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The Hamiltonian and Generating Functional for a Non-Relativistic Local Current Algebra*+

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Abstract

The non-relativistic current algebra with conserved current consisting of $\rho(\underset{\sim}{x})$, the particle number density and $\underset{\sim}{J}(\underset{\sim}{x})$, the flux density of particles, is studied. The Hamiltonian for any time reversal invariant system of spinless particles, interacting via a two-body interaction potential, is expressed as a hermitian form in the currents. This leads to a functional equation for the generating functional, which is the ground state expectation value of $\exp[i\int d\underset{\sim}{x}\,\rho(\underset{\sim}{x})f(\underset{\sim}{x})]$. In the N/V limit an expression for the generating functional in terms of correlation functions is given. Representations of the exponentiated current algebra which are translation invariant, satisfy the cluster decomposition property and which have different Hamiltonians are shown to be unitarily inequivalent.

1. Introduction.

Several physicists[1-5] have investigated the possibility of expressing field theory in terms of local currents instead of the canonical fields. To gain further insight into writing field theory in terms of local currents, we study in this paper the non-relativistic equal-time current algebra consisting of $\rho(\underset{\sim}{x})$, the particle number density and $\underset{\sim}{J}(\underset{\sim}{x})$, the flux density of particles. We seek to determine representations of the current algebra suitable for describing physical systems associated with a specific Hamiltonian H. A generating functional is used for this purpose. The representation incorporates certain general physical constraints on the system, such as current conservation, time reversal invariance and translation invariance. The dynamics, which is not studied here, would be obtained by considering the time dependent local currents, $\rho(\underset{\sim}{x},t) = e^{itH}\rho(\underset{\sim}{x})e^{-itH}$ and $\underset{\sim}{J}(\underset{\sim}{x},t) = e^{itH}\underset{\sim}{J}(\underset{\sim}{x})e^{-itH}$, in the representation determined by the equal-time current algebra and the Hamiltonian.

In this approach we start with non-relativistic quantum mechanics in second quantized form. Then $\rho(\underset{\sim}{x})$ and $\underset{\sim}{J}(\underset{\sim}{x})$ can be written in terms of the canonical annihilation and creation field operators, and their commutation relations computed. The commutation relations between $\rho(\underset{\sim}{x})$ and $\underset{\sim}{J}(\underset{\sim}{x})$ are taken as our starting point[1]. We will be especially interested in representations corresponding to the "N/V limit", since they describe systems with "an infinite number of degrees of freedom" and have many features similar to those of quantum field theory. In this case the quantum mechanics of N particles in a box of volume V is considered. The limit is taken as $N\to\infty$ and $V\to\infty$ in such a way that $N/V\to\bar{\rho}$, the average density of the system. In statistical mechanics this is known as the thermodynamic limit. It is applicable to systems with a large number of particles when surface effects can be neglected. In this paper we deal only with the case of zero temperature.

In section 2 the ρ, J current algebra is defined as in reference 1. For our purposes it is more convenient to deal with the group obtained by exponentiating the currents. This is reviewed along with its unitary representations as given by Goldin[6]. The generating functional L(f), the ground state expectation value of $\exp[i\int d\underset{\sim}{x}\rho(\underset{\sim}{x})f(\underset{\sim}{x})]$, is introduced and its use in defining a representation is discussed.

In section 3 we consider the Hamiltonian for a time reversal invariant system of spinless particles. Dashen and Sharp[1] have given a formal expression for the Hamiltonian in terms of currents as the sum of a kinetic energy term plus a potential energy term. A rigorous definition for the kinetic energy term has been given by Goldin and Sharp[7] for the Hamiltonian of a system of free bosons by considering it as a densely defined hermitian form. We generalize this form to obtain the Hamiltonian for a system of interacting particles. The resulting expression for the Hamiltonian combines the kinetic energy and potential energy into one factored term. Two points of view may be taken in this section:

(i) Given a representation in which a Hamiltonian exists, the Hamiltonian is expressed in terms of $\rho(\underset{\sim}{x})$ and $\underset{\sim}{J}(\underset{\sim}{x})$ as a densely defined hermitian form, or

(ii) Given a representation, an operator with all the properties of a Hamiltonian is defined from a densely defined hermitian form.

The form of the Hamiltonian leads in section 4 to a functional equation for the generating functional. Supplemented by the appropriate boundary conditions, this equation determines a representation associated with the Hamiltonian.

In section 5 the generating functional for a representation corresponding to a system of N particles is expressed in terms of correlation functions. This form of the generating functional is extended to the N/V limit representations. Next, we consider the consequences of translation invariance and the cluster decomposition property. The results are analogus to those in field theory[8]; the ground state is unique and is the only momentum eigenfunction. Furthermore, it is shown that representations corresponding to different Hamiltonians are unitarily inequivalent. Finally, the particle nature of the N/V limit representations is studied. The representation restricted to a finite volume is found to be the direct sum of N-particle representations. Thus the N/V limit representation is "locally Fock".

These results are illustrated by examples in the following paper where, in the N/V limit, the generating functional along with the Hamiltonian and functional equation are given exactly in the following cases: (i) Free Bose Gas, (ii) Non-interacting Bosons in an external potential, (iii) Free Fermi Gas, (iv) Bosons in one dimension with the two body interacting potential $U(x) = 2/x^2$.

2. Review of the Non-Relativistic Current Algebra.

This section contains a brief review of the non-relativistic current algebra and its representations. (For a more extensive review see Refs. 7 and 9.)

In terms of the canonical field operators $\psi(\underset{\sim}{x})$ and $\psi^\dagger(\underset{\sim}{x})$ which satisfy either the commutation (-) or anti-commutation (+) relations

$$[\psi(\underset{\sim}{x}), \psi(\underset{\sim}{y})]_\pm = [\psi^\dagger(\underset{\sim}{x}), \psi^\dagger(\underset{\sim}{y})]_\pm = 0$$
$$[\psi(\underset{\sim}{x}), \psi^\dagger(\underset{\sim}{y})]_\pm = \delta(\underset{\sim}{x}-\underset{\sim}{y}) \qquad (2.1)$$

the particle density and flux density are given by:

$$\rho(\underset{\sim}{x}) = \psi^\dagger(\underset{\sim}{x})\,\psi(\underset{\sim}{x})$$
$$\underset{\sim}{J}(\underset{\sim}{x}) = \frac{h}{2im}\,[\psi^\dagger(\underset{\sim}{x})\underset{\sim}{\nabla}\psi(\underset{\sim}{x}) - \underset{\sim}{\nabla}\psi^\dagger(\underset{\sim}{x})\,\psi(\underset{\sim}{x})] \qquad (2.2)$$

Henceforth the mass of the particles and $\hbar$ will be set equal to 1. Dashen and Sharp[1] showed that the equal-time commutation relations between $\rho(\underset{\sim}{x})$ and $\underset{\sim}{J}(\underset{\sim}{y})$ are given by:

$$[\rho(f_1), \rho(f_2)] = 0$$
$$[\rho(f), J(\underset{\sim}{g})] = i\,\rho(\underset{\sim}{g}\cdot\underset{\sim}{\nabla}f) \qquad (2.3)$$
$$[J(\underset{\sim}{g}_1), J(\underset{\sim}{g}_2)] = iJ(\underset{\sim}{g}_2\cdot\underset{\sim}{\nabla}\underset{\sim}{g}_1 - \underset{\sim}{g}_1\cdot\underset{\sim}{\nabla}\,\underset{\sim}{g}_2)$$

for both bosons and fermions. We have used the smeared currents $\rho(f) = \int d\underset{\sim}{x}\,\rho(\underset{\sim}{x})f(\underset{\sim}{x})$ and $J(\underset{\sim}{g}) = \int d\underset{\sim}{x}\,\underset{\sim}{J}(\underset{\sim}{x})\cdot\underset{\sim}{g}(\underset{\sim}{x})$, where $f(\underset{\sim}{x})$ and each component of $\underset{\sim}{g}(\underset{\sim}{x})$ belong to a suitable class of test functions; for example, Schwartz's space $\mathscr{S}$, the set of C^∞ functions of fast decrease at infinity.

The commutation relations (2.3) will be taken as the starting for the work of this paper. We will also assume current conservation, $\frac{d}{dt}\rho(\underset{\sim}{x},t) + \underset{\sim}{\nabla}\cdot\underset{\sim}{J}(\underset{\sim}{x},t) = 0$. This is expressed in terms of a Hamiltonian by: $[H,\rho(f)] = -iJ(\underset{\sim}{\nabla}f)$ (2.4)

Since the local currents correspond to physical observables we require them to be self-adjoint operators; $\rho(f)^\dagger = \rho(f)$ and $J(\underset{\sim}{g})^\dagger = J(\underset{\sim}{g})$. However, they may be unbounded operators. For this reason it is convenient to work with the unitary operators formed by exponentiating the currents[6],

$$U(f) = e^{i\rho(f)} \quad \text{and}$$

$$V(\varphi_t^g) = e^{itJ(g)} \tag{2.5}$$

where $\frac{d}{dt}\varphi_t^g(x) = g\circ\varphi_t^g(x)$, $\varphi_0^g(x) = x$, and "o" stands for composition, i.e. $g\circ\varphi(x) = g(\varphi(x))$.

Remark: $\varphi_t^g(x)$ is the flow corresponding to the vector field $g(x)$. This has the following physical interpretation. Imagine a fluid with velocity field $v = g(x)$. Then $\varphi_t^g(x)$ is the position of a particle which starts at point x, after a time t.

The exponentiated currents form a group with the following multiplication law:

$$\begin{aligned} U(f_1)U(f_2) &= U(f_1 + f_2) \\ V(\varphi)\, U(f) &= U(f\circ\varphi)\, V(\varphi) \\ V(\varphi_1)V(\varphi_2) &= V(\varphi_2 \circ \varphi_1) \end{aligned} \tag{2.6}$$

Throughout the rest of this paper we will be concerned with representations of the group of exponentiated currents. Goldin[6] has analyzed these representations using the Gel'fand-Vilenkin formalism for "nuclear Lie groups"[10]. The results listed below will be used in our study.

The Hilbert Space for every continuous representation of U(f) and $V(\varphi)$ is unitarily equivalent to one with direct sum decomposition,

$$\mathcal{H} = \int_{F\in\mathcal{S}'}^{\oplus} d\mu(F)\mathcal{H}_F$$

where μ is a cylindrical measure on $\mathcal{S}'$, the continuous dual of $\mathcal{S}$. (i.e. $\mathcal{S}'$ is the set of continuous real linear functionals on $\mathcal{S}$.) For physical reasons explained below we will only be concerned with the case when $\dim\mathcal{H}_F = 1$. The Hilbert Space is then the space of square integrable functions on $\mathcal{S}'$ with respect to the measure μ; i.e. $\mathcal{H} = L^2_\mu(\mathcal{S}')$.

U(f) acts as a multiplication operator on elements of $\mathcal{H}$, i.e.

$$U(f)\Psi(F) = e^{i(F,f)}\Psi(F) \quad , \forall\Psi(F)\in\mathcal{H} \tag{2.7}$$

In order to express the action of $V(\varphi)$ we need the mapping φ^* from $\mathcal{S}'$ onto $\mathcal{S}'$ defined by

$$(\varphi^* F, f) = (F, f\circ\varphi) \quad , \forall F\in\mathcal{S}' \text{ and } f\in\mathcal{S}$$

The action of $V(\varphi)$ is then given by

$$V(\varphi)\Psi(F) = \chi_\varphi(F)\Psi(\varphi^* F)\left[\frac{d\mu(\varphi^* F)}{d\mu(F)}\right]^{\frac{1}{2}} \quad , \forall\Psi(F)\in\mathcal{H} \tag{2.8}$$

where $d\mu(\varphi^* F)/d\mu(F)$ is the Radon-Nikodym derivative of $\mu(\varphi^* F)$ with respect to $\mu(F)$ and $\chi_\varphi(F)$, called the multiplier, is a complex valued function of modulus one. In order for the Radon-Nikodym derivative to exist the measure μ must be quasi-invariant with respect to the set of flows; i.e. for any measurable set $X\subset\mathcal{S}'$ and any flow φ, $\mu(X) = 0$ iff $\mu(\varphi^* X) = 0$.
The group law requires the multipliers to satisfy the equation,

$$\chi_{\varphi_2}(F)\chi_{\varphi_1}(\varphi_2^* F) = \chi_{\varphi_1\circ\varphi_2}(F) \text{ a.e.} \tag{2.9}$$

A representation of U(f) and $V(\varphi)$ is thus completely determined by a measure μ and a system of multipliers $\chi_\varphi(F)$.

The representation corresponding to the Quantum Mechanics of N identical particles has a measure concentrated on delta functions[6,11]; i.e. the measure is only non-zero on functionals of the form

$$F(x) = \sum_{j=1}^{N}\delta(x - x_j) \text{ and } d\mu(F) = d\sigma(x_1, x_2, \ldots, x_N) .$$

By a suitable choice of measure the ground state for a given Hamiltonian may be taken as $\Omega(F) = 1$. (In the N-particle representation the measure is given by $d\mu(F) = d\psi^*\psi(x_1 \ldots x_N)$ where $\psi(x_1 \ldots x_N)$ is the ground state wave function.)

Remarks: (1) The ground state $|\Omega\rangle$, is cyclic with respect to U(f).

In other words, the set of states of the form

$$\sum_{j=1}^{N} a_1\, U(f_1)\, |\Omega\rangle \text{ is dense in } \mathcal{H} = L^2_\mu(\mathcal{S}')\,.$$

The continuity of the representation then implies $\mathcal{H}$ is separable.

(2) Dicke and Goldin[12] have proposed a definition of statistics for representations of the exponentiated current algebra based on the multipliers. They found that the only "well behaved" irreducible representations of $U(f)$ and $V(\varphi)$ with $\dim\mathcal{H}_F = 1$ are those corresponding to either Bosons or Fermions.

(3) $\chi_\varphi(F) = 1$ always satisfies eq(2.9) . This corresponds to a representation for Bosons[12]. Thus a Boson representation can be completely defined by giving a measure μ and setting $\chi_\varphi(F) = 1$. There may be other systems of multipliers corresponding to Bosons.

(4) The representations with $\dim\mathcal{H}_F > 1$ have the following physical significance:

(i) If $U(f)$ and $V(\varphi)$ are reducible the representation can correspond to particles with different masses or with internal degrees of freedom (e.g. spin). In the latter case, additional local currents need to be added to obtain a complete set of observables (e.g. spin density). Spin has been treated briefly by Grodnik and Sharp[13] and Goldin[6].

(ii) If $U(f)$ and $V(\varphi)$ are "well behaved" and irreducible the representation corresponds to parastatistics[14].

Thus by restricting ourself to the case $\dim\mathcal{H}_F = 1$, we only will be considering identical spinless particles (either Bosons or Fermions).

