors $\vec{b}_i$ is described in Ref. 16 and also given in Appendix D. A flow chart of the computational procedure is given in Fig. 2. For the treatment of boundary conditions and application of external pressure, refer to Appendixes B and C, respectively.

Fig. 2

Flow Chart of Computational Procedure.
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F. Artificial Viscosity

In elastic-plastic problems, the response computed by the explicit temporal method is quite smooth. About the only problem that has appeared is the tendency of certain nodes to oscillate in a local snap-through mode whenever the residual membrane forces are high and no external loads are present, such as between two successive impulses. This has been eliminated quite nicely by a membrane artificial viscosity.

The artificial viscosity is of a linear type and is of the form of percentage critical damping, i.e.,

$$\{f_{IJ}^{vis}\} = 2\beta\{\dot{\delta}_{IJ}\}(KM)^{1/2}, \quad (30)$$

where β is the fraction of critical damping, K is the effective stiffness, and M is the effective mass. The effective stiffness is taken to be

$$K = \Delta f_{IJ} / \Delta \delta_{IJ}, \quad (31)$$

where Δf_{IJ} and $\Delta \delta_{IJ}$ are the changes in the deformation forces and elongations during the last time step. The effective mass is approximated by

$$M = \rho Ah / 3. \quad (32)$$

The viscous forces $\{f_{IJ}^{vis}\}$ are added to the forces computed by Eq. 18 in each time step. Values of β between 0.05 and 0.1 are generally used in the runs of longer durations.

III. SAMPLE RESULTS

Solutions for several problems are presented here. Most are thin structures for which experimental results are available; the last is a hexagonal shell used for packing fuel pins in liquid-metal fast breeder reactors. In all problems, an elastic-plastic stress-strain law with isotropic, linear

strain hardening was used; the elastic modulus is denoted by E , the plastic modulus by E_p , and the yield stress by σ_y .

A. Aluminum Beam

The first problem is a 10-in.-long aluminum beam clamped at both ends and loaded impulsively over a center portion, as shown in Fig. 3. Experimental results have been given for this problem by Balmer and Witmer.¹⁷ Figure 3 compares the deflection response with the experimental results.

Fig. 3. Comparison of Computed and Experimental Results for Central Deflection History of Impulsively Loaded Strip. ANL Neg. No. 900-3466 Rev. 1.

B. Mild-steel Wide Beam or Plate

The second problem is a mild-steel plate clamped at two ends and loaded impulsively over the entire surface, as shown in Fig. 4. Experimental results for this problem were

taken from Jones et al.¹⁸ The final deformations of the plate along the x and y axes, respectively, are given in Fig. 4. Corresponding deformations are also shown for the case in which the strain-rate effects were taken into account by the constitutive equation proposed by Cowper and Symonds.¹⁹ It can be seen that the first results overestimate the experimental data and the strain-rate-corrected results underestimate them almost by the same amount. This wide variation in the two sets of analytical results indicates that it may be difficult to correlate predictions with experimental data involving heavily strain-rate-dependent materials, at least until these effects in biaxial state of stress are better understood. It can be observed, further, that the analytical end deformations indicate a fixed boundary condition; however, the experimental results show more of a plastic hinge effect. In order to approach the hinge effect analytically, smaller elements would need to be taken close to the edge. This would, in turn, necessitate smaller time steps. The best approach would be the formulation of a special "hinge" boundary condition whereby the end moment would be proportional to rotation.

Fig. 4. Comparison of Computed and Experimental Results for Deflected Shapes of Impulsively Loaded Mild-steel Plate. ANL Neg. No. 900-3482 Rev. 1.

C. Aluminum Wide Beam or Plate

This problem is similar to that of Sec. III. B except that the material is aluminum. Dimensions and properties of the plate are given in Fig. 5, and final displacements are shown in Fig. 6. The agreement between experimental and analytical results is better than before. It is suspected that this rather close agreement results because aluminum is less strain-rate sensitive than mild steel. In general, better agreement is shown for the case in which hinged boundary conditions were assumed, especially close to the ends. Some values, like the maximum displacement at the free side, are in closer agreement if the fixed boundary condition is assumed. However, the best compromise would likely result if the proposed "hinge" boundary condition could have been used.

Fig. 5. Dimensions and Properties of Aluminum Specimen No. 10 (Ref. 18)

Fig. 6. Comparison of Permanent Profile of Aluminum Specimen No. 10

resulting from this assumption are shown in Fig. 8. It is not implied that this rather crude and somewhat arbitrary approximation represents the actual

D. Aluminum Plate with Fixed Supports

The experimental data for this problem, in which all the sides of the plate were clamped, are taken from Ref. 20. Dimensions and properties of this plate are given in Fig. 7. The predicted displacements are similar to the experimental deformation, as shown in Fig. 8. However, the effect of end constraints seems to have a greater influence in this case. It should be expected that a certain degree of slippage at the edges would take place during the experiment, while none was taken into account by the analytical model.

To estimate the effect of the in-plane constraints, a second run was made with certain nodes at the boundaries allowed to slip. Two nodes at the short side, and three nodes on the long side, approximately corresponding to the location of the clamps (see Ref. 20), were rigidly fixed, while the remaining nodes were permitted to move in the plane of the plate. The displacements

Fig. 7
Dimensions and Properties of Aluminum Specimen No. 19 (Ref. 20)

Fig. 8
Comparison of Permanent Profile of Aluminum Specimen No. 19

constraints at the sides. Nevertheless, the example shows the large effect that slippage may have on the final results.

An analytical model with one additional row of elements inside the fixed boundaries, permitting in-plane movement, could also have been used to estimate the slippage effect. In this case, a decision would need to be made as to how much of such in-plane displacement should be made available. In either case, accurate modeling of slippage at the boundary is very difficult, if not impossible.

E. Cylindrical Panel

The experimental results of this problem were taken from Ref. 21, where a similar comparison with analytical results was made. The experimental specimen is shown in Fig. 9. Due to symmetry, only one-half of the panel is used for the finite-element model, which is made up of 128 elements.

Fig. 9. Dimensions and Properties of Cylindrical Panel²¹

a hinged boundary condition was assumed in the model. A plastic hinge may or may not develop at the sides of the panel, so hinged and fixed sides were examined.

The final deformation of the panel at the crown line is shown in Fig. 10. The analytical results for both the fixed and hinged-side conditions underestimate the experimental data. The location of the maximum deflection is also displaced somewhat.

Although a plastic hinge at the near end of the panel is quite apparent, there seems to be no evidence of one on the far end. It

The area subjected to the pressure is indicated.

Previous examples have indicated that a hinged boundary condition yields closer approximations with experimental results if a plastic hinge develops and the final strain energy is many times that of the recoverable elastic energy. Since this condition is expected at the ends,

Fig. 10. Final Deformation of the Cylindrical Panel at the Crown Line

appears that the use of a fixed boundary condition at the far end of the panel may have been more appropriate. However, consideration of the slope of the deformation at that end indicates little difference as to what particular rotational constraint is imposed.

The cross-sectional deformation at $y = 6.28$ in. is shown in Fig. 11. Although the analytical results are underestimated, the agreement with experi-

mental data is quite good. For the experimental values given, it seems to make little difference what boundary conditions are used at the sides. Unfortunately, the experimental values do not extend to the sides; they would indicate the proper boundary condition to be used.

Fig. 11. Final Deformation of Cylindrical Panel at the Cross Section at $y = 6.28$ in.

As a test of slippage at the supports, two nodes at both ends of the panel were permitted to move freely in the y direction. Since the actual clamping of the ends was not known, the above assumption is rather arbitrary, and the apparent close agreement with experimental results is fortuitous. The final deformations due to this assumption are also shown in Figs. 10 and 11. It is clear again that axial slippage may have a large effect on the final results.

