On the Connection between Branching Processes and
Scale Invariant Field Theory for e^+e^- annihilation*

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MASTER

Abstract

Branching-process models for e^+e^- annihilation give the same moment sum-rules for the one-particle inclusive spectrum as those obtained in scale-invariant theories. In such models the one-particle inclusive spectrum is computed explicitly at high energies and is shown to be gaussian in a variable which is the analog of rapidity in hadronic reactions. Two-particle correlations, exclusive and prong cross sections are discussed and definite predictions are obtained.

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1. Introduction

The recent observation^(1,2) that the total hadronic annihilation cross section (σ_{had}) does not exhibit the theoretically predicted asymptotic behavior of Q^{-2} has confounded the physics community. Perhaps the most persuasive arguments which led to this prediction are based upon the assumptions of scale invariance together with the apparatus of the Wilson operator product expansion. However since this prediction is an asymptotic statement, we do not believe that there is yet reason to despair — at least until we have some notions as to how asymptotia is reached.

Our aim is to look for other predictions of scale invariant theories which might be testable at pre-asymptotic energies to give us some clues as to whether or not we are on the right track.

Throughout this paper all measurable quantities will be normalized to σ_{had} . In this way, we may avoid saying anything specific about σ_{had} in the hope that our predictions may be valid at currently measurable values of Q^2 .

To fix the notation, we define

$$P_n = P_n(Q) = \frac{\sigma_n}{\sigma_{\text{had}}} \quad (1.1)$$

the probability of producing n particles. In the first three sections we assume for convenience that the annihilation takes place only into neutral identical particles; in Sections 4 and 5 we relax this constraint. The average multiplicity and dispersion are defined as

$$\begin{aligned} \bar{n} &= \bar{n}(Q) = \sum_n n P_n \\ D^2 &= D^2(Q) = \sum_n n^2 P_n - \bar{n}^2 \end{aligned} \quad (1.2)$$

The one-particle and two-particle inclusive processes

$$\begin{aligned} e^+ e^- &\rightarrow a(p) + X \\ e^+ e^- &\rightarrow a_1(p_1) + a_2(p_2) + X \end{aligned} \quad (1.3)$$

have the cross-sections

$$\begin{aligned} \frac{E}{\sigma_{had}} \frac{d\sigma}{d^3p} &= f(E, Q) \\ \frac{E_1 E_2}{\sigma_{had}} \frac{d\sigma}{d^3p_1 d^3p_2} &= f^{(2)}(E_1, E_2, z, Q) \end{aligned} \quad (1.4)$$

We assume for simplicity that the photon is spinless. z is the cosine of the angle between the outgoing particles of mass m . We use center-of-mass notations and as usual put

$$m\gamma = q\beta, \quad x = \frac{Q^2}{2m\gamma} = \frac{Q}{2E}, \quad \left(1 \leq x \leq \frac{Q}{2m}\right) \quad (1.5)$$

(q is the four-momentum of the photon, $q^2 = Q^2$). We introduce a new variable, which we call rapidity,

$$y = \ln x, \quad \left(0 \leq y \leq \ln \frac{Q}{2m}\right) \quad (1.6)$$

The description of inclusive cross sections in terms of rapidity is simpler.

We also write

$$g(x, Q) = \frac{4\pi p E^2}{Q} f(E, Q) = \frac{1}{2\sigma_{had}} \frac{d\sigma}{dx} \quad (1.7a)$$

$$h(y, Q) = \frac{4\pi p E}{\bar{n}} f(E, Q) = \frac{1}{\bar{n}\sigma_{had}} \frac{d\sigma}{dy} \quad (1.7b)$$

Similarly, we define

$$g^{(2)}(x_1, x_2, z, Q) = \frac{(4\pi)^2}{Q^2} p_1 p_2 E_1^2 E_2^2 f^{(2)}(E_1, E_2, z, Q) \quad (1.8a)$$

$$h^{(2)}(y_1, y_2, z, Q) = \frac{(4\pi)^2}{n(n-1)} p_1 p_2 E_1 E_2 f^{(2)}(E_1, E_2, z, Q) \quad (1.8b)$$

The inclusive distribution functions satisfy the energy-momentum sum-rules given in Appendix A.

Using the similarity hypothesis, Polyakov⁽⁴⁾ has derived the following sum-rules for the one-particle inclusive distribution at large values of Q^2 :

$$\int_1^\infty g(x, Q) x^{-s} dx = A(s) \left(\frac{Q^2}{m^2} \right)^{-\alpha(s)} \quad (1.9)$$

where m^2 is an unknown mass-scale and

$$\alpha(0) < \alpha(1) < \alpha(2) \dots, \quad (\alpha(1) = 0, A(1) = 1) \quad (1.10)$$

Taking $s=0$ in Eq. (1.9) and using (A-1a) one obtains an average multiplicity which increases as a power of Q^2 .

Mueller⁽⁵⁾ has also derived the sum-rules (1.9) by solving the Callan-Symanzik equations in a neutral ϕ^4 theory assuming that the theory has an eigenvalue $\beta(g_\infty) = 0$.

Polyakov has also obtained that, in the limit of large Q and n , the exclusive probabilities satisfy the scaling property

$$p_n(Q) = p\left(\frac{n}{\bar{n}}, \bar{n}\right) \approx \frac{1}{\bar{n}} \varphi\left(\frac{n}{\bar{n}}\right) \quad (1.11)$$

which is the same as K-N-O scaling of hadronic interactions.⁽⁶⁾

Following an idea of Polyakov⁽⁴⁾ we have formulated a model of branching processes in order to describe the electron-positron annihilation into hadrons. This model is described in Section 2. In the framework of our model we recover the predictions (1.9) and (1.11). The model has the additional advantage, aside from its simplicity, of allowing us to build up expressions for $A(s)$ and $\alpha(s)$ which are transparent in their physical meaning. It also allows the calculation of various correlation functions, exclusive cross-sections, etc... Moreover the model can be extended to describe annihilation into several types of particles. We now state our main results:

1. Exclusive and prong cross-sections.

- a) The scaling law (1.11) is not valid only for the exclusive cross-sections but also for the prong cross-sections.
- b) For fixed n ,

$$\lim_{Q \rightarrow \infty} \frac{\sigma_n}{\sigma_2} = \lim_{Q \rightarrow \infty} \frac{p_n}{p_2} = \zeta_n \quad (1.12)$$

where ζ_n is a number. This result implies a universal fall-off in Q^2 of the cross-sections. The same is true for the prong cross-sections.

- c) The form factor (proportional to p_2) decreases like a power in Q^2 in the time-like region.