Much information about the representation can be obtained from the ground state expectation value of $U(f)$. This is known as the generating functional and is denoted by $L(f)$. Thus,

$$L(f) = (\Omega, U(f)\Omega) = \int_{\mathscr{S}'} d\mu(F)\, e^{i(F,f)} \tag{2.10}$$

The generating functional for any representation has the following properties:

(i) $L(f) = L(-f)^*$ (2.11)

This follows from the relation $U(f)^\dagger = U(-f)$.

(ii) $L(0) = 1$ (2.12)

Since the ground state is normalized, $(\Omega, \Omega) = 1$.

(iii) $L(f) \leq 1$ (2.13)

This follows from the condition that $U(f)$ be a unitary operator.

(iv) $L(f)$ is a positive functional. This means

$$\sum_{j,k=1}^{N} a_j^* a_k L(f_k - f_j) \geq 0, \ \forall a_j \varepsilon C, \ f_j \varepsilon \mathscr{S} \text{ and finite } N. \tag{2.14}$$

This property follows from the requirement that the inner product on $\mathcal{H}$ be positive: i.e. $\left(\sum_{j=1}^{N} a_j U(f_j)\Omega, \sum_{k=1}^{N} a_k U(f_k)\Omega\right) \geq 0$.

It can be shown that a continuous functional $L(f)$ satisfying the above four properties determines a measure μ for a representation of $U(f)$[6]. If μ is a quasi-invariant measure and the multipliers are known (e.g. this is the case for Bosons), a representation of both $U(f)$ and $V(\underset{\sim}{\varphi})$ is completely determined. Otherwise, it is necessary to know, $L(f,\underset{\sim}{\varphi}) = (\Omega, U(f)V(\underset{\sim}{\varphi})\Omega)$, in order to completely determine a representation of the exponentiated currents.

Remark: The exponentiated algebra and generating functional techniques we will be using are similar to those introduced by Araki[8] in studying the CCR's. They have been applied to find representations of the canonical commutation relations describing a non-relativistic infinite free bose gas by Araki and Woods[15]. A similar approach was used in a study of the CAR's by Araki and Wyss[16].

3. The Hamiltonian expressed in terms of Currents.

In this section we will express the Hamiltonian of a physical system in terms of the currents $\rho(\underset{\sim}{x})$ and $\underset{\sim}{J}(\underset{\sim}{x})$. A formal expression for the Hamiltonian abstracted from canonical field theory was given by Dashen and Sharp[1]. In terms of the canonical field operators (satisfying either the CCR's or CAR's) the Hamiltonian for a system of particles with a two body interaction potential $V(\underset{\sim}{x})$ is given by:

$$H = \tfrac{1}{2}\int d\underset{\sim}{x}\, \underset{\sim}{\nabla}\psi^{\dagger}(\underset{\sim}{x}) \cdot \underset{\sim}{\nabla}\psi(\underset{\sim}{x}) + \tfrac{1}{2}\iint d\underset{\sim}{x}d\underset{\sim}{y}\, \psi^{\dagger}(\underset{\sim}{x})\psi^{\dagger}(\underset{\sim}{y})V(\underset{\sim}{x}-\underset{\sim}{y})\psi(\underset{\sim}{y})\psi(\underset{\sim}{x}) \quad (3.1)$$

The potential energy term can be written as,

$$P.E. = \tfrac{1}{2}\iint d\underset{\sim}{x}d\underset{\sim}{y}\, \rho(\underset{\sim}{x})\,[\rho(\underset{\sim}{y}) - \delta(\underset{\sim}{x}-\underset{\sim}{y})]\, V(\underset{\sim}{x}-\underset{\sim}{y}) \quad (3.2)$$

To obtain the kinetic energy term we introduce the quantity

$\underset{\sim}{K}(\underset{\sim}{x}) = \underset{\sim}{\nabla}\rho(\underset{\sim}{x}) + 2i\underset{\sim}{J}(\underset{\sim}{x})$. In terms of the canonical fields

$\underset{\sim}{K}(\underset{\sim}{x}) = 2\,\psi^{\dagger}(\underset{\sim}{x})\underset{\sim}{\nabla}\psi(\underset{\sim}{x})$. Then formally the kinetic energy is given by,

$$K.E. = (1/8)\int d\underset{\sim}{x}\, \underset{\sim}{K}(\underset{\sim}{x})^{\dagger}\, \frac{1}{\rho(\underset{\sim}{x})}\, \underset{\sim}{K}(\underset{\sim}{x}) \quad (3.3)$$

Combining eqs. 3.2 and 3.3 the Hamiltonian is given by,

$$H = (1/8)\int d\underset{\sim}{x}\, \underset{\sim}{K}(\underset{\sim}{x})^{\dagger}\, \frac{1}{\rho(\underset{\sim}{x})}\, \underset{\sim}{K}(\underset{\sim}{x}) + \tfrac{1}{2}\iint d\underset{\sim}{x}d\underset{\sim}{y}\, \rho(\underset{\sim}{x})[\rho(\underset{\sim}{y})-\delta(\underset{\sim}{x}-\underset{\sim}{y})]V(\underset{\sim}{x}-\underset{\sim}{y}) \quad (3.4)$$

In the N/V limit there are two problems with writing H as the sum of the total K.E. plus the total P.E. :

(i) The K.E./particle and the P.E/particle are finite. However, the total K.E. and the total P.E. are infinite. Therefore, it is unclear just how each term in eq. 3.4 is to be defined.

(ii) From statistical mechanics the ground state energy is proportional to the number of particles; $E_0 \to \epsilon N$ as N becomes large. In the limit, $E_0 = \infty$.

Thus, the sum of the two terms in eq. 3.4 is also ill defined as it stands.

These problems lead us to consider an alternative expression for the Hamiltonian. First, it is necessary to define the quantity " $\frac{1}{\rho(\underset{\sim}{x})}$ " which appears in the kinetic energy term. In the representation corresponding to a Free Bose Gas, a rigorous definition has been given by Goldin and Sharp[7]. By extending their definition we can combine the K.E. and P.E. into one term and obtain a well defined expression for the Hamiltonian as a densely defined hermitian form.

We denote the Hamiltonain for a Free Bose Gas by
$H_0 = (1/8)\int d\underset{\sim}{x}\underset{\sim}{K}(\underset{\sim}{x})^{\dagger} \frac{1}{\rho(\underset{\sim}{x})} \underset{\sim}{K}(\underset{\sim}{x})$. It is defined as follows[7]:

Let $\mathcal{V}$ = span $\{w(\underset{\sim}{x})\rho(\underset{\sim}{x})\Phi; \Phi\in\mathcal{H}$ and $w(\underset{\sim}{x}) \in C^{\infty}$ functions of polynomial growth at infinity$\}$

$\mathcal{V}$ is a set of vector valued distributions "proportional" to $\rho(\underset{\sim}{x})$.

" $\frac{1}{\rho(\underset{\sim}{x})}$" is defined as a map from $\mathcal{V} \times \mathcal{V} \to \mathcal{S}'$ in the following way:

Let $v_1 = w_1(\underset{\sim}{x})\rho(\underset{\sim}{x})\Phi_1$ and $v_2 = w_2(\underset{\sim}{x})\rho(\underset{\sim}{x})\Phi_2$. Then

$$(v_1, \frac{1}{\rho(\underset{\sim}{x})} v_2) = (\Phi_1, w_1(\underset{\sim}{x})w_2(\underset{\sim}{x})\rho(\underset{\sim}{x})\Phi_2).$$

Let $\mathcal{D}$ = span $\{e^{i\rho(f)}\Omega; \forall f\in\mathcal{S}$ and Ω= the ground state$\}$.

$\mathcal{D}$ is a dense linear manifold in $\mathcal{H}$. For the Free Bose Gas it can be shown $\underset{\sim}{K}(\underset{\sim}{x})\mathcal{D}\subset\mathcal{V}$. As a result

$$(\Phi_1, H_0\, \Phi_2) = (1/8)\int d\underset{\sim}{x}\, (\underset{\sim}{K}(\underset{\sim}{x})\Phi_1, \frac{1}{\rho(\underset{\sim}{x})} \underset{\sim}{K}(\underset{\sim}{x})\Phi_2).$$

is a well defined hermitian from for all Φ_1 and $\Phi_2 \in \mathcal{D}$.

Remark: The seemingly natural operation of $\frac{1}{\rho(\underset{\sim}{x})}$ on $v = w_1(\underset{\sim}{x})\rho(\underset{\sim}{x})\Phi_1$,

$\frac{1}{\rho(\underset{\sim}{x})} v = w_1(\underset{\sim}{x})\Phi_1$, is not in fact well defined. Since, if v can also be written as $v = w_2(\underset{\sim}{x})\rho(\underset{\sim}{x})\Phi_2$ it does not necessarily follow that $w_1(\underset{\sim}{x})\Phi_1 = w_2(\underset{\sim}{x})\Phi_2$.

By generalizing the form of H_0 we will show for an interacting system that:

(1) H is defined as a bilinear form on the dense domain,

$\mathcal{D}$ = span $\{U(f)\Omega;\ f\in\mathcal{S}$ and Ω = the ground state$\}$, by

$$H = (1/8)\int dx\, \tilde{\underset{\sim}{K}}(\underset{\sim}{x})^{\dagger} \frac{1}{\rho(\underset{\sim}{x})} \cdot \tilde{\underset{\sim}{K}}(\underset{\sim}{x}) \quad \text{where } \tilde{\underset{\sim}{K}}(\underset{\sim}{x}) = \underset{\sim}{K}(\underset{\sim}{x}) - \underset{\sim}{A}(\underset{\sim}{x},\rho).$$

The operator $\underset{\sim}{A}(\underset{\sim}{x},\rho)$ will be defined precisely later.

(2) H is both hermitian and positive.

(3) $(\Phi, H\Omega) = 0$, $\forall\Phi\in\mathcal{D}$ where Ω = the ground state.

We start by assuming there is a representation of U(f) and $V(\underset{\sim}{\varphi})$ on a Hilbert Space $\mathcal{H}$ along with a Hamiltonian H satisfying the following conditions:

(i) There is a normalized state of lowest energy; the ground state Ω. We require, $H\geq 0$. Thus the zero of energy is choosen such that

$$H\ \Omega = 0\ . \tag{3.5}$$

(ii) $\mathcal{D}$ = span $\{U(f)\Omega;\ f\in\mathcal{S}\}$ is dense in $\mathcal{H}$ and $\mathcal{D}\subset$ the domain of H.

(iii) Current conservation

$$[H,\rho(f)] = -iJ(\underset{\sim}{\nabla}f) \tag{3.6}$$

(iv) There is an anti-unitary time reversal operator T such that,

$$T\rho(f)T^{-1} = \rho(f),$$
$$TJ(\underset{\sim}{g})T^{-1} = -J(\underset{\sim}{g}), \text{ and} \tag{3.7}$$
$$T\Omega = \Omega\ .$$

We will also make use of the relation

$$e^{A}Be^{-A} = \sum_{n=0}^{\infty} \frac{1}{n!}\ (ad^{n}A)B \tag{3.8}$$

where $(ad^{0}A)B = B$ and $(ad^{n}A)B = [A,(ad^{n-1}A)B]$.

Two simple results we will need can easily be derived from eqs. 2.3, 3.6 and 3.8. These are,

$$[e^{i\rho(f)},J(\underset{\sim}{g})] = -\tfrac{1}{2}\, i[e^{i\rho(f)},K(\underset{\sim}{g})] = -\rho(\underset{\sim}{g}\cdot\underset{\sim}{\nabla}f)e^{i\rho(f)} \tag{3.9}$$

and

$$[e^{i\rho(f)},H] = [-J(\nabla f) + \tfrac{1}{2}\rho(\nabla f\cdot\nabla f)]\, e^{i\rho(f)} \tag{3.10}$$

Our first theorem shows time reversal invariance and current conservation are sufficient to determine the matrix elements of $J(g)$ and H in terms of those for ρ.

Theorem 1: Suppose there is a representation of U(f) and $V(\phi)$ satisyfing conditions i-iv above. Let $|f\rangle = e^{i\rho(f)}\Omega$. Then,

$$\langle f_1|J(g)|f_2\rangle = \tfrac{1}{2}\langle f_1|\rho(g\cdot\nabla(f_1+f_2))|f_2\rangle \tag{3.11}$$

and

$$\langle f_1|H|f_2\rangle = \tfrac{1}{2}\langle f_1|\rho(\nabla f_1\cdot\nabla f_2)|f_2\rangle \tag{3.12}$$

Proof: Using time reversal invariance (eq. 3.7) we have

$$\langle f_1|J(g)|f_2\rangle = (TJ(g)e^{i\rho(f_2)}\,\Omega,\ Te^{i\rho(f_1)}\Omega)$$

$$= -(\Omega,\ e^{i\rho(f_2)}J(g)e^{-i\rho(f_1)}\Omega)$$

Substituting in eq. 3.9 twice and also using eq. 2.6 we obtain

$$\langle f_1|J(g)|f_2\rangle = -(\Omega,\ e^{-i\rho(f_1)}[J(g)-\rho(g\cdot\nabla(f_1+f_2))]e^{i\rho(f_2)}\Omega)$$

Therefore, $\langle f_1|J(g)|f_2\rangle = \tfrac{1}{2}\langle f_1|\rho(g\cdot\nabla(f_1+f_2))|f_2\rangle$

Next, by applying current conservation and using eqs. 3.5, 3.10 and 3.11 we have:

$$\langle f_1|H|f_2\rangle = (e^{i\rho(f_1)}\Omega,\ [H,e^{i\rho(f_2)}]\Omega)$$

$$= (e^{i\rho(f_1)}\Omega,\ [J(\nabla f_2) - \tfrac{1}{2}\rho(\nabla f_2\cdot\nabla f_2)]e^{i\rho(f_2)}\Omega)$$

$$= \tfrac{1}{2}\langle f_1|\rho(\nabla f_1\cdot\nabla f_2)|f_2\rangle$$

Remarks: (1) A hermitian form on a dense set of states does not necessarily determine an unbounded operator. If the form determines a hermitian operator it may have many (or no) self-adjoint extensions depending on the choice of its domain. Therefore, eqs. 3.11 and 3.12 are not sufficient to determine J and H as operators.

(2) As a result of eq. 3.12

$$(\rho(f_1)\Omega,\ H\ \rho(f_2)\Omega) = \tfrac{1}{2}(\Omega, \rho(\nabla f_1 \cdot \nabla f_2)\Omega)$$

In the N/V limit, for a translational invariant system

$$(\Omega, \rho(x)\Omega) = \bar{\rho}, \text{ the average density.}$$

Therefore, $(\rho(f_1)\Omega, H\ \rho(f_2)\Omega) = \frac{1}{2}\bar{\rho}\int dx\ \nabla f_1 \cdot \nabla f_2$

Now, using the matrix elements of H given by eq. 3.12 an expression for H will be derived in terms of $\rho(x)$ and $J(x)$. For this purpose, we first determine an operator $A(x,\rho)$ having the property that $K(g)\Omega = A(g,\rho)\Omega$. Consider a representation with Hilbert Space $\mathcal{H} = L^2_\mu(\mathcal{S}')$ and ground state $\Omega(F) = 1$. Let $A(g)$ be the operator of multiplication by $(K(g)\Omega)(F)$ defined by:

$$(A(g)\Phi)(F) = (K(g)\Omega)(F)\Phi(F) \text{ and}$$

Domain $A(g) = \{\Phi(F)\in\mathcal{H};\ \int d\mu(F)\ |(A(g)\Phi)(F)|^2 < \infty\}$

Since $e^{i\rho(f)}$ is multiplication by $e^{i(F,f)}$ we have $[A(g), e^{i\rho(f)}] = 0$.