Figures 12 and 13 show the histories of experimental and analytical results at two points on the panel. Four sets of analytical results are shown in each case. Two sets of results pertain to the 128-element model with fixed and hinged sides, respectively. The two remaining sets also refer to fixed and hinged sides, but to a different analytical model. Here, the number of rows of elements in the circumferential direction is six, instead of four, increasing the number of elements of the model to 192.

Peak displacements resulting from the predictions of the 192-element model, especially in the case with hinged sides, appear in good agreement with experimental results. Subsequent experimental data show pronounced structural damping, as opposed to the analytical results. In this respect, the analytical values could be adjusted if closer agreement were desired. Of greater interest, however, is the significant effect of element size within the results of the two models. Although this was not further investigated, it commands serious attention in subsequent studies.

Fig. 12. Variation of Displacement at Point $x = 0$, $y = 6.28$ in.

Fig. 13. Variation of Displacement at Point $x = 0$, $y = 9.42$ in.

F. Fuel-element Hexcan

The last problem involves the response of a subassembly fuel wrapper subjected to an internal transient pressure. Such fuel wrappers are used in the planned liquid-metal fast breeder reactors. The wrappers serve several functions: They provide a means of packing and supporting the fuel pins they provide a path for the sodium coolant, and they serve as a means of confining severe pressure releases due to accidents. Several two-dimensional studies have been made by modeling the cross section of the hexcan as a beam in plane strain.²² The principal aim of this study was to check the validity of the two-dimensional approximation.

The hexcan walls are irradiated CY-316 stainless steel. The material is assumed to operate at 1000°F. At this temperature, since strain-rate effects are 10% or less, they are neglected.

The loading is a uniform internal pressure over 12 in. of the hexcan, as shown in Fig. 14. Because of the symmetry of the load and the (x, y) plane section, a 1/12th section of the hexcan was used for the finite-element model. A uniform mesh with 5×31 nodes was used, involving a total of 240 elements. The time step was 4×10^{-4} msec.

Figure 15 shows the deformed configuration of the hexcan from the midplane of the pressurized region to one end at the time of 0.4 msec. It can

be seen that the midplane cross section has almost become circular, while the portions away from the load have hardly deformed. Figure 16 shows the deformed and original cross section.

Fig. 14. Hexagonal Subassembly Wrapper, Loading, and Material Behavior. ANL Neg. No. 900-3463 Rev. 1.

Fig. 15. Deformed Subassembly Wrapper. ANL Neg. No. 900-3467.

Fig. 16
Deformed and Undeformed Central Cross Section of Subassembly Wrapper. ANL Neg. No. 900-3468.

Figure 17 shows the axial variation of the normal displacements at the corner and middle of the flat at 0.4 msec. As can be seen, the deformation

varies little over the inner 6 in. of the loaded region (i.e., of the symmetric region). The transition between the deformed and undeformed regions is quite abrupt and is localized at the pressure discontinuity.

Fig. 17

Axial Variations of Normal Displacements of Corner and Center of Flat.
ANL Neg. No. 900-3465.

Figure 18 shows time histories of the displacements of various points on the x axis. Also shown is the response predicted by a two-dimensional, plane-strain analysis of the hexcan. As can be seen, the two-dimensional predictions agree quite well with the results of the three-dimensional analysis.

Fig. 18

Time Histories of Displacements.
ANL Neg. No. 900-3461 Rev. 1.

IV. DISCUSSION AND CONCLUSIONS

The work described here represents one of the first efforts to develop a three-dimensional finite-element program with explicit-lumped temporal integration for arbitrarily large displacements. Hence, there has been little time to investigate various alternative techniques. In particular, the mass-lumping procedures have not yet been critically studied, and it is not clear whether the rather cumbersome procedures used here are essential. Computational effort could be reduced if the moment-of-inertia tensor at all nodes were considered isotropic. Since studies with beams have shown that the magnitude of the rotational mass has little effect on the transient response calculations, the use of isotropic moment-of-inertia tensors may prove acceptable. Another questionable point is the inconsistency in that the element used here provides no resistance to rotations about the z axis, whereas the nodes are free to rotate about this axis. The distribution of kinetic energies was checked in the computations reported here, and it was found that the kinetic energy about the z axis increases almost monotonically with time, though for the 1000 and 3000 time-step computations made here, it is never more than 0.02% of the total energy. Thus, though this aspect of the model has not proven troublesome, it bears further investigation.

The computational speed of this method, as expected, is quite good. Presently, one element time step takes 2 msec on an IBM 360/195, so a 500-element mesh requires 1 sec for each time step; this is expensive, but not prohibitive. A large part of the computational effort goes into the evaluation of the integrals in Eqs. 18 and 20, as well as the attendant stress-strain computations, which must be made at 15 points in each element. No advantage is presently taken of the simplifications of Eqs. 18 and 20 when elements are elastic. Otherwise, it is unclear how the computational effort could be significantly reduced. Constant-curvature elements have been tried in this investigation, but the convergence is very slow and far more elements are required to achieve comparable accuracy.

The convected-coordinate procedure used here has proved quite attractive in that (a) the strain-nodal displacement and the nodal force and stress relations are considerably simpler than the Lagrangian counterparts, and (b) the displacement field is completely consistent with the Kirchhoff assumptions. The formulation is limited to small strains and small variations in rotations within an element. In most problems of reactor-safety analysis, these restrictions are quite acceptable. Furthermore, results obtained in Ref. 10 for beams show that modifying Eqs. 13 to account for moderate variations of rotations within an element affects the results very little for this class of problems.

The results obtained here for the three-dimensional hexcan problem differ little from two-dimensional predictions in the regions of maximum deformation, so two-dimensional analyses may be sufficient for this case. It is

precisely questions of this nature that the three-dimensional code is intended to answer; i.e., what are the limitations and ranges of validity of simpler models? If the surrounding coolant and adjacent hexcans are included in the analysis, as will be the case in planned experiments,²² it is expected that the three-dimensional effects will become more important because of the transverse pressure gradients induced in the coolant. Studies of these problems will require three-dimensional codes.

APPENDIX A

Formulation of Mass Matrix

The equations of motion are formulated in two coordinate systems. Translational equations are expressed in the fixed global system, while the rotational equations of motion for each mode are expressed in the node's body coordinate systems, which rotate with the nodes. The formulation of the mass matrix for translation and the mass-moment-of-inertia matrix for rotation follows.

1. Translational Mass

The mass M of each element of constant thickness h is

$$M = \rho h A, \quad (A.1)$$

where ρ is the mass density and A is the area of the element. The area of the element is determined from the vector product of two sides of the triangle, as shown in Fig. 19. Thus, in global coordinates we can write

$$M = \frac{\rho h}{2} \begin{vmatrix} 1 & 1 & 1 \\ x_{21} & y_{21} & z_{21} \\ x_{31} & y_{31} & z_{31} \end{vmatrix}, \quad (A.2)$$

where

$$\left. \begin{aligned} x_{21} &= x_2 - x_1; \\ y_{21} &= y_2 - y_1; \\ z_{21} &= z_2 - z_1; \end{aligned} \right\} \quad (A.3)$$

Fig. 19. Orientation of Unit Vectors $\vec{g}$ and $\vec{e}_3$ with Respect to the Element

and the nodes are numbered in a counterclockwise direction around the element. The mass determined in Eq. A.2 is divided equally and assigned to each of the three nodes. This is done for each element, and the contributions of all elements are summed at the respective nodes.

