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MODERN CONTROL APPLICATIONS TO NUCLEAR PROPULSION

UNIT PRICES

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ABSTRACT

Fuel-optimal control of a nuclear propulsion system is analyzed by means of the maximum principle, Lagrange multipliers and computers. Practical problems are shown to be most complicated by state constraints and high dimensionality. The optimal policy, a maximum allowable effort type of process, requires a time-optimal reactor control policy. With this in mind, a minimal-time, prompt-neutronic control process with reactivity rate and amplitude constraint is analyzed.

1.0 INTRODUCTION

A summary of optimal controls analysis for nuclear rocket propulsion is presented here. Though past work has been somewhat academic by necessity, this ground-work eventually should lead to the solution of practical optimal (or suboptimal) control problems.

1.1 Engine Operation and Dynamics

The advantages of nuclear rocket propulsion for space missions are well known¹. Perhaps the greatest disadvantages of such systems are their complexity, size, weight, and cost. By means of optimization techniques, these disadvantages can be attenuated. For example, a minimal propellant process would not only minimize propellant weight but also would minimize tankage weight and other accessory equipment weight.

The basic bleed-cycle engine and its classical control is discussed in reference 2. Following Fig. 1, a turbopump forces liquid hydrogen through a solid-core reactor where it is heated to high temperature and exhausted through a nozzle. Prior to entering the reactor core, the liquid propellant is preheated to a gas by cooling of certain structural elements, nozzle and reactor neutron reflector. In the process a small portion of hot gas is bled-off to drive the turbine.

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The engine thrust F (nearly proportional to nozzle chamber stagnation pressure P_c) and specific impulse (nearly proportional to the square root of chamber propellant temperature, $T_c^{1/2}$) are controlled by properly manipulating neutron multiplication (or reactivity) and propellant flow rate. As shown in Fig. 1, the latter is varied by means of the turbopump control valve and the former by means of control drums.

The mathematical model that accurately approximates the dynamics of this complex system involves about fifty nonlinear first-order ordinary differential equations. For this study, the model is greatly simplified so that analytical results are possible. The concepts may then be extended to the more complicated system by machine computations.

It is assumed: (1) that the pumping system dynamics are very fast compared to the heating process, (2) that the core inlet temperature is constant, and (3) that the heat-exchange is a lumped process with no axial heat transfer. Then the nuclear engine dynamics may be approximated by rate of reactor heat generation

$$\frac{dQ}{dt} = \frac{u_1 - \beta}{\ell} Q + \sum_{i=1}^6 \lambda_i C_i, \quad (1)$$

rate of precursor generation

$$\frac{dC_i}{dt} = \frac{\beta_i}{\ell} Q - \lambda_i C_i \quad (i=1, \dots, 6), \quad (2)$$

and rate of average core temperature change

$$\frac{dT}{dt} = \frac{Q}{c_m} - \frac{T}{\tau_h}, \quad (3)$$

where

$$\tau_h = c_m \left[\frac{\theta}{c_p u_2 (\theta + 1)} + \frac{1}{hA} \right] \approx 1/au_2. \quad (4)$$

Here a is a heat transfer constant, u_2 is propellant weight flow rate, c_m is effective core mass heat capacity, c_p is propellant specific heat at constant pressure, u_1 is total reactivity (u_1 = core temperature reactivity + propellant reactivity + control

reactivity), β is the portion of neutrons generated from precursors, λ_i is the decay constant of the i th precursor group, and ℓ is neutron generation time. θ is a weighting factor such that the average propellant temperature in the core is

$$T_g = \frac{T_i + \theta T_c}{1 + \theta},$$

where T_i is the core inlet propellant temperature, and T_c is the core exit propellant temperature (approximately nozzle chamber temperature). θ is often selected to be about 2, but its value depends on the flux distribution. h is a heat-transfer coefficient (with convection dominant), and A is a heat transfer area. This model and a design configuration are discussed in references 2 and 3.

Prior to the analysis of optimal control of nuclear rocket propulsion, optimal control theory is briefly discussed next.

