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SOME PROBLEMS IN THE ESTIMATION AND
APPLICATION OF WEIBULL STATISTICS

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September 1, 1967

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SOME PROBLEMS IN THE ESTIMATION AND APPLICATION OF WEIBULL STATISTICS*

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ABSTRACT

A. A Weibull-type distribution is derived from slightly modified premises which explains why size effects are usually overestimated.

B. A number of standardized forms are proposed for the Weibull distribution which enhance flexibility in application.

C. The empirical distribution of Weibull Modulus estimators was studied using computer-generated samples with known parent population parameters. It is shown that a rather large but similar variability exists for all estimators, even the maximum likelihood, and that the coefficient of variation seems to depend only on sample size. The precision of estimate (standard deviation) improves with increasing sample size and is, roughly, proportional to $n^{-\frac{1}{2}}$ (as would be expected).

D. The problem of fracture source distribution is solved, and an experiment on alumina beams is described which clearly demonstrates the strong influence of surface features. The experiment also provides additional information to aid in selecting the proper form of the Weibull.

E. Application of the Weibull to strength data of composites is demonstrated, and it is shown that the composite's strength is controlled by the weakest 10 – 15% of the fibers. Therefore, the upgrading of composites is best accomplished by proof-testing of the fiber prior to fabrication.

F. Finally, design problems being faced in large-scale application of graphite conversion machinery are described – scale up, design with arches and domes, effects of stress concentration, and fatigue studies. Some significant applications of fractography in design and diagnostics are described.

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INTRODUCTION

An attempt is made here to touch briefly on a variety of problems and areas of interest calling for the use of brittle materials and the analytical method pertaining to their characterization and application. The contents are divided into six essentially independent sections described in the abstract, although a standardized Weibull distribution developed in section B is used in subsequent sections.

In the first section, the often derived and re-derived Weibull Function is derived once again, in this case in terms of the distribution of supposed stress concentrations and the premise that local failure occurs when a unique limiting stress is reached, through the effect of a suitably severe stress concentration in the locality. A heuristic proposal for the necessary distribution of stress concentration is made, an approach for which there is ample precedent, but which emphasizes the need for a more fundamental analysis of load transmission through matter so that a logical quantitative association may develop between micro-mechanics and pragmatic continuum analysis.

In section B, some standardized forms of the Weibull are presented. A few of these have been used by the author and found to be convenient and instructive. Just as the standard form of the normal distribution broadens and eases its application, so also should be the result of a standardized Weibull. And just as the standard normal is defined by a single parameter, the coefficient of variation, the standard Weibull proposed here is defined by a single parameter, the Weibull Modulus. The graphical description, which the standardized form makes possible, reveals features of the distribution which tend to be obscured by the usual analytic expressions.

The third section describes a variety of empirical determinations of the Weibull Modulus using controlled, computer-generated populations. The results show a rather high variability in these estimates, even for the maximum likelihood estimator, which proves to be not outstandingly superior and therefore, for the usual purposes, not worth the effort it entails. The matter of "correct" plotting position is also considered in this section and it is shown that the mid rank position, given by $\frac{r - 1/2}{n}$, is nearer to observation than the commonly used mean rank $\frac{r}{n + 1}$. A slightly safer approach to plotting is suggested in which a probability interval, as a line, is used in preference to a unique point.

The fourth section deals with a specific practical problem, that of the location of fracture initiation. This distribution is implicit in the Weibull Distribution and may be used to determine in specific experiments the relative importance of surface features (stress concentrations) to the volume features. The study of fracture-source distribution can also aid in deciding whether or not to use a distribution containing a threshold.

Section E demonstrates the utility of the standardized Weibull Function in the analysis of composite structures, and some data are given to confirm the applicability of the analysis to such structures. Because of the wide variety of fibers and matrices

and the many many investigations, there is a confusing mass of published data with the inevitable disagreements and frequent lack of complete information. It is safe to say that no one theory will appear which can correlate it all. However, the characterization of fiber data by the Weibull and subsequent analysis as an idealized parallel model should be an excellent first step and will likely be perfectly adequate for a design when fiber properties are not dramatically altered by the matrix. Studies of fiber-reinforced graphite are being initiated at LRL which, it is hoped, will confirm these suppositions.

The last section touches on some actual design experiences with refractory brittle materials and in more detail on the matter of stress concentration, the unavoidable central issue in the behavior of "non-yielding" materials. The detailed stress distribution around a circular hole is examined, the characteristics of which are instructive in understanding the behavior of brittle materials subject to multiaxial stress. Thus, for example, the relative magnitudes of measured stress concentrations may be understood in terms of varying Weibull Modulus and size. A brief but significant experiment on stress concentration with creep is described. The fact that most brittle materials exhibit compressive strengths which are usually more than three times the tensile strength is an incentive to induce operational compressive stresses. This is entirely feasible through the use of pre-stressing, arch-like structures, and redundant supports. It is concluded that Weibull's concept of the "risk of rupture" is likely to remain a practical and useful basis for interpreting observation for a long time.

SECTION A

THE DISTRIBUTION OF STRESS CONCENTRATION

We assume that the material failure is characterized by a critical local stress which is equal to the maximum strength σ_L . There exist local features which raise the nominal stress by stress concentration so that even at moderate applied stresses the true local stress may reach this critical level and apparently premature failure results. The local stress concentration is K and deviates from unity at these features. Such stress concentrations are encountered with a frequency of ρ in unit volume. The magnitude of K is distributed such that $K \geq 1$ and $S(K)$ is the fraction of all the population values of K ranging from K up to $K = \infty$ (zero load-carrying capacity corresponding, say, to a microcrack), i. e., S is the cumulative distribution of higher ranks.

Now the probability that fracture does not occur at one of these perturbed regions under a given nominal applied stress σ is the probability that the highest local value of K is insufficient to raise the applied stress to σ_L , that is, the probability that K be less than σ_L/σ . The probability of survival S_s at this region is then

$$S_s = 1 - S\left(\frac{\sigma_L}{\sigma}\right) \equiv F\left(\frac{\sigma_L}{\sigma}\right) \quad (A-1)$$

By a simple extension, the probability that n concentrations survive is

$$S = \left[F\left(\frac{\sigma_L}{\sigma}\right) \right]^{\rho V} \quad (A-2)$$

since n is just the product of the number density ρ and the volume.

If we consider several segments of volume, V_j , within each of which a different stress is present, equation A-2 becomes the joint product of local cumulative probabilities:

$$S = \prod_{j=1}^n \left[F\left(\frac{\sigma_L}{\sigma_j}\right) \right]^{\rho V_j} = \prod_{j=1}^n F^{\rho V_j} \quad (A-3)$$

or, taking logs

$$\ln S = \rho \sum_{j=1}^n V_j \ln F_j$$

and, as $V_j \rightarrow dV \rightarrow 0$

$$n \rightarrow \infty$$

$$\ln S \rightarrow \rho \int \ln F dV.$$

Finally, then, equation A-3 becomes

$$S = \exp \rho \int \ln F \left(\frac{\sigma_L}{\sigma} \right) dV. \quad (A-4)$$

To proceed further, some assumptions are necessary regarding the lower-rank cumulative distribution function $F \left(\frac{\sigma_L}{\sigma} \right)$ of stress concentration.