2. Mass Moment of Inertia

Since the plate is assumed to be thin, the rotatory inertia of the cross section can be neglected. Consequently, the moment-of-inertia tensor for the triangle shown in Fig. 20 is given in the local coordinate system $(\hat{x}, \hat{y}, \hat{z})$ by

$$[\hat{I}] = \rho h \int_A \begin{bmatrix} \hat{I}_{xx} & \hat{I}_{xy} & 0 \\ \hat{I}_{yx} & \hat{I}_{yy} & 0 \\ 0 & 0 & \hat{I}_{zz} \end{bmatrix}, \quad (\text{A.4})$$

where

$$\left. \begin{aligned} \hat{I}_{xx} &= (\hat{y} - \hat{y}_c)^2, \\ \hat{I}_{yy} &= (\hat{x} - \hat{x}_c)^2, \\ \hat{I}_{xy} &= \hat{I}_{yx} = -(\hat{x} - \hat{x}_c)(\hat{y} - \hat{y}_c), \\ \hat{I}_{zz} &= \hat{I}_{xx} + \hat{I}_{yy} = (\hat{x} - \hat{x}_c)^2 + (\hat{y} - \hat{y}_c)^2, \end{aligned} \right\} \quad (\text{A.5})$$

and $\hat{x}_c$ and $\hat{y}_c$ are the centroidal coordinates given as

$$\text{and } \left. \begin{aligned} \hat{x}_c &= (\hat{x}_2 + \hat{x}_3)/3 = (a_3 - a_2)/3 \\ \hat{y}_c &= (\hat{y}_2 + \hat{y}_3)/3 = (b_2 - b_3)/3 \end{aligned} \right\}, \quad (\text{A.6})$$

where a_i and b_i are defined in Fig. 20. To simplify the integration, we introduce triangular coordinates $(\zeta_1, \zeta_2, \zeta_3)$, where

$$\left. \begin{aligned} \hat{x} &= \hat{x}_I \zeta_I \\ & \quad (\text{sum over } I, I = 1, 2, 3) \\ \hat{y} &= \hat{y}_I \zeta_I \end{aligned} \right\}. \quad (\text{A.7})$$

The integration formulas in triangular coordinates are given by Meek.²³ Thus,

$$\int_A \zeta_I^r \zeta_J^s = 2A r! s! / (r + s + 2)!. \quad (\text{A.8})$$

Consequently,

$$\int_A \zeta_I^r \zeta_J^s = \begin{cases} A/12 & \text{if } I \neq J, \ r \neq 0, \ s \neq 0 \\ A/6 & \text{if } I = J, \ r \neq 0, \ s \neq 0 \\ A/3 & \text{if } r = 0, \ s \neq 0 \\ A & \text{if } r = 0, \ s = 0 \end{cases}. \quad (\text{A.9})$$

Fig. 20. Definition of Element Coordinates

The integrals of the components defined by Eq. A.4 reduce to

$$\left. \begin{aligned} \int_A \hat{I}_{xx} &= A(\hat{y}_2 + \hat{y}_3 + \hat{y}_2 \hat{y}_3 - 6\hat{y}_c^2)/6; \\ \int_A \hat{I}_{yy} &= A(\hat{x}_2 + \hat{x}_3 + \hat{x}_2 \hat{x}_3 - 6\hat{x}_c^2)/6; \\ \int_A \hat{I}_{xy} &= -A[\hat{x}_2 \hat{y}_2 + \hat{x}_3 \hat{y}_3 + 0.5(\hat{x}_2 \hat{y}_3 + \hat{x}_3 \hat{y}_2) + 6\hat{x}_c \hat{y}_c]/6. \end{aligned} \right\} \quad (A.10)$$

In terms of the notation defined in Fig. 20, we have

$$\left. \begin{aligned} \int_A \hat{I}_{xx} &= A(b_2^2 + b_3^2 - b_2 b_3 - 6\hat{y}_c^2)/6; \\ \int_A \hat{I}_{yy} &= A(a_2^2 + a_3^2 - a_2 a_3 - 6\hat{x}_c^2)/6; \\ \int_A \hat{I}_{xy} &= A[a_2 b_2 + a_3 b_3 - 0.5(a_2 b_3 + a_3 b_2) + 6\hat{x}_c \hat{y}_c]/6. \end{aligned} \right\} \quad (A.11)$$

Equations A.10 or A.11 define the mass moment of inertia tensor of the element in the element's coordinates. This moment-of-inertia tensor is next transformed into the global coordinate system (x, y, z) by the transformation

$$[I] = [\mu]^T [\hat{I}] [\mu], \quad (A.12)$$

where $[\mu]$ is defined in Eq. 2 and derived in Appendix D. When the matrix multiplication of Eq. A.12 is performed, we get the following expressions for the individual components of $[I]$:

$$I_{IJ} = \hat{I}_{11} e_{Ix} e_{Jx} + \hat{I}_{22} e_{Iy} e_{Jy} + \hat{I}_{33} e_{Iz} e_{Jz} + \hat{I}_{12} (e_{Ix} e_{Jy} + e_{Iy} e_{Jx}), \quad (A.13)$$

where $I = 1, 2, 3$ and $J = 1, 2, 3$. Note that tensor $[I]$ is symmetric, so that $I_{21} = I_{12}$, $I_{31} = I_{13}$, and $I_{32} = I_{23}$.

The elements of the tensor $[I]$ are evaluated for each finite element, and one-third of their values is assigned to each node of the element. After the contributions from all elements are added, the eigenvalues and the respective eigenvectors of the resultant tensors are found at each node. This is accomplished by a standard computer-library routine which solves the cubic equation

$$\begin{vmatrix} \Lambda - I_{xx} & I_{xy} & I_{xz} \\ I_{xy} & \Lambda - I_{yy} & I_{yz} \\ I_{xz} & I_{yz} & \Lambda - I_{zz} \end{vmatrix} = 0 \quad (A.14)$$

The three roots of these equations in Λ are the eigenvalues $(I_{xx})_p$, $(I_{yy})_p$, and $(I_{zz})_p$, which are the principal moments of inertia. The corresponding eigenvectors establish the initial orientation of nodal coordinates $(\bar{x}, \bar{y}, \bar{z})$. The initial matrix $[\lambda^0]$ is thus established.

For the purposes of obtaining relative rotations at the nodes, the initial unit normal vector of each element is transformed to the initial nodal coordinates at each node as follows:

$$\begin{Bmatrix} \bar{e}_{3x}^0 \\ \bar{e}_{3y}^0 \\ \bar{e}_{3z}^0 \end{Bmatrix} = [\lambda^0]^T [\mu^0] \begin{Bmatrix} 0 \\ 0 \\ 1 \end{Bmatrix} = [\lambda^0]^T \begin{Bmatrix} e_{3x}^0 \\ e_{3y}^0 \\ e_{3z}^0 \end{Bmatrix}. \quad (A.15)$$

APPENDIX B

Boundary Conditions

The boundaries of the analytical model must simulate the boundary condition of the structure. This requires a means of prescribing necessary constraints at the boundary nodes. A large number of boundary conditions can be handled by specifying displacements and rotations (usually by setting them to zero) at the nodes with respect to a preselected system of coordinates.

As a matter of convenience, the system of nodal body coordinates is used for this purpose. The nodal coordinates of the boundary nodes are prescribed initially: The direction cosines of two nodal coordinates ($\bar{x}$ and $\bar{y}$) with respect to the global coordinates are specified as input, while the direction cosines of the $\bar{z}$ axis are generated internally by means of vector-product multiplication. These direction cosines form the elements of matrix $[\lambda]$ for the respective boundary node and thus define its nodal coordinates.

This set of coordinates of the boundary node supersedes the internally generated body coordinates assigned to all the nodes (see Appendix A).

