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EFFECTS OF SYNCHROTRON RADIATION ON THE COMPRESSION OF A
RELATIVISTIC ELECTRON LAYER

David A. Nowak

January, 1975

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EFFECTS OF SYNCHROTRON RADIATION ON THE COMPRESSION OF A RELATIVISTIC ELECTRON LAYER

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INTRODUCTION

In this paper we discuss the effect of a changing magnetic field on the equilibrium properties of an infinite electron layer. It is shown that the change in the total energy of the e-layer induced by a changing magnetic field is proportional to the change in the field, if the time scale for the change in the magnetic field is small compared to that for energy loss due to synchrotron radiation. Thus, in principle, it is possible to increase the total energy of an e-layer system by increasing the background magnetic field. The resulting system can be used as an intense source of relativistic electrons of total energy many times the initial e-layer energy. It also provides an intense synchrotron radiation source in the keV range.¹ The change in the field can be induced by compressing a metallic liner enclosing the e-layer or, through the use of a capacitor bank or similar device to increase the background magnetic field. The former has the advantage of confining the magnetic field to the neighborhood of the electron layer.

THEORY

Here we consider an infinitely long electron layer with thickness small in comparison with the radius, and consisting of n electrons per unit length with electron energy γmc^2 . Further, we assume the layer

to be charge neutralized but not current neutralized. Following G. Schmidt,² we write the energy of the electrons composing the layer as

$$\gamma^2 m^2 c^4 = m^2 c^4 + p_z^2 c^2 - \frac{2qB_0 p_\theta c \gamma^2}{(\gamma + v)^2} + p_r^2 c^2 \quad (1)$$

where q is the charge of the electron ($q = -e$), m is the mass, p_z and p_r are the momentum in the axial and radial directions respectively, p_θ is the canonical angular momentum, B_0 is the confining magnetic field and v is equal to $n(e^2/mc^2)$. p_z and p_θ are constants of the motion. p_r and p_z are assumed small in comparison with γmc^2 . The radius, R , of the layer is given by

$$R = - \frac{(\gamma + v) mcV_\theta}{qB_0} = \left(\frac{2p_\theta c}{eB_0} \right)^{1/2} \quad (2)$$

where V_θ is the azimuthal velocity, and the field inside the e-layer, B_{in} is given by

$$B_{in} = B_0 \frac{(\gamma - v)}{\gamma + v} \quad (3)$$

The field reversal parameter ζ is normally defined as

$$\zeta = \frac{B_0 - B_{in}}{B_v} \quad (4)$$

where B_v is the magnetic field present in the absence of the e-layer. In order to relate B_v to B_0 and B_{in} , it is necessary to know the volume of the system. For the purposes here, we will consider the e-layer to be enclosed in a perfectly conducting cylinder of radius R_w .

The energy per unit length of the electron layer consists of two parts; the kinetic energy per unit length of the layer itself, $E_L - nmc^2$, and the

magnetic energy per unit length stored in self-fields E_F . The total energy per unit length of the system E_T can be expressed as

$$E_T = E_L + E_F + E_V \quad (5)$$

where E_V is the energy per unit length of the vacuum magnetic field.

$$\begin{aligned} E_T &= n\gamma mc^2 + (R_W^2 - R^2) \frac{B_0^2}{8} + R^2 \frac{B_0^2}{8} \left(\frac{\gamma - v}{\gamma + v} \right)^2 \\ &= n\gamma mc^2 + \frac{R_W^2 B_0^2}{8} + \frac{R^2 B_0^2}{8} \left[\frac{-4\gamma v}{(\gamma + v)^2} \right] \end{aligned} \quad (6)$$

The confining field B_0 can be calculated by considering the flux contained within the cylindrical system.

$$R_W^2 B_V = (R_W^2 - R^2) B_0 + R^2 B_{in} \quad (7)$$

Substituting for R and B_{in} from Eqs. (2) and (3) respectively, we have

$$B_0 = \frac{B_V}{2} + \frac{B_V}{2} \left[1 + \frac{m^2 c^4}{e^2} \frac{8v(\gamma + v)(\gamma^2 - 1)}{\gamma^2 R_W^2 B_V^2} \right]^{1/2} \quad (8)$$

