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PROBABILITY DISTRIBUTIONS OF

TWO SINUSOIDS IN NOISE

AECU-3232

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ABSTRACT

Probability density functions and cumulative probability functions for the envelope of 2 sinusoidal signals with incommensurable frequencies and Rayleigh noise are given in tabular form and in curve form.

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Case No. 414.0

August 11, 1953

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PROBABILITY DISTRIBUTIONS OF TWO SINUSOIDS IN NOISE

In the pursuit of related studies, computations were performed in this group for the probability distributions of the envelope of Rayleigh noise plus two sinusoidal signals with incommensurable frequencies as given by Rice.¹ It was deemed of sufficient interest to others working in this field to make these calculations available in published form.

Rice develops the theory for the case where:

$$(1) \quad I = P \cos pt + Q \sin qt + I_n$$

where: I = total current

p, q = incommensurable radian frequencies

I_n = noise current.

If R is the envelope of the current as given by (1), the probability density of R is given by:

$$(2) \quad P(R) = R \int_0^{\infty} r J_0(Rr) J_0(Pr) J_0(Qr) e^{-\psi_0 r^2/2} dr$$

where:

ψ_0 = variance of the sine and of the cosine components of I_n .

This form was normalized by the substitution:

$$(3) \quad r = \frac{z}{\sqrt{\psi_0}}$$

$$dr = \frac{dz}{\sqrt{\psi_0}}$$

giving:

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¹S. O. Rice, Mathematical Analysis of Random Noise; Bell System Tech. Jr., July - August 1944, p. 104.

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(4)
$$P(a)da = a \int_0^{\infty} z J_0(az) J_0(bz) J_0(cz) e^{-z^2/2} dz da$$

where:

$$a = \frac{R}{\sqrt{\psi_0}}$$

$$b = \frac{P}{\sqrt{\psi_0}}$$

$$c = \frac{Q}{\sqrt{\psi_0}}$$

are normalized in terms of the standard deviation, $\sqrt{\psi_0}$.

Tables IA through Ik give values of $P(a)$ for all combinations of the parameters $(b,c) = (1/2, 1/2)$ to $(5,5)$ in half unit steps of each parameter. Figures 1 through 15 are the corresponding probability density curves for only unit steps. Half unit steps merely give comparable curves which add nothing of value. The general trend shows quite well in this set. Some of these tables and figures were left incomplete due to two reasons. The computing technique first used carried a values only up to 5 and further computations were not carried out in those cases where the $P(a)$ curves digressed too far from the types found in the experimental curves for which this study was to be a theoretical description.

In figures 16-20 these first 15 curves are grouped on 5 plots according to b values with c values running from 1-5 on each. Duplications will be noticed, and these are due to the fact that the curve for $(b,c) = (i,k)$ is identical to that for $(b,c) = (k,i)$ as is obvious in the form of equation (4). Certain peculiarities of these curves are quite striking in these grouped sets. The fact that the curves for larger values of b and c have two "humps" is seen clearly in the individual plots, but the effect of the progression of c values for each b value is best seen in the grouped figures. For $b = 1$ the progression is the type usually found in families of curves where each succeeding curve rises slower, reaches a lesser maximum and then falls slower than the preceding curve. But for larger values of b the c curves act differently. For values of c less than b , the curves rise more and more rapidly, reach greater maxima, and fall more rapidly. For values of c greater than b the complete opposite takes place.

Cumulative probability curves were computed by numerical integration of the $P(a)$ curves, and are given only in grouped sets and only for $c \geq b$ for each set. The curves for $c < b$ were omitted since they crossed over the other curves twice and made a confusing, jumbled

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picture. As they stand in Figures 21-23, each curve crosses over the others at the 50 per cent point so as to give a complete reversal of the order of occurrence of the curves on each side of that point. Conventional "S" curves occur in the plots with variations in the slope which reflect the two "humps" in the density curves. A Rayleigh distribution curve, $(b,c) = (0,0)$ was added to figure 21 for comparison purposes.

Figures 24-26 show these cumulative distributions plotted on a Rayleigh scale along with a Rayleigh straight line plot on each group. Again, the peculiarities previously described bring about distinct effects in these plots.