2.0 OPTIMAL CONTROL WITH STATE AND CONTROL CONSTRAINTS

Though the necessary conditions for optimal control trajectories is presented here, the reader is referred to reference 4 for a good discussion of the details. Represent the system dynamics by a set of n first-order ordinary differential equations of the form

$$\dot{\vec{x}} = \vec{f}(\vec{x}, \vec{u}), \quad (5)$$

with

$$\vec{x}(t_0) = \vec{x}_0.$$

Here $\vec{x}$ is a state vector composed of n state variables x_i and $\vec{u}$ is a control vector of m components u_j such that $u_j(t)$ is piecewise continuous and constrained to some closed set U . Further it is assumed that $\partial \vec{f} / \partial \vec{x}$ is a matrix whose components are continuous in $\vec{x}$ and $\vec{u}$. The special problem of interest here is to determine an admissible control $\vec{u}(t) \in U$ that transfers the process from some prescribed $\vec{x}(t_0) = \vec{x}_0$ to some terminal manifold set of points, defined by the intersection of q smooth surfaces, $\vec{g}(\vec{x}) = 0$, so as to minimize a performance index

$$J = \int_{t_0}^{t_1} f_0(\vec{x}, \vec{u}) dt. \quad (6)$$

In other words, if $\dot{\vec{x}}_0 = f_0(\vec{x}, \vec{u})$, the controlled process is to minimize $J = x_0(t_1)$ with $x_0(t_0) = 0$. Here, it is assumed that the terminal time t_1 is free. Then the maximum principle states that if $\vec{x}(t)$ is an optimal solution trajectory, a nonzero, continuous, $(n+1)$ -dimensional costate vector $\vec{p}(t)$ is related to the state by

$$\dot{\vec{p}} = - \frac{\partial R}{\partial \vec{x}}, \quad (7)$$

where the scalar R is an inner product

$$R = (\vec{p}, \vec{f}(\vec{x}, \vec{u})) = \sum_{i=0}^n p_i f_i. \quad (8)$$

Then the control that maximizes R with respect to all admissible controls is a candidate for the optimal. Pontryagin shows for this time-invariant free-end time problem that the maximum R , called the Hamiltonian, is an invariant equal to zero, that is

$$H(\vec{x}, \vec{p}) = \max_{\vec{u} \in U} R(\vec{x}, \vec{p}, \vec{u}) = 0. \quad (9)$$

Further, p_0 is a nonpositive constant. Also, the terminal manifold $\vec{g}(\vec{x}) = 0$ possesses a unique tangent plane $T(\vec{x}_1)$ at every $\vec{x}$, (where $\vec{x}(t_1) = \vec{x}_1$). Then, the terminal transversality condition is that $\vec{p}(t_1)$ must be orthogonal to $T(\vec{x}_1)$. If the terminal manifold is a point, this condition along with the manifold equations is replaced by $\vec{x}(t_1) = \vec{x}_1$.

If the state $\vec{x}$ is constrained, i.e., $\vec{x}(t) \in B$ (a closed region), as well as the control, then certain other conditions are necessary for optimality when trajectories are on a state-constraint boundary of B , ∂G defined by

$$s(\vec{x}) \equiv s(\vec{x}) \equiv 0. \quad (10)$$

First ∂G is a smooth n -dimensional surface such that the gradient, $\nabla s(\bar{x})$, does not vanish at any point on it.

In this study it is assumed that u , (the control required to satisfy (10)) can be computed from

$$\dot{s}(\bar{x}, \bar{u}) = \left[\frac{\partial s(\bar{x})}{\partial \bar{x}} \right]^T \bar{f}(\bar{x}, \bar{u}) = 0. \quad (11)$$

(T denotes a matrix transpose.)

If $\bar{u}_1$ is a boundary point of U , it is also assumed here that in the vicinity of $\bar{u}_1$, U can be defined by

$q(\bar{u}) \leq 0$ with $q(\bar{u}_1) = 0$. Such that $q(\bar{u})$ is smooth and $\partial q(\bar{u}_1)/\partial \bar{u}$ and $\partial \dot{s}(\bar{x}, \bar{u}_1)/\partial \bar{u}$ is nonzero. Assume these so-

called regularity conditions are satisfied by $\bar{u} \in \omega \subset U$.