If we set $R_W^2 = 2R_0^2$ where R_0 is the radius of gyration of an electron of energy γmc^2 in a magnetic field B_V^* , the total energy of the layer is then given by

$$\begin{aligned} E_L + E_F &= n\gamma mc^2 - \frac{\gamma v(\gamma^2 - 1)}{2\gamma^2} \frac{m^2 c^4}{e^2} \\ &\quad + \frac{m^2 c^4}{e^2} (\gamma^2 - 1) \left\{ \left(\frac{1}{16} \right) \left[1 + \left(1 + \frac{4v(\gamma + v)}{\gamma^2} \right)^{1/2} \right]^2 - \frac{1}{4} \right\} \\ &= n\gamma mc^2 \left[1 + \frac{v(\gamma^2 - 1)}{4\gamma^3} \right] \\ &= n\gamma_0 mc^2 \end{aligned} \quad (9)$$

*This is the special case of constant flux.

Thus, in principle, it is possible to convert a fraction of the kinetic energy of the layer to magnetic energy.

The energy of the e-layer system can be increased by compressing the system. For a single electron the change in energy is given by

$$\frac{d(\gamma mc^2)}{dt} = \frac{\gamma mc^2}{2B_v} \frac{dB_v}{dt} \quad (10)$$

This equation is also valid for a current neutralized e-layer in a changing magnetic field. However, in general we have, from differentiating Eq. (1)

$$\frac{d\gamma mc^2}{dt} = \frac{(\gamma^2 - 1)}{\gamma^3 + v} \frac{\gamma(\gamma + v)}{2B_0} \frac{mc^2}{dt} \frac{dB_0}{dt} \quad (11)$$

The above expression considers only the change in kinetic energy of the electrons due to a change in the confining field. The change in the energy of the electrons plus the energy in the self-fields is physically the important quantity. This can be related to the change in R_w by differentiating Eq. (6)

$$\begin{aligned} \frac{dE_L}{dR_w} &= \frac{dE_T}{dR_w} - \frac{d}{dR_w} \frac{R_w^2 B_v^2}{8} \\ &= \frac{d}{dR_w} \frac{m^2 c^4}{e^2} \gamma v \frac{(\gamma^2 + 1)}{2\gamma^2} + 2R_w \frac{(B_0^2 - B_v^2)}{8} \\ &\quad + \frac{R_w^2}{8} \frac{dB_0}{dR_w} - 2B_v \frac{dB_v}{dR_w} \\ &= - \frac{B_0^2 R_w}{4} - \frac{d}{dR_w} \frac{R_w^2 B_v^2}{8} \end{aligned} \quad (12)$$

where $-\frac{B_0^2 R_w}{4} dR_w$ is the work done by moving the outer boundary a distance dR_w . The second term on the R.H.S. of Eq. (12) can be evaluated by invoking flux conservation to yield

$$\frac{d}{dR_w} \frac{(R_w^2 B_v^2)}{8} = - \frac{R_w B_v^2}{4} \quad (13)$$

and the change in E_L with respect to time is obtained by multiplying Eq. (13) by dR_w/dt

$$\frac{dE_L}{dt} = - \left(\frac{B_0^2 - B_v^2}{4} \right) R_w \frac{dR_w}{dt} \quad (14)$$

The above equations are valid for both non-relativistic and relativistic electrons and thus can be used for ion layers as well as electron layers with the appropriate substitutions for q and m . However, if the electron layer is relativistic, it is necessary to consider also the energy loss due to synchrotron radiation. The radiated power, P , for a single electron in a magnetic field is given by³

$$P = \frac{-2}{3} \frac{e^2 c}{a^2} \beta^4 \gamma^4 = \frac{-2}{3} \frac{e^4 B^2}{m^2 c^3} (\gamma^2 - 1)$$

or

$$\frac{P}{mc^2} = - \frac{2}{3} \frac{e^4 B^2}{m^3 c^5} (\gamma^2 - 1) = -1.937 \times 10^{-9} \text{ sec}^{-1} \text{ gauss}^{-2} B^2 \gamma^2 \quad (15)$$

where a is the local radius of curvature and B is the local magnetic field.