An extension of this work to include a third sinusoid is contemplated if experimental curves indicate the need of this. Such a study would involve evaluating a normalized integral, an extension of (4),

$$(5) \quad P(a)da = a \int_0^{\infty} z J_0(az) J_0(bz) J_0(cz) J_0(dz) e^{-z^2/2} dz da$$

which obviously will constitute a prodigious volume of computation. Further extension could be done simply by inserting another Bessel function for each additional sinusoid. However, the increase in work with each new Bessel function would be warranted only in very special cases. The case for three sinusoids is the most involved case anticipated by this group, and the results will be published in a forthcoming report when completed.

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Figure 1

$(b, c) = (1, 1)$

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Figure 2
(b, c) = (1, 2)

7

Figure 3

$(b, c) = (1, 3)$

8

Figure 4

$(b, c) = (1, 4)$

9

Figure 5

$(b, c) = (1, 5)$

10

Figure 6

$(b, c) = (2, 2)$

//

Figure 7
(b, c) = (2, 3)

12

Figure 8

$(b, c) = (2, 4)$

13

Figure 9

$(b, c) = (2, 5)$

14

Figure 10
 $(b, c) = (3, 3)$

15

Figure 11

$(b, c) = (3, 4)$

16

Figure 12

$(b, c) = (3, 5)$

17

Figure 13

$(b, c) = (4, 4)$

Figure 14

$$(b, c) = (4, 5)$$

19

Figure 15
(b, c) = (5, 5)

20

Figure 16

$b = 1$

21

Figure 17

$$b = 2$$

22

Figure 18

$b = 3$

23

Figure 19

$$b = 4$$

24

Figure 20

$$b = 5$$

25

- 25 -

26

Figure 21

27

- 26 -

28

Figure 22

29

- 27 -

30

Figure 23

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Figure 24

Cumulative Plots on Rayleigh Scale
 $(p = 100e^{-x^2})$ plots as straight line
 identical with $(0,0)$

P = Percentage of time the values exceed the ordinate

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Figure 25

Cumulative Plots on Rayleigh Scale
($P = 100e^{-x^2}$ plots as straight line
identical with (0,0))

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Figure 26

Cumulative Plots on Rayleigh Scale
($P = 100e^{-x^2}$ plots as straight line
identical with (0,0))

P = Percentage of time value exceeds the ordinate

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TABLE I:A

$(b, c) = (\frac{1}{2}, \frac{1}{2})$		$(b, c) = (\frac{1}{2}, 1)$		$(b, c) = (\frac{1}{2}, 1\frac{1}{2})$		$(b, c) = (\frac{1}{2}, 2)$		$(b, c) = (\frac{1}{2}, 2\frac{1}{2})$	
a	P(a)	a	P(a)	a	P(a)	a	P(a)	a	P(a)
.125	.0970	.500	.2590	.500	.1565	.500	.0760	.500	.0300
.250	.1900	.750	.3578	1.000	.2990	1.000	.1780	1.000	.0890
.500	.3515	1.000	.4230	1.250	.3538	1.250	.2363	1.500	.1845
.625	.4144	1.125	.4421	1.500	.3885	1.500	.2940	2.000	.2940
.750	.4635	1.250	.4525	1.625	.3981	1.750	.3395	2.250	.3375
.875	.4979	1.375	.4510	1.750	.3990	2.000	.3700	2.500	.3625
1.000	.5160	1.500	.4440	1.875	.3956	2.125	.3783	2.625	.3675
1.125	.5209	1.750	.4043	2.000	.3860	2.250	.3780	2.750	.3658
1.250	.5112	2.000	.3480	2.250	.3488	2.375	.3705	2.875	.3622
1.375	.4895	2.500	.2120	2.500	.2925	2.500	.3600	3.000	.3480
1.500	.4620	3.000	.0921	3.000	.1770	2.750	.3245	3.250	.3172
1.750	.3815	3.500	.0214	3.500	.0700	3.000	.2790	3.500	.2695
2.000	.2960	4.000	.0000	4.000	.0160	3.500	.1610	4.000	.1520
2.500	.1385			4.500	.0000	4.000	.0640	4.500	.0630
3.000	.0417					4.500	.0090	5.000	.0105
3.500	.0000					5.000	.0000	5.500	.0000

