291
6-9-67

K-1695

AEC RESEARCH AND DEVELOPMENT REPORT

MASTER

THE NUMERICAL TREATMENT OF A THIN PLATE
HEAT TRANSFER PROBLEM

AUTHORS:

G. W. Stewart, III
Dale W. Lick

**UNION
CARBIDE**

UNION CARBIDE CORPORATION
NUCLEAR DIVISION

Operating the

OAK RIDGE GASEOUS DIFFUSION PLANT • OAK RIDGE NATIONAL LABORATORY
OAK RIDGE Y-12 PLANT • PADUCAH GASEOUS DIFFUSION PLANT

*For the Atomic Energy Commission
Under U.S. Government Contract W7405 eng 26*

DISTRIBUTION OF THIS DOCUMENT IS UNLIMITED

leg

DISCLAIMER

This report was prepared as an account of work sponsored by an agency of the United States Government. Neither the United States Government nor any agency Thereof, nor any of their employees, makes any warranty, express or implied, or assumes any legal liability or responsibility for the accuracy, completeness, or usefulness of any information, apparatus, product, or process disclosed, or represents that its use would not infringe privately owned rights. Reference herein to any specific commercial product, process, or service by trade name, trademark, manufacturer, or otherwise does not necessarily constitute or imply its endorsement, recommendation, or favoring by the United States Government or any agency thereof. The views and opinions of authors expressed herein do not necessarily state or reflect those of the United States Government or any agency thereof.

DISCLAIMER

Portions of this document may be illegible in electronic image products. Images are produced from the best available original document.

Printed in the United States of America. Available from Clearinghouse for Federal
Scientific and Technical Information, National Bureau of Standards,
U.S. Department of Commerce, Springfield, Virginia 22151
Price: Printed Copy \$3.00; Microfiche \$0.65

LEGAL NOTICE

This report was prepared as an account of Government sponsored work. Neither the United States, nor the Commission, nor any person acting on behalf of the Commission:

- A. Makes any warranty or representation, expressed or implied, with respect to the accuracy, completeness, or usefulness of the information contained in this report, or that the use of any information, apparatus, method, or process disclosed in this report may not infringe privately owned rights; or
- B. Assumes any liabilities with respect to the use of, or for damages resulting from the use of any information, apparatus, method, or process disclosed in this report.

As used in the above, "person acting on behalf of the Commission" includes any employee or contractor of the Commission, or employee of such contractor, to the extent that such employee or contractor of the Commission, or employee of such contractor prepares, disseminates, or provides access to, any information pursuant to his employment or contract with the Commission, or his employment with such contractor.

HC 3.00; MN-65

Report Date: May 31, 1967

Report Number: K-1695

Subject Category: Mathematics

and

Computers

LEGAL NOTICE

This report was prepared as an account of Government sponsored work. Neither the United States, nor the Commission, nor any person acting on behalf of the Commission:

A. Makes any warranty or representation, expressed or implied, with respect to the accuracy, completeness, or usefulness of the information contained in this report, or that the use of any information, apparatus, method, or process disclosed in this report may not infringe privately owned rights; or

B. Assumes any liabilities with respect to the use of, or for damages resulting from the use of any information, apparatus, method, or process disclosed in this report.

As used in the above, "person acting on behalf of the Commission" includes any employee or contractor of the Commission, or employee of such contractor, to the extent that such employee or contractor of the Commission, or employee of such contractor prepares, disseminates, or provides access to, any information pursuant to his employment or contract with the Commission, or his employment with such contractor.

THE NUMERICAL TREATMENT OF A THIN PLATE

HEAT TRANSFER PROBLEM

G. W. Stewart, III

Dale W. Lick

UNION CARBIDE CORPORATION

NUCLEAR DIVISION

Computing Technology Center

Oak Ridge Gaseous Diffusion Plant

Oak Ridge, Tennessee

THIS PAGE
WAS INTENTIONALLY
LEFT BLANK

ABSTRACT

This report considers the numerical solution of a system of linear equations resulting from a discrete approximation to a thin plate heat transfer problem. The slow convergence of a standard iterative method is analyzed and shown to be caused by one of the boundary conditions. A line iterative method which removes the difficulty is described.

THIS PAGE
WAS INTENTIONALLY
LEFT BLANK

1. Introduction

We will consider a two-dimensional heat transfer problem in a long thin rectangular plate which is itself generating heat. Three of the boundaries of the plate are insulated while heat is being transferred across the fourth boundary by forced convection.