The conditions which must be met by the function are

$$F(K) \rightarrow 1 \quad \text{as} \quad \frac{\sigma_L}{\sigma} \rightarrow \infty \quad \text{i.e. } K < \infty$$

$$F(K) \rightarrow 0 \quad \text{as} \quad \frac{\sigma_L}{\sigma} \rightarrow 1 \quad \text{i.e. } K > 1.$$

A flexible and convenient function having this property is

$$F(K) = \exp - \left(\frac{1}{K-1} \right)^m = \exp - \left(\frac{1}{\frac{\sigma_L}{\sigma} - 1} \right)^m.$$

Putting in (A-4), we get

$$S = \exp -\rho \int \left(\frac{\sigma}{\sigma_L - \sigma} \right)^m dV. \quad (A-5)$$

Note that when $\frac{\sigma}{\sigma_L} \ll 1$ this distribution becomes, approximately,

$$S = \exp -\rho \int \left(\frac{\sigma}{\sigma_L} \right)^m dV \quad (A-6)$$

which is the familiar form of the Weibull.

On the other hand, if we assumed, using equation A-4, as Weibull does,

$$\rho \ln F = - \left(\frac{\sigma}{\sigma_0} \right)^m$$

we get

$$F(K) = \exp -\frac{1}{\rho} \left(\frac{\sigma}{\sigma_0} \right)^m \approx \exp -\left(\frac{\sigma}{\sigma_L} \right)^m.$$

We find that the spatial flaw frequency ρ is implicit in Weibull's constant σ_0 and the stress σ_L :

$$\left(\frac{\sigma_L}{\sigma_0} \right)^m = \rho$$

Thus ρ may be expected to be enormous since σ_0 is of the order of the average value, and usually well over an order of magnitude below the theoretical strength.

Using Weibull's form we would need to allow $\sigma \rightarrow \infty$ for F to vanish, a physically awkward requirement but mathematically convenient. For the proposed stress concentration model with failure governed by a unique limiting value, the observed strength distribution is equation A-5, but when observed strength is well below the limiting value σ_L , the usual distribution, in equation A-6, is a satisfactory approximation.

The size effect is of primary importance in design and it is not infrequently observed that the usual predictions overestimate the observed effect of size difference. If we consider the distribution of equation A-5, the size effect corresponding to this distribution is easily obtained for the case of uniform tension, (but not for other cases of interest such as bending)

$$S = \exp -\rho \left(\frac{1}{\frac{\sigma_L}{\sigma} - 1} \right)^m V,$$

by comparing the strengths at the same probability of failure for two volumes we have

$$\frac{\frac{\sigma_L}{\sigma_2} - 1}{\frac{\sigma_L}{\sigma_1} - 1} = \left(\frac{V_2}{V_1} \right)^{1/m}.$$

This may be written in several useful ways,

$$\frac{\sigma_1}{\sigma_2} = \left(\frac{1 - \frac{\sigma_1}{\sigma_L}}{1 - \frac{\sigma_2}{\sigma_L}} \right) \left(\frac{V_2}{V_1} \right)^{1/m} \quad (A-7a)$$

or

$$\frac{\sigma_1}{\sigma_2} = \frac{\sigma_1}{\sigma_L} + \left(1 - \frac{\sigma_1}{\sigma_L}\right) \left(\frac{V_2}{V_1}\right)^{1/m} \quad (\text{A-7b})$$

or, as an estimate of the upper limit σ_L :

$$\sigma_L = \sigma_1 \left[\frac{1 - \left(\frac{V_2}{V_1}\right)^{1/m}}{\frac{\sigma_1}{\sigma_2} - \left(\frac{V_2}{V_1}\right)^{1/m}} \right] \quad (\text{A-7c})$$

Equation A-7a explains why, if scale-up (e. g. $V_1 \gg V_2$) is based upon very small specimens, for which σ_2 is a sensible fraction of σ_L , the predicted strength of larger samples by the common relation, $\left(\frac{V_2}{V_1}\right)^{1/m}$, will underestimate the actual strength of the large specimens. However, if as usual the highest strength is only a few percent of the theoretical, the usual size effect prediction is approached by this model. The "upper limit" σ_L may also be put to work as a parameter in constructing algebraic forms which give improved estimates of observed size effects.

There are other factors as well which cause a depression of the experimentally observed size effect. Experimental measurements of size effect in uniaxial tension are complicated by the presence of eccentricity. The presence of such unknown eccentricity tends to reduce the measured effect of size and leads to disagreement between Weibull Modulus estimates based on strength distribution and those based on size.

The sketch below shows a typical stress distribution when some eccentricity is present and gives the stress distribution in a rectangular beam with eccentricity parallel to a side:

$$\sigma = \sigma_T \left[\left(1 - 6 \frac{\epsilon}{h}\right) + 12 \frac{\epsilon}{h} \frac{x}{h} \right]$$

The risk of rupture is:

$$B = \left(\frac{\sigma_T}{\sigma_0}\right)^m V_T \left[\frac{\left(1 + 6 \frac{\epsilon}{h}\right)^{m+1} - \left(1 - 6 \frac{\epsilon}{h}\right)^{m+1}}{12 \frac{\epsilon}{h} (m+1)} \right]$$

Figure A-1 shows how the strength of the beam with eccentricity decreases as both the Weibull m and ϵ increase. If the grip geometry of a test specimen remains

Fig. A-1. The effect of eccentricity on measured tensile strength.

unchanged while the gage section is varied, the eccentricity as expressed above, $\frac{\epsilon}{h}$, will increase as h decreases and will depress the strength of the smaller specimens. The results will then give the appearance of a decreased size effect which, in itself, is not distinguishable from similar influences of upper and lower bounds. The effect may be reduced by: employing large tensile specimens with grip geometries which are proportional to the gage dimensions; studying size effect primarily by varying gage length; or using a variety of bending tests for such investigation.

The stress concentration model may be followed through with the assumption that the local stress concentration K is limited to values below a maximum K_L , giving rise to a threshold. In this case the proposed distribution function would take the form

$$F(k) = \exp - \left(\frac{K_L - K}{K - 1} \right)^m .$$

This is examined in the next section on standardized distributions. Other modifications of the function also suggest themselves.

Estimates of the limiting strength from size effects using equation A-7c lead to values which are well below those usually suggested for theoretical strength, and estimates of ρ may become intolerably large as demonstrated below. The risk of rupture in uniform tension is, with this model,

$$\rho \left(\frac{\sigma}{\sigma_L - \sigma} \right)^m V .$$

We take $V = 1 \text{ cc}$, $m = 10$ and assume that the average strength is $0.01 \sigma_L$. Since the failure fraction will be near 50% we can write the approximate relation

$$0.69 \cong \rho \left(\frac{1}{99} \right)^{10}$$

and

$$\rho \cong 0.69(10^{20})$$

which is within sight of the total number of atoms in that volume. The above computation also implies that, as the strength decreases a compensatory change in m is necessary to keep ρ within a physically possible bound.

As a final comment it is observed that efforts are needed to develop forms of the flaw intensity distribution function from physical considerations, and that elements with anisotropy will need to be considered since fracture may be initiated because specific local directional properties are exceeded. It is also abundantly clear that pure series interpretation of material failure is too severe. Even the most "brittle" materials

demonstrate capacities for partial and multiple fracture so that redundancy for load transmission is inevitably present, and it is only when such redundancy is exhausted that the weakest link becomes the deciding factor in ultimate collapse.

SECTION B STANDARDIZED FORMS

The Weibull Distribution may be standardized in several ways using various features, such as mean, mode etc., for reference. Weibull, for example, proposed a form analogous to the standardized normal using the standard deviation to normalize deviations from the sample mean. The use of the mean as a reference has been found very convenient in practice and is therefore recommended to other workers as a useful standard form.