$(b, c) = (\frac{1}{2}, 3)$		$(b, c) = (\frac{1}{2}, 3\frac{1}{2})$		$(b, c) = (\frac{1}{2}, 4)$		$(b, c) = (\frac{1}{2}, 4\frac{1}{2})$		$(b, c) = (\frac{1}{2}, 5)$	
a	P(a)	a	P(a)	a	P(a)	a	P(a)	a	P(a)
.500	.0085	.500	.0010	.500	.0000	.500	.0000	.500	.0000
1.000	.0360	1.000	.0094	1.000	.0010	1.000	.0000	1.000	.0000
1.500	.0060	1.500	.0376	1.500	.0104	1.500	.0015	1.500	.0000
2.000	.1940	2.000	.1002	2.000	.0402	2.000	.0140	2.000	.0000
2.500	.2975	2.500	.1990	2.500	.1040	2.500	.0425	2.500	.0120
2.750	.3410	3.000	.3000	3.000	.2019	3.000	.1080	3.000	.0432
3.000	.3630	3.250	.3413	3.500	.3059	3.500	.2062	3.500	.1088
3.125	.3688	3.375	.3544	3.750	.3428	4.000	.3080	4.000	.2072
3.250	.3640	3.500	.3605	3.875	.3557	4.250	.3421	4.500	.3078
3.375	.3578	3.625	.3625	4.000	.3600	4.375	.3544	4.750	.3420
3.500	.3430	3.750	.3649	4.125	.3642	4.500	.3600	4.875	.3530
3.750	.3098	3.875	.3561	4.250	.3608	4.625	.3617	5.000	.3595
4.000	.2588	4.000	.3400	4.375	.3509	4.750	.3567	5.250	.3412
4.500	.1444	4.250	.3030	4.500	.3375	4.875	.3462	5.500	.3135
5.000	.0585	4.500	.2475	5.000	.2485	5.000	.3350	6.000	.2220
5.500	.0110	5.000	.1415	5.500	.1500	5.500	.2490	6.500	.1170
6.000	.0000	5.500	.0580	6.000	.0560	6.000	.1430	7.000	.0350
		6.000	.0090	6.500	.0070	6.500	.0550	7.500	.0000
		6.500	.0000	7.000	.0000	7.000	.0015		
						7.500	.0000		

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TABLE I:B

(b, c) = (1, 1)		(b, c) = (1, 1 $\frac{1}{2}$)		(b, c) = (1, 2)		(b, c) = (1, 2 $\frac{1}{2}$)		(b, c) = (1, 3)	
a	P(a)	a	P(a)	a	P(a)	a	P(a)	a	P(a)
.500	.2090	.500	.1490	.500	.0895	.500	.0435	.500	.0160
1.000	.3560	1.000	.2670	1.000	.1780	1.000	.1040	1.000	.0513
1.250	.3925	1.250	.3088	1.500	.2640	1.500	.1830	1.500	.1114
1.375	.4015	1.375	.3231	1.750	.2958	2.000	.2600	2.000	.1860
1.500	.4030	1.500	.3345	2.000	.3160	2.250	.2880	2.500	.2600
1.625	.3981	1.625	.3445	2.125	.3230	2.500	.3075	2.750	.2860
1.750	.3885	1.750	.3465	2.250	.3240	2.625	.3124	3.000	.3000
2.000	.3540	2.000	.3440	2.375	.3230	2.750	.3163	3.125	.3106
2.500	.2500	2.500	.2950	2.500	.3200	2.875	.3134	3.250	.3107
3.000	.1350	3.000	.2010	3.000	.2730	3.000	.3090	3.375	.3088
3.500	.0490	3.500	.1120	3.500	.1890	3.250	.2925	3.500	.2975
4.000	.0040	4.000	.0440	4.000	.1040	3.500	.2622	3.750	.2835
4.500	.0000	4.500	.0045	4.500	.0392	4.000	.1816	4.000	.2528
		5.000	.0000	5.000	.0035	4.500	.0972	4.500	.1728
				5.500	.0000	5.000	.0360	5.000	.0900
						5.500	.0000	5.500	.0275
								6.000	.0000