2. The one-particle inclusive distribution. The sum rules (1.9) are obtained with no non-leading contributions. This uniformity property allows us to consider the inversion problem i.e. to solve (1.9) and find $g(x, Q)$. Working with the function $g(x, Q)$ and the variable x is useful in the parton model (where g is independent of Q) or in writing sum-rules like (1.9). It turns out, however, that our model makes simple predictions when one uses $h(y, Q)$ and the rapidity variable y . We obtain for large Q

$$h(y, Q) \approx \frac{1}{\sqrt{2\pi} \sigma_y} e^{-\frac{(y-\bar{y})^2}{2\sigma_y^2}} \quad (1.13)$$

i.e. the one-particle inclusive distribution is approximately Gaussian in y with $\bar{y} = \bar{y}(Q) \sim \ln Q$ and $\sigma_y = \sigma_y(Q) \sim (\ln Q)^{1/2}$. We consider this prediction much easier to test than the sum-rules (1.9).

3. The two-particle inclusive distribution. One can write sum-rules for $g^{(2)}(x_1, x_2, z, Q)$ similar to (1.9), but these sum-rules will be difficult to check experimentally. It turns out, however, that if the two observed particles have large rapidities, $h^{(2)}$ factorizes and is z independent:

$$h^{(2)}(y_1, y_2, z, Q) \approx h(y_1) h(y_2) \quad (1.14)$$

In physical terms this corresponds to "long range" correlations in rapidity.

This is a testable prediction of the model.

4. Production of charged and neutral pions. The average number of neutral pions $\langle n^0 \rangle_{n_{ch}}$ for a given number of charged pions (n_{ch}) increases linearly with n_{ch} .

A detailed description of the model is given in Section 2. The calculation of the exclusive cross-sections and inclusive distributions is performed in Sections 3 and 4. The annihilation into charged particles is considered in Section 5. The implications of the model and outlook are presented in Section 6.

2. A branching process model for e^+e^- annihilation.

We visualize the annihilation process as taking place as in Figure 1. In the center of mass of the e^+e^- system, the photon behaves like a fireball of mass Q which decays with a probability P_n (P_n are numbers independent of Q) into n fireballs of mass $\frac{Q}{n}$ which are at rest in our reference frame.

(Slowly-moving, large-mass fireballs are characteristic in scale invariant models, see Appendix B.) The distribution P_n is assumed to be strongly cut for large n . Each one of these fireballs then decays into other fireballs with the same probability distribution P_n . This process continues k times and is called a branching process (see Appendix C for more details). We assume that k is a function of Q

$$k = k(Q)$$

and that the dependence on Q enters only through $k(Q)$.

After the k^{th} step all the fireballs decay into particles. In Figure 1 we have assumed that the decay takes place only into two particles. This may be an oversimplified picture (see Section 4 for other possibilities). In any case, an important requirement in our model is that the probability of producing particles from the fireballs after the k^{th} step does not depend on the mass of the fireball.

It is clear that the set of numbers P_n determines the "short-distance" behavior of the annihilation process (they determine the exponents $\alpha(s)$ in Eq. (1.9)). The "long distance" behavior of our theory is determined by the final decay into particles (the final decay determines the coefficients $A(s)$ in Eq. (1.9)).

The function $k(Q)$ has to be postulated in our approach. As we will see in the next section, in order to obtain the sum-rules (1.9) we have to assume that for large Q^2

$$k = a \ln \left(\frac{Q^2}{m^2} \right) \quad (2.1)$$

An expression like

$$k = a \ln \ln \left(\frac{Q^2}{m^2} \right) \quad (2.2)$$

would give sum-rules similar to those obtained in asymptotically free theories for the electroproduction case. (7)

We have to confess that the physical interpretation of the function $k(Q)$ remains a mystery even to us. We still do not understand why the fireballs which decay independently into other fireballs decide all at once (at the same step k) to go into particles.†

We would like to stress that our model has nothing in common with the fireball models as used in strong interactions. (8) The basic differences are outlined in Appendix B.

In the next section we will consider in detail the fireball stage of the annihilation process while in Section 4 we discuss the complete annihilation process when we take into account the decay of the fireballs into particles.

3. The fireball production

In this section we confine ourselves to the branching process of fireball production. As we have mentioned above (see also Appendix C), the probability distribution P_n ($n=1,2,\dots$) completely determines the properties of the fireballs at the k^{th} step (see Figure 1).

It is useful to define the following quantities

$$Z(\beta) = \sum_{n=1} n^{1-\beta} P_n \quad (3.1)$$

where

$$Z(-1) = \sigma^2 + \mu^2, \quad Z(0) = \mu, \quad Z(1) = 1 \quad (3.2)$$

μ and σ are the average and dispersion of the probability distribution P_n . Since we expect the distribution P_n to be sharply cut-off at large n (see Appendix B), it is reasonable to assume that $Z(\beta)$ exists even for large negative values of β . (It will be seen later that this implies the existence of the sum-rules (1.9) even for negative values of s .)

We also consider the probability distributions (defined through P_n) which will be proven useful:

$$R_n = \frac{n P_n}{\mu} \quad (3.3)$$

and

$$P(\varrho) = n R_n, \quad (\varrho = \ln n) \quad (3.4)$$

where

$$\int_0^{\infty} P(\varrho) d\varrho = 1 \quad (3.5)$$

Obviously

$$\hat{P}(\beta) = \int_0^{\infty} P(\varrho) e^{-\beta \varrho} d\varrho = \frac{Z(\beta)}{\mu} \quad (3.6)$$

The distribution $P(\varrho)$ has an average $\bar{\mu}$ and a dispersion $\bar{\sigma}$:

$$\begin{aligned} \bar{\mu} &= \sum_n (\ln n) R_n = \frac{1}{\mu} \left(-\frac{\partial}{\partial \beta} \right) \ln Z(\beta) \Big|_{\beta=0} \\ \bar{\sigma}^2 + \bar{\mu}^2 &= \sum_n (\ln n)^2 R_n = \frac{1}{\mu} \left(-\frac{\partial}{\partial \beta} \right)^2 \ln Z(\beta) \Big|_{\beta=0} \end{aligned} \quad (3.7)$$

If the distribution P_n is narrow, using the method of steepest descent one obtains

$$\bar{\mu} \approx \ln \mu + \frac{\bar{\sigma}^2}{2}, \quad \bar{\sigma} \approx \frac{\sigma}{\mu} \quad (3.8)$$

3.1. The generating functionals

The generating functional corresponding to the first step ($k=1$) is

$$\begin{aligned} \Psi_1(\Phi(x)) &= \sum_{n_1} P_{n_1} \int \prod_{i=1}^{n_1} \delta(x_i - n_1) \Phi(x_i) dx_i \\ &= \sum_{n_1} P_{n_1} (\Phi(n_1))^{n_1} \end{aligned} \quad (3.9)$$

This corresponds to the decay of the fireball of mass Q into fireballs of mass M_i ($x_i = \frac{Q}{M_i}$), under the hypothesis that the mass of the incoming fireball or its energy (the fireballs do not carry three-momentum) is shared equally among the produced fireballs. The generating functional after k steps is (see (C-4)):

$$\Psi_k(\Phi(x)) = \sum_{n_1} \left[\sum_{n_2} \left[\dots \left[\sum_{n_k} P_{n_k} (\Phi(n_1 n_2 \dots n_k))^{n_k} \right] \dots \right] \right]^{n_1} \quad (3.10)$$

Eq. (3.10) contains all the properties of the branching process.

