RECEIVED BY DTIE SEP 1 11 1967

MASTER

K-1723

AEC RESEARCH AND DEVELOPMENT REPORT

EXISTENCE - UNIQUENESS THEOREMS FOR
NON - LINEAR DIRICHLET PROBLEMS

AUTHOR:

Dale W. Lick

**UNION
CARBIDE**

UNION CARBIDE CORPORATION
NUCLEAR DIVISION

Operating the

OAK RIDGE GASEOUS DIFFUSION PLANT • OAK RIDGE NATIONAL LABORATORY
OAK RIDGE Y-12 PLANT • PADUCAH GASEOUS DIFFUSION PLANT

*For the Atomic Energy Commission
Under U.S. Government Contract W7405 eng 26*

DISTRIBUTION OF THIS DOCUMENT IS UNLIMITED

DISCLAIMER

This report was prepared as an account of work sponsored by an agency of the United States Government. Neither the United States Government nor any agency Thereof, nor any of their employees, makes any warranty, express or implied, or assumes any legal liability or responsibility for the accuracy, completeness, or usefulness of any information, apparatus, product, or process disclosed, or represents that its use would not infringe privately owned rights. Reference herein to any specific commercial product, process, or service by trade name, trademark, manufacturer, or otherwise does not necessarily constitute or imply its endorsement, recommendation, or favoring by the United States Government or any agency thereof. The views and opinions of authors expressed herein do not necessarily state or reflect those of the United States Government or any agency thereof.

DISCLAIMER

Portions of this document may be illegible in electronic image products. Images are produced from the best available original document.

Printed in the United States of America. Available from Clearinghouse for Federal
Scientific and Technical Information, National Bureau of Standards,
U.S. Department of Commerce, Springfield, Virginia 22151
Price: Printed Copy \$3.00; Microfiche \$0.65

— LEGAL NOTICE —

This report was prepared as an account of Government sponsored work. Neither the United States, nor the Commission, nor any person acting on behalf of the Commission:

- A. Makes any warranty or representation, expressed or implied, with respect to the accuracy, completeness, or usefulness of the information contained in this report, or that the use of any information, apparatus, method, or process disclosed in this report may not infringe privately owned rights; or
- B. Assumes any liabilities with respect to the use of, or for damages resulting from the use of any information, apparatus, method, or process disclosed in this report.

As used in the above, "person acting on behalf of the Commission" includes any employee or contractor of the Commission, or employee of such contractor, to the extent that such employee or contractor of the Commission, or employee of such contractor prepares, disseminates, or provides access to, any information pursuant to his employment or contract with the Commission, or his employment with such contractor.

Report Date: August 23, 1967

Report Number: K-1723

Subject Category: Mathematics
&
Computers

CESTI PRICES

H.C. 3.00; MN 65

EXISTENCE-UNIQUENESS THEOREMS FOR NON-LINEAR
DIRICHLET PROBLEMS

Dale W. Lick
Consultant, Computing Technology Center

LEGAL NOTICE

This report was prepared as an account of Government sponsored work. Neither the United States, nor the Commission, nor any person acting on behalf of the Commission:

A. Makes any warranty or representation, expressed or implied, with respect to the accuracy, completeness, or usefulness of the information contained in this report, or that the use of any information, apparatus, method, or process disclosed in this report may not infringe privately owned rights; or

B. Assumes any liabilities with respect to the use of, or for damages resulting from the use of any information, apparatus, method, or process disclosed in this report.

As used in the above, "person acting on behalf of the Commission" includes any employee or contractor of the Commission, or employee of such contractor, to the extent that such employee or contractor of the Commission, or employee of such contractor prepares, disseminates, or provides access to, any information pursuant to his employment or contract with the Commission, or his employment with such contractor.