It is important to note that the radius, a , which enters into Eq. (15)

is not the radius of the layer but the instantaneous radius of curvature,

also γ^2 is assumed to be much greater than one. Thus, Eq. (15) must be

averaged over the distribution of $(1/a^2)$. Assuming a linear change in B across the layer, the average power radiated by a single electron is given

by (see Appendix B)

$$\langle P \rangle = - \frac{2}{3} \frac{e^4 B_0^2}{m^2 c^3} (\gamma^2 - 1) \frac{3\gamma^2 + v^2}{3(\gamma + v)^2} \quad (16)$$

Thus combining Eqs. (14) and (16) we have

$$\begin{aligned} \frac{dE_L}{dt} &= \left. \frac{dE_L}{dt} \right|_1 + \left. \frac{dE_L}{dt} \right|_2 \\ &= - (B_0^2 - B_v^2) \frac{R_w}{4} \frac{dR_w}{dt} \\ &\quad - \frac{2}{3} \frac{ne^4 B_0^2}{m^2 c^3} (\gamma^2 - 1) \frac{3\gamma^2 + v^2}{3(v + \gamma)^2} \end{aligned} \quad (17)$$

The first term on the R.H.S. of Eq. (17) represents the energy change due to compression of the system and is related to the changes in γ , B_0 and

R by

$$\begin{aligned} \left. \frac{dE_L}{dt} \right|_1 &= \frac{vm^2 c^4}{2e^2} \left(1 - \frac{1}{\gamma^2} \right) \frac{d\gamma}{dt} \Big|_1 + \frac{R_w}{4} B_0^2 \frac{dR_w}{dt} \\ &\quad + \frac{R_w^2}{4} B_0 \frac{dB_0}{dt} \Big|_1 - \frac{B_v^2 R_w}{4} \frac{dR_w}{dt} \end{aligned} \quad (18)$$

while the second term represents the change in energy due to radiation losses and is related to changes in γ and B_0 by

$$\left. \frac{dE_L}{dt} \right|_2 = \frac{vm^2 c^4}{e^2} \left(1 - \frac{1}{\gamma^2} \right) \frac{d\gamma}{dt} \Big|_2 + \frac{R_w^2}{4} B_0 \frac{dB_0}{dt} \Big|_2 \quad (19)$$

The change in γ is related to B_0 by Eq. (11), thus the change in γ or B_0 individually can be found by the appropriate substitution in Eqs. (18) and (19).

If the degradation of the e-layer energy due to radiation losses is neglected, the e-layer behaves like a gas with $\gamma_G = 3/2$ in the limit $v \ll \gamma$ and $\gamma_G = 2$ in the limit $v \gg \gamma$, where γ_G is the ratio of specific heats at constant pressure and constant volume.

For dE_L/dt to be greater than zero, it is necessary, from Eq. (17)

that

$$\begin{aligned} \frac{dR_w}{dt} &> \left(\frac{B_0^2}{B_0^2 - B_v^2} \right) \left(\frac{8}{3} \right) \left(\frac{3\gamma^2 + v^2}{3(\gamma + v)^2} \right) (\gamma^2 - 1) v \frac{r_e}{R_w} c \\ &> \left(\frac{R_w^2 (\gamma + v)^2}{R_w^2 (\gamma + v)^2 - R^2 v^2} \right) \left(\frac{2}{3} \right) \left(\frac{3\gamma^2 + v^2}{3(\gamma + v)^2} \right) (\gamma + v) (\gamma^2 - 1) \\ &\quad \times \left(\frac{r_e}{R_w} \right) c \left(\frac{R_w}{R} \right)^2 \end{aligned} \quad (20)$$