The probability $P_{n;k}$ of producing n fireballs after the k^{th} step ($P_{n;1} = P_n$) is obtained by putting $\Phi(x) = Z$ in Eq. (3.10):

$$P_{n;k} = \frac{1}{n!} \left. \frac{\partial^n \Psi_k(Z)}{\partial Z^n} \right|_{Z=0} \quad (3.11)$$

The average μ_k and the dispersion σ_k are defined from

$$\mu_k = \left. \frac{\partial \Psi_k(Z)}{\partial Z} \right|_{Z=1}, \quad \sigma_k^2 + \mu_k^2 - \mu_k = \left. \frac{\partial^2 \Psi_k(Z)}{\partial Z^2} \right|_{Z=1} \quad (3.12)$$

The one-fireball and two-fireball inclusive distributions after the k^{th} step are

$$G_k(x) = \frac{\delta \Psi_k(\Phi(x))}{\delta \Phi(x)} \Big|_{\Phi=1} \quad (3.13)$$

$$G_k^{(2)}(x_1, x_2) = \frac{\delta \Psi_k(\Phi(x))}{\delta \Phi(x_1) \delta \Phi(x_2)} \Big|_{\Phi=1} \quad (3.14)$$

3.2. Exclusive probabilities

Using Eqs. (3.10) and (3.12) it is easy to show that

$$\mu_k = \mu^k, \quad \sigma_k^2 = \frac{\sigma^2}{\mu(\mu-1)} \mu_k (\mu_k - 1) \quad (3.15)$$

We observe that the dispersion squared is quadratic in the average multiplicity; this suggests that we have long range correlations which will show up when we compute the two-fireball inclusive distribution.

Using Eqs. (3.10) and (3.11) we obtain

$$P_{1;k} = P_1^k, \quad P_{2;k} = \frac{P_2(1-P_1^k)}{P_1(1-P_1)} P_{1;k} \quad (3.16)$$

From Eq. (3.16) we have

$$\lim_{k \rightarrow \infty} \frac{P_{2;k}}{P_{1;k}} = \frac{P_2}{P_1(1-P_1)} = \zeta_2 \quad (3.17)$$

where ζ_2 is a number. One can show (see Eq. (C-13)) that the property (3.17) is general:

$$\lim_{k \rightarrow \infty} \frac{P_{n;k}}{P_{1;k}} = \zeta_n \quad (3.18)$$

i.e. the exclusive probabilities have a universal fall-off with k . As shown in Appendix C (see Eq. (C-6)) $P_{n;k}$ have also the scaling property

$$\lim_{\substack{k \rightarrow \infty \\ n \rightarrow \infty}} \frac{n}{\mu_k} \text{ fixed } P_{n;k} = \frac{1}{\mu_k} \varphi\left(\frac{n}{\mu_k}\right) \quad (3.19)$$

We will illustrate the properties (3.15), (3.18) and (3.19) in an explicitly computable example. Take $P_n = qp^{n-1}$ ($p+q=1$); the generating function

$$\begin{aligned} \Psi_1(Z) &= \frac{qZ}{1-pZ} \\ \Psi_k(Z) &= \frac{q^k Z}{1-p_k Z}, \quad (p_k = 1 - q^k) \end{aligned} \quad (3.20)$$

From Eq. (3.12) and (3.20) we have

$$\mu_k = q^{-k}, \quad \sigma_k^2 = \frac{1 - q^k}{q^{2k}} \quad (3.21)$$

which satisfy Eq. (3.15). Using (3.11) we obtain

$$P_{n;k} = \frac{1}{\mu_k} \left(1 - \frac{1}{\mu_k}\right)^{n-1}$$

which corresponds to (3.17) with $\zeta_n = 1$ and (3.19) with $\phi = e^{-\frac{n}{\mu_k}}$.

3.3. One-fireball inclusive distribution.

Using (3.3), (3.10) and (3.13), one has:

$$G_k(x) = \mu^k \sum_{n_1, \dots, n_k} R_{n_1} R_{n_2} \dots R_{n_k} \delta(x - n_1 n_2 \dots n_k) \quad (3.22)$$

Multiplying Eq. (3.22) by x^{-s} and integrating over x we obtain the sum rules:

$$\int_1^{\infty} G_k(x) x^{-s} dx = \left(\sum_n n^{-s+1} P_n \right)^k = \left(Z(s) \right)^k \quad (3.23)$$

These sum rules will be used in Section 4 in order to derive the sum-rules (1.9). We can go beyond the sum-rules and try to compute $G_k(x)$ explicitly at least in some region of x . We define

$$H_k(y) = \frac{G_k(x) x}{\mu_k}, \quad (y = \ln x) \quad (3.24)$$

where

$$\int_0^{\infty} H_k(y) dy = 1 \quad (3.25)$$

We take the Laplace transform of $H_k(y)$ and use Eq. (3.22):

$$\hat{H}_k(\beta) = \int_0^{\infty} H_k(y) e^{-\beta y} dy = \left(\sum_n n^{-\beta} P_n \right)^k = \left(\hat{P}(\beta) \right)^k \quad (3.26)$$

where we have used (3.6). The equation (3.26) expresses the fact that the probability distribution $H_k(y)$ is the k -fold convolution of the probability distribution $P(y)$. Since the probability distribution has an average $\bar{\mu}$ and a dispersion $\bar{\sigma}$ (see (3.7)), using the central limit theorem, $H_k(y)$ is gaussian for large k :

$$H_k(y) \approx \frac{1}{\sqrt{2\pi} \sigma_y} e^{-\frac{1}{2} \frac{(y - \bar{y})^2}{\sigma_y^2}} \quad (3.27)$$

where

$$\bar{y} = k \bar{\mu}, \quad \sigma_y = (k)^{1/2} \bar{\sigma} \quad (3.28)$$

We observe that this is a model independent result in the sense that it does not depend on the details of the probability distribution P_n .