With these assumptions, $\bar{x}(t)$ is an optimal trajectory for (5) when $\bar{x} \in \partial G$ if a continuous $\bar{p}(t)$ and a piecewise smooth $\rho(t)$ are defined by

$$\dot{\bar{p}} = - \frac{\partial R}{\partial \bar{x}} + \rho \frac{\partial \dot{s}}{\partial \bar{x}} \quad (12)$$

and

$$\frac{\partial R}{\partial \bar{u}} = \rho \frac{\partial \dot{s}}{\partial \bar{u}} + \nu \frac{\partial q}{\partial \bar{u}}. \quad (13)$$

Again, equations (8) and (9) must be satisfied, and p_0 is a nonpositive constant. Also, $\dot{\bar{p}}(t)$ is nonpositive.

If $\bar{x}(t) \in \partial G$ for an interval (t_a, t_b) , certain conditions need to be satisfied when trajectories enter the boundary at t_a and leave the boundary at t_b . These so-called jump conditions are: (1) at t_a

$$p(t_a-) = p(t_a+) + \mu \left. \frac{\partial S}{\partial \bar{x}} \right|_{t_a} \quad (14)$$

or

$$\bar{p}(t_a-) - \mu \frac{\partial S}{\partial \bar{x}} = \bar{0}$$

where μ is some nonzero constant, and (2) at t_b

$$\bar{p}(t_b+) = \bar{p}(t_b-). \quad (15)$$

A simple application of this theory is made next.

2.1 Prompt Neutron Kinetics Example

If the precursor neutrons and the heat-exchange dynamics are neglected, the system is represented by the prompt neutron kinetics,

$$\frac{dn}{dt} = \frac{\delta k}{\ell} n, \quad (16)$$

where n is neutron density, and δk is reactivity. Initially the system is in equilibrium and

$$n(t_0) = n_0.$$

The problem is to obtain the time optimal process that controls (16) from n_0 to n_1 in minimal time.

Reactivity is constrained for safety reasons so that

$$|\delta k| \leq \gamma \quad (17)$$

and

$$|\dot{\delta k}| \leq \alpha. \quad (18)$$

To formulate a problem that fits the above theory, let n and δk be state variables and let δk be a control variable. Then (17) is a state constraint, and (18) is a control constraint. f_0 is one for this case, and $S = |\delta k| - \gamma$.

Initially, $\delta k = 0$ and the system is not on a constraint boundary. Similarly at the terminal end $\delta k(t_1) = 0$. Application of the above theory shows that, if n_1 is appreciably larger than n_0 , then the optimal variations on reactivity ($\delta k = \alpha \text{ sign } p_2$) is a ramp increase until $\delta k = \gamma$ at $t = t_a$ then reactivity remains constant until $t = t_b$, at which time reactivity decreases until $\delta k = 0$ at $t = t_1$ and $n = n_1$. In other words the time-optimal control is a maximum allowable effort process. Naturally $t_a = t_0 + \gamma/\alpha$ and $t_b = t_1 - \gamma/\alpha$. Then from the solutions of (16) (i.e.,

$$n = n_0 \exp \int_{t_0}^t \delta k(\sigma) d\sigma / \ell$$

$$t_b = t_a + \frac{\ell}{\gamma} (\log n_1 - \log n_0) - \frac{\gamma}{\alpha}. \quad (19)$$

For the next interval $[t_0, t_a)$ the state is defined by

$$n = n_0 \exp [\alpha(t-t_0)^2 / 2\ell] \quad (20)$$

and

$$\delta k = \alpha(t - t_0) . \quad (21)$$

At t_a and t_b , the neutron density is

$$n(t_a) = n_a = n_0 \exp(\gamma^2/2\alpha l) \quad (22)$$

and

$$n(t_b) = n_b = n_1 \exp(-\gamma^2/2\alpha l) . \quad (23)$$

For the interval (t_a, t_b) ,

$$n = n_a \exp[\gamma(t - t_a)/l] \quad (24)$$

and

$$\delta k = \gamma . \quad (25)$$

Then for (t_b, t_1)

$$n = n_b \exp \int_{t_b}^t \delta k(\sigma)/l d\sigma$$

or

$$n = n_b \exp \left\{ \frac{(t - t_b)}{l} \left[\gamma - \frac{\alpha}{2} (t - t_b) \right] \right\} , \quad (26)$$

and

$$\delta k = \gamma - \alpha(t - t_b) . \quad (27)$$

Since $p_2(t)$ is piecewise constant of one value, $u(t)$ can only change sign once if the necessary conditions are satisfied at t_0 , t_a , t_b , and t_1 .