The Weibull Distribution of the form

$$S = \exp - \int_V \left(\frac{\sigma}{\sigma_0} \right)^m dV \quad (B-1)$$

S = survival probability (or more generally the cumulative probability of higher ranks)

σ = tensile stress of arbitrary spatial distribution

σ_0 = constant

m = Weibull Modulus

V = volume subject to tension

may be transformed to

$$S = \exp -\beta^m \Gamma^m. \quad (B-2)$$

β = normalized stress; a reference ultimate stress divided by its average

Γ = Gamma Function, $\Gamma\left(1 + \frac{1}{m}\right)$.

This form of the Weibull has many features to recommend it:

a) Data from a variety of studies may be pooled to yield large samples, since normalization eliminates the stress distribution integral.

b) A single chart (see Figure C-4) may be made to depict uniquely the family of Weibull Distributions.

c) This is a very convenient analytical form to use in extreme-value or other computations.

d) The constant σ_0 has been replaced by a much more convenient parameter, the mean value.

e) If the sample average is accepted, this form shows that, in fact, there is only one parameter left to define the distribution – m . It seems that on some occasions graphical estimates are made of σ_0 while the sample average is also referred to, without the realization that one defines the other.

f) In some material properties such as creep and fatigue the ratio of applied stress to strength is found to be a dominating parameter and its equivalent here, β , is a natural variable.

g) The kth moment of the distribution is given by

$$\mu_k = \frac{\Gamma\left(1 + \frac{k}{m}\right)}{\left[\Gamma\left(1 + \frac{1}{m}\right)\right]^k}$$

Before proceeding to the Weibull forms having thresholds we can quickly examine the results of extreme-value computations with the standard form of equation B-2.

As is well known, the equations which are satisfied by the modes of the extremes of a sample of size n are (with primes indicating differentiation):

$$SS'' = -(n - 1)(S')^2 \quad (B-3)$$

for the most probable least value, and,

$$(1 - S)S'' = (n - 1)(S')^2 \quad (B-4)$$

for the most probable largest value. Using the standardized form of S gives, for these modes:

$$\beta_{\text{least}} = \left(\frac{m - 1}{mn}\right)^{1/m} \frac{1}{\Gamma} \quad (B-5)$$

$$n = \left(1 - \frac{m - 1}{m\beta_{\text{max}}^m \Gamma^m}\right) \left(e^{\beta_{\text{max}}^m \Gamma^m} - 1\right) + 1 \quad (B-6)$$

The graphs of equations B-5 and B-6 are shown in Figure C-1. This chart may be used to assess quickly the size effect, to estimate the Weibull Modulus from observed data, or to test data for the presence of a lower threshold, the so-called zero strength.

The usual Weibull distribution containing a threshold is given as:

$$S = \exp - \int \left(\frac{\sigma - \sigma_u}{\sigma_0}\right)^m dv$$

and can be treated in an entirely analogous fashion to the previous case provided it is assumed that σ_u is distributed in space with geometric similarity to σ . This, admittedly, appears very artificial but may be rationalized with pragmatism if the data are well suited to the result, (the assumption is correct for uniform tension). The result is

$$S = \exp -\left(\frac{\beta - \alpha}{1 - \alpha}\right)^m \Gamma^m \quad (B-7)$$

where

$$\alpha = \sigma_u \text{ divided by the sample average, the same normalizing factor as used in } \beta$$

$$\beta \geq \alpha$$

The computation of the modal least value gives

$$\beta_{\text{least}} = (1 - \alpha) \left(\frac{m - 1}{mn} \right)^{1/m} \frac{1}{\Gamma} + \alpha \quad (B-8)$$

The coefficient of the first term is, of course, simply the modal minimum with $\alpha = 0$. It is suggested that α be regarded as representing a residual or prestress such as may exist at a heat treated surface which (when algebraically positive) retards failure. The threshold also depresses the size effect and thus observed size effect may give a clue to the presence of a lower bound. Recalling the results at the end of section A, we see now that any bound on the Weibull Distribution, whether above or below, depresses the effect of size. The frequently observed effect that experimentally measured size effects are not as strong as predicted by the conventional form

$$\frac{\sigma_2}{\sigma_1} = \left(\frac{V_1}{V_2} \right)^{1/m}$$

may simply result from the presence of a bound. It must be borne in mind that all conclusions are tempered by the uncertainty in m which may be more than sufficient to account for irregular size effects, as will be seen in section C. Another standardized form is that resulting from proof testing

$$S = \exp -(\beta^m - \psi^m) \Gamma^m \quad (B-9)$$

where the normalizing factor in the above equation is the sample average prior to the screening and ψ is the normalized proof stress. The modal least value of this distribution is given by equation B-5 until n , the sample size, exceeds the critical size,

$$n_{\text{crit}} = \left(\frac{m - 1}{m} \right) \frac{1}{\psi^m \Gamma^m}$$

above which the modal least value is simply ψ itself. Thus a study of the size effect can distinguish between this type of threshold (truncation) and the previously described type. A graph of typical least value behavior for each type of threshold is shown in Figure B-1. The data were generated by computer and a tabulation of the observed least values and their theoretical counterparts is presented in Table B-I. Since a sample which has been

Fig. B-1. Least value behavior with both types of thresholds, pre-stress and proof-stress.

Table B-I. Least value behavior of the Weibull with lower threshold parameters.
m = 4 (average of 5 samples of each size). See Figure B-1.

General form

$$S = \exp - \left[\frac{(\beta - \alpha)^m - (\psi - \alpha)^m}{(1 - \alpha)^m} \right] \Gamma^m$$

Modal least value with $\alpha = \psi = 0$ $\beta^* = \left(\frac{m-1}{mn} \right)^{1/m} \frac{1}{\Gamma}$

Modal least value with $\alpha = 0$ $\beta_{\psi}^* = \begin{cases} \beta^* & \beta^* > \psi \\ \psi & \beta^* < \psi \end{cases}$ approximation

Modal least value with $\psi = 0$ $\beta_{\alpha}^* = (1 - \alpha) \beta^* + \alpha$

Sample size	Theoretical modal least value	Avg "observed" least value	Absolute least value
$\alpha = \psi = 0$			
9	0.59	0.56	0.35 (0.39)*
19	0.49	0.48	0.4 (0.33)
49	0.385	0.31	0.19 (0.26)
99	0.325	0.32	0.25 (0.23)
$\psi = 0 \quad \alpha = 0.5$			
9	0.8	0.71	0.57
19	0.75	0.75	0.60
49	0.69	0.68	0.47
99	0.66	0.65	0.5
$\alpha = 0 \quad \psi = 0.5$			
9	0.59	0.56	0.50
19	0.49	0.61	0.48
49	0.48	0.50	0.48
99	0.48	0.53	0.50

*For $\alpha = \psi = 0$ the predicted absolute least value appears in parentheses.

truncated will have an increased average, using this sample mean as a normalizing factor will not result in a distribution of the form equation B-9 although, unless the truncation level is very high, the differences are not great. The enhancement of the mean value resulting from truncation is shown in Figure B-2.

Finally we have the upper bound form of the Weibull, discussed in section A, based on the idea of a single limiting strength. The standardized form for this distribution is expressed in terms of the survival probability at the sample average:

$$S = \exp \left[\beta \left(\frac{1 - \phi}{1 - \beta\phi} \right) \right]^m \ln S_a \quad (B-10)$$

where ϕ is the ratio of the sample average to the upper limiting strength, $\frac{\sigma_a}{\sigma_L}$, and S_a is the probability corresponding to σ_a , the sample average.