(b, c) = (1, 3 $\frac{1}{2}$)		(b, c) = (1, 4)		(b, c) = (1, 4 $\frac{1}{2}$)		(b, c) = (1, 5)	
a	P(a)	a	P(a)	a	P(a)	a	P(a)
.500	.0044	.500	.0000	.500	.0000	.500	.0000
1.000	.0182	1.000	.0048	1.000	.0000	1.000	.0000
1.500	.0544	1.500	.0216	1.500	.0056	1.500	.0000
2.000	.1164	2.000	.0582	2.000	.0238	2.000	.0060
2.500	.1900	2.500	.1188	2.500	.0625	2.500	.0250
3.000	.2634	3.000	.1932	3.000	.1230	3.000	.0630
3.250	.2883	3.500	.2611	3.500	.1995	3.500	.1260
3.500	.3024	3.750	.2833	4.000	.2640	4.000	.2040
3.625	.3060	4.000	.2980	4.250	.2890	4.500	.2655
3.750	.3056	4.125	.2978	4.500	.3015	4.750	.2898
3.875	.3011	4.250	.2992	4.625	.3006	5.000	.3000
4.000	.2968	4.375	.2914	4.750	.2992	5.500	.2915
4.250	.2741	4.500	.2862	4.875	.2974	6.000	.2400
4.500	.2453	4.750	.2660	5.000	.2500	6.500	.1560
5.000	.1600	5.000	.2365	5.500	.2365	7.000	.0770
5.500	.0880	5.500	.1540	6.000	.1560	7.500	.0150
6.000	.0240	6.000	.0840	6.500	.0845	8.000	.0000
6.500	.0000	6.500	.0195	7.000	.0210	8.500	.0000
		7.000	.0000	7.500	.0000		

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TABLE I:C

$(b, c) = (1\frac{1}{2}, 1\frac{1}{2})$		$(b, c) = (1\frac{1}{2}, 2)$		$(b, c) = (1\frac{1}{2}, 2\frac{1}{2})$		$(b, c) = (1\frac{1}{2}, 3)$	
a	P(a)	a	P(a)	a	P(a)	a	P(a)
.500	.1285	.500	.0980	.500	.0605	.500	.0310
1.000	.2220	1.000	.1740	1.000	.1200	1.000	.0730
1.500	.2730	1.250	.2000	1.500	.1725	1.500	.1260
1.750	.2888	1.500	.2235	1.750	.1943	2.000	.1740
1.875	.2906	1.750	.2380	2.000	.2120	2.250	.1958
2.000	.2940	2.000	.2520	2.250	.2250	2.500	.2125
2.125	.2911	2.250	.2610	2.500	.2375	2.750	.2255
2.250	.2925	2.500	.2756	2.750	.2475	3.000	.2340
2.500	.2825	2.750	.2668	3.000	.2490	3.250	.2438
3.000	.2460	2.875	.2645	3.250	.2503	3.500	.2450
3.500	.1715	3.000	.2580	3.500	.2415	3.750	.2438
4.000	.0960	3.250	.2405	3.750	.2250	4.000	.2360
4.500	.0360	3.500	.2170	4.000	.2040	4.250	.2210
5.000	.0050	4.000	.1560	4.500	.1440	4.500	.1980
5.500	.0000	4.500	.0810	5.000	.0800	5.000	.1400
		5.000	.0300	5.500	.0165	5.500	.0715
		5.500	.0000	6.000	.0000	6.000	.0180
						6.500	.0000