In summary, the complete solution is determined by n_0 and n_1 along with the maximum principle and its extensions for state constraints. If n_b is less than n_a in the above computations, the state constraint boundary is never attained and the optimal control is

$$\delta \dot{k} = \alpha \text{ for } (t_0, t_a) \quad (28)$$

and

$$\delta \dot{k} = -\alpha \text{ for } (t_a, t_1) . \quad (29)$$

For at $t = t_0$ and $t = t_1$, $\delta \dot{k} = 0$. The switching time t_a is computed from

$$(t_a - t_0)^2 = \frac{\ell}{\alpha} \log \frac{n_1}{n_0} . \quad (30)$$

Though in general the maximum principle only yields necessary conditions, they are also sufficient conditions for this problem since the trajectories are unique solutions.

A time-optimal neutronic process evolves as part of the solution to the next problem also.

2.2 Nuclear Rocket Minimal Propellant Startup

For this problem the system dynamics are represented by equations (1) to (4). The system is designed to operate at some specific impulse (or T_c) and some thrust F (or P_c). These two conditions specify a propellant flow rate $\dot{w}$ from the choked nozzle equation ($P_c \approx \text{constant } \dot{w} T_c^{1/2}$) and heat-exchange equations. The temperature and flow rate also determine reactor power.

Initially, the system is at equilibrium with reactivity $\delta k = u_1 = 0$ and $u_2 = \dot{w}$ at a necessary minimum value u_a . This minimum flow is required to cool structural shell and other accessory equipment. For this problem the control constraints are

$$|u_1| \leq \gamma \quad (31)$$

for reactivity, and

$$u_a \leq u_2 \leq u_b \quad (32)$$

for propellant flow rate. Reactor power cannot exceed its design value (also specified as the terminal power) Q_1 , i.e.,

$$0 \leq Q \leq Q_1 . \quad (33)$$

With these constraints it is desired to make a minimal propellant start ($f_0 = u_2$ in equation (6)) of the rocket engine from the specified initial equilibrium conditions to a specified temperature T_1 and power Q_1 . For these variables to remain at equilibrium specifies

$u_2(t) = Q_1/c_m a T_1$, and $u_1(t) = \ell \sum_{i=1}^6 \dot{c}_i(t)/Q_1$ for $t \geq t_1$.

While $Q(t) \leq Q_1$ (i.e., on the interval $[t_0, t_a]$)

$$R(\vec{x}, \vec{p}) = I(\vec{x}, \vec{p}) + \frac{p_1 Q u_1}{\ell} + (p_0 - a p_8 T) u_2, \quad (34)$$

where $I(\vec{x}, \vec{p})$ is not an explicit function of $\vec{u}$, and $\vec{p}(t)$ is defined by the adjoint system,

$$\dot{p}_1 = \frac{\beta - u_1}{\ell} p_1 - \sum_{i=1}^6 \frac{\beta_i}{\ell} p_{i+1} - \frac{p_8}{c_m}, \quad (35)$$

$$\dot{p}_j = \lambda_{j-1} (p_j - p_1) \quad (j=2, \dots, 7), \quad (36)$$

$$\dot{p}_8 = a u_2 p_8, \quad (37)$$

and

$$\dot{p}_0 = 0 \quad (p_0 = \text{nonpositive constant}). \quad (38)$$

The extremal control is

$$u_1 = \gamma \operatorname{sign} p_1, \quad (39)$$

and

$$u_2 = u_a \quad \text{for } F < 0 \quad (40)$$

$$u_2 = u_b \quad \text{for } F > 0,$$

where

$$F = p_0 - a T p_8.$$

Due to the nature of the solutions, it can be shown that $\vec{p}(t_0)$ may be selected so that the switching functions $F(t)$ and $p_1(t)$ do not change sign.