The problem is to establish the steady-state temperature distribution throughout the plate. Originally an attempt to solve the system of difference equations corresponding to this problem (§3) by successive over relaxation resulted in exceedingly slow convergence. Here we will give an informal analysis (§4) to show that the slow convergence was caused by the convective boundary conditions. In §5 we will describe how the problem was finally solved using a line iterative technique which treats simultaneously all finite difference equations and boundary conditions associated with a vertical line of the mesh.

2. Statement of the Problem

The partial differential equation for the steady-state temperature $T(x,y)$ in the plate (figure 1) in a rectangular coordinate system is the Poisson equation

$$\frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2} = -\frac{Q}{k},$$

where Q is the heat generation rate in the plate and k is the thermal conductivity of the plate. The left and right sides and the bottom of the plate are insulated, and so T has normal derivative zero along these boundaries. Cool air is forced over the upper surface with constant temperatures T_1 and T_2 over the left one-third (region 1) and right two-thirds (region 2) respectively.

If we let ℓ_x denote the length and ℓ_y the thickness of our plate, and h_v the film coefficient for region v ($v = 1, 2$), then our problem can be pictured by the following figure.

Figure 1

This is mathematically formulated as the following system.

$$(1) \quad \frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2} = -\frac{Q}{k}, \quad 0 < x < \ell_x, \quad 0 < y < \ell_y,$$

$$(2) \quad \frac{\partial T}{\partial n} = 0, \quad x = 0, \quad 0 \leq y \leq \ell_y$$

$$x = \ell_x, \quad 0 \leq y \leq \ell_y$$

$$y = 0, \quad 0 \leq x \leq \ell_x,$$

$$\begin{aligned}
 (3) \quad & -k \frac{\partial T}{\partial n} = h_1 (T - T_1) \quad , \quad y = \ell_y \quad , \quad 0 \leq x < \frac{\ell_x}{3} \quad , \\
 & -k \frac{\partial T}{\partial n} = h_2 (T - T_2) \quad , \quad y = \ell_y \quad , \quad \frac{\ell_x}{3} < x \leq \ell_x \quad ,
 \end{aligned}$$

where $\frac{\partial T}{\partial n}$ is the normal derivative of T along the appropriate boundary.

The orders of magnitude of the constants appearing in equations (1)-(3) are

$$\begin{aligned}
 (4) \quad & Q \sim 10^6 \\
 & k \sim 10 \\
 & \ell_x \sim 10^{-1} \\
 & \ell_y \sim 10^{-4} \\
 & h_v \sim 10 \\
 & T_v \sim 10^2 \quad .
 \end{aligned}$$

3. The Finite Difference Equations

The difference equations are obtained by subdividing the plate into rectangular regions (figure 2) and approximating the derivatives in equations (1)-(3) by differences of temperatures at the appropriate mesh points. These nodal temperatures are labeled T_{ij} ($i = 1, 2, \dots, m$; $j = 1, 2, \dots, n$) as indicated in figure 2. The spacings between points is $h_x = \ell_x / (n-1)$ and $h_y = \ell_y / (m-1)$ along the x and y axes, respectively. Using the standard approximations, the difference equations in the plate become

$$(5) \quad \frac{T_{i,j+1} + T_{i,j-1} - 2T_{ij}}{h_x^2} + \frac{T_{i+1,j} + T_{i-1,j} - 2T_{ij}}{h_y^2} = -\frac{Q}{k} \quad ,$$

$$i = 2, \dots, m-1; \quad j = 2, \dots, n-1 \quad .$$

At the insulated boundaries

$$\begin{aligned} \frac{T_{i,2} - T_{i,1}}{h_x} &= 0, \quad i = 2, \dots, m-1, \\ (6) \quad \frac{T_{i,n} - T_{i,n-1}}{h_x} &= 0, \quad i = 2, \dots, m-1, \\ \frac{T_{2,j} - T_{1,j}}{h_y} &= 0, \quad j = 2, \dots, n-1. \end{aligned}$$

At the convective boundaries

$$\begin{aligned} (7) \quad -k \frac{T_{m,j} - T_{m-1,j}}{h_y} &= h_v (T_{m,j} - T_{sv}), \\ j &= 2, \dots, n-1, \end{aligned}$$

where $v = 1$ or 2 as $T_{m,j}$ lies in the first or second boundary region.