$(b, c) = (1\frac{1}{2}, 3\frac{1}{2})$		$(b, c) = (1\frac{1}{2}, 4)$		$(b, c) = (1\frac{1}{2}, 4\frac{1}{2})$		$(b, c) = (1\frac{1}{2}, 5)$	
a	P(a)	a	P(a)	a	P(a)	a	P(a)
.500	.0125	.500	.0035	.500	.0005	.500	.0000
1.000	.0380	1.000	.0150	1.000	.0040	1.000	.0000
1.250	.0562	1.500	.0405	1.500	.0165	1.500	.0045
1.500	.0795	2.000	.0840	2.000	.0440	2.000	.0180
1.750	.1050	2.500	.1350	2.500	.0900	2.500	.0475
2.000	.1320	3.000	.1830	3.000	.1410	3.000	.0900
2.250	.1575	3.250	.2015	3.500	.1855	3.500	.1435
2.500	.1800	3.500	.2135	4.000	.2160	4.000	.1880
2.750	.1980	3.750	.2250	4.500	.2340	4.500	.2160
3.000	.2130	4.000	.2320	5.000	.2350	5.000	.2300
3.250	.2242	4.250	.2380	5.500	.2255	5.500	.2310
3.500	.2345	4.500	.2385	6.000	.1860	6.000	.2160
3.750	.2400	4.750	.2375	6.500	.1300	6.500	.1820
4.000	.2440	5.000	.2300	7.000	.0630	7.000	.1260
4.250	.2380	5.500	.1925	7.500	.0075	7.500	.0525
4.500	.2295	6.000	.1260	8.000	.0000	8.000	.0160
4.750	.2138	6.500	.0650			8.500	.0000
5.000	.1950	7.000	.0140				
5.500	.1320	7.500	.0000				
6.000	.0720						
6.500	.0325						
7.000	.0000						

37 TABLE I:D

$(b, c) = (2, 2)$ $(b, c) = (2, 2\frac{1}{2})$ $(b, c) = (2, 3)$ $(b, c) = (2, 3\frac{1}{2})$

a	P(a)	a	P(a)	a	P(a)	a	P(a)
.500	.0935	.500	.0740	.500	.0485	.500	.0255
1.000	.1580	1.000	.1300	1.000	.0950	1.000	.0600
1.250	.1775	1.500	.1635	1.500	.1335	1.500	.0990
1.500	.1905	1.750	.1715	1.750	.1470	2.000	.1360
2.000	.2060	2.000	.1780	2.000	.1580	2.250	.1485
2.500	.2225	2.500	.1900	2.500	.1700	2.500	.1575
3.000	.2340	3.000	.2040	3.000	.1800	2.750	.1650
3.250	.2372	3.500	.2170	3.500	.1960	3.000	.1780
3.500	.2310	3.625	.2175	3.750	.2025	3.500	.1715
4.000	.2000	3.750	.2175	4.000	.2080	4.000	.1880
4.500	.1395	4.000	.2160	4.500	.2070	4.250	.1955
5.000	.0800	4.500	.1845	5.000	.1750	4.500	.1980
5.500	.0165	5.000	.1300	5.500	.1210	4.750	.1995
6.000	.0000	5.500	.0715	6.000	.0660	5.000	.1950
		6.000	.0180	6.500	.0130	5.500	.1595
		6.500	.0000	7.000	.0000	6.000	.1080
						6.500	.0520
						7.000	.0140
						7.500	.0000

$(b, c) = (2, 4)$ $(b, c) = (2, 4\frac{1}{2})$ $(b, c) = (2, 5)$

a	P(a)	a	P(a)	a	P(a)
.500	.0105	.500	.0030	.500	.0005
1.000	.0300	1.000	.0130	1.000	.0040
1.250	.0450	1.250	.0212	1.500	.0135
1.500	.0630	1.500	.0345	1.750	.0228
2.000	.1040	1.750	.0508	2.000	.0360
2.500	.1375	2.000	.0680	2.250	.0518
2.750	.1512	2.500	.1075	2.500	.0700
3.000	.1560	3.000	.1410	3.000	.1110
3.250	.1625	3.500	.1575	3.500	.1435
3.500	.1680	4.000	.1680	3.750	.1538
4.000	.1760	4.500	.1755	4.000	.1600
4.250	.1785	4.750	.1805	4.250	.1658
4.500	.1845	5.000	.1850	4.500	.1665
5.000	.1950	5.500	.1815	5.000	.1700
5.500	.1870	6.000	.1740	5.500	.1705
6.000	.1560	6.500	.1495	6.000	.1680
6.500	.0975	7.000	.0980	6.500	.1625
7.000	.0490	7.500	.0450	7.000	.1470
7.500	.0075	8.000	.0080	7.500	.0975
8.000	.0000	8.500	.0000	8.000	.0480
				8.500	.0085
				9.000	.0000