where r_e is the classical electron radius. We can evaluate the above inequality in the limits $\gamma \gg v$ and $\gamma = v$. For $\gamma \gg v$ "weak layer limit" we have

$$\begin{aligned} \left(\frac{dR_w}{dt} \right)_{v=0} &> \frac{2}{3} \gamma (\gamma^2 - 1) \left(\frac{r_e}{R} \right) c \left(\frac{R_w}{R} \right) \\ &> 5.64 \times 10^{-3} \text{ (cm/sec)} \frac{\gamma (\gamma^2 - 1) R_w}{R^2} \end{aligned} \quad (21)$$

This can be related to the critical frequency of the synchrotron radiation spectrum, W_c , by

$$\left(\frac{dR_w}{dt} \right)_{v=0} > 0.95 \times 10^2 \left(\frac{R_w}{R} \right) W_c \text{ (eV)} \quad (22)$$

If we assume R_w to be of the order of R and dR_w/dt to be of order 10^6 cm/sec, we then find W_c to be of order 10^4 eV. In the limit $\gamma = v$ we have

$$\left(\frac{dR_w}{dt} \right)_{\gamma=v} > \left(\frac{4R_w^2}{4R_w^2 - R^2} \right) \left(\frac{4}{9} \right) \gamma (\gamma^2 - 1) \frac{r_e}{R} c \frac{R_w}{R} \quad (23)$$

$$> \left(\frac{4R_w^2}{4R_w^2 - R^2} \right) 0.63 \times 10^2 \left(\frac{R_w}{R} \right) W_c \text{ (eV)} \quad (24)$$

which yields approximately the same upper bound on W_c as in the limit $\gamma \gg v$. For the strong layer regime, $\gamma = v$, we have assumed the radius of the electron layer and γ to be the same as in the "weak layer limit" $\gamma \gg v$. The effect of the large v is to increase the power emitted per electron in the layer and also to shift the radiation spectrum toward higher frequencies. The radiation spectrum for an electron layer of $\gamma = 0$ and $\gamma = v$ is plotted in Fig. 1. Also plotted is the black-body spectrum of the same total power as the $\gamma = 0$ spectrum and with the peak intensity occurring at the same frequency.

Equation (17) cannot be evaluated without a knowledge of the time dependence of the magnetic field. If we assume the cylindrical wall to be moving inward with a constant velocity, v , it is possible to obtain a simple expression for γ as a function of time. Defining T_0 as $T_0 = r_0/v$ where r_0 is the initial radius of the system boundary, we have

$$\gamma = \frac{\gamma_0 \lambda}{[1 + \alpha(\lambda^4 - 1)]} \quad (25)$$

where $\alpha = \left(\frac{2}{3}\right) \frac{r_e^2 B_0^2 T_0 \gamma_0}{mc^2 4}$ and $\lambda = (1 - t/T_0)^{-1} = r_0/r$. In this idealized situation the critical frequency W_c , in the "weak layer limit" is given by

$$W_c = \frac{W_{c0} \lambda^4}{[1 + \alpha(\lambda^4 - 1)]^2} \quad (26)$$

In Fig. 2, the critical frequency, W_c , is plotted as a function of both time, t/T_0 , and the compression, λ , for various values of α . Evaluating the expression for α we find

$$\alpha = 4.85 \times 10^{-10} B_0^2 T_0 \gamma_0 \text{ sec}^{-1} \text{ gauss}^{-2} \quad (27)$$

Thus it is quite feasible for α to be much less than 1. If a many-fold increase in the energy of the electron-layer is to be achieved, it is necessary that α be much less than unity.