UNION CARBIDE CORPORATION

NUCLEAR DIVISION

Computing Technology Center
Oak Ridge Gaseous Diffusion Plant
Oak Ridge, Tennessee

**THIS PAGE
WAS INTENTIONALLY
LEFT BLANK**

A B S T R A C T

Existence-uniqueness theorems are proved for the Dirichlet problems for the non-linear equations

$$\nabla^2 u - \frac{c_1 u}{1 + c_2 u} = 0, u \geq 0,$$

$$\nabla^2 u + \frac{c_1}{1 + c_2 u} = 0, u \geq 0,$$

and

$$\nabla^2 u - f(u) = 0, u \geq 0$$

with boundary condition

$$u = \phi \geq 0.$$

**THIS PAGE
WAS INTENTIONALLY
LEFT BLANK**

EXISTENCE-UNIQUENESS THEOREMS FOR NON-LINEAR DIRICHLET PROBLEMS

In this paper we consider a Dirichlet problem for the equation

$$\nabla^2 u = f(u), \quad (1)$$

where ∇^2 is the Laplacian and f is non-linear. Most discussions of such problems involve the condition that u be defined and continuously differentiable for all u . However, in many applied problems, the physical meaning of u limits its range. The purpose of this report is to discuss such a restricted problem.

Let G be a bounded domain in R^n with boundary Γ . We consider the Dirichlet problems for the non-linear equations

$$\nabla^2 u = \frac{c_1 u}{1 + c_2 u}, \quad u \geq 0, \text{ in } G, \quad (2)$$

and

$$\nabla^2 u = - \frac{c_1}{1 + c_2 u}, \quad u \geq 0, \text{ in } G, \quad (3)$$

with boundary condition

$$u = \phi \geq 0 \text{ on } \Gamma, \quad (4)$$

where c_1 and c_2 are positive constants and ϕ is continuous.

Courant [1] has proved the following theorems.

Theorem 1. If f is defined, continuously differentiable and bounded for all u , then the Dirichlet problem for Equation (1) with boundary condition (4) has a solution in G .

Theorem 2. The Dirichlet problem for the equation

$$\nabla^2 u + cu = 0,$$

where $c \leq 0$ and continuous, has at most one solution.

With the aid of these two theorems and the Maximum Principle, we are able to establish the following results.

Theorem 3. The Dirichlet problem for Equation (2) with boundary condition (4) has one and only one solution.

Proof. Let

$$f(u) = \begin{cases} f_1(u) = \frac{c_1 u}{1 + c_2 u}, & u \geq 0 \\ f_2(u) = \frac{c_1 u}{1 - c_2 u}, & u < 0. \end{cases}$$

Then f is defined, continuously differentiable and bounded for all u .

By Theorem 1, the Dirichlet problem for Equation (1) with boundary condition (4) has a solution in G .

Suppose u is negative in $\bar{G}$. Since $u \geq 0$ on Γ , u assumes a negative minimum at some point, say P , in G , and there exists a neighborhood N of P in which $u \leq 0$. In N , u takes on its minimum at P and $\nabla^2 u \leq 0$, and the Maximum Principle implies that u is constant. Hence, $\nabla^2 u = 0$, $f(u) = 0$ and $u \equiv 0$ in N . This contradicts $u < 0$. Therefore, $u \geq 0$ in G , which establishes the existence of a solution.

Suppose u and v are two solutions of this problem. Let

$$w = u - v.$$

Then

$$\nabla^2 u - f_1(u) - [\nabla^2 v - f_1(v)] = 0$$

implies

$$\nabla^2 w - p(u, v)w = 0,$$

where

$$p(u, v) = \int_0^1 f_1'(tu + (1-t)v) dt.$$

Since p is continuous and positive, Theorem 2 gives the desired uniqueness.

Theorem 4. The Dirichlet problem for Equation (3) with boundary condition (4) has one and only one solution.

Proof. This theorem is verified by the arguments of Theorem 3 with

$$f(u) = \begin{cases} f_1(u) = -\frac{c_1}{1+c_2u}, & u \geq 0, \\ f_2(u) = -(2c_1 + f_1(-u)), & u < 0. \end{cases}$$

The problems discussed in Theorem 3 and Theorem 4 arise naturally in applied mathematics and so were explicitly considered. However, by reflecting f about $(0, f(0))$, the same techniques are sufficient to establish the following generalization.

Theorem 5. Let f be defined, bounded and continuously differentiable for $u \geq 0$. If f satisfies

- (i) $f(0) \leq 0$,
- (ii) $f'(u) \geq 0, u \geq 0$,
- (iii) $f'(0) > 0$,

then the Dirichlet problem for the non-linear equation

$$\nabla^2 u = f(u), \quad u \geq 0,$$

with boundary condition (4) has one and only one solution.

We observe that if $f(0) < 0$, then condition (iii) of Theorem 5 can be eliminated.

REFERENCES

- [1] Courant, R. and D. Hilbert, Methods of Mathematical Physics,
Interscience, 1962.