The boundary conditions of the nodes are imposed during every time cycle of the numerical computation. The translational displacements, calculated in the global coordinates, are first transformed into nodal coordinates by

$$\{\bar{u}\} = [\lambda]^T \{u\}, \quad (B.1)$$

where the elements of $[\lambda]^T$ are the prescribed direction cosines. The preselected ($\bar{x}, \bar{y}, \bar{z}$) displacement components are then adjusted, usually setting some of them to zero. Finally, the adjusted displacements are converted back into the global coordinate system by

$$\{u\} = [\lambda] \{\bar{u}\}. \quad (B.2)$$

Since the rotations of the nodes are treated in the nodal coordinate system, they are not subject to transformations by Eqs. B.1 and B.2. However, the rotations about the ($\bar{x}, \bar{y}, \bar{z}$) coordinates must still be adjusted during every cycle of computation.

APPENDIX C

External Pressure

A provision is incorporated within the program to automatically generate nodal forces equivalent to external pressures. The time-dependent pressure, to be supplied by the user, is multiplied by the area of the triangle to give the normal force acting on the element. This force is then subdivided equally among the three nodes. Contributions to each node are thus

$$\hat{f}_z^{\text{ext}} = pA/3. \quad (\text{C.1})$$

The vector product of the two element sides provides a direction normal to the plane of the element; the magnitude is twice the area of the triangular element. With reference to Fig. 19, Eq. C.1 becomes

$$\vec{f}_z^{\text{ext}} = \vec{i} f_x^{\text{ext}} + \vec{j} f_y^{\text{ext}} + \vec{k} f_z^{\text{ext}} = \frac{p}{6} \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ x_{JI} + u_{JIx} & y_{JI} + u_{JIy} & z_{JI} + u_{JIZ} \\ x_{IK} + u_{IKx} & y_{IK} + u_{IKy} & z_{IK} + u_{IKz} \end{vmatrix}, \quad (\text{C.2})$$

where I, J, and K are the nodes of the element. Equation C.2 decomposes the total force into the x, y, z components defined by the respective unit vectors $\vec{i}$, $\vec{j}$, $\vec{k}$. Contribution from all elements subjected to the pressure load are added to yield the total external force on each node. Note that the sense of the force in Eq. C.2 is in the negative direction, so that the external pressures are positive when applied from the negative $\hat{z}$ side of the elements.

APPENDIX D

Unit Vectors of Element Coordinates

The equation of the plane passing through the three nodal points of the element can be written as

$$Ax + By + Cz + D = 0, \quad (D.1)$$

where A, B, and C are the direction numbers of the normal to the plane and are defined as

$$\left. \begin{aligned} A &= \begin{vmatrix} y_1 + u_{1y} & z_1 + u_{1z} & 1 \\ y_2 + u_{2y} & z_2 + u_{2z} & 1 \\ y_3 + u_{3y} & z_3 + u_{3z} & 1 \end{vmatrix}, \\ B &= - \begin{vmatrix} x_1 + u_{1x} & z_1 + u_{1z} & 1 \\ x_2 + u_{2x} & z_2 + u_{2z} & 1 \\ x_3 + u_{3x} & z_3 + u_{3z} & 1 \end{vmatrix}, \\ C &= \begin{vmatrix} x_1 + u_{1x} & y_1 + u_{1y} & 1 \\ x_2 + u_{2x} & y_2 + u_{2y} & 1 \\ x_3 + u_{3x} & y_3 + u_{3y} & 1 \end{vmatrix}, \end{aligned} \right\} \quad (D.2)$$

and D is associated with the normal distance from the origin to the plane. The direction cosines of a unit normal vector $\vec{e}_3$, shown in Fig. 19, thus become

$$\left. \begin{aligned} e_{3x} &= A/(A^2 + B^2 + C^2)^{1/2}, \\ e_{3y} &= B/(A^2 + B^2 + C^2)^{1/2}, \\ e_{3z} &= C/(A^2 + B^2 + C^2)^{1/2}. \end{aligned} \right\} \quad (D.3)$$

We assume next a unit vector $\vec{g}$ in the plane of the plate with its base at point 1 and at an angle β from $\vec{e}_{21}$. If the direction cosines of this unit vector are defined as g_x , g_y , and g_z in the respective x, y, and z directions, then from normality we have

$$g_x^2 + g_y^2 + g_z^2 = 1. \quad (D.4)$$

Due to the orthogonality of vectors $\vec{e}_3$ and $\vec{g}$, we also have

$$e_{3x}g_x + e_{3y}g_y + e_{3z}g_z = 0. \quad (D.5)$$

In addition, the dot product of vectors $\vec{l}_{21}$ and $\vec{g}$ yields

$$g_x d_x + g_y d_y + g_z d_z = \cos \beta, \quad (\text{D.6})$$

where d_x , d_y , and d_z are the direction cosines of vector $\vec{l}_{21}$, which are given by

$$\left. \begin{aligned} d_x &= [(x_2 + u_{2x}) - (x_1 + u_{1x})]/l_{21}, \\ d_y &= [(y_2 + u_{2y}) - (y_1 + u_{1y})]/l_{21}, \\ d_z &= [(z_2 + u_{2z}) - (z_1 + u_{1z})]/l_{21}. \end{aligned} \right\} \quad (\text{D.7})$$

By solving Eqs. D.4-D.6 for the direction cosines g_x , g_y , and g_z , we have

$$\left. \begin{aligned} g_x &= d_x \cos \beta + (e_{3y} d_z - e_{3z} d_y) \sin \beta, \\ g_y &= d_y \cos \beta + (e_{3z} d_x - e_{3x} d_z) \sin \beta, \\ g_z &= d_z \cos \beta + (e_{3x} d_y - e_{3y} d_x) \sin \beta. \end{aligned} \right\} \quad (\text{D.8})$$

Equation D.8 thus yields the direction cosines of any vector in the plane of the triangle with respect to the global coordinates (x, y, z) . The direction cosines of the $\hat{x}$ axis, i.e., e_{1x} , e_{1y} , and e_{1z} , are determined by setting $\beta = \alpha/2$ in Eq. D.8 (see Fig. 19). Similarly, the direction cosines with respect to the $\hat{y}$ axis, namely, e_{2x} , e_{2y} , and e_{2z} , are given by Eq. D.8 when $\beta = \frac{1}{2}(\alpha + \pi)$. The direction cosines of the $\hat{z}$ axis are expressed by Eq. D.3. This completes the elements of matrix $[\mu]$ defined in Eq. 2 of the text.

APPENDIX E

Unit Vectors of Nodal Body Coordinates

The orientation of nodal masses can be described by the components of unit vectors $\vec{b}_i$, $i = 1, 2, 3$, in the global system, as defined in Eq. 1. The orientation of only two unit vectors need be specified, since the third is automatically defined by the vector product.

New components of $\vec{b}_3$ are computed at each time step in terms of angular velocities $\vec{\omega}$ and angular accelerations $\vec{\alpha}$, both of which are expressed in body coordinates $(\bar{x}, \bar{y}, \bar{z})$.

