BROOKHAVEN NATIONAL LABORATORY
Associated Universities, Inc.
Upton, New York

CRISP 73-16

ACCELERATOR DEPARTMENT
Informal Report

EFFECT OF A LARGE OTH HARMONIC OCTUPOLE ON
STABLE BETATRON OSCILLATIONS

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September 20, 1973

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Abstract

The effect of a 0th harmonic octupole (that is, constant in dependence on azimuth) on stable betatron oscillations is treated in smooth approximation. In this approximation, an exact invariant is obtained. Bounded motion results, except when the octupole component is very large. A condition for stability is obtained and, in the case of bounded motion, an expression for the amplitude variation is given. Some comments are made regarding the method of phase averaging.

1. Introduction

We consider the effect of a 0th harmonic octupole on a system of stable betatron oscillations. With the octupole component of the magnetic field an azimuthal constant, the smooth approximation leads to an exact invariant. Using this invariant, we show that the motion is bounded, except when the octupole component is very large. A stability criterion is obtained. In the case of bounded motion, an expression for the amplitude variation is given. Finally, we discuss the method of phase averaging and see how it contrasts with the exact solution.

2. The Invariant in Smooth Approximation

The equation of motion of a particle being focused around an equilibrium orbit and acted upon by an octupole field component can be written

$$x'' + K(s)x = 0(s)x^3, \quad (2.1)$$

where x is the particle position with respect to the equilibrium orbit, s is the

azimuthal distance, measured along the equilibrium orbit, $K(s)$ is the gradient forcing function ($= \frac{\text{gradient}}{\text{momentum}}$), $O(s)$ is the octupole forcing function, given by

$$O(s) = - \frac{1}{6p} \frac{\partial^3 B}{\partial x^3} \Big|_{x=0} . \quad (2.2)$$

Here, the magnetic field is assumed to be expanded in the series

$$B = B_0 + B'_0 x + \frac{1}{2} B''_0 x^2 + \frac{1}{6} B'''_0 x^3 \dots , \quad (2.3)$$

and
$$p = B_0 \rho , \quad (2.4)$$

with ρ the bending radius.

Our first assumption is the stability of the linear betatron system. We thus transform to betatron coordinates¹:

$$\eta = \frac{x}{\sqrt{\beta(s)}} , \quad (2.5)$$

$$\phi = \frac{1}{v} \int_0^s \frac{ds'}{\beta(s')} , \quad (2.6)$$

$$v = \frac{1}{2\pi} \int_0^C \frac{ds'}{\beta(s')} , \quad (2.7)$$

where C is the circumference of the equilibrium orbit. Equation (2.1) is then transformed to

$$\ddot{\eta} + v^2 \eta = v^2 \beta^3 O \eta^3 , \quad (2.8)$$

where differentiation is with respect to the angle variable, ϕ .

The smooth approximation consists of replacing $\beta(s)$ in this equation by some average value,

$$\beta(s) \rightarrow \langle \beta \rangle . \quad (2.9)$$

Now, since $O(s)$ is an azimuthal constant, we have in smooth approximation

$$\ddot{\eta} + v^2 \eta = v^2 \epsilon \eta^3, \quad (2.10)$$

where

$$\epsilon = \langle \beta \rangle^3 O, \text{ a constant.} \quad (2.11)$$

Thus, a first integral is obtained, giving

$$\frac{1}{2} (\dot{\eta}^2 + v^2 \eta^2) - \frac{1}{4} v^2 \epsilon \eta^4 = U, \quad (2.12)$$

a constant of the motion.

The quantity U is an invariant, and (2.12) defines a family of curves in the $[\eta, \dot{\eta}]$ phase plane. Any particle is constrained to lie on one of these curves, depending on its initial conditions.

We introduce phase and amplitude variables:

$$\eta = a \sin \alpha, \quad (2.13)$$

$$\dot{\eta} = a v \cos \alpha. \quad (2.14)$$

Then, in terms of a and α , (2.12) can be written

$$\frac{1}{2} a^2 v^2 - \frac{1}{4} v^2 a^4 \epsilon \sin^4 \alpha = U, \quad (2.15)$$

where,

$$a^2 = \eta^2 + \left(\frac{\dot{\eta}}{v}\right)^2, \quad (2.16)$$

$$\tan \alpha = \frac{v \dot{\eta}}{\eta}. \quad (2.17)$$

Define a_0 by

$$U = \frac{1}{2} v^2 a_0^2, \quad (2.18)$$

which can be taken as the value of U corresponding to the initial conditions:

$\alpha = 0, a = a_0$. Thus, writing

$$q = a^2/a_0^2, \quad (2.19)$$

(2.15) becomes

$$\frac{(q-1)}{q^2} = \frac{a_0^2 t}{2} \sin^4 \alpha . \quad (2.20)$$

Note that the existence of a positive quantity, a_0^2 , depends on the condition

$$\frac{1}{2} a_0^2 t < 1 , \quad (2.21)$$

which we will assume, for simplicity. The function

$$f(q) = \frac{q-1}{q^2} \quad (2.22)$$

is plotted in Fig. 1. From this figure, we see that:

for $t < 0$, bounded motion with

$$\frac{1}{2} a_0^2 t \leq f(q) \leq 0 ; \quad (2.23)$$

for $t > 0$, bounded motion with

$$0 \leq f(q) \leq \frac{1}{2} a_0^2 t , \quad (2.24)$$

$$\text{if } a_0^2 t < \frac{1}{2} ;$$

while for $t > 0$, unbounded motion if

$$a_0^2 t > \frac{1}{2} . \quad (2.25)$$

To find the amplitude variation in the case of bounded motion, we can solve

$$f(\hat{q}) = \frac{1}{2} a_0^2 t , \quad (2.26)$$

where

$$\hat{q} = \hat{a}^2 / a_0^2 , \quad (2.27)$$

and

$$\hat{a} < a_0 \quad \text{if } t < 0 , \quad (2.28a)$$

$$\hat{a} > a_0 \quad \text{if } t > 0 . \quad (2.28b)$$

The stability limit for the octupole strength is very large, in terms of standard values in synchrotrons. With

$$b_3 = \frac{3''}{6\beta_0} ,$$

$$t = - \frac{\langle \beta \rangle^3 b_3}{\rho} , \quad (2.29)$$

and noting that a_0^2 is just the emittance, ϵ , we have for stability,

$$b_3 < \frac{\rho}{\epsilon \langle \beta \rangle^3} . \quad (2.30)$$

Taking some representative values for high energy machines, $\rho = 400$ m, $\langle \beta \rangle = 20$ m, and $\epsilon = 10^{-6}$ rad. m, we have

$$b_3 < 5 \times 10^4 \text{ m}^{-3} .$$

Since, for real machines,

$$b_3 < 1 \text{ m}^{-3} ,$$

the effect of a 0th harmonic octupole is for practical purposes a bounded amplitude modulation.

To first order in t , the amplitude modulation, from (2.20), is

$$1 \leq q \leq \hat{q} , \quad (2.31)$$

where

$$\hat{q} = 1 + \frac{1}{2} a_0^2 t . \quad (2.32)$$

Since

$$\frac{1}{2} a_0^2 t = \frac{1}{2} \frac{\epsilon \langle \beta \rangle^3 b_3}{\rho} = 10^{-5} ,$$

where we have used $b_3 = 1 \text{ m}^{-3}$, we have a very small amplitude modulation in real cases.

3. The Method of Phase Averaging

From (2.10), (2.13) and (2.14) we can obtain equations of motion for a , α :

$$\dot{a} = v t a^3 \sin^3 \alpha \cos \alpha , \quad (3.1)$$

$$\dot{\alpha} = v - v t a^2 \sin^4 \alpha . \quad (3.2)$$

We also rewrite the first integral,

$$a^2 - a_0^2 = \frac{1}{2} t a^4 \sin^4 \alpha . \quad (3.3)$$

If we introduce a new phase variable

$$\delta = \alpha - v\phi , \quad (3.4)$$

then, the equations become

$$\dot{a} = v t a^3 \sin^3 (v\phi + \delta) \cos (v\phi + \delta) , \quad (3.5)$$

$$\dot{\delta} = -v t a^2 \sin^4 (v\phi + \delta) , \quad (3.6)$$

$$a^2 - a_0^2 = \frac{1}{2} t a^4 \sin^4 (v\phi + \delta) . \quad (3.7)$$

These equations depend on the independent variable ϕ . If a and δ are slowly varying over some interval in ϕ , say a few machine revolutions, then we can find new equations representing the motion of these averages. Defining

$$\langle \dot{a} \rangle = \int \dot{a} d\phi , \quad (3.8)$$

$$\langle \dot{\delta} \rangle = \int \dot{\delta} d\phi , \quad (3.9)$$

and assuming a and δ are constant over the averaging interval, we have (replacing a by a_0),

$$\langle \dot{a} \rangle = 0 , \quad (3.10)$$

$$\langle \dot{\delta} \rangle \sim -\frac{3}{4} \pi v a_0^2 , \quad (3.11)$$

where the averaging interval has been chosen to be one revolution or $\Delta\phi = 2\pi$. Thus, in this approximation we have a mean rate of phase change (or mean tune shift), and a mean amplitude. To find the mean amplitude, we average (3.7):

$$\langle a^2 \rangle = a_0^2 + \frac{3}{16} a_0^4 . \quad (3.12)$$

The actual motion consists of oscillations of "tune", i.e. phase, and amplitude about their means. The amplitude oscillations are pictured schematically in Fig. 1.

4. Conclusions

We have shown in smooth approximation that, for practical octupole strengths, the effect of a 0th harmonic octupole field on stable betatron motion is (1) to shift the mean tune and mean amplitude, and (2) to cause fluctuations about these means, with the fluctuations of the same order of magnitude as the average shifts.

References

1. E.D. Courant and H.S. Snyder, Ann. Phys. 3, 1 (1958).

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Fig. 1. Influence of Octupole on Stable Betatron Motion. Betatron Motion is, $q = 1$ (i.e., $t = 0$). The Function $f(q) = (q-1)/q^2$.