Also $A(g)\ e^{i\rho(f)}\Omega = e^{i\rho(f)}K(g)\Omega$. As a result the domain of $A(g)$ includes the set $\mathcal{D}$ and therefore it is a dense set. By time reversal invariance it follows that $(K(g)\Omega)(F)^* = (K(g)\Omega)(F)$. Thus $A(g)$ is hermitian. Moreover $A(g)$ is self-adjoint. To prove this it is sufficient to show that $[\text{Range}(A \pm i)]^\perp = \{0\}$. Let $\Phi(F) \in [\text{Range}(A \pm i)]^\perp$.

Then $\int_{\mathcal{S}'} d\mu(F)\Phi(F)(A(g)\pm i)\Psi(F) = 0,\ \forall\Psi(F)\in$ Domain $A(g)$. Pick $\Psi(F) = \chi_C(F) =$ the characteristic function for the set $C \subset \mathcal{S}'$. Then

$$\int_C d\mu(F)\ \Phi(F)\ [(K(g)\Omega)(F) \pm i] = 0,\ \forall C \subset \mathcal{S}' \text{ and therefore}$$

$$\Phi(F)\ [(K(g)\Omega)(F) \pm i] = 0,$$

Since $[(K(g)\Omega)(F) \pm i] \neq 0$ we have $\Phi(F) = 0$. Therefore, $A(g)$ is self-adjoint.

It will be useful to express $A(\underset{\sim}{g})$ as a function of ρ. This is possible since the ρ's are multiplication operators and polynomials in ρ applied to the ground state are dense. We proceed as follows:

Let $\mathcal{F}=\{f_j;\ j = 1,2, \ldots\}$ be a countable dense set of test functions. (e.g. In Schwartz's space, finite linear combinations with rational coefficients of the Hermite functions.)

Let $\mathcal{D}' = \text{span}\ \{e^{i\rho(f)}\Omega;\ f\epsilon\mathcal{F}\}$. Since $\mathcal{D}$ is dense in $\mathcal{H}$, by the continuity of the representaion it follows that $\mathcal{D}'$ is also dense. However, the states $\{e^{i\rho(f)}\Omega;\ f\epsilon\mathcal{F}\}$ are neither orthogonal nor linearly independent. It is therefore convenient to orthogonalize them using the Gram-Schmit procedure.

Let $|h_1\rangle = U(f_1)\Omega$;

$|h_n\rangle = \sum_{j=1}^{n} a_j^{(n)} U(f_j)\Omega$, such that $(h_i, h_j) = \delta_{i,j}$.

Clearly, span $\{h_j;\ j = 1,2, \ldots\} = \mathcal{D}'$. Since this set is dense we can write

$$K(\underset{\sim}{g})\Omega = \sum_{n=1}^{\infty} b_n(\underset{\sim}{g})\{\sum_{j=1}^{n} a_j^{(n)} U(f_j)\}\Omega$$

The desired operator $A(\underset{\sim}{g},\rho)$ is defined by

$$A(\underset{\sim}{g},\rho) = \sum_{n=1}^{\infty} b_n(\underset{\sim}{g})\{\sum_{j=1}^{n} a_j^{(n)} U(f_j)\}$$

Furthermore, $K(\underset{\sim}{g})\Omega$ depends linearly on $\underset{\sim}{g}$. As a result $b_n(\underset{\sim}{g})$ is a linear distribution; $b_n(\underset{\sim}{g}) = \int d\underset{\sim}{x}\, \underset{\sim}{b}_n(\underset{\sim}{x})\cdot \underset{\sim}{g}(\underset{\sim}{x})$. Therefore we can write, $A(\underset{\sim}{g},\rho) = \int d\underset{\sim}{x}\, \underset{\sim}{g}(\underset{\sim}{x})\cdot \underset{\sim}{A}(\underset{\sim}{x},\rho)$ where $\underset{\sim}{A}(\underset{\sim}{x},\rho) = \sum_{n=1}^{\infty} \underset{\sim}{b}_n(\underset{\sim}{x})\{\sum_{j=1}^{n} a_j^{(n)} U(f_j)\}$.

Next, define $\underset{\sim}{\tilde{K}}(\underset{\sim}{x}) = \underset{\sim}{K}(\underset{\sim}{x}) - \underset{\sim}{A}(\underset{\sim}{x},\rho)$. By construction we have

$$\underset{\sim}{\tilde{K}}(\underset{\sim}{x})\Omega = 0 \text{ and} \tag{3.14}$$

$$[e^{i\rho(f)}, \underset{\sim}{\tilde{K}}(\underset{\sim}{x})] = [e^{i\rho(f)}, \underset{\sim}{K}(\underset{\sim}{x})] = -2i\underset{\sim}{\nabla} f(\underset{\sim}{x})\rho(\underset{\sim}{x})e^{i\rho(f)}$$

<u>Theorem 2</u>: $(1/8)\int d\underset{\sim}{x}\, \tilde{K}(\underset{\sim}{x})^{\dagger} \frac{1}{\rho(\underset{\sim}{x})} \tilde{K}(\underset{\sim}{x})$ is a well defined hermitian form with domain $\mathcal{D}$. Furthermore,

$$\langle\Phi_1|(1/8)\int d\underset{\sim}{x}\,\tilde{\underset{\sim}{K}}(\underset{\sim}{x})^{\dagger}\frac{1}{\rho(\underset{\sim}{x})}\tilde{\underset{\sim}{K}}(\underset{\sim}{x})|\Phi_2\rangle = (\Phi_1, H\,\Phi_2),\ \forall\Phi_1,\Phi_2\ \varepsilon\ \mathcal{D} \tag{3.15}$$

Proof: Observe that

$$\langle f_1|(1/8)\int d\underset{\sim}{x}\tilde{\underset{\sim}{K}}(\underset{\sim}{x})^{\dagger}\frac{1}{\rho(\underset{\sim}{x})}\tilde{\underset{\sim}{K}}(\underset{\sim}{x})|f_2\rangle = (1/8)\int d\underset{\sim}{x}\langle\tilde{\underset{\sim}{K}}(\underset{\sim}{x})e^{i\rho(f_1)}\Omega,\ \frac{1}{\rho(\underset{\sim}{x})}\tilde{\underset{\sim}{K}}(\underset{\sim}{x})e^{i\rho(f_2)}\Omega\rangle$$

$$= (1/8)\int d\underset{\sim}{x}\langle -2i\underset{\sim}{\nabla}f_1(\underset{\sim}{x})\rho(\underset{\sim}{x})e^{i\rho(f_1)}\Omega,\ \frac{1}{\rho(\underset{\sim}{x})}(-2i)\underset{\sim}{\nabla}f_2(\underset{\sim}{x})\rho(\underset{\sim}{x})e^{i\rho(f_2)}\Omega\rangle$$

$$= \tfrac{1}{2}(e^{i\rho(f_1)}\Omega,\ \rho(\underset{\sim}{\nabla}f_1\cdot\underset{\sim}{\nabla}f_2)e^{i\rho(f_2)}\Omega)$$

$$= \langle f_1|H|f_2\rangle$$

This can be extended by linearity to the domain $\mathcal{D}$.

Formal manipulations can easily be performed with this form of H. For example, we can verify current conservation:

$$[(1/8)\int d\underset{\sim}{x}\,\tilde{\underset{\sim}{K}}(\underset{\sim}{x})^{\dagger}\frac{1}{\rho(\underset{\sim}{x})}\tilde{\underset{\sim}{K}}(\underset{\sim}{x}),\rho(f)] = (1/8)\int d\underset{\sim}{x}\{\tilde{\underset{\sim}{K}}(\underset{\sim}{x})^{\dagger}\frac{1}{\rho(\underset{\sim}{x})}[\tilde{\underset{\sim}{K}}(\underset{\sim}{x}),\ \rho(f)]$$

$$+ [\tilde{\underset{\sim}{K}}(\underset{\sim}{x})^{\dagger},\ \rho(f)]\frac{1}{\rho(\underset{\sim}{x})}\tilde{\underset{\sim}{K}}(\underset{\sim}{x})\}$$

$$= \tfrac{1}{4}\int d\underset{\sim}{x}\{\tilde{\underset{\sim}{K}}(\underset{\sim}{x})^{\dagger}\frac{1}{\rho(\underset{\sim}{x})}\rho(\underset{\sim}{x})\underset{\sim}{\nabla}f(\underset{\sim}{x}) - \underset{\sim}{\nabla}f(\underset{\sim}{x})\rho(\underset{\sim}{x})\frac{1}{\rho(\underset{\sim}{x})}\tilde{\underset{\sim}{K}}(\underset{\sim}{x})\}$$

$$= \tfrac{1}{4}\int d\underset{\sim}{x}\,\underset{\sim}{\nabla}f(\underset{\sim}{x})[\tilde{\underset{\sim}{K}}(\underset{\sim}{x})^{\dagger} - \tilde{\underset{\sim}{K}}(\underset{\sim}{x})] = -iJ(\underset{\sim}{\nabla}f)$$

In the last step we used $\underset{\sim}{A}(\underset{\sim}{x},\rho)^{\dagger} = \underset{\sim}{A}(\underset{\sim}{x},\rho)$, which follows from time reversal invariance.

These manipulations can be cast into a rigorous form by showing that $(\Phi_1, H\rho(f)\Phi_2) - (\rho(f)\Phi_1,\ H\Phi_2) = -i(\Phi_1, J(\underset{\sim}{\nabla}f)\Phi_2),\ \forall\Phi_1,\ \Phi_2\ \varepsilon\ \mathcal{D}$ follows from eqs. 3.11 and 3.12.

In an alternative approach, only a representation of U(f) and $V(\underset{\sim}{\varphi})$ is assumed. Then the hermitian form in eq. 3.15 is used to define an operator with all the properties of a Hamiltonian. It is necessary to show the hermitian form is positive. This can be done if one assumes $(\Phi,\rho(f)\Phi)\geq 0,\ \forall\ f\varepsilon\mathcal{S}$ such that $f(\underset{\sim}{x}) \geq 0$ and $\Phi\varepsilon$ Domain of $\rho(f)$. This is physically necessary since the expectation value of the density in any state must be positive. In the representation with the Hilbert Space

$\mathcal{H} = L^2_\mu\ (\mathcal{S}')$,

$$(\Phi, \rho(f)\Phi) = \int d\mu(F)\ (F,f)\,|\Phi(F)|^2$$

Therefore, the measure is concentrated on functionals $F\varepsilon\mathcal{S}'$ such that $(F,f) \geq 0,\ \forall f(\underset{\sim}{x}) \geq 0$.

Theorem 3: $(1/8)\int d\underset{\sim}{x}\ \tilde{K}(\underset{\sim}{x})^\dagger \frac{1}{\rho(\underset{\sim}{x})} \tilde{K}(\underset{\sim}{x})$ is a positive hermitian form.

Proof: Let $\Phi = \sum_{j=1}^{n} a_j U(f_j)\Omega$.

$$\langle \Phi, (1/8)\int d\underset{\sim}{x}\ \tilde{K}(\underset{\sim}{x})^\dagger \frac{1}{\rho(\underset{\sim}{x})} \tilde{K}(\underset{\sim}{x})\Phi\rangle = \tfrac{1}{2}\sum_{j,k=1}^{n} a_k^*\, a_j \langle f_k|\rho(\underset{\sim}{\nabla} f_k \cdot \underset{\sim}{\nabla} f_j)|f_j\rangle$$

$$= \tfrac{1}{2}\int d\mu(F)\ (F, |\sum_{j=1}^{n} a_j \underset{\sim}{\nabla} f_j e^{i(F,f_j)}|^2) \geq 0$$

The following theorem of Friedrichs[17] tells us the hermitian form in eq. 3.15 defines a positive self-adjoint operator.

Friedrichs' Theorem: A positive semidefinite hermitian form $\{\psi_1, \psi_2\}$ defined on a dense linear set R in a Hilbert Space $\mathcal{H}$ can be extended by continuity to a positive semidefinite hermitian form on a larger linear set $R' > R$ which consists of elements $\psi\varepsilon\mathcal{H}$ such that, for some sequence $\psi_n \varepsilon R$, $||\psi - \psi_n|| \to 0$ and $\{\psi_n - \psi_m, \psi_n - \psi_m\} \to 0$. Furthermore, there exists a unique positive self-adjoint operator A such that $\mathcal{D}(A) \subset R'$ and $\{\psi_1, \psi_2\} = (\psi_1, A\psi_2)$, $\forall \psi_1 \varepsilon R'$ and $\psi_2 \varepsilon \mathcal{D}(A)$.

Therefore the expression $(1/8)\int d\underset{\sim}{x}\tilde{K}(\underset{\sim}{x})^\dagger \frac{1}{\rho(\underset{\sim}{x})} \tilde{K}(\underset{\sim}{x})$ can be used to define an operator with all the properties of a Hamiltonian. If we had begun with a Hamiltonian, it is not clear whether this would be the same as the one constructed from Friedrichs' Theorem due to the technical question concerning the domain of H. We will not pursue this matter further here.

Remarks: (1) Eqs. 3.11 and 3.12 and the result that eq. 3.12 defines a positive hermitian form have been obtained independently by Aref'eva[18] using different methods.

(2) Coester and Haag[19] have discussed a similar form for the Hamiltonian in terms of the canonical relativistic scalar fields $\varphi(\underset{\sim}{x})$ and $\pi(\underset{\sim}{x})$.

(3) There is an interesting similarity between the form of the Hamiltonian derived above and the Hamiltonian for a particle in a magnetic field.

$$H_0 = (1/8)\int d\underset{\sim}{x}\, \underset{\sim}{K}(\underset{\sim}{x})^\dagger \frac{1}{\rho(\underset{\sim}{x})} \underset{\sim}{K}(\underset{\sim}{x}) \longleftrightarrow H_c = p^2/2m$$

$$H = (1/8)\int d\underset{\sim}{x}[\underset{\sim}{K}(\underset{\sim}{x}) - \underset{\sim}{A}(\underset{\sim}{x},\rho)]^\dagger \frac{1}{\rho(\underset{\sim}{x})} [\underset{\sim}{K}(\underset{\sim}{x}) - \underset{\sim}{A}(\underset{\sim}{x},\rho)] \longleftrightarrow H = \frac{1}{2m}(p - \frac{e}{c} A)^2$$

In our case, for an interaction the free Hamiltonian is modified by $\underset{\sim}{K}(\underset{\sim}{x}) \to \underset{\sim}{K}(\underset{\sim}{x}) - \underset{\sim}{A}(\underset{\sim}{x},\rho)$ while in Q.M. the free Hamiltonian is modified by $p \to p - (e/c)A$. There is also a difference. In Q.M. $\dot{x}_{free} = \frac{p}{m} \to \dot{x} = \frac{1}{m}(p - \frac{e}{c} A)$ while in our case $\dot{\rho} = -\underset{\sim}{\nabla}\cdot\underset{\sim}{J}$ remains true for both the free case and the interaction.