The Newmark- β method (with $\beta = 0$) of time integration can be expressed as

$$\vec{b}^{i+1} = \vec{b}^i + \Delta t(d\vec{b}^i/dt) + \frac{1}{2}\Delta t^2(d^2\vec{b}^i/dt^2), \quad (\text{E.1})$$

where the superscripts denote the time step. By vector analysis we know

$$\left. \begin{aligned} (d\vec{b}/dt) &= \vec{\omega} \times \vec{b}; \\ (d^2\vec{b}/dt^2) &= \vec{\omega} \times (\vec{\omega} \times \vec{b}) + (\vec{\alpha} \times \vec{b}). \end{aligned} \right\} \quad (\text{E.2})$$

Consider the nodal coordinates $(\bar{x}, \bar{y}, \bar{z})$ to be temporarily fixed. By definition, the $\bar{x}$ and $\bar{y}$ components of $\vec{b}_3^i$ are zero, while $\vec{b}_3^i$ is unity, as shown in Fig. 21a. The $\vec{b}_{3\bar{x}}^{i+1}$ component is then given by the dot product of $\vec{b}_1^i$ and $\vec{b}_3^{i+1}$. Thus, from Eq. E.1 we have

$$\begin{aligned} \vec{b}_{3\bar{x}}^{i+1} &= \vec{b}_1^i \cdot \vec{b}_3^{i+1} = \vec{b}_1^i \cdot \vec{b}_3^i + \Delta t \vec{b}_1^i \cdot (\vec{\omega} \times \vec{b}_3^i) \\ &\quad + \frac{1}{2}\Delta t^2 \left\{ \left[\vec{\omega} \times (\vec{\omega} \times \vec{b}_3^i) \right] \cdot \vec{b}_1^i + (\vec{\alpha} \times \vec{b}_3^i) \cdot \vec{b}_1^i \right\}. \end{aligned} \quad (\text{E.3})$$

The first right-hand term of Eq. E.3 vanishes by definition. If we drop superscripts and bars for convenience, Eq. E.3 becomes

$$\vec{b}_{3\bar{x}} = \Delta t \vec{b}_1 \cdot (\vec{\omega} \times \vec{b}_3) + \frac{1}{2}\Delta t^2 \left\{ \vec{b}_1 \cdot [\vec{\omega} \times (\vec{\omega} \times \vec{b}_3)] + \vec{b}_1 \cdot (\vec{\alpha} \times \vec{b}_3) \right\}. \quad (\text{E.4})$$

Note that $\vec{b}_1 \equiv \delta_{1i}$ and $\vec{b}_3 \equiv \delta_{3i}$ where δ_{1i} and δ_{3i} are Kröner deltas. Thus, expanding Eq. E.4 in index notation, we find

$$\vec{b}_{3\bar{x}} = \Delta t \epsilon_{ijk} \delta_{1i} \omega_j \delta_{3k} + \frac{1}{2}\Delta t^2 (\epsilon_{lmn} \epsilon_{ijm} \omega_i \omega_l \delta_{3j} \delta_{ln} + \epsilon_{ijk} \delta_{1i} \alpha_j \delta_{3k}), \quad (\text{E.5})$$

where ϵ_{ijk} is the permutation tensor. Equation E.5 reduces to

$$b_{3\bar{x}} = \Delta t \omega_2 + \frac{1}{2}(\Delta t)^2(\omega_1 \omega_3 + \alpha_2). \quad (\text{E.6})$$

Similarly, from Figs. 21a and 21b, the two other components are obtained:

$$\left. \begin{aligned} b_{3\bar{y}} &= -\Delta t \omega_1 + \frac{1}{2}\Delta t^2(\omega_2 \omega_3 - \alpha_1); \\ b_{1\bar{y}} &= \Delta t \omega_3 + \frac{1}{2}\Delta t^2(\omega_1 \omega_2 + \alpha_3). \end{aligned} \right\} \quad (\text{E.7})$$

Fig. 21
Orientation of Unit Vectors $\vec{b}_3^{i+1}$ and $\vec{b}_1^{i+1}$

Normality and orthogonality of the unit vectors are used to find the remaining components of $\vec{b}_3$ and $\vec{b}_1$. By the normality of $\vec{b}_3$, it follows that

$$b_{3\bar{z}} = (1 - b_{3\bar{x}}^2 - b_{3\bar{y}}^2)^{1/2}. \quad (\text{E.8})$$

From the orthogonality of $\vec{b}_1^{i+1}$ and $\vec{b}_3^{i+1}$, we have

$$b_{1\bar{x}}b_{3\bar{x}} + b_{1\bar{y}}b_{3\bar{y}} + b_{1\bar{z}}b_{3\bar{z}} = 0. \quad (\text{E.9})$$

Since all the components of $\vec{b}_3^{i+1}$ are known and $b_{1\bar{x}} \approx 1$, Eq. E.9 yields

$$b_{1\bar{z}} = -(b_{3\bar{x}} + b_{1\bar{y}}b_{3\bar{y}})/b_{3\bar{z}}. \quad (\text{E.10})$$

The integration formulas tacitly assume that rotations during the time interval are small so that $b_{3\bar{z}}$ is always close to unity, and Eq. E.10 should be well conditioned. Finally, the condition of normality is used to find $b_{1\bar{x}}$. Thus,

$$b_{1\bar{x}} = (1 - b_{1\bar{y}}^2 - b_{1\bar{z}}^2)^{1/2}. \quad (\text{E.11})$$

The components of $\vec{b}_1^{i+1}$ and $\vec{b}_3^{i+1}$ derived above are expressed in terms of components in the nodal coordinate system. To transform them to the global coordinate system so that they correspond to Eq. 1 of the text, we have

$$\left. \begin{aligned} \begin{Bmatrix} b_{1x} \\ b_{1y} \\ b_{1z} \end{Bmatrix} &= [\lambda] \begin{Bmatrix} b_{1\bar{x}} \\ b_{1\bar{y}} \\ b_{1\bar{z}} \end{Bmatrix}; \\ \begin{Bmatrix} b_{3x} \\ b_{3y} \\ b_{3z} \end{Bmatrix} &= [\lambda] \begin{Bmatrix} b_{3\bar{x}} \\ b_{3\bar{y}} \\ b_{3\bar{z}} \end{Bmatrix}. \end{aligned} \right\} \quad (E.12)$$

The vector product of b_1^{i+1} and b_3^{i+1} yields the remaining components of the new matrix $[\lambda]$ as follows:

$$\vec{b}_2^{i+1} = \vec{b}_3^{i+1} \times \vec{b}_1^{i+1} = \begin{vmatrix} \vec{b}_1 & \vec{b}_2 & \vec{b}_3 \\ b_{3x} & b_{3y} & b_{3z} \\ b_{1x} & b_{1y} & b_{1z} \end{vmatrix}. \quad (E.13)$$

APPENDIX F
Formulation of Matrix $[E^m]$

The relationship between the strains at the nodes of the element and the Cartesian strains were derived by Argyris et al.¹¹ Thus,

$$\begin{Bmatrix} \epsilon_{12} \\ \epsilon_{23} \\ \epsilon_{31} \end{Bmatrix} = [H] \begin{Bmatrix} \hat{\epsilon}_x \\ \hat{\epsilon}_y \\ 2\hat{\epsilon}_{xy} \end{Bmatrix}, \quad (F.1)$$

where

$$[H] = \begin{bmatrix} (\hat{x}_{21}/l_{21})^2 & (\hat{y}_{21}/l_{21})^2 & (\hat{x}_{21}/l_{21})(\hat{y}_{21}/l_{21}) \\ (\hat{x}_{32}/l_{32})^2 & (\hat{y}_{32}/l_{32})^2 & (\hat{x}_{32}/l_{32})(\hat{y}_{32}/l_{32}) \\ (\hat{x}_{13}/l_{13})^2 & (\hat{y}_{13}/l_{13})^2 & (\hat{x}_{13}/l_{13})(\hat{y}_{13}/l_{13}) \end{bmatrix}. \quad (F.2)$$

For the particular coordinate system chosen, $[H]$ can be expressed in terms of only two direction cosines and the lengths of the triangle sides. We observe from Fig. 19 that, because $\hat{x}$ is defined to bisect the angle at node 1,

$$\left. \begin{aligned} \hat{x}_{13}/l_{13} &= -\hat{x}_{21}/l_{21}; \\ \hat{y}_{13}/l_{13} &= \hat{y}_{21}/l_{21}. \end{aligned} \right\} \quad (F.3)$$