If the lower threshold of the form of equation B-7 is also included, we get

$$S = \exp \left[\left(\frac{\beta - \alpha}{1 - \alpha} \right) \left(\frac{1 - \phi}{1 - \beta\phi} \right) \right]^m \ln S_a \quad (B-11)$$

the normalized upper limit ϕ is defined above and the normalized lower limit α is, as before, $\frac{\sigma_u}{\sigma_a}$. When ϕ tends to zero $\ln S_a$ goes to $-\Gamma^m$ and the form of equation B-2 or B-7 results.

Experience has shown equation B-2 to be by far the most useful and convenient form. On rare occasions the invocation of a lower limit seemed appropriate (see section D) and equation B-7 was used. Because a variety of values of m and α may be used to give similar fits to the data, no great significance was attached to the estimate of α , though efforts were made to find fits with low values of α . On even rarer occasions data was analyzed as a composite of sub-samples each characterized by a mean value and Weibull Modulus. An example of such a duplex distribution is shown in Figure B-3.

Fig. B-2. Effect of truncation on the mean value.

Fig. B-3. Complex-distribution of strength for
Poco EP1924 graphite, avg = 15,000 psi.

SECTION C
PROPERTIES OF ESTIMATORS FOR THE WEIBULL MODULUS
USING ARTIFICIALLY GENERATED STATISTICAL SAMPLES

There are many possible estimators for the Weibull Modulus, and in this section some of these are described, as well as certain experiments designed to test their accuracy. Samples representing known Weibull distributions were generated and then analyzed by various methods used to deduce the Weibull Modulus. In all cases the parent distribution was of the simplest standardized form

$$S = \exp -\beta^m \Gamma^m. \quad (C-1)$$

Random samples from specified Weibull populations were generated by the following method: A random number, uniformly distributed between 0 and 1, was generated by a standard technique and identified with S. The above equation, for which m was specified, was then solved for the "observed" value of β . These samples were then analyzed by a variety of methods described below to estimate the Weibull Modulus and other properties of the distribution, and it is from such estimates that we obtain the empirical observations of the estimator distributions. The computer experiment was undertaken to evaluate several aspects of this statistical distribution:

- 1) The distribution of some Weibull Modulus estimators.
- 2) The relative accuracy of the various estimators.
- 3) The effect of sample size.
- 4) The confirmation of design guide-lines.
- 5) The "correct" plotting position.

The estimators examined in some detail were: (a) The least value; (b) the greatest value; (c) the standard deviation; (d) the range; (e) the average of the estimates by least and greatest values; and (f) the maximum likelihood and, to a lesser extent, the log-log slope estimator and the semi-log graphical estimator. Each of these estimates is described briefly below.

The Weibull Modulus in the distribution of equation C-1 is related to the most probable least value by equation B-5:

$$\beta_{\text{least mode}} = \left(\frac{m-1}{mn} \right)^{1/m} \frac{1}{\Gamma}$$

and to the most probable largest value by equation B-6:

$$n = \left[1 - \frac{m-1}{m\beta_{\text{max}}^m \Gamma^m} \right] \left(e^{\beta_{\text{max}}^m \Gamma^m} - 1 \right) + 1.$$

It's relation to the standard deviation, expressed as coefficient of variation, CV, is

$$CV = \sqrt{\frac{\Gamma(1 + 2/m)}{[\Gamma(1 + 1/m)]^2} - 1}$$

Finally the range parameter is the difference between the modes of the extremes. If the extreme values are independent of one another, the mode of the range will be the difference of the modes of the extremes. Epstein (1) also gives a one-sided confidence interval for m, based on the extremes in a sample of size n:

$$\text{then } m \leq \frac{\ell n(1 + R)}{\ell n \frac{\max}{\min}} \quad \text{with prob} \left(\frac{n}{R} \right) \frac{\Gamma\left(\frac{n}{R}\right) \Gamma(n)}{\Gamma\left(\frac{n}{R} + n\right)}$$

where R is a parameter.

Figures C-1, C-2, and C-3 show the extreme values, range parameter, and the coefficient of variation as functions of the Weibull Modulus, derived from the equations above.

To get estimator (e) above from a given sample, the estimates of m from the two extremes were averaged. For estimator (f) a computer program was written to produce maximum likelihood estimates of the Weibull Modulus and the threshold u in the equation

$$S = \exp - \left(\frac{x-u}{x_0} \right)^m \quad (C-2)$$

By setting u = 0, a maximum likelihood estimate may be determined for m alone, or the parameter may be left undefined and an estimate obtained for m and u, the latter value of m being, of course, different from the former. Equation C-2 with u = 0 can be transformed to equation C-1. The details of this estimate are described in Ref. 2 and it should be noted that when both m and u are being estimated, a unique maximum likelihood estimator may not exist, as will be seen in the experiment.

The log-log slope estimator is the one most familiar to those interested in the Weibull Distribution. In the ideal case a sample from the Weibull Distribution produces a straight line of slope m when plotted to suitable logarithmic coordinates. A few of the generated samples were treated this way and fitted by eye. The semi-log estimate uses the chart shown in Figure C-4, a useful plot of equation C-1. To use it, the normalized data may be plotted at the selected probability levels and the best vertical line (if appropriate) gives the estimated Weibull Modulus. However, to avoid selecting a unique probability point, one may plot the observations as probability intervals. An interval of 1/n is convenient and also makes curve fitting easier. Again only a few data samples were examined this way.

There are a number of distinct advantages to the semi-log chart of Figure C-4. First, it may be used in making plots for any Weibull Modulus; secondly, it gives an

Fig. C-1. Extreme values of the Weibull Distribution vs sample size.

Fig. C-2. Range of the Weibull Distribution vs sample size.

Fig. C-3. Coefficient of variation vs Weibull Modulus.

Fig. C-4.

immediate perspective of the m -value appropriate to each observation. This allows conservative design estimates of m to be made which are on a firmer basis than those derived from the log-log plot; thirdly, a straight line on the log-log plot is not a sufficient condition for the applicability of a Weibull Distribution of the type in equation C-1, since the intercept of the line is also implicit in the distribution. There is no analogous ambiguity in the semi-log plot of Figure C-4; and finally, it demonstrates the reason why the central portion of the data is very insensitive for estimation of m . In this region minute changes in the probability (plotting position) imply large variations in m . The chart also suggests that the most effective probability regions for resolving m are the upper and lower 25% for m greater than about 3.

The Computer Populations

In the Series I experiments, 20 samples of size 10 and size 50 were generated from parent populations described by the m -values 3, 6, 15, and 30. In Series II the population characterized by $m = 10$ was sampled 20 times for each of the sizes 9, 19, 49, and 99, and that of $m = 3$ sampled for sizes 10, 33, and 100. In a third series (Series III) of experiments, sample sizes 9, 19, 49, and 99 were generated, five of each size, for another variety of m -values: 2, 4, 10, and 20. These were analyzed graphically.

Tables C-I, C-II, and C-III summarize the results of each series. Before focusing on each aspect of the estimation problem it is necessary to note that only the standard deviation estimate and the maximum likelihood estimate were essentially free of subjective factors. In order to achieve conditions typical of the practical situations where these estimators might be applied, the graphs included in this section were used on the generated data (it would have been personally much more convenient to have the computer back-out the exact m -values implied by the estimators) and consequently visual interpolation was needed and fairly gross rounding-off was done. The uncertainties, therefore, are representative in part of the subjective visual use of these graphs. This is also true for the cases where graphical estimates were made by hand, as discussed previously. The several objects of this investigation may now be examined.