(4) In terms of the canonical fields both the currents (eq. 2.2) and the Hamiltonian (eq. 3.1) have the same form for both Bosons and Fermions. In terms of the currents (as we will see in the following paper) the free Hamiltonian has a different form for Bosons and Fermions. This is not as surprising as it might appear at first sight. In Quantum Mechanics the free Hamiltonian for Bosons and Fermions is formally the same; $H = -\frac{1}{2}\sum_{j=1}^{n} \frac{\partial^2}{\partial x_j^2}$. However, the domains are different; symmetric functions for Bosons and antisymmetric functions for Fermions. As a result the Free Bose Hamiltonian and the Free Fermi Hamiltonian are different operators with distinct spectra[(13)].

(5) Hopefully there will be a systematic method for determining $\underset{\sim}{A}(\underset{\sim}{x},\rho)$ for a given potential. Eq. 3.4 might be used as a guide towards this end.

4. Functional Differential Equation for L(f).

Using the results of the previous section we will derive a functional differential equation for the generating functional L(f). When supplemented by the appropriate boundary conditions this equation can be used to determine L(f) and hence a representation corresponding to a given physical system. This has been done in great detail for the Free Bose Gas in ref. 20 (see also Goldin and Sharp[21]).

We start with the ground state condition (eq. 3.14), $\tilde{K}(x)\Omega = 0$. Forming the inner product of $\tilde{K}(x)\Omega$ with $e^{-i\rho(f)}\Omega$ we find $0 = (\Omega, e^{i\rho(f)}\tilde{K}(x)\Omega)$. Using the definition of $\tilde{K}(x)$ and eq. 3.11 we then have

$$0 = (\Omega, e^{i\rho(f)}[\nabla\rho(x) - i\nabla f(x)\rho(x)]\Omega) - (\Omega, e^{i\rho(f)}A(x,\rho)\Omega), \quad (4.1)$$

Both terms can be evaluated using functional derivatives of L(f). Since $\frac{1}{i}\frac{\delta}{\delta f(x)}L(f) = (\Omega, e^{i\rho(f)}\rho(x)\Omega)$, eq. 4.1 can be written

$$[\nabla - i\nabla f(x)]\frac{1}{i}\frac{\delta}{\delta f(x)}L(f) = A(x, \frac{1}{i}\frac{\delta}{\delta f})L(f) \quad (4.2)$$

The solutions of this equation which are physically admissable are restricted by several conditions. These include the general properties (eqs. 2.11 - 2.14) of a generating functional, namely:

(1) $L(f) = L(-f)^*$

(2) $L(0) = 1$

(3) $|L(f)| \leq 1$

(4) L(f) is a positive functional.

Other conditions may include,

(5) L(f) is an extremal solution in the sense that it cannot be written as a convex linear combination of two other solutions. This has the effect of requiring the representation of U(f) and $V(\varphi)$ to be irreducible (see ref. 20, Th. 3.4).

In the N/V limit we can also use translational invariance or the cluster decomposition property. (These will be explained further in the next section.)

(6) $\frac{1}{i} \frac{\delta}{\delta f(\underset{\sim}{x})} L(f)\Big|_{f=0} = (\Omega, \rho(\underset{\sim}{x})\Omega) = \overline{\rho}$

(7) $L(f) = L(f_{\underset{\sim}{a}})$, where $f_{\underset{\sim}{a}}(\underset{\sim}{x}) = f(\underset{\sim}{x}-\underset{\sim}{a})$

(8) $\lim_{\underset{\sim}{a}\to\infty} L(f + h_{\underset{\sim}{a}}) = L(f)\, L(h)$, where $h_{\underset{\sim}{a}}(\underset{\sim}{x}) = h(\underset{\sim}{x} - \underset{\sim}{a})$

For the Free Bose Gas eq. 4.2 becomes, $(\underset{\sim}{\nabla} - i\underset{\sim}{\nabla} f(\underset{\sim}{x}))\frac{1}{i} \frac{\delta}{\delta f(\underset{\sim}{x})} L(f) = 0$. In this case it is known[20] that conditions (2) - (6) uniquely determine L(f). It is not known whether these conditions are sufficient in other cases. Furthermore, it is not yet known how to determine the $\underset{\sim}{A}(\underset{\sim}{x},\rho)$ corresponding to a specific interaction. However, in the following paper, $\underset{\sim}{A}(\underset{\sim}{x},\rho)$ and L(f) are given explicitly in the N/V limit and eq. 4.2 is verified for three additional cases:

(1) Bosons in an external potential, $\underset{\sim}{A}(\underset{\sim}{x},\rho) = \rho(\underset{\sim}{x})\underset{\sim}{\nabla} \ln\overline{\rho}(\underset{\sim}{x})$

(2) Free Fermi Gas in 1 dimension, $A(x,\rho) = 2\rho(x)\int \frac{dy}{x-y}\,\rho(y)$

(3) $2/x^2$ interaction in 1 dimension, $A(x,\rho) = 4\rho(x)\int \frac{dy}{x-y}\,\rho(y)$.

5. L(f) in the N/V Limit.

In this section we discuss some general properties of the generating functional L(f) in the N/V limit. First, for an N-particle representation we find an expression for L(f) in terms of correlation functions. This form of L(f) is extended to the N/V limit when the correlation functions satisfy appropriate bounds. Next, we consider the consequences of translational invariance and the cluster decomposition property. It is shown that different generating functionals give rise to unitarily inequivalent representations of U(f). Finally, the particle nature of the N/V limit representation is examined.

A. Expansion of L(f) in terms of Correlation Functions.

The N-particle representations of the current algebra (eq. 2.3) have been studied by Grodnik and Sharp[11], and Goldin[6]. We will use the correspondence between these representations and conventional Quantum Mechanics to obtain an expression for L(f) in terms of correlation functions. An N-particle representation is defined on the Hilbert Space,

$$\mathcal{H} = \begin{cases} L^2_S(R^N) & \text{; The totally symmetric functions for Bosons.} \\ L^2_A(R^N) & \text{; The totally anti-symmetric functions for Fermions.} \end{cases}$$

Acting on $\Psi(\underset{\sim}{x}_1, \ldots \underset{\sim}{x}_N) \in \mathcal{H}$,

$$\begin{aligned} \rho(\underset{\sim}{x})\Psi(\underset{\sim}{x}_1, \ldots \underset{\sim}{x}_N) &= \sum_{k=1}^{N} \delta(\underset{\sim}{x}-\underset{\sim}{x}_k)\Psi(\underset{\sim}{x}_1, \ldots \underset{\sim}{x}_N), \text{ or} \\ \rho(f)\Psi(\underset{\sim}{x}_1, \ldots \underset{\sim}{x}_N) &= \sum_{k=1}^{N} f(\underset{\sim}{x}_k)\ \Psi(\underset{\sim}{x}_1, \ldots \underset{\sim}{x}_N) \end{aligned} \tag{5.1}$$

and

$$\underset{\sim}{J}(\underset{\sim}{x})\Psi(\underset{\sim}{x}_1, \ldots \underset{\sim}{x}_N) = \frac{1}{2i}\sum_{k=1}^{N}(-\underset{\sim}{\nabla}_x\delta(\underset{\sim}{x}-\underset{\sim}{x}_k)+2\delta(\underset{\sim}{x}-\underset{\sim}{x}_k)\underset{\sim}{\nabla}_{x_k})\Psi(\underset{\sim}{x}_1, \ldots \underset{\sim}{x}_N)$$

or, (5.2)

$$J(\underset{\sim}{g})\Psi(\underset{\sim}{x}_1, \ldots \underset{\sim}{x}_N) = \frac{1}{2i}\sum_{k=1}^{N}(2\underset{\sim}{g}(\underset{\sim}{x}_k)\cdot\underset{\sim}{\nabla}_{x_k}+(\underset{\sim}{\nabla}\cdot\underset{\sim}{g})(\underset{\sim}{x}_k))\Psi(\underset{\sim}{x}_1, \ldots \underset{\sim}{x}_N)$$

The generating functionals are given by

$$L(f)=(\Omega, e^{i\rho(f)}\Omega) = \int d\underset{\sim}{x}_1 \ldots \int d\underset{\sim}{x}_N e^{if(\underset{\sim}{x}_1)} \ldots e^{if(\underset{\sim}{x}_N)}\Omega^*\Omega(\underset{\sim}{x}_1, \ldots \underset{\sim}{x}_N) \tag{5.3}$$

where $\Omega(\underset{\sim}{x}_1 \ldots \underset{\sim}{x}_N)$ = The ground state wave function, and

$$\begin{aligned} L(f,\underset{\sim}{g}) &= (\Omega, e^{i\rho(f)}e^{iJ(\underset{\sim}{g})}\Omega) \\ &= \int d\underset{\sim}{x}_1 \ldots \int d\underset{\sim}{x}_n \Omega^*(\underset{\sim}{x}_1 \ldots \underset{\sim}{x}_N)\prod_{k=1}^{N} e^{if(\underset{\sim}{x}_k)}e^{ij(\underset{\sim}{x}_k,\underset{\sim}{g})}\Omega(\underset{\sim}{x}_1 \ldots \underset{\sim}{x}_N) \end{aligned} \tag{5.4}$$

where $j(\underset{\sim}{x},\underset{\sim}{g}) = \frac{1}{2i}\left[2\underset{\sim}{g}(\underset{\sim}{x})\cdot\underset{\sim}{\nabla} + (\underset{\sim}{\nabla}\cdot\underset{\sim}{g})(\underset{\sim}{x})\right]$

Remarks:

(1) One can write,

$$e^{ij(x,g)}\psi(x)=\psi\circ\varphi(x)\left[\det \frac{\partial}{\partial x_m}\varphi_n(x)\right]^{\frac{1}{2}}$$

where φ is the flow corresponding to the vector field g. The factor

$\left[\det \frac{\partial}{\partial x_m}\varphi_n(x)\right]$ is the Jacobian of the transformation, $x \to \varphi(x)$,

and is necessary in order for $e^{iJ(g)}$ to be unitary. (See ref. 6)

(2) $\mathcal{H}$ is unitarily equivalent to $L^2_\mu(\mathscr{S}')$ where the measure is concentrated on $\{F\in\mathscr{S}';\ F=\sum_{k=1}^{N}\delta(x-x_k)\}$ and $d\mu(F)=d\Omega^*\Omega(x_1\ldots x_N)$.

Furthermore, the ground state is given by $\Omega(F)=1$. Boson and Fermion representations are distinguished by the multipliers $\chi_\varphi(F)$.

For a representation defined by $L(f,g)$ it is convenient to think in terms of the n-point functions, $(\Omega,\rho(x_1)\ldots\rho(x_m)J(x_{m+1})\ldots J(x_n)\Omega)$, instead of the measure and multipliers on $\mathcal{H}=L^2_\mu(\mathscr{S}')$. By the Reconstruction Theorem (see ref. 22) the n-point functions determine a representation of the current algebra. All the n-point functions can be obtained by taking functional derivatives of $L(f,g)$. Therefore, $L(f,g)$ determines a representation of the current algebra.

Remarks:

(1)There is a slight complication in determining the n-point functions from $L(f,g)$. The ρ's are obtained directly by taking functional derivatives.

$$\frac{1}{i}\frac{\delta}{\delta f(x_1)}\cdots\frac{1}{i}\frac{\delta}{\delta f(x_m)}L(f,0)\Big|_{f=0}=(\Omega,\rho(x_1)\ldots\rho(x_m)\Omega)$$

Since the J's do not commute

$$\frac{1}{i}\frac{\delta}{\delta g(x_1)}\cdots\frac{1}{i}\frac{\delta}{\delta g(x_m)}L(0,g)\Big|_{g=0}=\frac{1}{m!}\sum_\pi(\Omega,J(x_{\pi_1})\ldots J(x_{\pi_m})\Omega)$$

where $\sum_\pi$ = the sum over all permutations of (1,2,...m).

However, by using the commutation relations (2.3), $(\Omega,J(x_1)\ldots J(x_m)\Omega)$

can be obtained inductively from $\frac{1}{i}\frac{\delta}{\delta g(x_1)}\cdots\frac{1}{i}\frac{\delta}{\delta g(x_m)}L(0,g)\big|_{g=0}$ plus the n-point functions of lower order ($n < m$).

(2) The J's (in the n-point functions) can be replaced by ρ's using the operator $A(x,\rho)$ defined in section 3.

$$J(x_1)\Omega = -\tfrac{1}{2}i[A(x_1,\rho) - \nabla\rho(x_1)]\Omega$$

$$J(x_1)J(x_2)\Omega = -\tfrac{1}{2}i\{[J(x_1),(A(x_2,\rho)-\nabla\rho(x_2))] + (A(x_2,\rho)-\nabla\rho(x_2))J(x_1)\}\Omega$$

Using the functional representation[11] $J(x) = \rho(x)\frac{1}{i}\nabla\frac{\delta}{\delta\rho(x)} + F(\rho(x))$,

$$[J(x_1), A(x_2,\rho)] = \rho(x_1)\frac{1}{i}\nabla_{x_1}\frac{\delta}{\delta\rho(x_1)}A(x_2,\rho)$$

Thus $J(x_1)J(x_2)\Omega$ can be obtained from a function of ρ on Ω. This procedure can be extended to $J(x_1)\ldots J(x_n)\Omega$ Therefore, a representation of the current algebra is determined by $A(x,\rho)$ and $L(f)$, provided the derivatives of $A(x,\rho)$ are well behaved. Goldin[6] used an expression similar to $A(x,\rho)$ to give rigorous sufficient conditions for recovering a representation of the current algebra from that of the exponentiated currents.

The n-point functions of ρ can be related to the correlation functions, which are defined as follows (for the N-particle representation):

$$R_n(x_1,\ldots,x_n) = \begin{cases} 1 & \text{for } n=0 \\ N!/(N-n)!\int dx_{n+1}\cdots\int dx_N\,\Omega^*\Omega(x_1,\ldots,x_N), & 0<n\leq N \\ 0 & N<n \end{cases}$$

Using the symmetry of the wave function and eq. 5.1 we obtain

$$(\Omega,\rho(x_1)\Omega) = R_1(x_1)$$

$$(\Omega,\rho(x_1)\rho(x_2)\Omega) = R_2(x_1,x_2) + \delta(x_1-x_2)R_1(x_1)$$

$$(\Omega,\rho(x_1)\rho(x_2)\rho(x_3)\Omega) = R_3(x_1,x_2,x_3) + \sum_{\text{perm}}\delta(x_1-x_2)R_2(x_2,x_3) + \delta(x_1-x_2)\delta(x_1-x_3)R_1(x_1)$$

Thus $(\Omega,\rho(x_1)\ldots\rho(x_n)\Omega)$ is the sum of n terms, each term being the sum over permutations of the variables $x_1, \ldots, x_n$ of the product of m delta functions multiplied by R_{n-m}.