Furthermore, with the help of the identities

$$\left. \begin{aligned} \hat{x}_{21} + \hat{x}_{32} + \hat{x}_{13} &= 0 \\ \hat{y}_{21} + \hat{y}_{32} + \hat{y}_{13} &= 0 \end{aligned} \right\}, \quad (F.4)$$

Eqs. F.3 become

$$\left. \begin{aligned} \hat{x}_{32}/l_{32} &= [(\hat{l}_{13} - \hat{l}_{21})/\hat{l}_{32}](\hat{x}_{21}/l_{21}) \\ \hat{y}_{32}/l_{32} &= -[(\hat{l}_{13} + \hat{l}_{21})/\hat{l}_{32}](\hat{y}_{21}/l_{21}) \end{aligned} \right\}. \quad (F.5)$$

Substituting Eqs. F.3 and F.5 into Eq. F.2, we get

$$[H] = \begin{bmatrix} (\hat{x}_{21}/\ell_{21})^2 & (\hat{y}_{21}/\ell_{21})^2 & (\hat{x}_{21}/\ell_{21})(\hat{y}_{21}/\ell_{21}) \\ [(\ell_{13} - \ell_{12})/\ell_{32}]^2 (\hat{x}_{21}/\ell_{21})^2 & [(\ell_{13} + \ell_{21})/\ell_{32}]^2 (\hat{y}_{21}/\ell_{21})^2 & -[(\ell_{13}^2 - \ell_{12}^2)/\ell_{32}^2] (\hat{x}_{21}/\ell_{21})(\hat{y}_{21}/\ell_{21}) \\ (\hat{x}_{21}/\ell_{21})^2 & (\hat{y}_{21}/\ell_{21})^2 & -(\hat{x}_{21}/\ell_{21})(\hat{y}_{21}/\ell_{21}) \end{bmatrix}. \quad (F.6)$$

From Eq. F.1, it follows that

$$\begin{Bmatrix} \hat{e}_x \\ \hat{e}_y \\ 2\hat{e}_{xy} \end{Bmatrix} = [H]^{-1} \begin{Bmatrix} e_{12} \\ e_{23} \\ e_{31} \end{Bmatrix}, \quad (F.7)$$

and the inversion of $[H]$ yields

$$[H]^{-1} = \begin{bmatrix} [(\ell_{13} + \ell_{21})/(4\ell_{13})](\ell_{21}/\hat{x}_{21})^2 & -[\ell_{32}^2/(4\ell_{21}\ell_{13})](\ell_{21}/\hat{x}_{21})^2 & [(\ell_{13} + \ell_{21})/(4\ell_{21})](\ell_{21}/\hat{x}_{21})^2 \\ [(\ell_{13} - \ell_{21})/(4\ell_{13})](\ell_{21}/\hat{y}_{21})^2 & [\ell_{32}^2/(4\ell_{21}\ell_{13})](\ell_{21}/\hat{y}_{21})^2 & -[(\ell_{13} - \ell_{21})/(4\ell_{21})](\ell_{21}/\hat{y}_{21})^2 \\ (\ell_{21}/\hat{x}_{21})(\ell_{21}/\hat{y}_{21})/2 & 0 & -(\ell_{21}/\hat{x}_{21})(\ell_{21}/\hat{y}_{21})/2 \end{bmatrix}. \quad (F.8)$$

We note also that the elongations $\{\delta\}$ of the sides are related to the strains at the sides as follows:

$$\begin{Bmatrix} \delta_{12} \\ \delta_{23} \\ \delta_{31} \end{Bmatrix} = [\mathcal{L}^0] \begin{Bmatrix} e_{12} \\ e_{23} \\ e_{31} \end{Bmatrix}, \quad (F.9)$$

where

$$[\mathcal{L}^0] = \begin{bmatrix} \ell_{21}^0 & 0 & 0 \\ 0 & \ell_{32}^0 & 0 \\ 0 & 0 & \ell_{13}^0 \end{bmatrix}. \quad (F.10)$$

Then from Eqs. F.7 and F.9, we find that

$$\begin{Bmatrix} \hat{e}_x \\ \hat{e}_y \\ 2\hat{e}_{xy} \end{Bmatrix} = [H]^{-1} [\mathcal{L}^0]^{-1} \begin{Bmatrix} \delta_{12} \\ \delta_{23} \\ \delta_{31} \end{Bmatrix}, \quad (F.11)$$

where the inversion of $[\ell^0]$ yields

$$[\ell^0]^{-1} = \begin{bmatrix} (1/\ell_{21}^0) & 0 & 0 \\ 0 & (1/\ell_{32}^0) & 0 \\ 0 & 0 & (1/\ell_{13}^0) \end{bmatrix}. \quad (\text{F.12})$$

It is apparent from Eqs. F.11 and 15 that

$$[\mathbf{E}^{\mathbf{m}}] = [\mathbf{H}]^{-1}[\ell^0]^{-1}. \quad (\text{F.13})$$

Matrix multiplication of Eqs. F.8 and F.12 yields the final expression for $[\mathbf{E}^{\mathbf{m}}]$:

$$[\mathbf{E}^{\mathbf{m}}] = \begin{bmatrix} [(\ell_{13} + \ell_{21})/(4\ell_{13}\ell_{21}^0)](\ell_{21}/\hat{x}_{21})^2 & -[\ell_{32}^2/(4\ell_{21}\ell_{13}\ell_{32}^0)](\ell_{21}/\hat{x}_{21})^2 & [(\ell_{13} + \ell_{21})/(4\ell_{21}\ell_{13}^0)](\ell_{21}/\hat{x}_{21})^2 \\ [(\ell_{13} - \ell_{21})/(4\ell_{13}\ell_{21}^0)](\ell_{21}/\hat{y}_{21})^2 & [\ell_{32}^2/(4\ell_{21}\ell_{13}\ell_{32}^0)](\ell_{21}/\hat{y}_{21})^2 & -[(\ell_{13} - \ell_{21})/(4\ell_{21}\ell_{13}^0)](\ell_{21}/\hat{y}_{21})^2 \\ (\ell_{21}/\hat{x}_{21})(\ell_{21}/\hat{y}_{21})/(2\ell_{21}^0) & 0 & -(\ell_{21}/\hat{x}_{21})(\ell_{21}/\hat{y}_{21})/(2\ell_{13}^0) \end{bmatrix}. \quad (\text{F.14})$$

APPENDIX G

Shape Functions of Transverse Deflection

In terms of triangular coordinates, Zienkiewicz¹⁴ gives the transverse deflection of a triangular element as follows:

$$\begin{aligned}\hat{u}_Z^{\text{def}} = & \beta_1 \zeta_1 + \beta_2 \zeta_2 + \beta_3 \zeta_3 + \beta_4 (\zeta_2^2 \zeta_1 + 0.5 \zeta_1 \zeta_2 \zeta_3) + \beta_5 (\zeta_2^2 \zeta_3 + 0.5 \zeta_1 \zeta_2 \zeta_3) \\ & + \beta_6 (\zeta_3^2 \zeta_1 + 0.5 \zeta_1 \zeta_2 \zeta_3) + \beta_7 (\zeta_3^2 \zeta_2 + 0.5 \zeta_1 \zeta_2 \zeta_3) + \beta_8 (\zeta_1^2 \zeta_3 + 0.5 \zeta_1 \zeta_2 \zeta_3) \\ & + \beta_9 (\zeta_1^2 \zeta_2 + 0.5 \zeta_1 \zeta_2 \zeta_3),\end{aligned}\quad (\text{G.1})$$

where ζ_1 , ζ_2 and ζ_3 are the triangular coordinates, and $\beta_1, \beta_2, \dots, \beta_9$ are constants that can be solved in terms of the boundary conditions. We define the nodal rotations as

$$\left. \begin{aligned}\hat{\theta}_x &= \partial \hat{u}_Z^{\text{def}} / \partial \hat{y}; \\ \hat{\theta}_y &= -\partial \hat{u}_Z^{\text{def}} / \partial \hat{x}.\end{aligned}\right\} \quad (\text{G.2})$$