1. The Distribution of the Estimators

Figure C-5 summarizes the results of Series I, showing the coefficient of variation obtained for each estimator applied to 20 samples. Some of the irregularity may be attributable to subjective influences discussed earlier. There is a consistency in the behavior and the estimate does, in fact, improve with increasing sample size. The essential observation insofar as the estimator distributions are concerned is that all coefficients of variation are sufficiently similar at each sample size to suggest that sample size alone, and not the m -value, is determining, without much difference

Table C-I. Weibull Modulus estimators. Results of Series I computer experiments (20 samples for each case).

Parent population parameter m	Sample size	Estimators of m and their coefficients of variation						Avg. CV
		Std. dev.	Least value	Max. value	Range parameter	Avg. of least max.	Max. lik. (u = 0)	
3	10	3.4	3.6	3.5	3.2	3.6	3.6	
Coeff. of var.		0.29	0.42	0.31	0.3	0.3	0.31	0.32
	50	4.6	3.0	3.1	3.0	3.1	3.1	
Coeff. of var.		0.12	0.21	0.19	0.14	0.12	0.092	0.15
6	10	6.7	6.4	6.8	6.2	6.4	7.1	
Coeff. of var.		0.32	0.41	0.27	0.32	0.3	0.3	0.32
	50	5.8	5.8	6.0	6.0	5.9	6.0	
Coeff. of var.		0.15	0.31	0.22	0.16	0.23	0.15	0.18
15	10	17	13.9	16.6	15.1	15.0	18.1	
Coeff. of var.		0.36	0.44	0.26	0.35	0.29	0.29	0.33
	50	14.5	13.4	15.8	14.3	14.6	14.6	
Coeff. of var.		0.15	0.17	0.2	0.17	0.15	0.13	0.16
30	10	39.6	28.6				29.9	
Coeff. of var.		0.33	0.27				0.22	0.28
	50	30	27				30.5	
Coeff. of var.		0.12	0.17				0.11	0.13

Table C-II. Weibull Modulus estimators. Results of Series II computer experiments (20 samples for each case).

$m = 3$ n	S.D.	Least	Max.	Range	Avg.	Max. lik.*	Avg. CV
10	3.4	3.6	3.5	3.2	3.6	3.6	
C. V.	0.29	0.42	0.31	0.3	0.3	0.31	0.32
33	3.64	3.11	3.1	3.0	3.11		
C. V.	0.29	0.27	0.23	0.18	0.17		0.24
50	4.6	3.0	3.1	3.0	3.1	3.1	
C. V.	0.12	0.21	0.19	0.14	0.12	0.092	0.15
100	6.5	3.2	2.8	2.9	3.0	3.0	
C. V.	0.61	0.16	0.12	0.10	0.094	0.07	0.11**
$m = 10$							
n							
9	11.69	11.45	10.75	10.67	11.36		
C. V.	0.353	0.442	0.322	0.395	0.35		0.37
19	9.9	9.35	10.55	9.45	9.93		
C. V.	0.184	0.216	0.232	0.190	0.182		0.20
49	9.61	10.65	9.75	10.22	10.18		
C. V.	0.094	0.161	0.112	0.123	0.097		0.12
99	10.1	9.32	10.27	10.1	9.78		
C. V.	0.10	0.247	0.08	0.17	0.136		0.14

* Assuming location $u = 0$.

** Excluding S.D. estimate.

Table C-III. Weibull Modulus estimators. Results of computer experiments.
(20 samples for each case.)

Parent distribution $u = 0, m = 2$								
Sample size	m-value by semi-log graph		Maximum likelihood estimates			m-value by single point estimates		
	Best fit	Design	u	m	$u = 0$ m	Least value	Highest value	Std. dev.
9	2.5	2.0	0.33	2.2	3.5	3.5	3	3.3
	1.4	1.0	-0.09	1.8	1.4	1	1	2.1
	1.5	1.1	-1.67u + 2.18		2.0	2.2	1	2.7
	1.5	1.2	-1.27	5.8	2.0	1.5	2	2.2
	2.5	2.2	0.31	1.9	3.0	3	2	2.8
Averages	1.9	1.5	-0.1	2.6	2.4	2.3	1.8	2.6
19	1.5	1.0	0.0	1.8	1.8	1.5	1.8	3.7
	2.1	1.3	0.25	1.7	2.6	2.5	3	2.4
	2.0	1.5	0.26	1.5	2.4	2.5	1.8	3.8
	1.8	1.0	-0.30	2.9	1.5	1	2	2.1
	1.4	1.0	-1.26u + 2.07		1.5	1	1	2.5
Averages	1.8	1.2	0.05	2.0	1.9	1.7	1.9	2.9
49	2.0	1.4	-1.74u + 2.6		2.2	1		5.5
	3.0	2.3	0.07	2.7	3.0	2		9
	1.7	1.3	-1.67u + 2.1		1.9	1.5		5.8
	1.8	1.3	-0.06	2.0	1.6	3		4.9
	2.3	2.0	-1.70u + 2.7		2.5	1		5.1
Averages	2.2	1.7	0	2.3	2.2	1.7		6.1
99	2.0	1.9	0.18	1.7	2.4	3	2	7
	2.0	1.5	-0.03	2.4	2.3	3	1.8	10
Averages	2	1.7	0.07	2.1	2.1	3	1.9	8.5
Parent distribution $u = 0, m = 4$								
9	4.8	3.0	-6.78u + 6.53		6.3	4	6.5	5.7
	4.6	3.0	-5.35u + 5.72		5.7	6	4.5	5.7
	2.8	1.9	-0.04	3.3	3.2	3.5	2.5	3.1
	4.7	4.0	-0.14	5.2	6.1	5	6	5.8
	3.3	2.2	-4.7u + 4.6		4.5	4	5	4.0
Averages	4.0	2.8	0.05	4.2	5.2	4.5	4.9	4.8
19	3.5	3	0.46	2.1	4.1	4.5	3	4.8
	2.8	2	0.46	1.4	3.4	4.2	3.8	3.3
	3.9	2.9	-8.51u + 6.0		4.5	3.4	4.5	4.5
	2.7	2	0.30	1.9	3.0	3.5	2	2.9
	3	2	-3.6	18.8	3.8	3.8	4.5	3.6
Averages	3.2	2.4	-2.5	5.8	3.8	3.9	3.6	3.8
49	3	2.2	-0.16	3.9	3.3	3.5		3.3
	3	2.6	-0.13	3.8	3.2	2.5		3.3
	4.5	3			4.0	5		4
	5	3.8	-0.02	5.5	5.4	4.7		5.6
	3.9	3	-0.71	7.7	4.4	3.5		4.5
Averages	3.9	2.9	-0.25	5.1	4.1	3.8		4.1

No unique pair of m and u.

Table C-III. Weibull Modulus estimators. Results of computer experiments.
(20 samples for each case) (Continued).