Remark: The above expressions are independent of the number of particles in the representation. As we will see they are also true in the N/V limit. If $\rho(\underset{\sim}{x})$ can be written in terms of the canonical field operators as, $\rho(\underset{\sim}{x}) = \psi(\underset{\sim}{x})^{\dagger}\psi(\underset{\sim}{x})$ (eq. 2.2), the correlation functions are the n-point functions for the canonical fields. $R_n(\underset{\sim}{x}_1,\ldots,\underset{\sim}{x}_n) = (\Omega,\psi^{\dagger}(\underset{\sim}{x}_1)\ldots\psi^{\dagger}(\underset{\sim}{x}_n)\psi(\underset{\sim}{x}_n)\ldots\psi(\underset{\sim}{x}_1)\Omega)$.

The correlation functions have the physical interpretation,

$$(1/n!)R_n(\underset{\sim}{x}_1,\ldots,\underset{\sim}{x}_n) = \begin{cases}\text{The probability of finding n particles at} \\ \text{the points } \underset{\sim}{x}_1,\ldots,\underset{\sim}{x}_n \text{ regardless of the positions} \\ \text{of the remaining particles.}\end{cases}$$

We can now obtain an expression for L(f) in terms of the correlation functions. Let $F(\underset{\sim}{x}) = e^{if(\underset{\sim}{x})}-1$, and note that

$$e^{if(\underset{\sim}{x}_1)} = F(\underset{\sim}{x}_1) + 1,$$

$$e^{if(\underset{\sim}{x}_1)}e^{if(\underset{\sim}{x}_2)} = F(\underset{\sim}{x}_1)F(\underset{\sim}{x}_2) + F(\underset{\sim}{x}_1) + F(\underset{\sim}{x}_2) + 1,$$

$$e^{if(\underset{\sim}{x}_1)}\ldots e^{if(\underset{\sim}{x}_n)} = \sum_{\text{perm}}\sum_{j=1}(1/(j!(n-j)!))\prod_{k=1}^{j}F(\underset{\sim}{x}_{\pi_k}) \qquad (5.5)$$

Substituting eq. 5.5 into eq. 5.3 for L(f) and using the symmetry of the wave function Ω and appropriate change of variable lables in the integrals, we obtain

$$L(f) = \sum_{n=0}^{N}(1/n!)\int d\underset{\sim}{x}_1\ldots\int d\underset{\sim}{x}_n F(\underset{\sim}{x}_1)\ldots F(\underset{\sim}{x}_n)R_n(\underset{\sim}{x}_1,\ldots,\underset{\sim}{x}_n) \qquad (5.6)$$

(As a check notice the leading term, the one without any δ functions, in the n-point function $(\Omega,\rho(\underset{\sim}{x}_1)\ldots\rho(\underset{\sim}{x}_n)\Omega) = \frac{1}{i}\frac{\delta}{\delta f(\underset{\sim}{x}_1)}\cdots\frac{1}{i}\frac{\delta}{\delta f(\underset{\sim}{x}_n)}$ $L(f)|_{f=0}$ is just $R_n(\underset{\sim}{x}_1,\ldots,\underset{\sim}{x}_n)$.)

In order to carry out the N/V limit we introduce the following notation:

Let $R_n^{(N)}$ = The n^{th} correlation function for N particles in a box of volume V, and

Let $a_n^{(N)} = \int_V d\underset{\sim}{x}_1 \ldots \int_V d\underset{\sim}{x}_n F(\underset{\sim}{x}_1) \ldots F(\underset{\sim}{x}_n) R_n^{(N)}(\underset{\sim}{x}_1, \ldots, \underset{\sim}{x}_n)$

The generating functional for the N-particle representation can now be written as, $L_N(f) = \sum_{n=0}^{\infty} (1/n!) a_n^{(N)}$.

If the N/V limit is to exist we might expect $R_n^{(N)} \to R_n$ $\forall n$ and

$L_N(f) \to L(f) = \sum_{n=0}^{\infty} (1/n!) a_n$ where $a_n = \int_{-\infty}^{\infty} d\underset{\sim}{x}_1 \ldots \int_{-\infty}^{\infty} d\underset{\sim}{x}_n F(\underset{\sim}{x}_1) \ldots F(\underset{\sim}{x}_n) R_n(\underset{\sim}{x}_1 \ldots \underset{\sim}{x}_n)$.

In the next theorem we give sufficient conditions for the N/V limit of L(f) to exist. These conditions are probably adequate for most physical systems. (They will be used in the following paper to explicitly calculate L(f) in the N/V limit for several examples.)

Theorem 4: If $R_n^{(N)}(\underset{\sim}{x}_1, \ldots, \underset{\sim}{x}_n) \to R_n(\underset{\sim}{x}_1, \ldots, \underset{\sim}{x}_n)$ and $|R_n^{(N)}| \le c^n n^{n/2}$ $\forall n, N$ for some constant c, Then $L_N(f) \to L(f)$.

Remark: Girrard[23] used an expression similar to eq. 5.6 in studying the thermodynamics of a Free Bose Gas in terms of the local current algebra. The proof given below is essentially the same as the one he used.

Lemma: The series $S(c) = \sum_{n=0}^{\infty} (1/n!) c^n n^{n/2}$ converges for all c.

Proof: We use the ratio test. Let S_n = the ratio of the n+1th term to the nth term. Then

$$\begin{aligned} S_n &= \frac{(1/(n+1)!)\; c^{n+1}\; (n+1)^{(n+1)/2}}{(1/n!)\; c^n\; n^{n/2}} \\ &= c(n+1)^{(n-1)/2} / n^{n/2} \\ &= c(n+1)^{-\frac{1}{2}} (1 + 1/n)^{n/2} \\ &\to c\; 0\; e^{\frac{1}{2}} = 0 \end{aligned}$$

Therefore the series for S converges.

Proof of Theorem: Since $R_n^{(N)} \to R_n$ and $|R_n^{(N)}| \le c^n n^{n/2}$ it follows that $|R_n| \le c^n n^{n/2}$. As a result

$$|a_n| \le \int dx_1 \ldots \int dx_n |F(x_1)\ldots F(x_n)R_n|$$
$$\le (c\int dx|e^{if(x)}-1|)^n n^{n/2}$$

Let $\bar{c} = c\int dx|e^{if(x)}-1|$. The series for L(f) is bounded term by term by the series for $S(\bar{c})$. Therefore L(f) converges. Furthermore, the series for $L_N(f)$ and L(f) converge uniformly. We now show $L_N(f) \to L(f)$. First notice that there exists an n_0 such that for $N > n_0$, $|S(\bar{c}) - \sum_{n=1}^{N} (1/n!)\, \bar{c}^n n^{n/2}| < \varepsilon/4$. Furthermore, there exists an N_0 such that for $N > N_0$, $(1/n!)|a_n^{(N)} - a_n| < \varepsilon/2n_0$ for $n \le n_0$.

Then, for $N > n_0$ and N_0, we have

$$|L_N(f)-L(f)| \le |L_N(f) - \sum_{n<n_0} (1/n!)a_n^{(N)}| + |L(f) - \sum_{n<n_0} (1/n!)a_n| + \sum_{n<n_0} (1/n!)|a_n^{(N)}-a_n|$$

$$\le \varepsilon/4 + \varepsilon/4 + n_0(\varepsilon/2n_0) = \varepsilon$$

Since ε is arbitrary $L_N(f) \to L(f)$.

Remark: In order for L(f) to be a generating functional for a representation of U(f), it must satisfy eqs. 2.11-2.14. These equations are preserved when limits are taken. Since the $L_N(f)$ satisfy them, it follows that L(f) also satisfies them. Therefore, L(f) defines a generating functional.

An alternative expression for L(f) can be obtained in terms of the cluster functions[24] of the correlation functions. These are defined as, $T_n(x_1,\ldots,x_n) = \sum_G (-)^{m-n}(m-1)! \prod_{j=1}^{m} R_{G_j}(x_k \in G_j)$ where G = a partition of (1,2,...,n) into subsets $(G_1,G_2,\ldots,G_m)$. L(f) can be expressed in terms of T_n as follows:

$$L(f) = \exp \sum_{n=1}^{\infty} (-)^{n-1}(1/n!)\int dx_1 \ldots \int dx_n F(x_1)\ldots F(x_n)T_n(x_1\ldots x_n) \quad (5.7)$$

Remark: T_n is the non-relativistic analogue of the truncated n-point functions[25] in relativistic field theory.

B. Translational Invariance and the Cluster Decomposition Property.

Translational invariance and the cluster decomposition property play an important role in determining representation of the local currents in the N/V limit. A representation of U(f) and $V(\underset{\sim}{\varphi})$ is translational invariant if there is a set of unitary operators $Q(\underset{\sim}{a})$, continuous in $\underset{\sim}{a}$, such that:

(i) $Q(\underset{\sim}{a}_1)Q(\underset{\sim}{a}_2) = Q(\underset{\sim}{a}_1 + \underset{\sim}{a}_2)$ (5.8)

(ii) $Q(\underset{\sim}{a})U(f)Q(\underset{\sim}{a})^{-1} = U(f_{\underset{\sim}{a}})$, where $f_{\underset{\sim}{a}}(\underset{\sim}{x}) = f(\underset{\sim}{x}-\underset{\sim}{a})$ (5.9)

(iii) $Q(\underset{\sim}{a})V(\underset{\sim}{\varphi})Q(\underset{\sim}{a})^{-1} = V(\underset{\sim}{\varphi}_{\underset{\sim}{a}})$, where $\underset{\sim}{\varphi}_{\underset{\sim}{a}}(\underset{\sim}{x}) = \underset{\sim}{\varphi}(\underset{\sim}{x}-\underset{\sim}{a}) + \underset{\sim}{a}$ (5.10)

(iv) $Q(\underset{\sim}{a})\Omega = \Omega$ (5.11)

These conditions are equivalent to the requirement that the generating functional is translational invariant, i.e.

$$L(f_{\underset{\sim}{a}},\underset{\sim}{\varphi}_{\underset{\sim}{a}}) = L(f,\underset{\sim}{\varphi}). \qquad (5.12)$$

Also, the correlation functions are translational invariant, i.e.

$$R_n(\underset{\sim}{x}_1 + \underset{\sim}{a},\underset{\sim}{x}_2+\underset{\sim}{a},\ldots,\underset{\sim}{x}_n+\underset{\sim}{a}) = R_n(\underset{\sim}{x}_1,\underset{\sim}{x}_2,\ldots \underset{\sim}{x}_n) \qquad (5.13)$$

Furthermore, $R_1(\underset{\sim}{x}) = (\Omega,\rho(\underset{\sim}{x})\Omega) = \bar{\rho}$, the average density.

The cluster decomposition property is based on the physical idea that as particles get far apart their interaction becomes negligible. This condition can be expressed in terms of correlation functions by requiring

$$\lim_{\lambda\to\infty}\{R_{n+m}(\underset{\sim}{x}_1\ldots\underset{\sim}{x}_n,\underset{\sim}{y}_1+\lambda\underset{\sim}{a},\ldots\underset{\sim}{y}_m+\lambda\underset{\sim}{a}) - R_n(\underset{\sim}{x}_1\ldots\underset{\sim}{x}_n)R_m(\underset{\sim}{y}_1+\underset{\sim}{a},\ldots\underset{\sim}{y}_m+\lambda\underset{\sim}{a})\} = 0 \qquad (5.14)$$

Combined with translational invariance we then have

$$\lim_{\lambda\to\infty} R_{n+m}(\underset{\sim}{x}_1\ldots\underset{\sim}{x}_n,\underset{\sim}{y}_1+\lambda\underset{\sim}{a},\ldots\underset{\sim}{y}_m+\lambda\underset{\sim}{a}) = R_n(\underset{\sim}{x}_1\ldots\underset{\sim}{x}_n)R_m(\underset{\sim}{y}_1\ldots\underset{\sim}{y}_m) \qquad (5.15)$$

Using eq. 5.6 this implies

$$\lim_{\lambda\to\infty} L(f+h_{\lambda\underset{\sim}{a}}) = L(f)L(h)\ , \text{ where } h_{\lambda\underset{\sim}{a}}(\underset{\sim}{x}) = h(\underset{\sim}{x}-\lambda\underset{\sim}{a}) \qquad (5.16)$$

This relation can be used as a boundary condition in determining physical solutions of the functional equation (4.2) for L(f). (See Appendix 1 for an example.)

Remark: The cluster decomposition property can also be expressed in terms of the cluster functions of the correlation functions as follows:

Let $r(\underset{\sim}{x}_1,\ldots,\underset{\sim}{x}_n) = \left(\begin{array}{l}\text{The radius of the smallest ball containing}\\ \text{the point; } \underset{\sim}{x}_1,\ldots,\underset{\sim}{x}_n.\end{array}\right)$

Then $T_n(\underset{\sim}{x}_1,\ldots,\underset{\sim}{x}_n) \to 0$ as $r(\underset{\sim}{x}_1,\ldots,\underset{\sim}{x}_n) \to \infty$

Translational invariance and the cluster decomposition property have important consequences in relativistic quantum field theory. We will discuss the corresponding results for the non-relativistic local current algebra. This discussion is greatly facilitated by the application of some results of Araki[8]. The next theorem shows that the ground state is unique.

Theorem: Suppose the generating functional $L(f) = (\Omega, U(f)\Omega)$ defines a continuous unitary representation of U(f) satisfying the cluster decomposition property and translational invariance. Then any state Ω' invariant under $Q(\underset{\sim}{a})$ up to a factor (i.e. $Q(\underset{\sim}{a})\Omega' = W(\underset{\sim}{a})\Omega'$, where $W(\underset{\sim}{a})$ is a complex number) is a multiple of Ω.

Proof: (See Araki[8], Th. 6.1)

Thus the ground state is the only translational invariant state. The generators of the translation operators are the momentum operators; i.e. $Q(\underset{\sim}{a}) = \exp i\underset{\sim}{a}\cdot\underset{\sim}{P}$, where $\underset{\sim}{P}$ = The total momentum operator. Suppose the state $|\underset{\sim}{p}>$ is a momentum eigenstate, then $Q(\underset{\sim}{a})|\underset{\sim}{p}> = e^{i\underset{\sim}{a}\cdot\underset{\sim}{p}}|\underset{\sim}{p}>$. By the above theorem $|\underset{\sim}{p}>$ is a multiple of Ω. Therefore Ω is the only momentum eigenstate. (Furthermore, $\underset{\sim}{P}\Omega = 0$.)

The above theorem has an additional consequence.