Since the deformation part of the transverse displacements are zero at the nodes, the unknowns of Eq. G.1 are determined in terms of nodal rotations $\hat{\theta}_{xI}$ and $\hat{\theta}_{yI}$. With the nodal rotations defined by Eq. G.2, the transverse deflection can be written as

$$\hat{u}_Z^{\text{def}} = [\varphi_I^f] \{\hat{\theta}\}, \quad (\text{G.3})$$

where $I = 1$ to 6, $\hat{\theta}$ is defined in Eq. 6,

$$\left. \begin{aligned}\varphi_1^f &= b_2 (\zeta_1^2 \zeta_3 + 0.5 \zeta_1 \zeta_2 \zeta_3) - b_3 (\zeta_1^2 \zeta_2 + 0.5 \zeta_1 \zeta_2 \zeta_3), \\ \varphi_2^f &= a_2 (\zeta_1^2 \zeta_3 + 0.5 \zeta_1 \zeta_2 \zeta_3) - a_3 (\zeta_1^2 \zeta_2 + 0.5 \zeta_1 \zeta_2 \zeta_3), \\ \varphi_3^f &= b_3 (\zeta_2^2 \zeta_1 + 0.5 \zeta_1 \zeta_2 \zeta_3) - b_1 (\zeta_2^2 \zeta_3 + 0.5 \zeta_1 \zeta_2 \zeta_3), \\ \varphi_4^f &= a_3 (\zeta_2^2 \zeta_1 + 0.5 \zeta_1 \zeta_2 \zeta_3) - a_1 (\zeta_2^2 \zeta_3 + 0.5 \zeta_1 \zeta_2 \zeta_3), \\ \varphi_5^f &= b_1 (\zeta_3^2 \zeta_2 + 0.5 \zeta_1 \zeta_2 \zeta_3) - b_2 (\zeta_3^2 \zeta_1 + 0.5 \zeta_1 \zeta_2 \zeta_3), \\ \varphi_6^f &= a_1 (\zeta_3^2 \zeta_2 + 0.5 \zeta_1 \zeta_2 \zeta_3) - a_2 (\zeta_3^2 \zeta_1 + 0.5 \zeta_1 \zeta_2 \zeta_3),\end{aligned}\right\} \quad (\text{G.4})$$

and a and b are defined in Fig. 20. The above six terms are the flexural shape functions in Eq. 5.

In the formulation of $[E^f]$ (see Eq. 17), we need the second partials of the shape functions given in Eq. G.4. Differentiation formulas in triangular coordinates given by Meek²³ are

$$\text{and } \left. \begin{aligned} \partial \zeta_i / \partial \hat{x} &= b_i / (2A) \\ \partial \zeta_i / \partial \hat{y} &= a_i / (2A) \end{aligned} \right\}. \quad (\text{G.5})$$

By using these formulas, we find elements of $[E^f]$. Since three integration points (points 4-6 shown in Fig. 20) are used, we specialize these expressions and obtain

1. For point 4 ($\zeta_1 = 0$, $\zeta_2 = 0.5$, $\zeta_3 = 0.5$):

$$\left. \begin{aligned} \partial^2 \varphi_1^f / \partial \hat{x}^2 &= 0.5b_1(b_3^2 - b_2^2) / (2A)^2 \\ \partial^2 \varphi_2^f / \partial \hat{x}^2 &= 0.5b_1^2(a_2 - a_3) / (2A)^2 \\ \partial^2 \varphi_3^f / \partial \hat{x}^2 &= b_1[0.5b_2(3b_3 - b_2) + b_3^2] / (2A)^2 \\ \partial^2 \varphi_4^f / \partial \hat{x}^2 &= [0.5b_1^2(a_1 - a_3) - b_2(a_1b_2 + 4A)] / (2A)^2 \\ \partial^2 \varphi_5^f / \partial \hat{x}^2 &= b_1[0.5b_3(b_3 - 3b_2) - b_2^2] / (2A)^2 \\ \partial^2 \varphi_6^f / \partial \hat{x}^2 &= [0.5b_1^2(a_2 - a_1) + b_3(a_1b_3 - 4A)] / (2A)^2 \\ \partial^2 \varphi_1^f / \partial \hat{y}^2 &= 0.5a_1^2(b_2 - b_3) / (2A)^2 \\ \partial^2 \varphi_2^f / \partial \hat{y}^2 &= 0.5a_1(a_3^2 - a_2^2) / (2A)^2 \\ \partial^2 \varphi_3^f / \partial \hat{y}^2 &= [0.5a_1^2(b_1 - b_3) - a_2(a_2b_1 - 4A)] / (2A)^2 \\ \partial^2 \varphi_4^f / \partial \hat{y}^2 &= a_1[0.5a_2(3a_3 - a_2) + a_3^2] / (2A)^2 \\ \partial^2 \varphi_5^f / \partial \hat{y}^2 &= [0.5a_1^2(b_2 - b_1) + a_3(a_3b_1 + 4A)] / (2A)^2 \\ \partial^2 \varphi_6^f / \partial \hat{y}^2 &= a_1[0.5a_3(a_3 - 3a_2) - a_2^2] / (2A)^2 \\ \partial^2 \varphi_1^f / \partial \hat{x} \partial \hat{y} &= 0.5a_1b_1(b_2 - b_3) / (2A)^2 \\ \partial^2 \varphi_2^f / \partial \hat{x} \partial \hat{y} &= 0.5a_1b_1(a_2 - a_3) / (2A)^2 \\ \partial^2 \varphi_3^f / \partial \hat{x} \partial \hat{y} &= a_1(b_1^2 - b_2^2 + 0.5b_1b_2) / (2A)^2 \\ \partial^2 \varphi_4^f / \partial \hat{x} \partial \hat{y} &= b_1(a_1^2 - a_2^2 + 0.5a_1a_2) / (2A)^2 \\ \partial^2 \varphi_5^f / \partial \hat{x} \partial \hat{y} &= a_1(b_3^2 - b_1^2 - 0.5b_1b_3) / (2A)^2 \end{aligned} \right\}. \quad (\text{G.6})$$