Sample size	m-value by semi-log graph		Maximum likelihood estimates			m-value by single point estimates		
	Best fit	Design	u	m	u = 0 m	Least value	Highest value	Std. dev.
99	3.9	3	0.33	2.5	4.0	4.8		4.0
	4.1	3	0.0	4.2	4.2	3		4.3
	4.1	3	0.17	3.4	4.1	4		4
	4	3.2	0.0	4.1	4.1	4		4
	4.8	3	-0.36	6.7	4.8	3.5		4.2
Averages	4.2	3	0.02	4.2	4.2	3.9		4.1
Parent distribution u = 0, m = 10								
9	12	10	-13.6u + 13.8		16	14	13	14.5
	10	6	-0.85	24	12.5	11	14	12
	9	6.5	-10.2u + 10.5 ^c		8.6	10.5	10	10.7
	7.6	4.5	-5.8	63	14.1	5.5	8.5	7.6
	10	8	-12.1u + 12.6		7.6	14	12	13.7
Averages	9.7	7	-3.3	43	11	11	11	12
19	14	6	-17.3u + 16.4		13.9	6.5	20	12.5
	10.5	8	-0.28	16	12.8	10	15	12.5
	7.1	6	-0.13	9.8	10.5	8	9	8.4
	11	7	0.03	13.8	8.9	16	15	14
	6.4	5	-7.8u + 7.7		12.6	5.5	9.5	7.1
Averages	9.8	6.4	-0.13	13	11.7	9.2	13.7	10.8
49	10.5	8	0.4	6.7	11.3	10	10.5	12.4
	9.2	8	0.4	6.0	10.0	10	10	10.4
	12	9	-10.6u + 11.2		11.3	20	10	13.5
	9	7	0.59	3.8	9.3	11.5	8	10
	8.2	7	0.28	6.5	9.0	9	9	9.2
Averages	9.7	7.8	0.42	5.6	10.2	12	9.5	11.1
Parent distribution u = 0, m = 20								
9	16	10	-19.5u + 19.3		19.2	11	20	15
	14.5	11	0.78	4.2	19.6	18	17	20
	20	18	-64.8u + 33.4 ⁱ		29.8	20	14	30
	10	6	-14.6u + 14.3 [*]		14.2	11	13	12.5
	23	15	-0.91u + 29.5		32.6	18	25	30
Averages	16.7	12	0.78	4.2	22.9	15.6	17.8	21.4
19	22	10	-2.5	82	23	14	Significantly greater than 20.	23
	20	8	-2.8	91	22.9	11		18.5
	32	20	-29.6u + 30.3		30.4	25		42
	21	15	0.84	3.6	22.7	25		26
	18	11	0.88	2.2	19.1	25		22
Averages	22.6	12.4	-0.9	45	23.6	20		26

No unique pair of m and u.

Fig. C-5. Coefficient of variation of m -estimators vs sample size.

between the estimators tried. The standard deviation estimate behaves badly for $m = 2$ and 3 where accuracy and precision deteriorated with increasing sample size. In other respects, for the broad purposes at hand, the estimators may be treated as being of the same distribution. If the coefficients of variation are averaged for each sample size, the results are: at $n = 10$, $CV = 0.31$; and at $n = 50$, $CV = 0.15$.

Graphical studies of the shape of the distribution of the estimators show them to be similar and suggest that the estimators are about normally distributed. For example, Figure C-6 shows the distribution of the pooled standard deviation estimates from the m -values at the size 10 and 50 samples. They are quite satisfactorily approximated by normal distributions with standard deviations corresponding to the above tabulated average values. One of the surprising outcomes is the degree of variability of the maximum likelihood estimate and its behavior when two parameters (m and u) are available for this estimate. In some of these cases unique values of m and u were not produced; instead, within the limits of accuracy imposed by the computational process (5% of the estimated m) there occurred a rather wide region where various pairs of m and u gave equally acceptable fits to the data. In all such cases observed, the relation between m and u was linear:

$$m = -Au + B.$$

This is very significant behavior because in the graphical estimation of a threshold u and parameter m only one pair m and u would be used, and the assumption that u estimated a unique threshold might be very gravely in error. Some of these relations may be seen in Table C-III.

Clearly, when in doubt the best policy is to take $u = 0$.

2. The Relative Accuracy of the Various Estimators

The results indicate that none of the estimators are outstandingly accurate nor does the precision of estimate differ sharply and consistently between the various estimates tried. The average of the extreme value estimates is a convenient one since a similar estimate with each extreme lends confidence to the use of a Weibull approximation. The graphs of Figure C-5 show that the maximum likelihood estimate may yield slightly less variable results for the larger samples. In any case it is a good idea to plot the data and have a look at the fit throughout the sample. Remarks have already been made regarding the accuracy of the threshold estimates by the maximum likelihood method. It will be noted that on occasion the method produces a negative value of u . This has been rejected by some workers as untenable, but a moment's reflection is sufficient to deny this viewpoint. In the first place, the choice of a negative u obviously produces an improved fit to the observed sample and for that reason may yield improved probability estimates, at least in the region near the observations. Secondly, a negative value of u implies a pre-existing disposition to failure such as may accompany the

Fig. C-6. Pooled standard dev. estimator of m ($m = 3, 6, 15, 30 - 20$ groups of $n = 10$ and $n = 50$ for each m).

presence of tensile residual or pre-stress. Finally, the premises and logical operations upon which the Weibull Distribution is founded may not be unique nor are they likely to be particularly accurate descriptions of the real world. It may therefore be necessary to invoke pragmatism when a substantial improvement in the analytical approximation of the data results.

It is recommended that samples be estimated by the average of the extreme value estimates and the range for purposes of quick approximation, but more deliberate analyses should rely on graphical displays.

3. The Effect of Sample Size

The Series II tests explored the effect of sample size for $m = 3$ and $m = 10$ to supplement the previous studies. The results are graphed on Figure C-7, and in general confirm the previously described trends.

No explanation is made for the apparent minimum in the $m = 10$ case. It may be supposed that at very large sample size the least-value estimation becomes poorer because it is bounded, and its sensitivity to small changes in the observed value increases; however, Figure C-1 shows that even with 1000 samples the squeeze on the least-value estimate is not, relatively, pronounced and also explains the increasing reliability of the maximum value estimation, showing the manner of divergence and decreasing sensitivity of the estimate to small changes in the observed largest value. It may be significant that two estimates which do not explicitly involve the least-value (the CV and largest) do not exhibit the minimum.

It is concluded that there is little advantage, insofar as estimating m is concerned, to use samples in excess of about 50, and that for the most part samples of 30 should be sufficient, provided a coefficient of variation on the estimate of about 20% is acceptable.

Once again the practical difficulties of obtaining a coefficient of variation better than 10% are evident. A summary of all measured coefficients of variation is plotted against sample size on Figure C-8, showing the averages and total range of the estimates (excluding the standard deviation estimates for $m = 2, 3$). The CV for any estimate may be predicted roughly by

$$CV = n^{-0.5}$$

4. The Confirmation of Design Guide Lines

The experimental series in Table C-III was aimed at testing design estimates. In these cases a graphical fit using Figure C-4 was made, and on this basis two estimates were produced. The first, a so-called best fit, produced what appeared to the eye to be the best fit while the least value of m which was at all consistent with the data, particularly in the region of the "weakest" observations, was regarded as a conservative basis for design. This underscores the two distinct requirements faced by the designer:

Fig. C-7. Effect of sample size on Weibull Modulus estimate (20 samples of each).

Fig. C-8. Effect of sample size on precision of Weibull Modulus estimates.

A statistical distribution is sought which gives a good fit to observed data and reliably predicts effects of scale-up and therefore stress distribution changes; but, even in the absence of a "good" fit it is still necessary to undertake analytical design estimates using conservative values of the Weibull Modulus. The designer, after all, needs to know that he is conservative and this may be achieved without knowing the "real" distribution function, although having a decent approximation to the "real" distribution gives the designer a quantitative measure of conservatism.

Thus, in Table C-III the column marked design, estimated as described above, is seen never to overestimate the true population parameter when m exceeds 2. This design estimate seems to fall consistently around 70% of the estimated "best" value of m . The table shows many cases where the "best" estimated m substantially exceeds the true value, as would be expected from the information described in the previous sections, and in these cases scale-up predictions could be risky; however, for purposes of design it is suggested that a lesser value of m than that estimated, be used.