3.4. The two-fireball inclusive distribution.

We use Eqs. (3.10) and (3.14) in order to derive the two-fireball inclusive distribution

$$G_k^{(2)}(x_1, x_2) = \sum_{n_1, n_2, \dots, n_k} \mu^k (n_1 - 1) R_{n_1} R_{n_2} \dots R_{n_k} \delta(x_1 - n_1 n_2 \dots n_k) \cdot \dots$$

$$\cdot \left(\delta(x_2 - n_1 n_2 \dots n_k) + \sum_{n'_k} \delta(x_2 - n_1 n_2 \dots n_{k-1} n'_k) \mu R_{n'_k} + \dots \right. \quad (3.29)$$

$$\left. + \sum_{n'_2, \dots, n'_k} \delta(x_2 - n_1 n'_2 \dots n'_k) \mu^{k-1} R_{n'_2} \dots R_{n'_k} \right)$$

Multiplying Eq. (3.29) by $x_1^{-s_1} x_2^{-s_2}$ and integrating over x_1 and x_2 we derive the sum-rules:

$$\int G_k^{(2)}(x_1, x_2) x_1^{-s_1} x_2^{-s_2} dx_1 dx_2 = C(s_1, s_2) \left(\left(Z(s_1) Z(s_2) \right)^k - \left(Z(s_1 + s_2) \right)^k \right) \quad (3.30)$$

$$= C(s_1, s_2) \left(\int G_k(x_1) x_1^{-s_1} dx_1 \int G_k(x_2) x_2^{-s_2} dx_2 - \int G_k(x) x^{-s_1 - s_2} dx \right)$$

where

$$C(s_1, s_2) = \frac{Z(s_1 + s_2 - 1) - Z(s_1 + s_2)}{Z(s_1) Z(s_2) - Z(s_1 + s_2)} \quad (3.31)$$

The sum-rules (3.30) are k independent, and they relate the two-fireball distribution to the one-fireball distribution.

If $0 \leq s_2 \leq 1$, using (3.1) we get

$$Z(s_1) Z(s_2) > Z(s_1 + s_2)$$

which implies

$$\int G_k^{(2)}(x_1, x_2) x_1^{-s_1} x_2^{-s_2} dx_1 dx_2 \approx C(s_1, s_2) \int G_k(x_1) x_1^{-s_1} dx_1 \times \int G_k(x_2) x_2^{-s_2} dx_2 \quad (3.32)$$

The significance of the factorization property (3.32) will become clearer when we compute explicitly the two-fireball distribution (if s_1 is also in the interval $0 \leq s_1 \leq 1$ then $C(s_1, s_2) \approx \frac{\sigma^2}{\mu(\mu-1)} + 1$). Let us now consider the function

$$H_k^{(2)}(y_1, y_2) = \frac{G_k^{(2)}(x_1, x_2) x_1 x_2}{\sigma_k^2 + \mu_k^2 - \mu_k}, \quad (y_i = \ln x_i) \quad (3.33)$$

Obviously

$$\int_0^\infty H_k^{(2)}(y_1, y_2) dy_1 dy_2 = 1 \quad (3.34)$$

We take a double Laplace transform of Eq. (3.33) and use (3.29); we obtain:

$$\begin{aligned} H_k^{(2)}(\beta_1, \beta_2) &= \int H_k^{(2)}(y_1, y_2) e^{-\beta_1 y_1 - \beta_2 y_2} dy_1 dy_2 \\ &= \frac{\mu-1}{\mu^k-1} l(\beta_1+\beta_2) \frac{(\mu \hat{P}(\beta_1) \hat{P}(\beta_2))^k - (\hat{P}(\beta_1+\beta_2))^k}{\mu \hat{P}(\beta_1) \hat{P}(\beta_2) - \hat{P}(\beta_1+\beta_2)} \end{aligned} \quad (3.35)$$

where

$$l(\beta) = \sum_{n=1}^{\infty} \frac{n^{1-\beta} (n-1) P_n}{\sigma^2 + \mu^2 - \mu}$$

If β_1 and β_2 are small (which corresponds to large y_1 and y_2), we make the approximation of replacing $l(\beta_1+\beta_2)$ by $l(0)$ to get

$$\hat{H}_k^{(2)}(\beta_1, \beta_2) \approx \frac{1}{\mu^k - 1} \left((\mu \hat{P}(\beta_1) \hat{P}(\beta_2))^k - (\hat{P}(\beta_1 + \beta_2))^k \right) \quad (3.36)$$

Using (3.26) and (3.35), we obtain

$$H_k^{(2)}(y_1, y_2) \approx \frac{1}{\mu^k - 1} \left(\mu^k H_k(y_1) H_k(y_2) - \delta(y_1 - y_2) H_k(y_1) \right) \quad (3.37)$$

This approximation satisfies the normalization condition (3.34). If y_1 and y_2 do not coincide (we are looking for "long range" correlations), and k is large enough,

$$H_k^{(2)}(y_1, y_2) \approx H_k(y_1) H_k(y_2) \quad (3.38)$$

And hence the two-particle distribution will be a product of two gaussians. We also deduce

$$G_k^{(2)}(x_1, x_2) \approx \left(\frac{\sigma^2}{\mu^2 - \mu} + 1 \right) G_k(x_1) G_k(x_2) \quad (3.39)$$

which for $\sigma \neq 0$ expresses the existence of "long range" correlations.

4. Particle production

4.1. The decay of the fireballs into particles

In the last section we have examined the fireball production after k steps (see Figure 1). We are now in a position to derive the particle production. In order to do so we have to devise a model which describes the decay of the fireball into particles (this corresponds to the "long distance" behavior of the theory).

The simplest situation corresponds to the case in which a fireball (at the k^{th} step) of mass $M = \frac{Q}{x}$ decays just into two particles of mass m . This

decay is described by the generating function

$$\Phi(M, \varphi(p)) = \frac{M}{2\pi\sqrt{M^2 - 4m^2}} \int \delta(M - E_1 - E_2) \delta(\vec{p}_1 + \vec{p}_2) \varphi(p_1) \varphi(p_2) \frac{d^3p_1}{E_1} \frac{d^3p_2}{E_2} \quad (4.1)$$

We will treat this example in full detail. For the more general decay mechanism, to be defined below, we consider only the inclusive distributions. Let us assume that the fireball M decays into different kinds of particles, each species being labelled by "c" (since we have not introduced quantum numbers, they can differ only by their mass). The one-particle inclusive distribution in the fireball rest frame is

$$E_c \frac{dN_c}{d^3p_c} = f_c(M, E_c) \quad (4.2)$$

We consider a new function

$$\chi_c(x, M) = \frac{4\pi}{M\lambda_c} E_c^2 p_c f_c(M, E_c), \quad \left(x = \frac{M}{2E_c}\right) \quad (4.3)$$

where for convenience we have introduced λ_c defined as the average fraction of M carried by the particle of type "c" ($\sum_c \lambda_c = 1$).