<u>Corollary</u>: Suppose the generating functional $L(f) = (\Omega, U(f)\Omega)$ determines a continuous unitary representation of $U(f)$ satisfying the cluster decomposition property and translational invariance. Then the set of operators $B = \{U(f), Q(\underset{\sim}{a})\}$ is irreducible. (i.e. Any bounded operator that commutes with every operator in the set B is a multiple of the identity.)

<u>Proof</u>: (See Araki[8], section 6.)

In bounded regions the translation operators are similar to the operators $V(\underset{\sim}{\varphi})$. In fact, if the flow $\underset{\sim}{\xi}_a(\underset{\sim}{x}) = \underset{\sim}{x}+\underset{\sim}{a}$, is a valid test function then it follows from the multiplication law (eq. 2.6) that,

$V(\underset{\sim}{\xi}_a)V(\underset{\sim}{\xi}_b)=V(\underset{\sim}{\xi}_{a+b})$, $V(\underset{\sim}{\xi}_a)U(f)V(\underset{\sim}{\xi}_a)^{-1}=U(f_a)$ and $V(\underset{\sim}{\xi}_a)V(\underset{\sim}{\varphi})V(\underset{\sim}{\xi}_a)^{-1}=V(\underset{\sim}{\varphi}_a)$.

Thus $V(\underset{\sim}{\xi})$ behaves like a translation operator (except for $V(\underset{\sim}{\xi})\Omega = \Omega$). However, we have been considering only continuous representations. Therefore it is necessary to impose a topology on the set of flows. Goldin[26] has discussed this point. He suggests a topology on a restricted set of flows $\underset{\sim}{\varphi}$ for which $\underset{\sim}{\varphi}(\underset{\sim}{x}) \to \underset{\sim}{x}$ as $|\underset{\sim}{x}| \to \infty$. Thus $\underset{\sim}{\xi}_a$ would not be in the set of test functions. In order to obtain the translation operators from $V(\underset{\sim}{\varphi})$ we are led to consider a sequence of flows $\underset{\sim}{\varphi}_n$ converging to $\underset{\sim}{\xi}_a$. The next theorem gives a sufficient condition for $V(\underset{\sim}{\varphi}_n) \to Q(\underset{\sim}{a})$.

<u>Theorem 5</u>: Let $\underset{\sim}{\varphi}_n$ be a sequence of flows such that $f \circ \underset{\sim}{\varphi}_n \to f_a, \forall f \in \mathscr{S}$ and $\underset{\sim}{\varphi}_n^{-1} \circ \underset{\sim}{\varphi} \circ \underset{\sim}{\varphi}_n \to \underset{\sim}{\varphi}_a$ for all flows $\underset{\sim}{\varphi}$. If $(\Omega, V(\underset{\sim}{\varphi}_n)\Omega) \to 1$, then $V(\underset{\sim}{\varphi}_n) \to Q(\underset{\sim}{a})$.

<u>Proof</u>: Since $(\Omega, V(\underset{\sim}{\varphi}_n)\Omega) \to 1$, it follows $V(\underset{\sim}{\varphi}_n)\Omega \to \Omega$

Let $D = \text{span}\,\{U(f)V(\underset{\sim}{\varphi})\Omega ; f \in \mathscr{S}$ and $\underset{\sim}{\varphi} \in$ flows$\}$. D is a dense set for any representation defined from a generating functional $L(f, \underset{\sim}{\varphi})$.

Let $\psi \in D$. We will show $V(\underset{\sim}{\varphi}_n)\psi \to Q(\underset{\sim}{a})\psi$

$$||V(\underset{\sim}{\varphi}_n)\psi - Q(\underset{\sim}{a})\psi||^2 = ||V(\underset{\sim}{\varphi}_n)\psi||^2 + ||Q(\underset{\sim}{a})\psi||^2 - (V(\underset{\sim}{\varphi}_n)\psi, Q(\underset{\sim}{a})\psi) - (Q(\underset{\sim}{a})\psi, V(\underset{\sim}{\varphi}_n)\psi)$$

Since $V(\underset{\sim}{\varphi})$ and $Q(\underset{\sim}{a})$ are unitary, $||V(\underset{\sim}{\varphi}_n)\psi|| = ||\psi|| = ||Q(\underset{\sim}{a})\psi||$

Since $\psi \in D$ we can write $\psi = \sum_{j=1}^{m} b_j U(f_j) V(\underset{\sim}{\varphi}_j)\Omega$. Then,

$$(V(\underset{\sim}{\varphi}_n)\psi, Q(\underset{\sim}{a})\psi) = \sum_{j=1}^{m} b_j^* (V(\underset{\sim}{\varphi}_n)U(f_j)V(\underset{\sim}{\varphi}_j)\Omega, Q(\underset{\sim}{a})\psi)$$

$$= \sum_{j=1}^{m} b_j^{*} \; (V(\varphi_n)\Omega, V(\varphi_n^{-1} \circ \varphi_j \circ \varphi_n)^{-1} U(\qquad f_j \circ \varphi_n)^{-1} Q(a)\psi)$$

Since the representations we are considering are strongly continuous,

$$V(\varphi_n^{-1} \circ \varphi_j \circ \varphi_n)^{-1} U(\qquad f_j \circ \varphi_n)^{-1} Q(a)\psi \to V(\varphi_{ja})^{-1} U(f_{ja})^{-1} Q(a)\psi$$

and since $V(\varphi_n)\Omega \to \Omega$, we have

$$\begin{aligned}(V(\varphi_n)\psi, Q(a)\psi) &\to \sum_{j=1}^{n} b_j^{*}(\Omega, V(\varphi_{ja})^{-1} U(f_{ja})^{-1} Q(a)\psi) \\ &= \sum_{j=1}^{n} b_j^{*}(U(f_{ja})V(\varphi_{ja})\Omega, \; Q(a)\psi) \\ &= (Q(a)\psi, \; Q(a)\psi) = ||\psi||^2\end{aligned}$$

Therefore $||V(\varphi_n)\psi - Q(a)\psi|| \to 0$. Since D is dense it follows $V(\varphi_n) \to Q(a)$

Remark: Theorem 5 has a physical interpretation. Since $J(x)$ is the momentum density, we expect $\int_{-\infty}^{\infty} J(x)\cdot a = a\cdot P$ where P = the total momentum operator. Thus, $e^{it\int J(x)\cdot a dx} = e^{ita\cdot P} = Q(ta)$.

But $e^{itJ(g)} = V(\varphi_t)$ where φ_t is the flow corresponding to the vector field g; i.e. $\frac{d}{dt}\varphi_t(x) = g \circ \varphi_t(x)$ and $\varphi_{t=0}(x) = x$. For $g(x) = a$, $\varphi_t(x) = x + ta$. Thus we expect $Q(a) = V(\xi_a)$ where $\xi_a(x) = x + a$. However, $\int_{-\infty} J(x)\cdot a \, dx$ may not be well defined since it is an integral over all space. Thus we must take an appropriate limit to make the integral well defined.

In appendix 2 it will be shown for the representation of $U(f)$ and $V(\varphi)$ corresponding to a Free Bose Gas, there is a sequence φ_n satisfying the conditions of theorem 5. Therefore the translation operators are in the closure of the algebra generated by the set $\{V(\varphi)\}$. Then by the previous corollary it follows the set of operators $\{U(f), V(\varphi)\}$ are irreducible.

(This result was proved by different means in ref. 20.) It is not yet known whether this result is true for other representations of physical interest.

Next, we will show that different Hamiltonians give rise to unitarily inequivalent representations of the local current algebra. In order to do this we need the following theorem.

<u>Theorem</u>: Suppose the generating functional $L(f) = (\Omega, U(f)\Omega)$ determines a continuous unitary representation of $U(f)$ satisfying the cluster decomposition property and translational invariance. If there is a set of unitary operators $Q'(\underset{\sim}{a})$ and a cyclic vector Ω' (i.e. Span $\{U(f)\Omega'; f\in\mathscr{I}\}$ is dense) satisfying equations 5.8, 5.9 and 5.11 then there exists a unitary operator S such that; $SU(f)S^{-1} = U(f)$, $SQ(\underset{\sim}{a})S^{-1} = Q'(\underset{\sim}{a})$, and $S\Omega = \Omega'$.

<u>Proof</u>: (See Araki[8], Th. 6.2)

<u>Corollary 1</u>: Suppose the generating functionals $L_1(f) = (\Omega_1, U_1(f)\Omega_1)$ and $L_2(f) = (\Omega_2, U_2(f)\Omega_2)$, each satisfying translational invariance, define two continuous unitary representations of $U(f)$. Furthermore, suppose $L_2(f)$ satisfies the cluster decomposition property. Then the representations are unitarily equivalent iff $L_1(f) = L_2(f)$.

<u>Proof</u>: Let $\mathcal{H}_1$ and $\mathcal{H}_2$ be the Hilbert Spaces and Ω_1 and Ω_2 the cyclic vectors for the two representations. Suppose the representations are unitarily equivalent. Then there exists a unitary operator S_1 such that; $S_1 : \mathcal{H}_1 \to \mathcal{H}_2$ and $S_1U_1(f)S_1^{-1} = U_2(f)$. Let $\Omega_2' = S_1\Omega_1$ and $Q_2'(\underset{\sim}{a}) = S_1Q_1(\underset{\sim}{a})S_1^{-1}$. It is easily shown that Ω_2' is cyclic in $\mathcal{H}_2$ and eqs. 5.8, 5.9 and 5.11 are satisfied for Ω_2' and $Q_2'(\underset{\sim}{a})$. By the above theorem there exists a unitary operator S_2 such that; $S_2 : \mathcal{H}_2 \to \mathcal{H}_2$, $S_2U_2(f)S_2^{-1} = U_2(f)$ and $S_2\Omega_2 = \Omega_2'$. Let $S = S_2^{-1}S_1 : \mathcal{H}_1 \to \mathcal{H}_2$. Then $S\Omega_1 = \Omega_2$ and $SU_1(f)S^{-1} = U_2(f)$.

Therefore $L_1(f) = (\Omega_1, U_1(f)\Omega_1)_1$

$$= (S\Omega_1, SU_1(f)\Omega_1)_2$$

$$= (\Omega_2, U_2(f)\Omega_2)_2 = L_2(f)$$

Conversely, if $L_1(f) = L_2(f)$ the representations are clearly unitarily equivalent.

Remark: The last two theorems have used only L(f). They are important for representations of U(f) and $V(\underset{\sim}{\varphi})$ in which span $\{U(f)\Omega\}$ is dense. Furthermore, they can be generalized using $L(f,\underset{\sim}{\varphi})$ for representations in which span $\{U(f)V(\underset{\sim}{\varphi})\Omega\}$ is dense.

Now suppose there are two representations of U(f) and $V(\varphi)$ with Hamiltonians H_1 and H_2 of the form $H = (1/8)\int d\underset{\sim}{x}\tilde{\underset{\sim}{K}}(\underset{\sim}{x})^{\dagger}\frac{1}{\rho(\underset{\sim}{x})}\tilde{\underset{\sim}{K}}(\underset{\sim}{x})$ with $\tilde{\underset{\sim}{K}}^{(1)}(\underset{\sim}{x}) = \underset{\sim}{K}(\underset{\sim}{x}) - \underset{\sim}{A}^{(1)}(\underset{\sim}{x},\rho)$ and $\tilde{\underset{\sim}{K}}^{(2)}(\underset{\sim}{x}) = \underset{\sim}{K}(\underset{\sim}{x}) - \underset{\sim}{A}^{(2)}(\underset{\sim}{x},\rho)$. If the representations are unitarily equivalent then by corollary 1, $L_1(f) = L_2(f)$. Therefore we may take $\mathcal{H}_1 = \mathcal{H}_2$. Consider the following identity;

$$(\Omega, e^{i\rho(f)}[\rho(-\nabla\cdot\underset{\sim}{g}) - i\rho(\nabla f\cdot\underset{\sim}{g})]\Omega) = (\Omega, e^{i\rho(f)}K(\underset{\sim}{g})\Omega)$$

Since $K(\underset{\sim}{g})\Omega = A^{(1)}(\underset{\sim}{g},\rho)\Omega = A^{(2)}(\underset{\sim}{g},\rho)\Omega$ we have,

$$(\Omega, e^{i\rho(f)}A^{(1)}(\underset{\sim}{g},\rho)\Omega) = (\Omega, e^{i\rho(f)}A^{(2)}(\underset{\sim}{g},\rho)\Omega), \text{ for all } f\in\mathcal{S}.$$

Therefore, $A^{(1)}(\underset{\sim}{g},\rho)\Omega = A^{(2)}(\underset{\sim}{g},\rho)\Omega$. Since $[A(\underset{\sim}{g},\rho), e^{i\rho(f)}] = 0$ and span $\{e^{i\rho(f)}\Omega;\ f\in\mathcal{S}\}$ is dense, it follows $A^{(1)}(\underset{\sim}{g},\rho) = A^{(2)}(\underset{\sim}{g},\rho)$. We have proved the following theorem.

<u>Theorem 6</u>: Suppose ther are two continuous unitary representations of U(f) and $V(\varphi)$ (denoted by i = 1,2) with Hamiltonians

$$H_i = (1/8)\int d\underset{\sim}{x}\ \tilde{\underset{\sim}{K}}^{(i)}(\underset{\sim}{x})^{\dagger}\ \frac{1}{\rho(\underset{\sim}{x})}\ \tilde{\underset{\sim}{K}}^{(i)}(\underset{\sim}{x}) \text{ where } \tilde{\underset{\sim}{K}}^{(i)}(\underset{\sim}{x}) = \underset{\sim}{K}(\underset{\sim}{x}) - \underset{\sim}{A}^{(i)}(\underset{\sim}{x},\rho)$$

and satisfying the cluster decomposition property, translational invariance and time reversal invariance. If $\underset{\sim}{A}^{(1)}(\underset{\sim}{x},\rho) \neq \underset{\sim}{A}^{(2)}(\underset{\sim}{x},\rho)$ then the representations are unitarily inequivalent.

Remarks: (1) Roughly speaking theorem 6 states, different Hamiltonians correspond to inequivalent representations. Two important questions remain unanswered at this time. First, given a system of particles (Boson or Fermion) with an interaction potential $V(\underset{\sim}{x})$, is $\underset{\sim}{A}(\underset{\sim}{x}, \rho)$ uniquely determined? Second, does a Hamiltonian with a given $\underset{\sim}{A}(\underset{\sim}{x},\rho)$ uniquely determine the representation? The second question is equivalent to asking whether the functional equation (4.2) for L(f) has a unique solution. This is known to be the case for a Free Bose Gas[20] ($\underset{\sim}{A}(\underset{\sim}{x},\rho) = 0$) but uniqueness has not been established for other cases.

(2) Since $\underset{\sim}{A}(\underset{\sim}{x},\rho)$ considered as a function of ρ, may be an unbounded operator, its definition is representation dependent. For some representations it may not even be defined. In some representations two operators $\underset{\sim}{A}^{(1)}(\underset{\sim}{x},\rho)$ and $\underset{\sim}{A}^{(2)}(\underset{\sim}{x},\rho)$ may be equal while in others they may be unequal.