2. For point 5 ($\zeta_1 = 0.5$, $\zeta_2 = 0$, $\zeta_3 = 0.5$):

$$\left. \begin{aligned}
 \partial^2 \varphi_1^f / \partial \hat{x}^2 &= b_2 [0.5b_1(b_1 - 3b_3) - b_3^2] / (2A)^2 \\
 \partial^2 \varphi_2^f / \partial \hat{x}^2 &= [0.5b_2^2(a_3 - a_2) + b_1(a_2b_1 - 4A)] / (2A)^2 \\
 \partial^2 \varphi_3^f / \partial \hat{x}^2 &= 0.5b_2(b_1^2 - b_3^2) / (2A)^2 \\
 \partial^2 \varphi_4^f / \partial \hat{x}^2 &= 0.5b_2^2(a_3 - a_1) / (2A)^2 \\
 \partial^2 \varphi_5^f / \partial \hat{x}^2 &= b_2 [0.5b_3(3b_1 - b_3) + b_1^2] / (2A)^2 \\
 \partial^2 \varphi_6^f / \partial \hat{x}^2 &= [0.5b_2^2(a_2 - a_1) - b_3(a_2b_3 + 4A)] / (2A)^2 \\
 \partial^2 \varphi_1^f / \partial \hat{y}^2 &= [0.5a_2^2(b_3 - b_2) + a_1(a_1b_2 + 4A)] / (2A)^2 \\
 \partial^2 \varphi_2^f / \partial \hat{y}^2 &= a_2 [0.5a_1(a_1 - 3a_3) - a_3^2] / (2A)^2 \\
 \partial^2 \varphi_3^f / \partial \hat{y}^2 &= 0.5a_2^2(b_3 - b_1) / (2A)^2 \\
 \partial^2 \varphi_4^f / \partial \hat{y}^2 &= 0.5a_2(a_1^2 - a_3^2) / (2A)^2 \\
 \partial^2 \varphi_5^f / \partial \hat{y}^2 &= [0.5a_2^2(b_2 - b_1) - a_3(a_3b_2 - 4A)] / (2A)^2 \\
 \partial^2 \varphi_6^f / \partial \hat{y}^2 &= a_2 [0.5a_3(3a_1 - a_3) + a_1^2] / (2A)^2 \\
 \partial^2 \varphi_1^f / \partial \hat{x} \partial \hat{y} &= a_2(b_1^2 - b_2^2 - 0.5b_1b_2) / (2A)^2 \\
 \partial^2 \varphi_2^f / \partial \hat{x} \partial \hat{y} &= b_2(a_1^2 - a_2^2 - 0.5a_1a_2) / (2A)^2 \\
 \partial^2 \varphi_3^f / \partial \hat{x} \partial \hat{y} &= 0.5a_2b_2(b_3 - b_1) / (2A)^2 \\
 \partial^2 \varphi_4^f / \partial \hat{x} \partial \hat{y} &= 0.5a_2b_2(a_3 - a_1) / (2A)^2 \\
 \partial^2 \varphi_5^f / \partial \hat{x} \partial \hat{y} &= a_2(b_2^2 - b_3^2 + 0.5b_2b_3) / (2A)^2 \\
 \partial^2 \varphi_6^f / \partial \hat{x} \partial \hat{y} &= b_2(a_2^2 - a_3^2 + 0.5a_2a_3) / (2A)^2
 \end{aligned} \right\} \quad (G.7)$$

3. For point 6 ($\zeta_1 = 0.5$, $\zeta_2 = 0.5$, $\zeta_3 = 0$):

$$\left. \begin{aligned}
 \partial^2 \varphi_1^f / \partial \hat{x}^2 &= b_3 [0.5b_1(3b_2 - b_1) + b_2^2] / (2A)^2 \\
 \partial^2 \varphi_2^f / \partial \hat{x}^2 &= [0.5b_3^2(a_3 - a_2) - b_1(a_3b_1 + 4A)] / (2A)^2 \\
 \partial^2 \varphi_3^f / \partial \hat{x}^2 &= b_3 [0.5b_2(b_2 - 3b_1) - b_1^2] / (2A)^2 \\
 \partial^2 \varphi_4^f / \partial \hat{x}^2 &= [0.5b_3^2(a_1 - a_3) + b_2(a_3b_2 - 4A)] / (2A)^2 \\
 \partial^2 \varphi_5^f / \partial \hat{x}^2 &= 0.5b_3(b_2^2 - b_1^2) / (2A)^2 \\
 \partial^2 \varphi_6^f / \partial \hat{x}^2 &= 0.5b_3^2(a_1 - a_2) / (2A)^2 \\
 \partial^2 \varphi_1^f / \partial \hat{y}^2 &= [0.5a_3^2(b_3 - b_2) - a_1(a_1b_3 - 4A)] / (2A)^2 \\
 \partial^2 \varphi_2^f / \partial \hat{y}^2 &= a_3 [0.5a_1(3a_2 - a_1) + a_2^2] / (2A)^2 \\
 \partial^2 \varphi_3^f / \partial \hat{y}^2 &= [0.5a_3^2(b_1 - b_3) + a_2(a_2b_3 + 4A)] / (2A)^2 \\
 \partial^2 \varphi_4^f / \partial \hat{y}^2 &= a_3 [0.5a_2(a_2 - 3a_1) - a_1^2] / (2A)^2 \\
 \partial^2 \varphi_5^f / \partial \hat{y}^2 &= 0.5a_3^2(b_1 - b_2) / (2A)^2 \\
 \partial^2 \varphi_6^f / \partial \hat{y}^2 &= 0.5a_3(a_2^2 - a_1^2) / (2A)^2 \\
 \partial^2 \varphi_1^f / \partial \hat{x} \partial \hat{y} &= a_3(b_3^2 - b_1^2 + 0.5b_1b_3) / (2A)^2 \\
 \partial^2 \varphi_2^f / \partial \hat{x} \partial \hat{y} &= b_3(a_3^2 - a_1^2 + 0.5a_1a_3) / (2A)^2 \\
 \partial^2 \varphi_3^f / \partial \hat{x} \partial \hat{y} &= a_3(b_2^2 - b_3^2 - 0.5b_2b_3) / (2A)^2 \\
 \partial^2 \varphi_4^f / \partial \hat{x} \partial \hat{y} &= b_3(a_2^2 - a_3^2 - 0.5a_2a_3) / (2A)^2 \\
 \partial^2 \varphi_5^f / \partial \hat{x} \partial \hat{y} &= 0.5a_3b_3(b_1 - b_2) / (2A)^2 \\
 \partial^2 \varphi_6^f / \partial \hat{x} \partial \hat{y} &= 0.5a_3b_3(a_1 - a_2) / (2A)^2
 \end{aligned} \right\} \quad (G.8)$$

APPENDIX H

Integration across Thickness of the Element

The integrations of Eqs. 18 and 20 are performed numerically. To integrate, across the thickness of the element, the element is divided into n layers and the stresses are assumed to vary linearly with the thickness coordinate $\hat{z}$ within each layer and to be continuous from layer to layer. The contribution of the membrane force f_i and moment m_i of layer i with respect to the central plane, as shown in Fig. 22, are then

$$\left. \begin{aligned} f_i &= \frac{1}{2} q_i (\sigma_i + \sigma_{i+1}) \\ \text{and} \\ m_i &= \frac{1}{2} q_i \{ \sigma_i [\xi_i + (q_i/3)] + \sigma_{i+1} [\xi_i + (2q_i/3)] \} \end{aligned} \right\}, \quad (\text{H.1})$$

where q_i is the thickness of layer i , ξ_i is the $\hat{z}$ coordinate measured from the central plane of the element, and σ_i and σ_{i+1} are the stresses at the top and bottom of layer i . If all layers are assumed to be of equal thickness, then ξ_i and q_i are given by

$$\left. \begin{aligned} q_i &= h/n \\ \text{and} \\ \xi_i &= h[2(i-1) - n]/(2n), \quad i = 1, \dots, n+1 \end{aligned} \right\}, \quad (\text{H.2})$$

where h is the element thickness. By substituting Eqs. H.2 into H.1, we get

$$\left. \begin{aligned} f_i &= h(\sigma_i + \sigma_{i+1})/(2n) \\ \text{and} \\ m_i &= h^2 [\sigma_i (6i - 3n - 4) + \sigma_{i+1} (6i - 3n - 2)]/(12n^2) \end{aligned} \right\}. \quad (\text{H.3})$$

Fig. 22
Definition of Symbols Used
in Trapezoidal Integration

The contribution of all layers is summed to yield the total force f and total moment m across the element thickness:

$$\left. \begin{aligned} f &= \frac{1}{2n} \sum_{i=1}^n (\sigma_i + \sigma_{i+1}) \\ \text{and} \\ m &= \frac{h^2}{12n^2} \sum_{i=1}^n [\sigma_i(6i - 3n - 4) + \sigma_{i+1}(6i - 3n - 2)] \end{aligned} \right\}. \quad (\text{H.4})$$

Note that Eqs. H.4 require $n + 1$ values of stress. For example, if the number of layers is four, we need five values of stress across the thickness of the element.

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