Based on the previous experiments it now appears that, regardless of estimation method in a sample of size 10, we are over 80% sure that the true modulus exceeds 0.7 times the estimated expected value. These results confirm the conservatism of the so-called design estimate. With the present results giving distribution of the estimators it is possible to proceed in a less arbitrary fashion to choose m -values for design to which a confidence statement may be attached.

5. The "Correct" Plotting Position

The computer experiments were used to "backout" an estimate of the correct average plotting positions. From the Series I data, size 10 and 50 samples, the average value of each ranked variable was found for the 20 groups of $m = 6$. The size 10 groups with $m = 15$ were similarly treated. That is, the 20 least values were averaged, then the 20 next-ranked values, and so on. The probability position of the average value was found with the distribution equation

$$S = \exp -\beta^m \Gamma^m$$

using the parent distribution value of m .

The results in Table C-IV give average value of each ranked variable from the twenty samples and, in the row below, the theoretical probability from the above equation. Three suggested plotting positions are also tabulated, the common mean rank position given by $\frac{r}{n+1}$, the median rank position so strongly advocated by Johnson (3), and the interval mid-rank position given by $\frac{r - \frac{1}{2}}{n}$. The absolute mean deviation for each of the recommended plotting positions is also shown and although all three are pretty good the consistently inferior one is the commonly used mean rank position. Since the median rank position requires tables, the best of the three plotting positions will be the mid-rank, but the probability interval plot is still considered best

Table C-IV. Plotting position.

Rank	<u>m = 6 n = 10</u>		3	4	5	6	7	8	9	10	Mean absolute deviation (from theoretical rank)
	1	2									
Avg of 20	0.6604	0.8073	0.8904	0.9501	0.9837	1.0350	1.0897	1.1354	1.1926	1.2575	
Theoretical failure probability	0.05	0.16	0.27	0.375	0.44	0.545	0.63	0.74	0.84	0.92	
By mean rank											
$F = \frac{r}{n+1}$	0.09	0.18	0.27	0.363	0.45	0.545	0.63	0.73	0.82	0.91	0.0122
By median rank	0.067	0.163	0.26	0.356	0.452	0.548	0.644	0.74	0.837	0.933	0.0084
By interval mid-mark	0.05	0.15	0.25	0.35	0.45	0.55	0.65	0.75	0.85	0.95	0.0120
Fraction within 1/n ranked interval	0.65	0.65	0.60	0.45	0.5	0.5	0.3	0.35	0.45	0.60	
	<u>m = 15 n = 10</u>										
Avg of 20	0.8450	0.9220	0.9502	0.9854	0.9984	1.0189	1.040	1.0573	1.0782	1.1011	
Theoretical failure probability	0.045	0.16	0.27	0.375	0.44	0.535	0.655	0.74	0.835	0.92	
By mean rank											
$F = \frac{r}{n+1}$	0.09	0.18	0.27	0.363	0.45	0.545	0.63	0.73	0.82	0.91	0.0157
By median rank	0.067	0.163	0.26	0.356	0.452	0.548	0.644	0.741	0.837	0.933	0.0106
By interval mid-mark	0.05	0.15	0.25	0.35	0.45	0.55	0.65	0.75	0.85	0.95	0.0125
Fraction within 1/n ranked interval	0.65	0.65	0.45	0.50	0.60	0.45	0.15	0.35	0.40	0.55	

Table C-IV. Plotting position (Continued).

Rank	<u>m = 6</u> <u>n = 50</u> (First ten)										Mean absolute deviation (from theoretical rank)
	1	2	3	4	5	6	7	8	9	10	
Avg of 20	0.5063	0.6044	0.6572	0.6921	0.7230	0.7530	0.7775	0.7956	0.8140	0.8312	
Theoretical failure probability	0.013	0.028	0.047	0.067	0.088	0.11	0.13	0.15	0.17	0.19	
By mean rank											
$F = \frac{r}{n+1}$	0.02	0.04	0.06	0.078	0.098	0.118	0.137	0.157	0.176	0.196	0.0085
*By median rank	0.014	0.034	0.053	0.073	0.093	0.112	0.132	0.152	0.172	0.192	0.0034
By interval mid-mark	0.01	0.03	0.05	0.07	0.09	0.11	0.13	0.15	0.17	0.19	0.0013
Fraction within 1/n ranked interval	0.75	0.50	0.25	0.30	0.10	0.20	0.15	0.10	0.25	0.25	
Rank	<u>m = 6</u> <u>n = 50</u> (Last ten)										
	41	42	43	44	45	46	47	48	49	50	
Avg of 20	1.1714	1.1900	1.2092	1.2225	1.2395	1.2565	1.2762	1.3132	1.3394	1.3981	
Theoretical failure probability	0.81	0.84	0.86	0.885	0.90	0.915	0.935	0.96	0.97	0.99	
By mean rank											
$F = \frac{r}{n+1}$	0.805	0.825	0.84	0.86	0.88	0.9	0.92	0.94	0.96	0.98	0.0155
By median rank	0.806	0.824	0.844	0.864	0.884	0.904	0.924	0.944	0.966	0.984	0.0121
By interval mid-mark	0.81	0.83	0.85	0.87	0.89	0.91	0.93	0.95	0.97	0.99	0.0065
Fraction within 1/n ranked interval	0.25	0.30	0.15	0.15	0.10	0.35	0.10	0.10	0.10	0.65	

* $F_{\text{median rank}} \approx 0.0198r - 0.006$

as it avoids the implication of uniqueness for a statistical quantity. As noted earlier, the recommended method of plotting the data here is to use a probability interval rather than a specified point. Using the interval has several advantages: a) it includes all of the proposed plotting positions, b) it is an instantaneous measure of sample size, c) it emphasizes the fact that an observation does not represent a unique point from the parent population, d) makes curve fitting a much easier task. It has the apparent disadvantage of implicitly tending to raise the importance of the interval midpoint as well as having some arbitrariness since the number of observations which may be expected to fall within this interval depend on sample size as shown also in Table C-IV. The row labeled "Fraction within $1/n$ " was found by determining the limits on β corresponding to the ranked probability interval and counting the number of observations within this β -interval which are of the same rank.

SECTION D

THE DISTRIBUTION OF FRACTURE-SOURCE LOCATION

According to Weibull's statistical theory of failure, the cause of fracture in brittle materials is associated with tensile stress interacting with flaws distributed within the material. One of the consequences of the model is that fracture may, in principle, be initiated anywhere within the material where a sufficient tensile stress exists in combination with a sufficiently severe flaw. In addition, therefore, to a statistical description of strength, there is inherent in the model a statistical description of fracture location.

The prediction of the fracture-location distribution can contribute to further assessment of the Weibull Model as an approximate description of real behavior. In the following, some statistical properties of fracture location implicit in the Weibull Distribution will be developed and a limited experiment described wherein the fracture-source distribution was experimentally determined for aluminum oxide specimens.

The statistical problem is to find the probability that the fracture source is located within some specified but arbitrary volume of a test specimen.