(3) Consider the N/V limit of interacting physical systems characterized by a coupling constant λ and for which the assumptions in theorem 6 are valid. If the systems are described by unitarily equivalent representations, then by theorem 6 and eq. 3.12 the Hamiltonians H_λ are identical as hermitian forms. Therefore, the Hamiltonian operators would be different self-adjoint extensions of the same hermitian form. Furthermore, the ground states are the same since there is a unique translational invariant state. On the other hand, if the systems are described by unitarily inequivalent representations, then solving the systems by perturbation theory is more difficult since it is no longer possible to express the ground states as a convergent series in λ. This point will be discussed in the next paper in connection with a specific example.

C. Restriction of the N/V representation to a Finite Volumn.

We can gain further insight about L(f) in the N/V limit by restricting the test functions to have support in a bounded set ν.

If supp $f \subset \nu$, then $L(f) = \sum_{n=0}^{\infty} (1/n!) \int_\nu dx_1 \cdots \int_\nu dx_n F(x_1) \cdots F(x_n) R_n(x_1 \cdots x_n)$.

Since $F(x_1) \cdots F(x_n) = \sum_{\text{perm}} \sum_{j=0}^{n} (-)^{n-j}/(j!(n-j)!) \prod_{k=1}^{j} \exp if(x_{\pi_k})$ and R_n is a symmetric function, we have

$$L(f) = \sum_{n=0}^{\infty} (1/n!) \int_\nu dx_1 \cdots \int_\nu dx_n e^{if(x_1)} \cdots e^{if(x_n)} P_n(\nu; x_1 \cdots x_n) \qquad (5.17)$$

$$\text{where } P_n(\nu; x_1 \cdots x_n) = \sum_{j=0}^{\infty} (-)^j/j! \int_\nu dx_{n+1} \cdots \int_\nu dx_{n+j} R_{n+j}(x_1 \cdots x_{n+j}) \qquad (5.18)$$

P_n has the physical interpretation,

$$(1/n!) P_n(\nu; x_1 \cdots x_n) = \begin{pmatrix} \text{The probability for finding n particles at points} \\ x_1, \ldots, x_n \text{ and the remaining particles outside } \nu. \end{pmatrix}$$

To prove this we consider N particles in a box of volume V, in this case

$$(1/n!) P_n^{(N)}(\nu; x_1 \cdots x_n) = \frac{N!}{n!(N-n)!} \int_{V-\nu} dx_{n+1} \cdots \int_{V-\nu} dx_N \Omega^* \Omega (x_1 \cdots x_N)$$

$$= \frac{N!}{n!(N-n)!} \sum_{j=0}^{N-n} (-)^j \frac{(N-n)!}{j!(N-j-n)!} \int_\nu dx_{n+1} \cdots \int_\nu dx_{n+j} \int_V dx_{n+j+1} \cdots \int_V dx_N \Omega^* \Omega$$

$$= \frac{N!}{n!(N-n)!} \sum_{j=0}^{N-n} (-)^j \frac{(N-n)!}{j!(N-j-n)!} \int_\nu dx_{n+1} \cdots \int_\nu dx_{n+j} \frac{(N-j-n)!}{N!} R_{j+n}$$

$$= \frac{1}{n!} \sum_{j=0}^{N-n} \frac{(-)^j}{j!} \int_\nu dx_{n+1} \cdots \int_\nu dx_{n+j} R_{n+j}(x_1 \cdots x_{n+j})$$

As $N \to \infty$ we obtain the expression in eq. 5.18.

Remarks: (1) Formally eq. 5.18 can be inverted. The R's are given in terms of the P's by the equation: For $x_1, \ldots x_n \in \nu$

$$R_n(x_1 \cdots x_n) = \sum_{j=0}^{\infty} (1/j!) \int_\nu dx_{n+1} \cdots \int_\nu dx_{n+j} P_{n+j}(\nu; x_1 \cdots x_{n+j}) \qquad (5.19)$$

If the sum in eq. 5.19 converges and is consistent (i.e. the same value of R_n is obtained for points $x_1, \ldots, x_n$ in overlapping volumns), then the R's can be determined from the local probability distributions. Since the R's determine L(f) this implies L(f) can be determined by its local behavior.

(2) If the volumn ν is not bounded each term in the expansion for P_n (eq. 5.18) will be infinite.

(3) As a result of the probability interpretation for P_n:

$$\text{(i) } P_n(\nu; x_1 \ldots x_n) \geq 0 \text{ and (ii) } \sum_{n=0}^{\infty} (1/n!) \int_\nu dx_1 \ldots \int_\nu dx_n P_n(\nu; x_1 \ldots x_n) = 1$$

Property (ii) also follows from L(0) = 1.

(4) If $R_n \leq c^n n^{n/2} \forall n$, then the lemma to theorem 4 can be extended to show P_n exists (i.e. the series for P_n converges). However, this is not sufficient to imply $P_n \geq 0$.

From eq. 5.17 we see for supp $f \subset \nu$, L(f) is the sum of terms which have the form of N-particle generating functions (eq. 5.3) with ground state given by $\Omega^*_N \Omega_N(x_1 \ldots x_N) = (1/N!) P_N(\nu; x_1 \ldots x_N)$. As a result the N/V limit representation restricted to finite volumns (this is a representation of the subalgebra formed by restricting the test functions) can be represented in the Hilbert Space formed by the direct sum of N-particle spaces (Fock Space). However, the ground state for this restriction would not have a definite number of particles. Thus, locally the N/V limit can be considered as "Fock Space". This is the "particle like" nature of the N/V limit.

For a Free Bose Gas $P_n(\nu; x_1 \ldots x_n)$ can be calculated exactly. It has been shown for this case that[7]

$$L(f) = \exp \bar{\rho} \int dx \, (e^{if(x)} - 1)$$

$$= \sum_{n=0}^{\infty} (1/n!) \int dx_1 \ldots \int dx_n F(x_1) \ldots F(x_n) \bar{\rho}^n$$

Therefore, $R_n(\underset{\sim}{x}_1 \cdots \underset{\sim}{x}_n) = \bar{\rho}^n$. As a result,

$$P_n(\nu; \underset{\sim}{x}_1 \cdots \underset{\sim}{x}_n) = (1/n!) \sum_{j=0}^{\infty} (-)^j/j! \int_\nu d\underset{\sim}{x}_{n+1} \cdots \int_\nu d\underset{\sim}{x}_{n+j} \bar{\rho}^{n+j}$$

$$= (1/n!) \sum_{j=0}^{\infty} (-)^j/j! \, \nu^j \bar{\rho}^{n+j} = (\bar{\rho}^n/n!) \exp(-\bar{\rho}\,\nu)$$

This is a Poisson Distribution with mean equal to $\bar{\rho}\,\nu$. This is to be expected since we have taken the limit of a large number of non-interacting particles ($N \to \infty$) with the probability of finding a given particle in a given unit volumn (prob. = $1/V$) approaching zero such that the product ($N \cdot$prob. = $N/V = \bar{\rho}$) is a constant.

Remark: The Hilbert Space, $\mathcal{H} = L^2_\mu(\mathscr{S}')$, can be used to represent the N/V limit. The measures for the N-particle representations and eq. 5.17 suggest the measure in the N/V limit is concentrated on functionals consisting of a countably infinite number of delta functions; $F = \sum_{j=1}^{\infty} \delta(\underset{\sim}{x} - \underset{\sim}{x}_j)$ such that if $n_F(\nu)$ is the number of delta functions with support in volumn ν then $\lim_{\nu \to \infty} n_F(\nu)/\nu = \bar{\rho}$. The functionals can be characterized by the sequence of points $\{\underset{\sim}{x}_1, \underset{\sim}{x}_2 \cdots\}$ which can be interpreted as the positions of the particles. The measure μ can be considered as a measure on these sequences. In this context there are similarities with recent work of Lenard[27] in which he discussed the state in classical statistical mechanics in terms of correlation functions. The present formalism becomes distinctly quantum mechanical in nature only when the J's are considered.

Also, representations corresponding to different average densities $\bar{\rho}_1$ and $\bar{\rho}_2$ will have measures μ_1 and μ_2 with different sets of measure zero (in $\mathscr{S}'$). As a result, representations corresponding to different average densities are unitarily inequivalent.

The same methods may be used to obtain expressions for $L(f, \underset{\sim}{g})$ similar to those for L(f). The results are:

$$L(f,\underset{\sim}{g}) = \sum_{n=0}^{\infty} (1/n!)\int d\underset{\sim}{x}_1\int d\underset{\sim}{y}_1\cdots\int d\underset{\sim}{x}_n\int d\underset{\sim}{y}_n\,\delta(\underset{\sim}{x}_1-\underset{\sim}{y}_1)\;\cdots\;\delta(\underset{\sim}{x}_n-\underset{\sim}{y}_n)$$

$$\prod_{k=1}^{n}[e^{if(\underset{\sim}{x}_k)}\;e^{ij(\underset{\sim}{x}_k,\underset{\sim}{g})}-1]\;R_n(\underset{\sim}{y}_1\cdots\underset{\sim}{y}_n;\underset{\sim}{x}_1\cdots\underset{\sim}{x}_n) \qquad (5.20)$$

where $j(\underset{\sim}{x},\underset{\sim}{g}) = \frac{1}{2}i\,[2\underset{\sim}{g}(\underset{\sim}{x})\cdot\underset{\sim}{\nabla} + (\underset{\sim}{\nabla}\cdot\underset{\sim}{g})(\underset{\sim}{x})]$

In the N-particle representations $R_n(\;\;;\;\;)$ is given by:

$$R_n(\underset{\sim}{y}_1\cdots\underset{\sim}{y}_n;\underset{\sim}{x}_1\cdots\underset{\sim}{x}_n) = \frac{N!}{(N-n)!}\int d\underset{\sim}{x}_{n+1}\cdots\int d\underset{\sim}{x}_N\,\Omega^*\,(\underset{\sim}{y}_1\cdots\underset{\sim}{y}_n,\underset{\sim}{x}_{n+1},\cdots\underset{\sim}{x}_N)\,\Omega\,(\underset{\sim}{x}_1\cdots\underset{\sim}{x}_N)$$

In terms of the canonical field operators,

$$R_n(\underset{\sim}{y}_1\cdots\underset{\sim}{y}_n;\underset{\sim}{x}_1\cdots\underset{\sim}{x}_n) = (\Omega,\psi^\dagger(\underset{\sim}{y}_n)\cdots\psi^\dagger(\underset{\sim}{y}_1)\psi(\underset{\sim}{x}_1)\cdots\psi(\underset{\sim}{x}_n)\Omega)$$

Clearly $R_n(\underset{\sim}{x}_1\cdots\underset{\sim}{x}_n;\underset{\sim}{x}_1\cdots\underset{\sim}{x}_n) = R_n(\underset{\sim}{x}_1\cdots\underset{\sim}{x}_n)$. Also, as a consequence of Schwartz's inequality, $|R_n(\underset{\sim}{y}_1\cdots\underset{\sim}{y}_n;\underset{\sim}{x}_1\cdots\underset{\sim}{x}_n)|^2 \le R_n(\underset{\sim}{x}_1\cdots\underset{\sim}{x}_n)R_n(\underset{\sim}{y}_1\cdots\underset{\sim}{y}_n)$

An alternative expression for $L(f,\underset{\sim}{g})$ can be obtained in terms of the cluster functions defined as

$$T_n(\underset{\sim}{y}_1\cdots\underset{\sim}{y}_n;\underset{\sim}{x}_1\cdots\underset{\sim}{x}_n) = \sum_G (-)^{m-n}(m-1)!\prod_{j=1}^{m} R_{G_j}(\underset{\sim}{y}_k\varepsilon G_j;\underset{\sim}{x}_k\varepsilon\, G_j)$$

Where G = a partition of (1,2,...n) into subsets $(G_1,\ldots,G_m)$.

$L(f,\underset{\sim}{g})$ can now be expressed as

$$L(f,\underset{\sim}{g}) = \exp\sum_{n=1}^{\infty}\frac{(-)^{n-1}}{n!}\int d\underset{\sim}{x}_1\int d\underset{\sim}{y}_1\cdots\int d\underset{\sim}{x}_n\int d\underset{\sim}{y}_n\,\delta(\underset{\sim}{x}_1-\underset{\sim}{y}_1)\cdots\delta(\underset{\sim}{x}_n-\underset{\sim}{y}_n)$$

$$\prod_{k=1}^{n}[e^{if(\underset{\sim}{x}_k)}e^{ij(\underset{\sim}{x}_k,\underset{\sim}{g})}-1]\;T_n(\underset{\sim}{y}_1\cdots\underset{\sim}{y}_n;\underset{\sim}{x}_1\cdots\underset{\sim}{x}_n) \qquad (5.21)$$

Finally, if supp $f\subset\nu$ and supp $\underset{\sim}{g}\subset\nu$

$$L(f,\underset{\sim}{g}) = \sum_{n=0}^{\infty}(1/n!)\int_\nu d\underset{\sim}{x}_1\cdots\int_\nu d\underset{\sim}{x}_n\int d\underset{\sim}{y}_1\cdots\int d\underset{\sim}{y}_n\,\delta(\underset{\sim}{x}_1-\underset{\sim}{y}_1)\cdots\delta(\underset{\sim}{x}_n-\underset{\sim}{y}_n)$$

$$\prod_{k=1}^{n}[e^{if(\underset{\sim}{x}_k)}\;e^{ij(\underset{\sim}{x}_k,\underset{\sim}{g})}]\;P_n(\nu;\underset{\sim}{y}_1\cdots\underset{\sim}{y}_n;\underset{\sim}{x}_1\cdots\underset{\sim}{x}_n) \qquad (5.22)$$

where $P_n(\nu;y_1\cdots y_n;x_1\cdots x_n) = \sum_{k=0}^{\infty}(-)^k/k!\int_\nu dx_{n+1}\cdots\int_\nu dx_{n+j}$

$$R_{n+j}(y_1\cdots y_n,x_{n+1}\cdots x_{n+j};x_1\cdots x_{n+j}) \qquad (5.23)$$

Thus the generating functional for a representation of U(f) and $V(\phi)$ in the N/V limit restricted to a finite volumn is the sum of terms similar to L(f,g) for an N-particle representation (eq. 5.4). (If $P_n(\nu;y_1\cdots y_n;x_1\cdots x_n) = W_n(y_1\cdots y_n)W_n(x_1\cdots x_n)$ then the restriction is the direct sum of N-particle representations.)

6. Summary.

We have shown the Hamiltonian, considered as a densely defined hermitian form, can be written

$$H = (1/8)\int dx\, \tilde{K}(x)^{\dagger}\frac{1}{\rho(x)}\tilde{K}(x), \text{ where } \tilde{K}(x) = [\nabla\rho(x)+2iJ(x)] - A(x,\rho).$$

The generating functional in the N/V limit can be expressed as,

$$L(f) = \sum_{n=0}^{\infty}(1/n!)\int dx_1\cdots\int dx_n F(x_1)\ldots F(x_n)R_n(x_1\cdots x_n)$$

where $F(x) = e^{if(x)}-1$ and R_n = the n^{th} correlation function.