STABILITY

The results of the prior section are valid as long as the momentum in the radial direction is small in relation to γ_{mc} . The thickness of the layer is related to the maximum radial momentum (see Appendix A) by

$$\frac{p_r^2}{2\gamma m} = 1/2 \gamma m \omega^2 d^2 \quad (28)$$

where $\omega^2 = (c/R)^2 [1 + v/\gamma (1 + R/d)]$, and $2d$ is the thickness of the layer, R being the radius. Then the restriction on the thickness of the layer becomes

$$\omega^2 d^2 < c^2$$

or for $d \ll R$ $v \geq \gamma$ we have

$$(d/R) (v/\gamma) < 1 \quad (29)$$

ω in Eq. (28) is the frequency of radial oscillations about the mean radius R , which is induced by the self-fields of the layer. When the quantity $(d/R)(v/\gamma)$ exceeds one, the electrons can no longer propagate in the azimuthal direction.

Thus there is no upper limit on the current of an infinitely long e-layer, as long as the ratio of c/R is less than γ/v . However, when the end effects are taken into account, there exists an upper limit to the current per unit length that can be carried by electron layer of finite

length. (See Appendix A.) This limit can be expressed as

$$v/\gamma < 2 \quad (30)$$

If we allow the plasma to be partially current neutralized, Eqs. ((1)-(4)) still hold if we replace v by $v_{\text{net}} = v(1 - f)$ where f is the fraction of current neutralization. The use of current neutralization allows for e-layer configurations without the presence of large self-fields. However, the plasma supporting the return current necessary for current neutralization will drift outward due to the $\underline{J}_p \times \underline{B}$ force acting on the plasma. For an equilibrium configuration it is necessary to counteract this force by inducing a pressure gradient in the background plasma across the e-layer. For a fully current neutralized relativistic e-layer, the change in pressure across the e-layer, ΔP , is given by

$$\Delta P = \frac{J_p B_v 2d}{2\pi R} = \frac{necB_v}{2v} = \frac{B_v^2}{2v} \frac{v}{\gamma} \quad (31)$$

where $2d$ is the thickness of the e-layer, and $\frac{necB_v}{2\pi R}$ is the current per unit length in the e-layer. Thus for $v \geq \gamma$ the change in plasma pressure across the e-layer is of the order of the pressure of the vacuum magnetic field. This situation raises serious questions of the stability of such a current neutralized e-layer. One can compare this with the magnetic pressure gradient across a non-neutralized e-layer of the same radius and γ as the fully current neutralized e-layer. The change in pressure ΔP given by

$$\Delta P = \frac{B_0^2}{8\pi} - \frac{B_{in}^2}{8\pi}$$

using Eq. (3) we have

$$\Delta P = \frac{B^2}{2\pi} \frac{v}{\gamma}$$

where B is the field that exists in the fully current neutralized e-layer.

DISCUSSION

In the previous sections we have discussed the compression of a relativistic electron layer and the effect of the synchrotron radiation on the change of energy of the layer induced by the changing magnetic field. In theory, it is possible to increase the total energy many-fold. The limit on the energy increase was shown to be a function of the time for the compression and the rate of the synchrotron radiation losses.

The resulting system can be considered as an intense source of relativistic electrons of much higher energy than presently available from pulsed relativistic electron beam machines. Alternatively, the compressed system would also be an intense source of synchrotron radiation, capable of converting large fractions of the energy of the electron layer into keV photons. It should be noted that the radiation emitted in such a system would be in the plane perpendicular to the axis of the electron layer and thus some care in system design would be required to extract the radiation from the system.

The total energy density capable of being stored in such a system was found to be limited by the effect of the self-fields on the propagation of the current in the azimuthal direction. This is the same phenomenon as the Lawson limit for the maximum current capable of being propagated in a relativistic electron beam machine.⁴

APPENDIX A

Following G. Schmidt,² we write the Hamiltonian as

$$H = \left[m^2 c^4 + p_z^2 c^2 + \left(\frac{p_\theta}{r} - \frac{q}{c} A \right)^2 c^2 + p_r^2 c^2 \right]^{1/2} \quad (A1)$$

for $p_r^2 c^2 \ll \gamma^2 m^2 c^4$ we have

$$H \approx \psi + \frac{p_r^2}{2\gamma m} = \gamma m c^2 \quad (A2)$$

where ψ is given by

$$\psi = \left[m^2 c^4 + p_z^2 c^2 + \left(p_\theta / r - \frac{q}{c} A \right)^2 c^2 \right]^{1/2}.$$