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TABLE I:E

$(b, c) = (2\frac{1}{2}, 2\frac{1}{2})$		$(b, c) = (2\frac{1}{2}, 3)$		$(b, c) = (2\frac{1}{2}, 3\frac{1}{2})$	
a	P(a)	a	P(a)	a	P(a)
.500	.0740	.500	.0505	.500	.0400
1.000	.1240	1.000	.1060	1.000	.0780
1.250	.1375	1.250	.1200	1.500	.1080
1.500	.1470	1.500	.1290	1.750	.1190
1.750	.1505	1.750	.1348	2.000	.1260
2.000	.1520	2.000	.1360	2.250	.1305
2.250	.1530	2.250	.1372	2.500	.1300
2.500	.1550	2.500	.1375	2.750	.1320
2.750	.1595	2.750	.1403	3.000	.1320
3.000	.1650	3.000	.1440	3.250	.1332
3.250	.1722	3.250	.1495	3.500	.1365
3.500	.1820	3.500	.1540	3.750	.1425
3.750	.1912	4.000	.1720	4.000	.1480
4.000	.2000	4.250	.1828	4.250	.1572
4.250	.1998	4.500	.1890	4.500	.1665
4.500	.1980	4.750	.1900	4.750	.1758
4.750	.1900	5.000	.1900	5.000	.1800
5.000	.1750	5.500	.1595	5.500	.1705
5.500	.1210	6.000	.1020	6.000	.1500
6.000	.0540	6.500	.0520	6.500	.1040
6.500	.0130	7.000	.0140	7.000	.0490
7.000	.0000	7.500	.0000	7.500	.0150
				8.000	.0000

$(b, c) = (2\frac{1}{2}, 4)$		$(b, c) = (2\frac{1}{2}, 4\frac{1}{2})$		$(b, c) = (2\frac{1}{2}, 5)$	
a	P(a)	a	P(a)	a	P(a)
.500	.0210	.500	.0090	.500	.0025
.750	.0345	1.000	.0260	1.000	.0100
1.000	.0490	1.500	.0540	1.500	.0285
1.500	.0825	2.000	.0860	2.000	.0560
1.750	.0980	2.500	.1125	2.500	.0900
2.000	.1100	3.000	.1260	3.000	.1170
2.250	.1192	3.250	.1300	3.250	.1235
2.500	.1250	3.500	.1295	3.500	.1295
2.750	.1292	3.750	.1275	3.750	.1313
3.000	.1290	4.000	.1280	4.000	.1280
3.500	.1295	4.250	.1317	4.500	.1305
4.000	.1320	4.500	.1350	5.000	.1300
4.250	.1402	4.750	.1378	5.500	.1375
4.500	.1440	5.000	.1400	6.000	.1500
4.750	.1520	5.500	.1540	6.500	.1625
5.000	.1600	6.000	.1680	7.000	.1680
5.500	.1705	6.500	.1690	7.500	.1500
6.000	.1740	7.000	.1470	8.000	.1120
6.500	.1560	7.500	.1050	8.500	.0425
7.000	.1050	8.000	.0480	9.000	.0000
7.500	.0525	8.500	.0085		
8.000	.0160	9.000	.0000		
8.500	.0000				