L(f) satisfies the functional differential equation,

$$(\nabla - i\nabla f(x))\frac{1}{i}\frac{\delta}{\delta f(x)}L(f) = A(x,\frac{1}{i}\frac{\delta}{\delta f})\,L(f).$$

Furthermore, under the assumption of translational invariance and the cluster decomposition property, inequivalent representations are needed for different Hamiltonians.

There remains two problems in determining representations of physical interest:

(1) Given a potential V(x) determine $A(x,\rho)$ and

(2) Given $A(x,\rho)$ solve the functional equation (subject to the appropriate boundary conditions) for L(f).

Undoubtably these tasks can be accomplished in general only by using approximation methods. Once a representation for a given system has been determined its dynamics can be studied. Extending this approach to study the thermodynamics of a system is also of interest.

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Appendix 1

In this section we will show how the cluster decomposition property can be used as a boundary condition for the functional differential equation (4.2) to uniquely determine the generating functional for a Free Bose Gas in the N/V limit. (In ref. 20 other boundary conditions were used for this purpose.) We will assume we already know that the generating functional for a Free Bose Gas satisfies the equation,

$$(\nabla - i\nabla f)\,\frac{1}{i}\,\frac{\delta}{\delta f(x)}\,L(f) = 0 \tag{A.1}$$

The first method for solving this equation is based on the use of integrating factors. Eq. A.1 can be rewritten as,

$$\nabla\left\{e^{-if(x)}\,\frac{1}{i}\,\frac{\delta}{\delta f(x)}\,L(f)\right\} = 0 \tag{A.2}$$

Integrating between point x and ∞ $\left(\int_x^\infty dr\,\cdot\right)$ we obtain

$$\left[e^{-if(x)}\,\frac{1}{i}\,\frac{\delta}{\delta f(x)}\,L(f)\right]\Big|_{x=\infty} - e^{-if(x)}\,\frac{1}{i}\,\frac{\delta}{\delta f(x)}\,L(f) = 0 \tag{A.3}$$

Using the cluster decomposition property,

$$\frac{1}{i}\,\frac{\delta}{\delta f(x)}\,L(f) = (\Omega,\rho(x)e^{if(x)}\Omega) \to (\Omega,\rho(x)\,\Omega)(\Omega,\, e^{i\rho(f)}\Omega) \text{ as } |x| \to \infty,$$

translational invariance, $(\Omega,\rho(x)\Omega) = \bar{\rho}$ = the average density, and the fact $f(x) \to 0$ as $|x| \to \infty$, Eq. A.3 becomes

$$\bar{\rho}\,L(f) - e^{-if(x)}\,\frac{1}{i}\,\frac{\delta}{\delta f(x)}\,L(f) = 0 \tag{A.4}$$

This can be written as,

$$\frac{1}{i}\,\frac{\delta}{\delta f(x)}\left\{\exp\left[-\bar{\rho}\int(e^{if(x)}-1)\,dx\right]L(f)\right\} = 0 \tag{A.5}$$

Therefore, $\exp\left[-\bar{\rho}\int(e^{if(x)}-1)\,dx\right]L(f) = \text{constant}$ (A.6)

The constant can be determined from the requirement $L(0) = 1$.

The result is, $L(f) = \exp\,\bar{\rho}\int(e^{if(x)}-1)\,dx$. (A.7)

An alternative method for solving eq. A.1 uses the cluster decomposition property for the correlation functions. We have shown in the N/V limit that L(f) has the form (eq. 5.6),

$$L(f) = \sum_{n=0}^{\infty} (1/n!) \int d\underset{\sim}{x}_1 \ldots \int d\underset{\sim}{x}_n (e^{if(\underset{\sim}{x}_1)} -1) \ldots (e^{if(\underset{\sim}{x}_n)} -1) R_n(\underset{\sim}{x}_1 \ldots \underset{\sim}{x}_n) \quad (A.8)$$

Substituting eq. A.8 into eq. A.1 we obtain,

$$\sum_{n=0}^{\infty} \frac{e^{if(\underset{\sim}{x}_1)}}{(n-1)!} \int d\underset{\sim}{x}_2 \ldots \int d\underset{\sim}{x}_n (e^{if(\underset{\sim}{x}_2)} -1) \ldots (e^{if(\underset{\sim}{x}_n)} -1) \underset{\sim}{\nabla}_1 R_n(\underset{\sim}{x}_1 \ldots \underset{\sim}{x}_n) = 0 \quad (A.9)$$

Since eq. A.9 is true for all f, each term separately must be zero,

$$\underset{\sim}{\nabla}_1 R_n(\underset{\sim}{x}_1 \ldots \underset{\sim}{x}_n) = 0 \quad \forall n \quad (A.10)$$

Furthermore, R_n is a symmetric function. Therefore, R_n = const. The cluster decomposition property can be used to relate the different constants as follows:

$$R_1(\underset{\sim}{x}) = (\Omega, \rho(\underset{\sim}{x}) \Omega) = \bar{\rho}$$

$$\lim_{\underset{\sim}{a} \to \infty} R_n(\underset{\sim}{x}_1 \ldots \underset{\sim}{x}_{n-1}, \underset{\sim}{x}_n + \underset{\sim}{a}) = R_1(\underset{\sim}{x}_n)\, R_{n-1}(\underset{\sim}{x}_1 \ldots \underset{\sim}{x}_{n-1})$$

$$= \bar{\rho}\, R_{n-1}(\underset{\sim}{x}_1 \ldots \underset{\sim}{x}_{n-1})$$

By induction we have $R_n = \bar{\rho}^n$. Therefore,

$$L(f) = \sum_{n=0}^{\infty} (1/n!) \int d\underset{\sim}{x}_1 \ldots \int d\underset{\sim}{x}_n (e^{if(\underset{\sim}{x}_1)} -1) \ldots (e^{if(\underset{\sim}{x}_n)} -1) \bar{\rho}^n$$

$$= \exp \bar{\rho} \int d\underset{\sim}{x} (e^{if(\underset{\sim}{x})} -1).$$

Appendix 2

In ref. 20 it was shown the generating functional for the representation of U(f) and $V(\varphi)$ corresponding to the Free Bose Gas is given by:

$$L(f,\varphi) = (\Omega,\ U(f)\ V(\varphi)\ \Omega)$$
$$= \exp\ \bar{\rho} \int dx\ (e^{if(x)}[\det \frac{\partial \varphi(x)^r}{\partial x_s}]^{\frac{1}{2}} - 1)$$

In this section we will show there is a sequence of test fuctions φ_n such that $V(\varphi_n) \to Q(a)$, the translation operator. First, it is necessary to define which flows are to be used as test tunctions. Goldin[26] has suggested a topology on the flows in analogy to the topology on Schwartz's Space. His topology is defined by a countable number of metrics,

$$\ll \varphi, \psi \gg_n = \max_{0 \le |m| \le n} \ \sup_{x} \ |(1+|x|^2)^n (\varphi^{(m)}(x) - \psi^{(m)}(x))|$$

Since we want the test functions to include the identity flow $\varphi_o(x) = x$ and to have an inverse, we will take the test functions to be the set of flows φ such that $\ll \varphi, \varphi_o \gg_n < \infty$ and $\ll \varphi^{-1}, \varphi_o \gg_n < \infty$ for all n.

By theorem 5, If $\varphi_n(x) \to x + a$ and $(\Omega,\ V(\varphi_n)\Omega) \to 1$ then $V(\varphi_n) \to Q(a)$. We will first consider the one dimensional case.

Let

$$\varphi_n(x) = \begin{cases} x & 2n < x \\ x + a\,\dfrac{2n-x}{n} & n < x < 2n \\ x + a & |x| < n \\ x + a\,\dfrac{2n + x}{n} & -2n < x < -n \\ x & x < -2n \end{cases}$$

(Fig. 1) ⟶

φ_n would be a test function except that its derivative is discontinuous at 4 points. By changing φ_n in a small region about each discontinuity it can be made into a smooth function (and hence a test function) without changing the subsequent arguments.

Clearly $\varphi_n(x) \to x + a$ as $n \to \infty$.

In order to verify $(\Omega, V(\varphi_n)\Omega) \to 1$ we must show

$$\int_{-\infty}^{\infty} dx \, [(\frac{d\varphi_n}{dx})^{\frac{1}{2}} -1] \to 0 \text{ as } n \to \infty.$$

Let $I_n = \int_{-\infty}^{\infty} [(\frac{d\varphi_n}{dx})^{\frac{1}{2}} -1] \, dx$

$$= \int_{-2n}^{-n} [(1 - \frac{a}{n})^{\frac{1}{2}} -1] \, dx + \int_{n}^{2n} [(1 + \frac{a}{n})^{\frac{1}{2}} -1] dx$$

For n large $(1 \pm \frac{a}{n})^{\frac{1}{2}} = 1 \pm \frac{1}{2}\frac{a}{n} + 0(\frac{1}{n^2})$

$$I_n = \int_{-2n}^{-n} [-\tfrac{1}{2}\frac{a}{n} + 0(\frac{1}{n^2})] \, dx + \int_{n}^{2n} [\tfrac{1}{2}\frac{a}{n} + 0(\frac{1}{n^2})] dx$$

$$= -\tfrac{1}{2} a + n \, 0(\frac{1}{n^2}) + \tfrac{1}{2} a + n \, 0(\frac{1}{n^2})$$

$$= 0(\frac{1}{n}) \to 0 \text{ as } n \to \infty$$

Therefore $V(\varphi_n) \to Q(a)$ by theorem 5.

In 2 dimenions consider a translation in the x direction by a distance a. Let

$$\varphi_n(x,y)_y = y$$

$$\varphi_n(x,y)_x = x + a \, \alpha_n(x) \, \beta_n(y)$$

where

$$\alpha_n(x) = \begin{cases} \frac{2n - x}{n} & n < x < 2n \\ 1 & -n < x < n \\ \frac{2n + x}{n} & -2n < x < -n \\ 0 & 2n < |x| \end{cases}$$

and

$$\beta_n(y) = \begin{cases} \dfrac{(n+\Delta)-y}{\Delta} & n < y < n+\Delta \\ 1 & -n < y < n \\ \dfrac{(n+\Delta)+y}{\Delta} & -(n+\Delta) < y < -n \\ 0 & n+\Delta < |y| \end{cases}$$

Δ = an arbitrary positive constant

To prove $(\Omega, V(\varphi_n)\Omega) \to 1$, it is necessary to show

$$\int dx \int dy \left(\left[\det \frac{\partial \varphi_n(x,y)_r}{\partial x_s} \right]^{\frac{1}{2}} - 1 \right) \to 0.$$

This can be verified by a calculation similar to the 1-dimensional case. In fact a similar argument works for any number of space dimensions.

Therefore, for the Free Bose Gas representation there is a sequence of test functions φ_n such that $Q(a) = \lim V(\varphi_n)$.

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Figure Captions

Fig. 1. The flow $\varphi_n(x)$ vs. x, in one dimension.

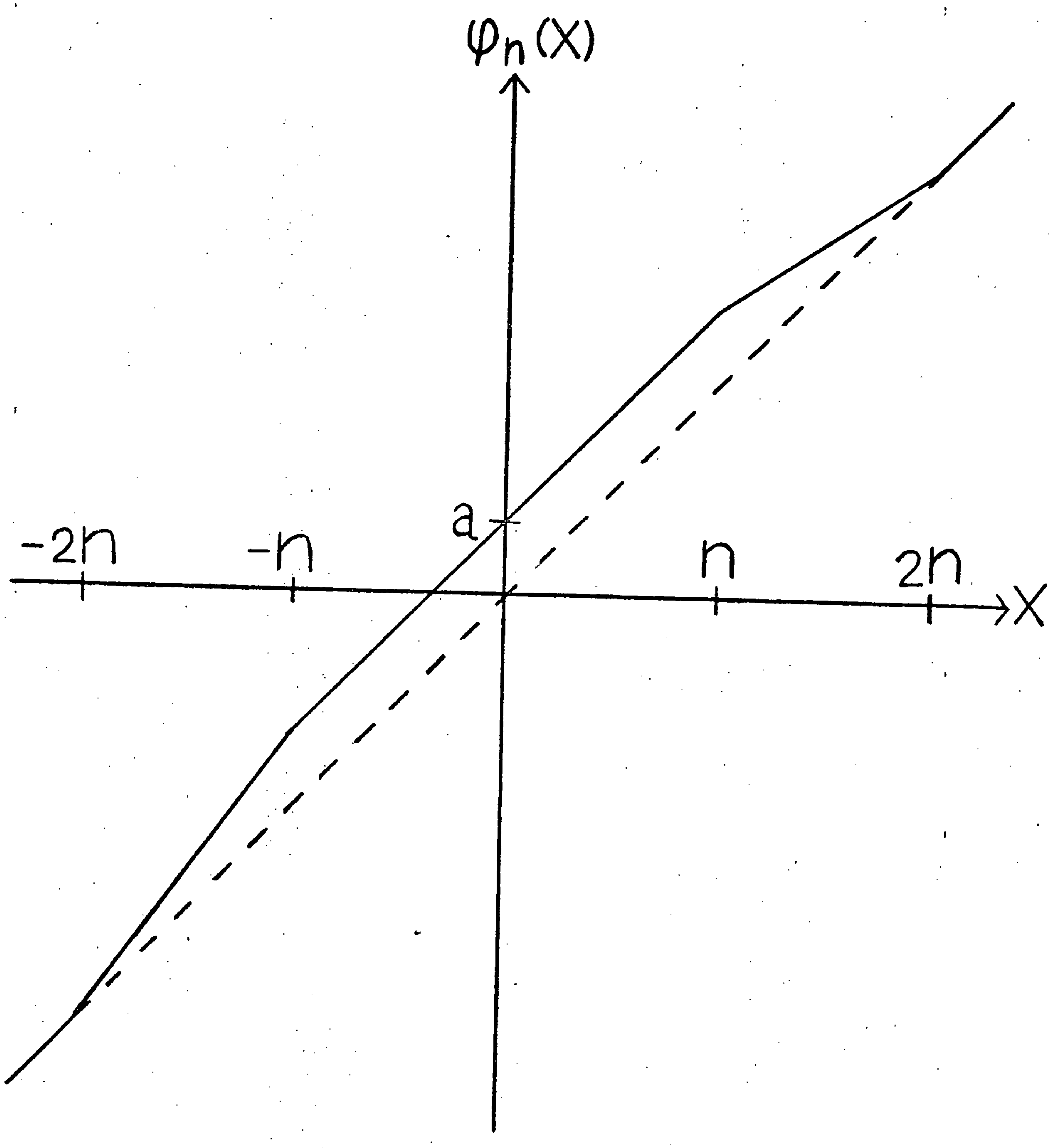

Fig. 1. Ralph Menikoff

The Hamiltonian and Generating Functional for a Non-Relativistic Local Current Algebra.