Figure 2. $m = 4, n = 9$

Equations (5)-(7) form a system of linear equations which may be solved for T_{ij} . The order of the system is $mn - 4$ (the corner points do not appear in the equations). For moderately small mesh spacings the system is too large to be solved directly, and so we apply an iterative method. The first method tried was the Gauss-Siedel iteration. To this

end the equations were rewritten in the form:

$$(8) \quad 2(h_x^2 + h_y^2)T_{ij} = h_x^2(T_{i-1,j} + T_{i+1,j}) + h_y^2(T_{i,j-1} + T_{i,j+1}) \\ + \frac{Q}{k} h_x^2 h_y^2 ,$$

$$(9) \quad \begin{aligned} T_{i,1} &= T_{i,2} , \\ T_{i,n} &= T_{i,n-1} , \\ T_{1,j} &= T_{2,j} , \end{aligned}$$

$$(10) \quad T_{m,j} = \lambda_v T_{m-1,j} + (1 - \lambda_v) T_{sv} ,$$

where

$$\lambda_v = (1 + h_y h_v/k)^{-1} .$$

These equations suggest the iterative scheme of solving each equation in succession for its left hand side while holding its right hand side constant. This is in fact the Gauss-Seidel iteration. Its convergence may be established by standard methods and it may be over-relaxed in the hope of improving convergence.

In practice, the convergence of this iterative scheme was very slow, whatever over-relaxation factor was used. An analysis of the entire problem was performed which showed why this slow convergence could be expected. Rather than present the complete analysis, which was tedious, we will instead analyze a modified form of the original problem which shares its salient features.

4. A One-Dimensional Analogue

The modified problem is a one-dimensional analogue of the original problem along a vertical line. It is motivated by the fact that the vertical temperature profiles all have about the same shape.

The problem is

$$\begin{aligned} \frac{d^2 T}{dy^2} &= -\frac{Q}{k}, \quad 0 < y < \ell_y \\ (11) \quad \left. \frac{dT}{dy} \right|_{y=0} &= 0 \\ -k \left. \frac{dT}{dy} \right|_{y=\ell_y} &= h(T - T_s) \end{aligned}$$

Its solution is

$$(12) \quad T(y) = -\frac{Q}{2k} (y^2 - \ell_y^2) + \frac{Q}{h} \ell_y + T_s.$$

Proceeding as in the last section, we form a difference approximation to this problem by subdividing the interval $[0, \ell_y]$ into $m-1$ equal subintervals. The resulting linear system is

$$\begin{aligned} 2T_i &= T_{i-1} + T_{i+1} + h_y^2 Q/k, \quad i = 2, \dots, m-1, \\ (13) \quad T_1 &= T_2 \\ T_m &= \lambda T_{m-1} + (1 - \lambda) T_s, \end{aligned}$$

The point Jacobi matrix is $B = -D^{-1}(L + U)$, where $A = D+L+U$ with D diagonal, L and U strictly lower and upper triangular. In this case

$$B = \begin{pmatrix} 0 & 1 & & & \\ \frac{1}{2} & 0 & \frac{1}{2} & \bigcirc & \\ & \diagdown & \diagdown & \diagdown & \\ & \bigcirc & \frac{1}{2} & 0 & \frac{1}{2} \\ & & & \lambda & 0 \end{pmatrix} .$$

Let $\rho(B)$ be the spectral radius of B (i.e., $\rho(B) = \max |\mu_i|$, where the μ_i are the eigenvalues of B). Then it is known [1] that the optimum over-relaxation factor, ω_b , for the Gauss-Seidel method is given by

$$\omega_b = \frac{2}{1 + \sqrt{1 - \rho^2(B)}} ,$$

and the asymptotic convergence rate of the associated successive over-relaxation method is

$$R_\infty = -\ln(\omega_b - 1).$$

In the limit the number of iterations to reduce the error in the solution by a factor of e^{-1} will be about $1/R_\infty$.

Since B is a non-negative irreducible matrix $\rho(B)$ can be strictly bounded by the minimum and maximum of its row sums. But the greatest row sum of B is unity and its smallest row sum is λ . Hence

$$\lambda < \rho(B) < 1 .$$

Since

$$\omega_b = \frac{2}{1 + \sqrt{1 - \rho^2(B)}}$$

is a monotonically increasing function of $\rho(B)$, the result of replacing $\rho(B)$ by λ will be an underestimate for ω_b . For example, with $m = 11$, we have

$$\lambda \sim .99999$$

$$\omega_b \geq 1.991$$

$$R_\infty \leq .009$$

$$1/R_\infty \geq 110$$

Hence, we may expect slow convergence of the successive over-relaxation method.