We assume that χ_c is a function of x only and not of M . This corresponds to assuming the parton results⁽⁹⁾ for the fireball decay into particles. When studying the one-particle inclusive spectrum, we will use the Mellin transform of $\chi_c(x)$:

$$\bar{\Sigma}_c(\beta) = \int_1^\infty \chi_c(x) x^{-\beta} dx \quad (4.4)$$

where

$$\overline{Z}_c(1) = 1, \quad \overline{Z}_c(0) = \frac{\overline{N}_c}{2} \quad (4.5)$$

$\overline{N}_c$ is the average number of particles "c" produced by the fireball (this number is independent of M , i.e. the integral (4.4) converges for $\beta=0$). The two-particle decay scheme mentioned above corresponds to

$$\chi(x) = \delta(1-x), \quad \overline{Z}_c(\beta) = 1 \quad (4.6)$$

4.2. Exclusive particle cross-sections.

We consider for simplicity the case in which the fireball decays into 2 particles. Then (see (1.1), (1.2) and (3.15))

$$\begin{aligned} p_n(Q) &= P_{\frac{n}{2}; k}, \quad (k = a \ln \frac{Q^2}{m^2}) \\ \bar{n}(Q) &= 2 \mu^k = 2 \left(\frac{Q^2}{m^2} \right)^{a \ln \mu} \\ D^2(Q) &= 4 \sigma_k^2 = \frac{\sigma^2}{\mu(\mu-1)} \bar{n}(\bar{n}-2) \end{aligned} \quad (4.7)$$

From (4.7) we find the constraint

$$a \ln \mu \leq \frac{1}{2}. \quad (4.8)$$

Using now (3.16), (3.18) and (3.19) we obtain

$$p_2(Q) = P_1^k = \left(\frac{Q^2}{m^2} \right)^{a \ln P_1} \quad (4.9)$$

$$\lim_{Q \rightarrow \infty} \frac{p_n(Q)}{p_2(Q)} = \zeta_n \quad (4.10)$$

$$\lim_{\substack{Q, n \rightarrow \infty \\ n/\bar{n} = \text{fixed}}} P_n(Q) = \frac{1}{\bar{n}} \varphi\left(\frac{n}{\bar{n}}\right) \quad (4.11)$$

From Eq. (4.9) we learn that the form factor decreases like a power in Q^2 , Eq. (4.10) expresses the universal fall-off of the exclusive cross-sections, and Eq. (4.11) implies K-N-O scaling.

4.3. The one-particle inclusive distribution

We use (1.7a), (3.10) and (4.3) to derive the one particle inclusive distribution

$$g_c(x, Q) = \lambda_c \int G_k(x) \chi_c\left(\frac{x}{x_1}\right) \frac{dx_1}{x_1} \quad (4.12)$$

Multiplying (4.12) by x^{-s} and using (3.23) and (4.4) we derive

$$\begin{aligned} \int_1^\infty g_c(x, Q) x^{-s} dx &= \lambda_c \bar{Z}_c(s) (\bar{Z}(s))^k \\ &= A_c(s) \left(\frac{Q^2}{m^2}\right)^{-\alpha(s)} \end{aligned} \quad (4.13)$$

where we have used (2.1),

$$\begin{aligned} A_c(s) &= \lambda_c \bar{Z}_c(s) \\ \alpha(s) &= -a \ln \bar{Z}(s) \end{aligned} \quad (4.14)$$

The special case in which the fireball decays into two particles corresponds to $A_c(s) = 1$. The sum rules (4.13) coincide indeed with those derived by Polyakov and Mueller^(4,5) and their physical interpretation becomes now clearer. The "short distance" structure of the theory (anomalous dimensions) are determined by the constant "a" in (2.1) and the mechanism of the fireball decay into fireballs ($Z(s)$ is determined by P_n through Eq. (3.1)) while the constants $A_c(s)$ depend on the type of the observed particle and the "long-distance" behavior of the model which depends on the fireball decay into particles.

A very important feature of our model is the uniformity property of the sum-rules (4.13) (non-leading contributions can be present in $k(Q)$ only, we can have for instance $k = a \ln \left(\frac{Q^2}{m^2} + 1 \right)$).

We would like to stress that the sum-rules (4.13) can be checked experimentally not only for positive s but also for negative s . In this way we can get some insight into the properties of $A_c(s)$ since $Z(s)$ exists for any real s (this is a consequence of the sharp fall-off of P_n for large n , (see Eq. (3.1)).

In the two particle decay scheme

$$g(x, Q) = G_k(x) \quad (4.15)$$

and we can then use (3.27) to derive

$$h(y, Q) \approx \frac{1}{\sqrt{2\pi} \sigma_y} e^{-\frac{(y - \bar{y})^2}{\sigma_y^2}} \quad (4.16)$$

where

$$\begin{aligned} \bar{y} &= k \bar{\mu} = \ln \left(\frac{Q^2}{m^2} \right)^{\bar{\mu} a} \\ \sigma_y^2 &= k \bar{\sigma}^2 = \ln \left(\frac{Q^2}{m^2} \right)^{\bar{\sigma}^2 a} \end{aligned} \quad (4.17)$$

If the distribution P_n is narrow, we can use the approximation (3.8) for $\bar{\mu}$ and $\bar{\sigma}$. In this case the approximate expression in Eq. (4.16) satisfies both sum-rules (A-1b). The more general decay scheme gives also (4.16) but with

$$\bar{y} = \bar{\mu}_c + k \bar{\mu} \quad (4.18)$$

$$\sigma_y^2 = \bar{\sigma}_c^2 + k \bar{\sigma}^2$$

where $\bar{\mu}_c$ and $\bar{\sigma}_c^2$ are defined from $\bar{Z}_c(s)$ in the same way as $\bar{\mu}$ and $\bar{\sigma}$ are

derived from $Z(\beta)$ (see Eq. (3.7)).

4.4. The two-particle inclusive distribution

We start with the two-particle decay model. Using Eq. (3.10) and Eq. (4.1) we get

$$g^{(2)}(x_1, x_2, z, Q) = G_k^{(2)}(x_1, x_2) + \frac{4\pi E_1 p_1}{Q} \delta(\vec{p}_1 + \vec{p}_2) G_k(x_1) \quad (4.19)$$

The second term in (4.19) represents the "short range" correlation related to the fireball decay into particles. If we choose to observe particles which are noncolinear, then

$$g^{(2)}(x_1, x_2, z, Q) = G_k^{(2)}(x_1, x_2), \quad (z \neq -1) \quad (4.20)$$

and we can use the sum-rules (3.30) for $g^{(2)}(x_1, x_2, z, Q)$. In the more general decay scheme (which corresponds to assuming scaling for the two-particle inclusive distribution of the fireball decay), similar fixed angle sum-rules may be derived. In any case, the approximate expression valid for large y 's

$$h^{(2)}(y_1, y_2, z, Q) \approx h(y_1)h(y_2) \quad (4.21)$$

which is derived from Eq. (3.39) in the two-particle decay scheme, remains valid.

Eq. (4.21) expresses the existence of long-range correlations in rapidity. Therefore, we can see now how long-range correlations, suggested by the K-N-0 scaling property of the exclusive cross-sections Eq. (4.11) manifest themselves in momentum distributions.

5. Charged particle production

We have considered up to now the annihilation into neutral particles only. In this section we want to show that the introduction of quantum numbers in the model, although not a trivial complication, can be done. In order to illustrate this point, we consider an explicitly computable model. The model allows for the production of neutral and charged fireballs during the branching stage and for charge conservation at each splitting of a fireball into others. We deal only with the exclusive and semi-inclusive (prong) cross-sections and do not consider the problem of inclusive spectra.