The problem can thus be treated as a simple harmonic oscillator in a potential well ψ . Assuming a uniform current distribution across an e-layer with inner radius a and outer radius b , we can write the magnetic potential A as

$$A = \frac{1}{2} B_0 r + \frac{qnV_\theta}{c} \frac{\ln(b/a)}{b-a} r \quad \text{for } r \leq a \quad (A3a)$$

$$A = \frac{1}{2} B_0 r + \frac{qnV_\theta}{c} \left[r \frac{\ln(b/r)}{b-a} + \frac{1}{2(b-a)} \frac{r^2 - a^2}{r} \right] \text{ for } a \leq r \leq b \quad (A3b)$$

$$A = \frac{1}{2} B_0 r + \frac{qnV_\theta}{c} \frac{b+a}{2r} \quad \text{for } r > b \quad (A3c)$$

when $P_r = 0$, $a = b$ and the potential ψ has a minimum at R given by Eq. (2).

Thus we expand ψ about R , and remembering that $\left. \frac{\partial \psi}{\partial r} \right|_R = 0$, we have

$$H = H_0 + H_1 = \psi(R) + \frac{1}{2} \frac{\partial^2 \psi}{\partial r^2} x^2 + \frac{p_r^2}{2\gamma m}$$

or

$$H_1 = \frac{p_r^2}{2\gamma m} + \frac{1}{2} \frac{\partial^2 \psi}{\partial r^2} x^2 \quad (A4)$$

where x is the displacement for the equilibrium radius R . Taking derivatives of ψ we have

$$\frac{\partial \psi}{\partial r} = \frac{1}{\gamma m c^2} \left(\frac{P_\theta}{r} - \frac{q}{c} A \right) \left(\frac{-P_\theta}{r^2} - \frac{q}{c} \frac{\partial A}{\partial r} \right) \quad (A5)$$

$$\frac{\partial^2 \psi}{\partial r^2} = \frac{1}{\gamma m c^2} \left[\left(\frac{-P_\theta}{r^2} - \frac{q}{c} \frac{\partial A}{\partial r} \right)^2 + \left(\frac{P_\theta}{r} - \frac{q}{c} A \right) \left(\frac{2P_\theta}{r^3} - \frac{q}{c} \frac{\partial^2 A}{\partial r^2} \right) \right] \quad (A6)$$

Now since $\left. \frac{\partial \psi}{\partial r} \right|_R = 0$ we have

$$\left(\frac{P_\theta}{R^2} + \frac{q}{c} \frac{\partial A}{\partial r} \right)_R = 0 \quad (A7)$$

Thus

$$\left. \frac{\partial^2 \psi}{\partial r^2} \right|_R = \frac{1}{\gamma m c^2} \left(\frac{P_\theta}{r} - \frac{q}{c} A \right)_R \left(\frac{2P_\theta}{r^3} - \frac{q}{c} \frac{\partial^2 A}{\partial r^2} \right)_R \quad (A8)$$

where

$$\left. \frac{\partial A}{\partial r} \right|_R = \frac{B_0}{2} + \frac{q n V_\theta}{c} \left(\frac{1}{b-a} \right) \left(\ln(b/R) - \frac{1}{2} + \frac{a^2}{2R^2} \right) \quad (A9)$$

$$\left. \frac{\partial^2 A}{\partial r^2} \right|_R = \frac{q n V_\theta}{c(b-a)} \left(-\frac{1}{R} - \frac{a^2}{R^3} \right) \quad (A10)$$

in the limit $b \rightarrow a$; $R \rightarrow \frac{b+a}{2}$. Substituting Eq. (2), (A3b) in the limit