The main point to note is that the rate of convergence depends on the parameter λ which arises from the convective boundary condition. If λ is near unity, as it must be when the plate is thin or the subdivision is fine, then the convergence must be slow. All this suggests that if some way could be found to remove λ from the original problem, then the convergence would be much faster. In the next section we will do this by resorting to a line iterative method.

One final result may be obtained from the reduced problem. If the above difference system is solved for T_m and the appropriate substitutions are made, the result is

$$T_m = T_s + \frac{m-2}{m-1} \frac{\ell y^Q}{h}$$

Comparing the actual solution and this approximate solution, we see

$$\frac{T_m - T_s}{T(l_y) - T_s} = \frac{m - 2}{m - 1}.$$

This gives us an idea of how rapidly the finite difference approximation converges to the analytic solution as the mesh is refined.

5. The Method of Solution

From the work of the previous section, it is clear that the slow convergence of the successive over-relaxation method is due to the presence of the convective boundary condition. We can satisfactorily overcome this difficulty by using a line iterative technique which simultaneously treats all finite difference equations and boundary conditions for each vertical line of the mesh. This is done as follows.

Let

$$T_j = (T_{1,j}, \dots, T_{m,j})^T$$

Then the equations (8) - (10) can be written in the block matrix form

$$\begin{pmatrix} D_v & -P & & & \\ -P & D_v & -P & & \\ & -P & D_v & -P & \\ & & & & \text{O} \\ \text{O} & & & & \end{pmatrix} \begin{pmatrix} T_2 \\ T_3 \\ \vdots \\ \vdots \\ T_{n-1} \end{pmatrix} = \begin{pmatrix} C_2 \\ C_3 \\ \vdots \\ \vdots \\ C_{n-1} \end{pmatrix},$$

where

$$D_v = \begin{pmatrix} h_x^2 & -h_x^2 & & \bigcirc \\ -h_x^2 & q & -h_x^2 & \\ & & -h_x^2 & q \\ \bigcirc & & -h_x^2 & -h_x^2 / \lambda_v \end{pmatrix},$$

$$q = 2(h_x^2 + h_y^2),$$

and

$$P = \begin{pmatrix} 0 & & & \\ & h_y^2 & & \bigcirc \\ & & h_y^2 & \\ \bigcirc & & & h_y^2 \\ & & & 0 \end{pmatrix}.$$

Observe that the boundary conditions on the left and right sides of the plate have not been included.

This block matrix equation generates a natural block iterative scheme (analogous to the Gauss-Siedel method). Namely,

$$D_v T_2^{(k+1)} = PT_3^{(k)} + C_2$$

$$D_v T_3^{(k+1)} = PT_2^{(k+1)} + PT_4^{(k)} + C_3$$

$$D_v T_{n-2}^{(k+1)} = PT_{n-3}^{(k+1)} + PT_{n-1}^{(k)} + C_{n-2}$$

$$D_v T_{n-1}^{(k+1)} = PT_{n-2}^{(k+1)} + C_{n-1}$$

The missing boundary conditions may be adjusted after each application of these equations.

The proposed iterative scheme requires that the system of equations involving the D_v be solved. Since the D_v are positive definite [1] (D_v is symmetric and irreducibly diagonally dominant), the solution of the above equations by Gaussian elimination without pivoting is stable [2]. Since the D_v are tri-diagonal, the number of operations involved in their solution will be within the range of practicability. A further savings of computations can be effected by taking advantage of the fact that the D_v are identical except for their last diagonal element.

The procedure outlined above was tried on the problem, and converged in 250 iterations. This fast convergence was to be expected since we incorporated the convective boundary conditions into the equations involving the D_v , which were solved directly. No attempt was made to over-relax the method.

One mildly surprising point is that although the D_v have a rather high condition number ($\|D_v\| \cdot \|D_v^{-1}\|$), the solutions obtained from these

equations were sufficiently accurate for practical purposes. As a tentative explanation of this situation, we note that most of the elements of D_v are zero and our method of solution above leaves these zeros undisturbed. Under mesh refinements the numerical solutions behaved approximately as suggested by the study of the one-dimensional analogue. The original problem was also solved directly using Fourier series. The comparable numerical values of this solution agreed very well with the values of our numerical method.

REFERENCES

1. Varga, R. S., Matrix Iterative Analysis, Printice Hall, 1962.
2. Wilkinson, J. H., Error Analysis of Direct Methods of Matrix Inversion, J. ACM 8 (1961), 281-330.