This model generalizes the model considered in Section 3 (see Eq. (3.20)). We will again make the distinction between fireball production and particle production.

For the first stage, we assume the following generating functions for the fireball production:

$$\begin{aligned}\Psi_1^0(Z_+, Z_-, Z_0) &= \frac{q Z_0}{1 - p(\alpha Z_0 + \beta Z_+ Z_-)} \\ \Psi_1^\pm(Z_+, Z_-, Z_0) &= \frac{(q)^{1/2} Z_\pm}{[1 - p(\alpha Z_0 + \beta Z_+ Z_-)]^{1/2}}\end{aligned}\quad (5.1)$$

$$(\alpha + \beta = 1, \quad p + q = 1)$$

which by iteration (see Eq. (C-4)) gives:

$$\begin{aligned}\Psi_k^0(Z_+, Z_-, Z_0) &= \frac{q^k Z_0}{1 - p_k(\alpha Z_0 + \beta Z_+ Z_-)} \\ \Psi_k^\pm(Z_+, Z_-, Z_0) &= \frac{q^{k/2} Z_\pm}{[1 - p_k(\alpha Z_0 + \beta Z_+ Z_-)]^{1/2}}\end{aligned}\quad (5.2)$$

$$(p_k = 1 - q^k)$$

Ψ_k^0 represents the generating function for charged ($Z_\pm$) and neutral (Z_0) fireball production from a neutral fireball after k steps; $\Psi_k^\pm$ represents the generating function for fireball production from a positively (negatively)

charged fireball ($k = a \ln \frac{Q^2}{m^2}$).

For the decay of the fireball into particles, we assume only two-body decays:

$$Z_{\pm} = \zeta_0 \zeta_{\pm}; \quad Z_0 = u \zeta_0^2 + v \zeta_+ \zeta_- \quad (u+v=1) \quad (5.3)$$

The average number of neutral and positive particles are:

$$\begin{aligned} \bar{n}^0 &= \left. \frac{\partial \Psi_k^0}{\partial \zeta_0} \right|_{\zeta_0 = \zeta_{\pm} = 1} = 2 \left(-\beta v + \frac{1-v\alpha}{q^k} \right) \\ \bar{n}^{\pm} &= \left. \frac{\partial \Psi_k^0}{\partial \zeta_{\pm}} \right|_{\zeta_0 = \zeta_{\pm} = 1} = -\beta u + \frac{\beta + v\alpha}{q^k} \end{aligned} \quad (5.4)$$

The ratio of neutral to charged particles

$$\frac{\bar{n}^0}{2\bar{n}^+} = \frac{1-v\alpha - q^k \beta v}{\beta + v\alpha - q^k \beta u} \quad (5.5)$$

varies for small values of k and reaches an asymptotic constant value. This observation is very important if we want to build a model which is supposed not to break isospin bounds.⁽¹¹⁾ We notice that if $u < \frac{1}{2}$, the asymptotic value is smaller than one and is reached from below; if $u > \frac{1}{2}$, the asymptotic value is larger than one and is reached from above. We now compute the prong cross-sections:

$$P_{n_+, k}^{(pr)} = \frac{1}{n_+!} \left. \frac{\partial^{n_+} \Psi_k^0}{\partial \zeta_+^{n_+}} \right|_{\zeta_+ = 0, \zeta_- = \zeta_0 = 1} = \frac{q^k}{\beta + v\alpha} \left(1 - \frac{q^k}{\beta + v\alpha} \right)^{n_+} \quad (5.6)$$

The exclusive cross-sections for charged particles for example are:

$$P_{m_+;k} = \frac{1}{n_+!} \left. \frac{\partial^{n_+} \Psi_k^0}{\partial \zeta_+^{n_+}} \right|_{\zeta_+ = \zeta_0; \zeta_- = 1} = q^k v [(1-q^k) \alpha v]^{n_+ - 1} \quad (5.7)$$

We thus derive

$$\lim_{Q \rightarrow \infty} \frac{P_{m_+;k}^{(pr)}}{P_{1;k}^{(pr)}} = \xi_{n_+}^{(pr)} \quad (5.8)$$

$$\lim_{Q \rightarrow \infty} \frac{P_{m_+;k}}{P_{1;k}} = \xi_{n_+} \quad (5.9)$$

where $\xi_{n_+}^{(pr)} = 1$ and $\xi_{n_+} = (\alpha v)^{n_+ - 1}$. From Eq. (5.8) and (5.9) we deduce that the prong-cross-sections and the exclusive cross-sections have a universal fall-off with Q .

From Eq. (5.6) we also derive

$$\lim_{\substack{Q, n_+ \rightarrow \infty \\ \frac{n_+}{Q} \text{ fixed}}} P_{m_+;k}^{(pr)} = \frac{1}{n_+} \varphi\left(\frac{n_+}{Q}\right) \quad (5.10)$$

where $\phi(x) = e^{-x}$, i.e. the prong-cross-sections have the K-N-0 scaling property.

An interesting quantity is the average number of neutral particles for a given number of positive particles:

$$\begin{aligned} \langle n_0 \rangle_{n_+} &= \frac{\frac{\partial}{\partial \zeta_0} \frac{\partial^{n_+}}{\partial \zeta_+^{n_+}} \Psi_k^0}{\frac{\partial^{n_+}}{\partial \zeta_+^{n_+}} \Psi_k^0} \bigg|_{\zeta_+ = 0; \zeta_0 = \zeta_- = 1} \\ &= \frac{2}{\beta + \alpha v} (1 - 2v\beta + n_+) = c_1 + c_2 n_+ \end{aligned} \quad (5.11)$$

In Eq. (5.11) we have given only the expression valid for large values of k .

Thus for large Q , we expect the average number of neutral particles to increase linearly with the number of positive particles.

We conjecture that the properties (5.8-11) are not specific to our model but are generally valid for branching processes. We base our conjecture on the observation that all the properties derived in Section 3 from the same model in the neutral case were proven to be true in the general case.

We would like to point out that there are techniques available⁽¹²⁾ to study branching processes with nonidentical particles, and we may return to this subject in a later publication if the basic features of our model are proven to be correct by experiment.

6. Discussion

We have formulated a branching process model for e^+e^- annihilation which makes a number of specific predictions which are listed in the Introduction. The connection with scale invariant theories is realized through the sum-rules of Eq. (4.13).

In our model

$$\lim_{s \rightarrow \infty} \alpha(s) = \text{finite}$$

and we find that $\alpha(s)$ is nonsingular along the whole real s axis. The reader should keep in mind that it is possible⁽⁵⁾ to construct field theoretical models in which $\alpha(s)$ has poles for negative integer values so that it is not obvious that any scale invariant theory can be described in terms of branching processes.