$b \rightarrow a$, and (A10) into (A8) we get

$$\left. \frac{\partial^2 \psi}{\partial r^2} \right|_R = \frac{1}{\gamma m} \left[\frac{(\gamma + v)^2 m^2 c^2}{R^2} - \frac{v^2 m^2 c^2}{R d} - \frac{(\gamma + v) v m^2 c^2}{R} \left(\frac{1}{R} - \frac{1}{d} \right) \right] \quad (A11)$$

$$= \gamma m \left[\left(\frac{c}{R} \right)^2 + \left(\frac{v}{\gamma} \right) \left(\frac{c}{R} \right) \left(\frac{c}{R} + \frac{c}{d} \right) \right] \quad (A12)$$

where $2d$ is the thickness of the e-layer. Thus the Hamiltonian for the radial motion becomes

$$H_1 = \frac{p_r^2}{2\gamma m} + \frac{\gamma m}{2} \omega^2 x^2 \quad (A13)$$

where

$$\omega^2 = \left(c/R\right)^2 \left[1 + \frac{v}{Y} \left(1 + \frac{R}{d}\right)\right] \quad (A14)$$

The condition that $P_r^2 c^2 \ll \gamma^2 m^2 c^2$ becomes

$$\omega_d^2 \ll c^2$$

or

$$\left(d/R\right)^2 \left[1 + \frac{v}{Y} \left(1 + R/D\right)\right] \ll 1 \quad (A15)$$

when $\omega_d^2 \sim c^2$ then the energy in the radial motion is comparable to that in the azimuthal direction and we will no longer have a well defined e-layer.

The same procedure can be followed to obtain the energy associated with the axial oscillations for an e-layer of finite length. Thus starting with

$$H_1 = \frac{p_z^2}{2\gamma m} + 1/2 \frac{\partial^2 \psi}{\partial z^2} \delta^2 \quad (A16)$$

where

$$\frac{\partial^2 \psi}{\partial z^2} = -q \frac{\partial^2 A_\phi}{\partial z^2} \quad (A17)$$

and δ is the distance from the center of the layer, with $2L$ being the length of the layer. A_ϕ differs from the A given previously in that A_ϕ includes the effects of finite length. Substituting in (A16) we have

$$H_1 = \frac{p_z^2}{2\gamma m} - \frac{q}{2} \frac{\partial^2 A_\phi}{\partial z^2} \delta^2 \quad (A18)$$

The above expression is valid only for δ small. What is really involved in (A18) is not the second derivative of the potential but the potential difference $A_\phi(\delta) - A_\phi(0)$. Thus the energy associated with axial oscillations

is given by

$$\frac{p_z^2}{2\gamma m} = -q[A_\phi(L) - A_\phi(0)] \quad (A19)$$

If we consider an infinitely thin e-layer the potential can be expressed as follows.

$$A_\phi(z) = \frac{I}{2L} \frac{R}{c} \int_{-L}^L dz' \int_0^{2\pi} \frac{d\phi \cos \phi}{[2R^2(1 - \cos \phi) + (z' - z)^2]^{1/2}} \quad (A20)$$

Where I is the total current, integrating over z' we have

$$A_\phi(z) = \frac{I}{2Lc} \int_0^{2\pi} d\phi \cos \phi \log \left[\frac{L - z + \sqrt{(L - z)^2 + 2R^2(1 - \cos \phi)}}{L + z + \sqrt{(L + z)^2 + 2R^2(1 - \cos \phi)}} \right] \quad (A21)$$

Evaluating (A21) at $z = 0$, and L we have

$$A_\phi(0) \equiv 0$$

$$A_\phi(L) = \frac{IR}{2Lc} \int_0^{2\pi} d\phi \cos \phi \log \left[\frac{\sqrt{2R^2(1 - \cos \phi)}}{2L + \sqrt{4L^2 + 2R^2(1 - \cos \phi)}} \right]$$