Particularly, select

$$0 < p_8(t_0) \ll p_1(t_0), \quad (41)$$

and

$$p_j(t_1) = 0 \quad (j=2, \dots, 7). \quad (42)$$

The latter equation is the necessary transversality condition.

Thus flow rate is at its minimal allowable value u_a and reactivity is at its maximum allowable value until $Q(t) = Q_1$ at $t = t_a$. Then for $t > t_a$ reactivity is equal to

$$u = l \sum_{i=1}^6 \dot{c}_i / Q_1 \quad (43)$$

to keep $Q(t) = Q_1$. That is, $s = Q - Q_1$, and $\dot{s} = 0$ stipulates (43). Then from (12), for the interval $(t_a, t_1]$, the costate is represented by

$$\dot{p}_1 = \frac{\beta - u_1}{l} (p_1 - l) - \sum_{i=1}^6 \frac{\beta_i}{l} p_{i+1} - \frac{p_8}{c_m}, \quad (44)$$

$$\dot{p}_j = \lambda_{j-1} (p_j - p_1) + \lambda_{j-1} \rho \quad (j=2, \dots, 7), \quad (45)$$

$$\dot{p}_8 = au_2 p_8 \quad (46)$$

and

$$\dot{p}_0 = 0. \quad (47)$$

From equations (13) and (34),

$$\rho = p_1 \quad (48)$$

and

$$v = -F. \quad (49)$$

Since there is no exit corner one can let $\mu = 0$, and

$$p_1(t_a^-) = p_2(t_a^+). \quad (50)$$

If all these conditions are satisfied, it is possible to select p_0 and $p_1(t_0)$ so that the Hamiltonian is zero.

Also, it is readily seen that all the accessory conditions for optimal state-constrained trajectories are satisfied. Hence, $u_2 = u_a$ until $t = t_1$, (determined by $T(t_1) = T_1$) at which time $u_2 = Q_1 / c_m a T_1$ to maintain the desired maximum temperature T_1 .

Such a fuel-optimal start is presented in Fig. 2 and was computed digitally. All necessary conditions were satisfied by these trajectories for a hypothetical nuclear rocket designed for operation at $T_c = 4500^\circ R$ and $F = 100,000$ lb. of thrust. It is of interest to note that the fuel-optimal control utilizes a minimal-time reactor control policy.

For an actual system, additional constraints and higher order complicate the problem considerably. For example, thermal stress limitation due to the mechanical design of the reactor roughly imply some $\dot{T}$ constraint. At low flow rate, this can be approximated by a power constraint similar to the above problem. Also, the turbopump is constrained to operate outside of regions of stall and cavitation. Such an allowable performance region is shown in Fig. 3. These undesirable regions of operation are separated from the admissible operating conditions by lines of constant specific speed.

For example, in Fig. 3, the optimal trajectory without such pump constraint would proceed from point o to D to C. The pump stall constraint, however, would require that flow rate u_2 increase at point S to maintain a constant specific speed N_{ss} until point E is reached ($T(t) = T_1$). Then flow rate would be increased to $Q_1/c_m T_1$. To complicate this, however, the minimum flow constraint really depends on core temperature and would have to increase slightly as temperature increases. Also, reactivity and flow rate cannot change value instantly. All these arguments make analytical solutions most difficult, and indicate the significance of computer algorithms. By means of perturbation theory and linear programming it appears possible to obtain solutions to practical optimal control problems.⁶

Another approach has been to generate suboptimal (nearly optimal) trajectories by means of a classical feedback control scheme such as shown by Fig. 1. In any event the optimization analysis is necessary to judge system performance and to gain insight into optimal policies.

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FIG. 1 A CLOSED-LOOP CONTROLLED NUCLEAR ROCKET ENGINE

FIG. 2 FUEL-OPTIMAL NUCLEAR ENGINE START

WEIGHT FLOW RATE

N_s = SPECIFIC-SPEED

T_1 = TERMINAL TEMPERATURE

T_0 = INITIAL TEMPERATURE

FIG. 3 PUMP PERFORMANCE MAP