A general conclusion of our paper is that e^+e^- annihilation physics should be studied using the rapidity variables $y_i = \ln \omega_i$ in the same way as the p_i and the rapidity variable $y = \frac{1}{2} \ln \frac{E-p_{||}}{E+p_{||}}$ are used in order to describe

strong interaction phenomena.

A first attempt in this direction was done by us⁽¹⁰⁾ in a study of the one-pion inclusive spectrum where we have shown that the present SPEAR data are in qualitative agreement with our prediction (1.13).

We also believe that the model may allow one to do phenomenology in e^+e^- annihilation by determining empirically the probability distribution P_n . In this way different processes could be correlated in a quantitative way.

Finally, in this paper we have assumed for simplicity that the photon is spinless. The same picture could be done, in principle, separately for the longitudinal and transverse part of the cross-section.

We would like to acknowledge useful discussions with our colleagues at the Rockefeller University and with A.H. Mueller.

Appendix A: Energy momentum sum-rules

With our definitions Eqs. (1.4), (1.7), the inclusive distributions satisfy the following sum-rules: (13)

The one-particle inclusive distribution:

$$(A-1a) \quad \int_0^1 g(x, Q) \frac{x}{p} dx = 1, \quad \int_0^1 g(y, Q) dy = 1$$

$$(A-1b) \quad \int_0^1 g(y, Q) e^{-\frac{y}{2}} dy = \frac{n}{2}, \quad \int_0^1 g(y, Q) dy = 1$$

The two particle inclusive distribution:

$$(A-2a) \quad \int_0^1 \int_0^1 g^{(2)}(x_1, x_2, Q) \frac{x_1}{p} \frac{x_2}{p} dx_1 dx_2 = \frac{n(n-1)}{2}$$

$$(A-2b) \quad \int_0^1 \int_0^1 g^{(2)}(x_1, x_2, Q) \frac{x_1}{p} \frac{x_2}{p} dx_1 dx_2 \approx 1$$

Appendix B: A scale invariant model of fireball production

We want to present a scale invariant fireball production model which has the properties required by our branching process model (see Section 2). We confine ourselves to the problem of one fireball decaying into other fireballs; this corresponds to the vertex P_n in Figure 1. The exercise is interesting also in order to understand the difference between such a picture and the fireball production mechanism realized for instance in the statistical bootstrap model of Hagedorn and Frautschi. (8)

For simplicity we consider a one-space one-time world in which a fireball

of mass Q decays into other fireballs of mass $M_i^2 = E_i^2 - p_i^2$. (We use c.m. notations.)

Our model is defined by the generating functional:

$$\Psi(Q, \Phi(E, p)) = c(\alpha, \lambda) \sum_{n=1}^{\infty} \frac{(2\lambda)^n}{4(Q^2)^{n(\alpha+1)-1}} \int \delta(Q - \sum_{i=1}^n E_i) \delta(\sum_{i=1}^n p_i) \cdot \prod_{i=1}^n (E_i^2 - p_i^2)^{\alpha} \theta(E_i^2 - p_i^2) \Phi(E_i, p_i) dE_i dp_i \quad (B-1)$$

where λ is a dimensionless constant, α is a large positive number, and $c(\alpha, \lambda)$ is a normalization constant to be determined later (the smallest allowed mass is assumed to be zero).

We introduce the light-cone variables:

$$\eta_{\pm, i} = \frac{E_i \pm p_i}{Q} \quad (B-2)$$

and consider the generating functional derived from (B-1):

$$\Psi(a_+, a_-; \Phi(\eta_{\pm})) = c(\alpha, \lambda) \sum_{n=1}^{\infty} \lambda^n \int_0^{\infty} \delta(a_+ - \sum_{i=1}^n \eta_{+, i}) \cdot \delta(a_- - \sum_{i=1}^n \eta_{-, i}) \prod_{i=1}^n \eta_{+, i}^{\alpha} \eta_{-, i}^{\alpha} \Phi(\eta_{\pm, i}) d\eta_{+, i} d\eta_{-, i} \quad (B-3)$$

From (B-3) we can derive all the information about the decay. The condition

$$\Psi(1, 1; 1) = 1 \quad (B-4)$$

determines $c(\alpha, \lambda)$.

$$P_n = \frac{1}{n!} \left. \frac{\partial^n \Psi(1, 1; z)}{\partial z^n} \right|_{z=0} \quad (B-5)$$

gives the probability to produce n fireballs.

$$E \frac{dN}{dP} = \frac{E}{Q} \left. \frac{\delta \Psi(1,1; \Phi(\eta_{\pm}))}{\delta \Phi(\eta_{\pm})} \right|_{\Phi=1} \quad (\text{B-6})$$

gives the one-fireball inclusive distribution.

Using Laplace transforms and the identity

$$\frac{e^{-bs}}{s^n} = \frac{1}{\Gamma(n)} \int_0^{\infty} \theta(t-b) (t-b)^{n-1} e^{-st} dt \quad (\text{B-7})$$

we derive the following results:

$$C(\alpha, \lambda) = \left[\sum_{n=1}^{\infty} \frac{u^n}{(\Gamma(n(\alpha+1)))^2} \right]^{-1} \quad (\text{B-8})$$

where $u = \lambda(\Gamma(\alpha+1))^2$;

$$P_n = C(\alpha, \lambda) \frac{u^n}{(\Gamma(n(\alpha+1)))^2} \quad (\text{B-9})$$

$$E \frac{dN}{dP} = C(\alpha, \lambda) \frac{(\eta_+ + \eta_-)}{2} \sum_{n=1}^{\infty} \frac{\lambda n u^{n-1} \eta_+^{\alpha} \eta_-^{\alpha}}{(\Gamma((n-1)(\alpha+1)))^2} \left[(1-\eta_+)(1-\eta_-) \right]^{(n-1)(\alpha+1)-1} \quad (\text{B-10})$$

We illustrate the calculation deriving Eq. (B-8):

$$\begin{aligned} \Psi(a_+, a_-; 1) &= C(\alpha, \lambda) \sum_{n=1}^{\infty} \lambda^n \int_0^{\infty} \delta(a_+ - \sum_{i=1}^n \eta_{+,i}) \delta(a_- - \sum_{i=1}^n \eta_{-,i}) \\ &\quad \cdot \prod_{i=1}^n \eta_{+,i}^{\alpha} \eta_{-,i}^{\alpha} d\eta_{+,i} d\eta_{-,i} \end{aligned} \quad (\text{B-11})$$

We take the double Laplace transform of $\Psi(a_+, a_-; 1)$:

$$\begin{aligned}
\hat{\Psi}(s, t) &= \int_0^\infty \Psi(a_+, a_-; 1) e^{-sa_+ - ta_-} ds dt \\
&= c(\alpha, \lambda) \sum_{n=1}^\infty \lambda^n \left(\int_0^\infty \eta_+^\alpha e^{-s\eta_+} d\eta_+ \right)^n \left(\int_0^\infty \eta_-^\alpha e^{-t\eta_-} d\eta_- \right)^n \\
&= c(\alpha, \lambda) \sum_{n=1}^\infty \frac{u^n}{(st)^{n(\alpha+1)}}
\end{aligned} \tag{B-12}$$

Using now the identity (B-7) with $b=0$, we derive (B-8).