For $L \gg R$ we have⁵

$$A_\phi(L) = \frac{IR}{4Lc} \int_0^{2\pi} d\phi \cos \phi \log(1 - \cos \phi)$$

$$A_\phi(L) = \frac{IR\pi}{2Lc} = -\frac{\gamma mc^2}{4q} \quad (A22)$$

Thus we have for P_z

$$\frac{p_z^2}{2\gamma m} = \frac{\gamma mc^2}{4} < \frac{(\gamma mc)^2}{2\gamma m} = \frac{\gamma mc^2}{2}$$

Thus the restriction of v in order that the kinetic energy associated with axial oscillations be less than γmc^2 is

$$v < 2\gamma \quad (A23)$$

APPENDIX B

Here we consider the power spectrum of the synchrotron radiation emitted from a single electron in a non-current-neutralized e-layer. If we assume a linear change in the B-field across the e-layer we can express the magnitude of the B-field as a function of its position in the layer as

$$B(x) = B_0 \left(\frac{\gamma - v}{\gamma + v} + \frac{2vx}{\gamma + v} \right) \quad (B1)$$

the power radiated, P , is proportional to B^2 thus we have

$$P(x) = P_0 \left(\frac{\gamma - v + 2vx}{\gamma + v} \right)^2 \quad (B2)$$

and the average power is given by

$$\begin{aligned} \langle P \rangle &= P_0 \int_0^1 \left(\frac{\gamma - v + 2vx}{\gamma + v} \right)^2 dx \\ &= P_0 \frac{\gamma^2 + v^3/3}{(\gamma + v)^2} \end{aligned} \quad (B3)$$

where the subscript 0 refers to the quantity subscripted in the current-neutralized system of the same e-layer energy and with magnetic field B_0 .

To calculate the spectrum it is necessary to average the spectrum for an electron in a given magnetic field over the range of the magnetic field in the e-layer. This is done as follows, define

$$I_0(\omega) = \int_0^1 x dx I(\omega, \omega_c(x)) \quad (B4)$$

where $I(\omega, \omega_c(x))$ is given by^{3,5}

$$I(\omega, \omega_c(x)) = (2\sqrt{3})(\omega/\omega_c)(e^2\gamma/c) \int_{2(\omega/\omega_c)}^{\infty} K_{5/3}(y) dy \quad (B5)$$

and

$$\omega_c(x) = \frac{3e B_0 \gamma^3}{mc \sqrt{\gamma^2 - 1}} \left[\frac{\gamma - \nu}{\gamma + \nu} + \frac{2\nu x}{\gamma + \nu} \right] \quad (B6)$$

$I(\omega, \omega_c)$ is the energy radiated per unit frequency interval per revolution, as a function of the local ω_0 and ω_c . We can compare the effects of current neutralization on the radiation spectrum by fixing γ and R , and allowing ν to vary in Eq. (B6). Defining B_{CN} as the magnetic field in the current neutralized system, B_0 is given by

$$B_0 = B_{CN} \left(\frac{\gamma + \nu}{\gamma} \right) \quad (B7)$$

and $\omega_c(x)$ in terms of the current neutralized critical frequency ω_{CN} is given by

$$\omega_c(x) = \omega_{CN} [1 + \nu/\gamma (2x - 1)] \quad (B8)$$

The average power $\langle P \rangle$ in terms of P_{CN} is given by

$$\begin{aligned} \langle P \rangle &= P_{CN} \int_0^1 [1 + \nu/\gamma (2x - 1)]^2 dx \\ \langle P \rangle &= P_{CN} \frac{\gamma^2 + \nu^2/3}{\gamma^2} \end{aligned} \quad (B9)$$

Equation (B4) was integrated numerically to yield the radiation spectrum in Fig. 1.⁶

Fig II

REFERENCES

1. Nowak, D. and Shearer, J, COPD 74-59.
2. Schmidt, G., The Phys. of Fluids, Vol. 15, No. 8, 1972.
3. See for example Jackson, J. D., Classical Electrodynamics.
4. See for example Alfven, H., Phys. Rev, Vol. 55, No. 5, 1939.
5. Garelis, E., private communication.
6. Garelis, E., private communication.

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