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TABLE I:F

(b, c) = (3, 3)		(b, c) = (3, 3½)		(b, c) = (3, 4)		(b, c) = (3, 4½)		(b, c) = (3, 5)	
a	P(a)	a	P(a)	a	P(a)	a	P(a)	a	P(a)
.500	.0610	.500	.0510	.500	.0340	.500	.0185	.500	.0075
1.000	.1020	1.000	.0880	1.000	.0660	1.000	.0430	1.000	.0220
1.250	.1125	1.500	.1065	1.500	.0915	1.500	.0705	1.500	.0450
1.500	.1185	2.000	.1120	2.000	.1060	2.000	.0940	2.000	.0740
1.750	.1208	2.500	.1100	2.500	.1075	2.250	.1012	2.500	.0975
2.000	.1200	3.000	.1080	3.000	.1080	2.500	.1050	3.000	.1080
2.500	.1175	3.500	.1120	3.500	.1050	3.000	.1080	3.500	.1085
3.000	.1200	4.000	.1200	4.000	.1080	3.500	.1050	4.000	.1040
3.250	.1202	4.500	.1350	4.500	.1170	4.000	.1040	4.500	.1035
3.500	.1260	5.000	.1500	5.000	.1300	4.500	.1080	5.000	.1050
4.000	.1400	5.500	.1650	5.500	.1430	5.000	.1150	5.500	.1155
4.500	.1575	6.000	.1680	6.000	.1560	5.500	.1265	6.000	.1260
5.000	.1750	6.500	.1495	6.500	.1690	6.000	.1380	6.500	.1365
5.500	.1760	7.000	.1050	7.000	.1470	6.500	.1560	7.000	.1470
6.000	.1560	7.500	.0450	7.500	.0975	7.000	.1610	7.500	.1575
6.500	.1105	8.000	.0080	8.000	.0400	7.500	.1425	8.000	.1360
7.000	.0490	8.500	.0000	8.500	.0085	8.000	.0960	8.500	.1020
7.500	.0075			9.000	.0000	8.500	.0425	9.000	.0450
8.000	.0000					9.000	.0000	9.500	.0000

TABLE I:G

(b, c) = (3½, 3½)		(b, c) = (3½, 4)		(b, c) = (3½, 4½)		(b, c) = (3½, 5)	
a	P(a)	a	P(a)	a	P(a)	a	P(a)
.500	.0520	.500	.0440	.500	.0300	.500	.0160
1.000	.0870	1.000	.0760	1.000	.0580	1.000	.0370
1.500	.1005	1.500	.0915	1.500	.0795	1.500	.0615
2.000	.1020	2.000	.0960	2.000	.0900	2.000	.0820
2.500	.0975	2.500	.0925	2.500	.0925	2.500	.0925
3.000	.0960	3.000	.0900	3.000	.0900	3.000	.0930
3.500	.0980	3.500	.0910	3.500	.0875	3.500	.0875
4.000	.1040	4.000	.0920	4.000	.0880	4.000	.0840
4.500	.1125	4.500	.0990	4.500	.0900	4.500	.0855
5.000	.1250	5.000	.1050	5.000	.0950	5.000	.0850
5.500	.1375	5.500	.1210	5.500			
6.000	.1500	6.000	.1380				
6.500	.1625	6.500	.1495				
7.000	.1470	7.000	.1610				
7.500	.1050	7.500	.1350				
8.000	.0640	8.000	.0880				
8.500	.0170	8.500	.0425				
9.000	.0000	9.000	.0180	9.500	.0190		
		9.500	.0000	10.000	.0000		
						10.000	.0100
						10.500	.0000

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TABLE I:H

$(b, c) = (4, 4)$ $(b, c) = (4, 4\frac{1}{2})$ $(b, c) = (4, 5)$

a	P(a)	a	P(a)	a	P(a)
.500	.0455	.500	.0385	.500	.0260
1.000	.0750	1.000	.0670	1.000	.0510
1.500	.0870	1.500	.0810	1.500	.0705
1.750	.0875	2.000	.0840	2.000	.0800
2.000	.0880	2.500	.0800	2.250	.0810
2.500	.0825	3.000	.0780	2.500	.0800
3.000	.0810	3.500	.0770	3.000	.0780
3.500	.0805	4.000	.0760	3.500	.0735
4.000	.0800	4.500	.0765	4.000	.0720
4.500	.0855	5.000	.0800	4.500	.0720
5.000	.0900			5.000	.0750

TABLE I:J

$(b, c) = (4\frac{1}{2}, 4\frac{1}{2})$ $(b, c) = (4\frac{1}{2}, 5)$

a	P(a)	a	P(a)
.500	.0400	.500	.0340
1.000	.0670	1.000	.0590
1.500	.0765	1.500	.0720
2.000	.0760	2.000	.0740
2.500	.0725	2.500	.0700
3.000	.0690	3.000	.0690
3.500	.0700	3.500	.0665
4.000	.0680	4.000	.0640
4.500	.0720	4.500	.0675
5.000	.0750	5.000	.0650