Let us now remind the reader of the results of the statistical bootstrap model⁽⁸⁾; in this model, the distribution P_n has a maximum for $n=2$ ($P_2 \approx 0.7$), and the one fireball is heavy while the other one has the smallest allowed mass. One obtains then a linear chain decay which leads to short-range correlations.

Examining the probability distribution P_n given by Eq. (B-9), one sees that

$$n_{\max} \sim \frac{1}{\frac{2(\alpha+1)}{\alpha+1}} \tag{B-13}$$

which does not have to be equal to two. A characteristic feature of the distribution P_n , however, is a sharp fall-off at large n .

The n -fireball contribution to the inclusive distribution (B-10) is sharply peaked at

$$\eta_\pm \sim \frac{\alpha}{n(\alpha+1)-2} \sim \frac{1}{n} \quad (\text{for large } \alpha) \tag{B-14}$$

which implies

$$E^2 \sim E^2 - p^2 \sim \frac{Q^2}{n} \tag{B-15}$$

i.e. slow fireballs in the c.m. and the energy of the incoming fireball is shared approximately equally among the outgoing fireballs. This is exactly the assumption used in Section 2 and is basically different from the thermodynamic fireball model used in strong interactions.

Appendix C: Definitions and some properties of branching processes

In this appendix we present some general results concerning branching processes. More can be found in Ref. 12.

In a branching process we start with one "particle" which randomly splits into other "particles". These constitute the "particles" in the first "step" or first "generation". Each of these now splits randomly and independently of the others (but with the same probabilities as originally) into other "particles". These constitute the second "step" or the second "generation". This process can continue k times and the particles after the k^{th} step form the $(k+1)$ st-step.

The basic quantity in a branching process is the probability P_n ($n=1,2,\dots$) of producing n particles at the first step. The assumption that each particle splits in the same way and independently of all the others that are present means that if there are m such particles present at any given step, then the probability of these particles resulting into n particles in the next step is given by the m -fold convolution of P_n i.e.

$$\sum_{n_1+n_2+\dots+n_m=n} P_{n_1} P_{n_2} \dots P_{n_m}$$

To find the probability $P_{n;k}$ that there exist n particles at the k^{th} step, we can use the recursion relation (in this notation $P_{n;1} = P_n$).

$$P_{n;k+1} = \sum_{m=1} P_{m;k} \left(\sum_{n_1+n_2+\dots+n_m=n} P_{n_1} P_{n_2} \dots P_{n_m} \right) \quad (\text{C-1})$$

Introducing the generating function

$$\Psi_k(z) = \sum_n P_{n;k} z^n, \quad (C-2)$$

Eq. (C-1) becomes

$$\Psi_{k+1}(z) = \Psi_k(\Psi(z)) = \Psi(\Psi_k(z)) \quad (C-3)$$

and so $\Psi_k(z)$ is the k -fold iterate of $\Psi(z)$:

$$\Psi_k(z) = \Psi(\Psi(\dots(\Psi(z)))). \quad (C-4)$$

This implies that the average number of particles μ_k at the k^{th} step is related to the average μ of the first step:

$$\mu_k = \left. \frac{\partial \Psi_k(z)}{\partial z} \right|_{z=1} = \left(\frac{\partial \Psi(z)}{\partial z} \right)^k = \mu^k \quad (C-5)$$

The number of particles increases geometrically with k and each individual probability $P_{n;k}$ tends to zero as $k \rightarrow \infty$, keeping n fixed. However, re-scaling n by its average μ^k , one obtains the K-N-0 scaling property:

$$\lim_{K \rightarrow \infty} \mu^k P_{n;k} = \phi(x) \quad (C-6)$$

$$\frac{n}{\mu^k} = x$$

To see this⁽¹²⁾ and to find the rate of decrease of each $P_{n;k}$, we define the functions

$$\hat{\phi}_k(\beta) = \Psi_k(e^{-\frac{\beta}{\mu^k}}) \quad (\beta \geq 0, \quad k=1,2,\dots) \quad (C-7)$$

Eq. (C-3) becomes

$$\hat{\phi}_{k+1}(\beta\mu) = \Psi(\hat{\phi}_k(\beta)). \quad (\text{C-8})$$

Taking the $k \rightarrow \infty$ limit in (C-8) we get⁽¹²⁾

$$\hat{\phi}(\beta) = \lim_{k \rightarrow \infty} \hat{\phi}_k(\beta) \quad (\text{C-9})$$

i.e. the limit exists and $\hat{\phi}(\beta)$ satisfies the functional equation:

$$\hat{\phi}(\mu\beta) = \Psi(\hat{\phi}(\beta)). \quad (\text{C-10})$$

Furthermore, $\hat{\phi}(\beta)$ is the Laplace transform of $\phi(x)$ which appears in Eq. (C-6):

$$\hat{\phi}(\beta) = \int_0^{\infty} e^{-\beta x} \phi(x) dx. \quad (\text{C-11})$$

By construction the K-N-O function $\phi(x)$ is continuous, positive, and normalized to one distribution on $0 \leq x \leq \infty$ with average 1. The solution of Eq. (C-10) is uniquely determined by $\Psi(z)$.

The correction terms to Eq. (C-6) turn out to be powers in k :⁽¹²⁾

$$\mu^k P_{n;k} = \phi\left(\frac{n}{\mu_k}\right) + O\left(\frac{1}{a^k}\right) \quad (\text{C-12})$$

where

$$a = \mu^{\frac{\delta}{3+\delta}}, \quad \delta = -\frac{\ln P_1}{\ln \mu}.$$

The ratio $\frac{P_{n;k}}{P_{1;k}}$ (note $P_{1;k} = P_1^k$) is an increasing function of k and tends to a finite limit: (12)

$$\lim_{k \rightarrow \infty} \frac{P_{n;k}}{P_{1;k}} = \xi_n \quad (n=1,2,\dots). \quad (C-13)$$

This formalism and result can be extended to multi-type branching processes. (12)

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Footnote

† A very qualitative argument for Eq. (2.1) can be given as follows: If P_n is a narrow distribution, then the average mass of the fireballs produced at the first step is Q/μ (μ is defined by Eq. (3.2)), at the second step $\frac{Q}{\mu^2}$ etc...., at the k^{th} step $\frac{Q}{\mu^k}$ and k is determined by requiring that the process stops as soon as $\frac{Q}{\mu^k}$ becomes of the order of some hadronic mass m^2 , then $k = \frac{1}{\log \mu} \log \frac{Q}{m^2}$. (In this case the average multiplicity increases like Q .)

Figure caption

Figure 1. Branching process model for e^+e^- annihilation. The solid lines represent fireballs, the dashed lines particles.

Fig. 1