TABLE I:K

$(b, c) = (5, 5)$

a	P(a)
.500	.0360
1.000	.0600
1.500	.0690
2.000	.0680
2.500	.0650
3.000	.0630
3.500	.0595
4.000	.0600

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TABLE II:A

(b, c) = (1, 1)		(b, c) = (1, 2)		(b, c) = (1, 3)		(b, c) = (1, 4)		(b, c) = (1, 5)	
a	F(a)	a	F(a)	a	F(a)	a	F(a)	a	F(a)
.5	.059	.5	.023	.5	.003	.5	.000	.5	.000
1.0	.221	1.0	.097	1.0	.022	1.0	.001	1.0	.000
1.5	.441	1.5	.221	1.5	.067	1.5	.008	1.5	.000
2.0	.659	2.0	.386	2.0	.150	2.0	.031	2.0	.001
2.5	.829	2.5	.567	2.5	.276	2.5	.080	2.5	.009
3.0	.936	3.0	.734	3.0	.436	3.0	.169	3.0	.033
3.5	.986	3.5	.866	3.5	.610	3.5	.302	3.5	.085
4.0	.999	4.0	.948	4.0	.769	4.0	.464	4.0	.177
4.1	1.000	4.5	.987	4.5	.891	4.5	.634	4.5	.311
		5.0	.999	5.0	.964	5.0	.786	5.0	.473
		5.2	1.000	5.5	.995	5.5	.900	5.5	.642
				6.0	1.000	6.0	.968	6.0	.794
						6.5	.997	6.5	.907
						7.0	1.000	7.0	.974
								7.5	.998
								7.8	1.000

TABLE II:B

(b, c) = (2, 2)		(b, c) = (2, 3)		(b, c) = (2, 4)		(b, c) = (2, 5)	
a	F(a)	a	F(a)	a	F(a)	a	F(a)
.5	.028	.5	.013	.5	.003	.5	.000
1.0	.100	1.0	.053	1.0	.014	1.0	.001
1.5	.198	1.5	.117	1.5	.040	1.5	.006
2.0	.310	2.0	.199	2.0	.089	2.0	.020
2.5	.429	2.5	.292	2.5	.159	2.5	.052
3.0	.559	3.0	.389	3.0	.245	3.0	.106
3.5	.691	3.5	.494	3.5	.340	3.5	.182
4.0	.812	4.0	.609	4.0	.439	4.0	.273
4.5	.909	4.5	.728	4.5	.543	4.5	.372
5.0	.970	5.0	.837	5.0	.653	5.0	.473
5.5	.998	5.5	.922	5.5	.764	5.5	.573
5.8	1.000	6.0	.976	6.0	.865	6.0	.675
		6.5	.998	6.5	.940	6.5	.773
		6.8	1.000	7.0	.983	7.0	.866
				7.5	.999	7.5	.939
				7.8	1.000	8.0	.983
						8.5	.999
						8.7	1.000

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TABLE II:C

$(b, c) = (3, 3)$ $(b, c) = (3, 4)$ $(b, c) = (3, 5)$

a	F(a)	a	F(a)	a	F(a)
.5	.018	.5	.010	.5	.002
1.0	.066	1.0	.039	1.0	.010
1.5	.131	1.5	.084	1.5	.029
2.0	.200	2.0	.142	2.0	.063
2.5	.268	2.5	.204	2.5	.114
3.0	.336	3.0	.266	3.0	.173
3.5	.406	3.5	.328	3.5	.236
4.0	.482	4.0	.388	4.0	.297
4.5	.567	4.5	.453	4.5	.355
5.0	.663	5.0	.524	5.0	.415
5.5	.766	5.5	.602	5.5	.478
6.0	.862	6.0	.689	6.0	.547
6.5	.940	6.5	.783	6.5	.622
7.0	.985	7.0	.876	7.0	.703
7.5	.999	7.5	.948	7.5	.791
7.9	1.000	8.0	.986	8.0	.878
		8.5	.999	8.5	.946
		8.8	1.000	9.0	.989
				9.5	1.000

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