Beginning with the Weibull Distribution below,

$$S = \exp \left[- \int \left(\frac{\sigma}{\sigma_0} \right)^m dV \right] \quad (D-1)$$

we can alter it by choosing normalized variables describing the stress distribution and the volume, that is,

$$\begin{aligned} \sigma &= \sigma_T f(\xi) & \sigma_T &= \text{reference stress, usually maximum tension} \\ V &= C V_T d\xi & V_T &= \text{volume in tension, } C = \text{constant} \\ & & \xi &= \text{normalized position variable} \end{aligned}$$

this gives

$$S = \exp \left[- \left(\frac{\sigma_T}{\sigma_0} \right)^m V_T \int C f(\xi)^m d\xi \right].$$

The integral will be a function of m and the limits of integration, say $G(\xi_1, \xi_2, m)$, giving

$$S = \exp \left[- \left(\frac{\sigma_T}{\sigma_0} \right)^m V_T G(\xi_1, \xi_2, m) \right] \quad (D-2)$$

The question of probability of fracture within some specified region of a test specimen relates to the relative effective size of the region in question to the total effective specimen volume. We may say, for a simple example, that the probability of fracture within a given region of a specimen subject to uniform tension is found by the ratio of the given region's volume to that of the whole specimen. This process may be generalized for any stress distribution by finding the size of specimen having the same probability of fracture but subject to a uniform tension only. The ratios of such "equivalent" volumes will represent the relative frequency of fracture expected to occur in the respective volumes.

Consider equation D-2 above as representing a portion of a specimen whose overall distribution integral G_T is given by the total limits as $G(u, v, m)$. Using σ_T as a convenient reference stress, we may find the volume subject to a uniform tension σ_T having the same probability of failure as the whole specimen by

$$S = \exp \left[- \left(\frac{\sigma_T}{\sigma_0} \right)^m V_T G_T \right] = \exp \left[- \left(\frac{\sigma_T}{\sigma_0} \right)^m V_{EQ_T} \right]$$

For the subsidiary portion within the limits (ξ_1, ξ_2) , the same process yields:

$$S_S = \exp \left[- \left(\frac{\sigma_T}{\sigma_0} \right)^m V_T G(\xi_1, \xi_2, m) \right] = \exp \left[- \left(\frac{\sigma_T}{\sigma_0} \right)^m V_{EQ_S} \right].$$

The probability, F_S , that fracture will initiate in the subsidiary volume is simply the ratio of its "equivalent" volume V_{EQ_S} to the equivalent volume of the whole specimen V_{EQ_T} or

$$F_S = \frac{V_{EQ_S}}{V_{EQ_T}} = \frac{G(\xi_1, \xi_2, m)}{G(u, v, m)} \quad (D-3)$$

Thus we need pay attention only to the distribution integral G . For a useful example, consider the rectangular beam subject to uniform moment. Here the dimensionless functions are:

for stress

$$\sigma = \sigma_T \xi$$

$$f(\xi) = \xi$$

where ξ is the dimensionless transverse distance from the neutral plane

and volume under tension is introduced by:

$$dV = V_T d\xi$$

so that

$$G(\xi, m) = \int \xi^m d\xi.$$

Putting in limits for the whole specimen we get

$$G_T(0, 1, m) = \frac{1}{m+1}$$

and for the subsidiary portion from $\xi = 0$ to $\xi = \xi$

$$G_S(0, \xi, m) = \frac{\xi^{m+1}}{m+1}.$$

Now the probability of fracture initiating between the neutral axis and the fraction ξ of the beam half-height is simply

$$F_{0-\xi} = \xi^{m+1} \quad (D-4)$$

It is frequently desired to invoke a condition which is tantamount to saying that a finite stress exists for which no failure occurs or, analogously, that there exists a region, within a specimen subject to varying stress, where fracture will definitely not occur. Such a condition may be included in the present analysis by putting a lower limit in the space integral, that is, $G_T(0, 1, m)$ as defined above becomes $G_T = G(k, 1, m)$ and $k \leq \xi \leq 1$. For example, in the uniform bending specimen fracture may be assumed to occur only between the tensile surface and the surface which is kc from the neutral axis (c is the usual beam half-height and $0 \leq k \leq 1$).

The general form of the function G becomes now

$$G = \int_{\xi_1}^{\xi_2} \xi^m d\xi \quad k \leq \xi_1 \leq \xi_2 \leq 1. \quad (D-5)$$

and for the beam under uniform bending, we have

$$G = \frac{\xi_2^{m+1} - \xi_1^{m+1}}{m+1}. \quad (D-6)$$

Again the probability that the fracture will occur between the surface kc and a tensile surface at ξ is the ratio of the stress distribution integrals,

$$F_{k-\xi} = \frac{\xi^{m+1} - k^{m+1}}{1 - k^{m+1}}. \quad (D-7)$$

While it appears that the difference between equations D-4 and D-7 may be an aid in deciding the appropriate statistical model to be applied to a specific material, provided reliable determinations have been made of the fracture-source locations, the graph of Figure D-1 shows that a trade-off exists between k and m so that resolution between the statistical models may, in fact, be difficult.

These computations may be applied to 3-pt bending where fracture sources vary in transverse position as well as depth. If ξ_x is the ratio of transverse position from the zero stress support to half-span and ξ_y is as defined previously, the probability that fracture occurs between the zero-stress loci and the positions ξ_x and ξ_y is

$$F_{0 \text{ to } \xi_x \xi_y} = (\xi_x \xi_y)^{m+1}$$

Some interesting probability contours may be determined if we take $\xi_x = \xi_y$ as shown in Figure D-2.

The remainder of this section describes an experiment in which square alumina beams subject to uniform bending tests were examined photomicrographically to determine the fracture-source location. The determinations were made by Dr. V. D. Frechette of Alfred University. The observed distribution is compared graphically with that expected by the above analysis.

Table D-I summarizes the data for the 1/4 in. $\times$ 1/4 in. alumina beams, giving the measured modulus of rupture, the depth of the fracture from the tensile surface and its transverse location from a vertical edge, in the order of actual testing. In Table D-II the data are ranked according to transverse fracture-source location. Since we assume that there are no constraints on transverse location, the probability distribution should be linear, the result of uniform probability density. This statistical test should let us see whatever effect the transverse boundary may exert. The resulting graph in Figure D-3 shows that the edge has in fact been a strong influence on failure and about half the specimens were caused to fail by the transverse edge. This is not unexpected, as it is well known that the surface condition of a brittle material is of profound importance in strength measurements. In order to examine the volumetric distribution of flaws, it is necessary to remove from consideration those observations which result from the predominating effects of surface features. Thus, the distribution of Figure D-3 suggests that no more than about 16 observations are independent of the transverse boundaries.

Fig. D-1. Fracture-source distribution in pure bending.

$\xi = 0$	$P = 1$	$m = 5$	$m = 10$	$m = 15$
$\xi = 0.5$	P_a	> 0.98	~ 1	~ 1
$\xi = 0.75$	P_b	0.82	0.96	$\sim .99$
$\xi = 0.95$	P_c	0.26	0.4	0.54

Fig. D-2. Fracture location in a simple beam.

Table D-I. Location of fracture origins.

Specimen no.	Strength M-R ksi	Distance from tension edge (microns)	Distance from side - mm
1	30.67	40	0.04
2	32.26	8	0.04
4	30.67	30	0.5
5	31.54	0	0
6	30.24	350	3.0
7	30.1	165	1.7
8	32.26	12	0.2
11	31.39	20	0.02
12	32.11	65	0.025
13	31.69	90	0.09
14	32.26	340	4.0
16	32.40	Corner chip at region of origin	
17	32.11	<10	0.5
18	34.57	<5	2.0
22	30.38	<10	0.6
24	32.83	<10	0.5
25	32.54	160	0.01
26	31.25	Corner chip at region of origin	
30	33.67	<5	0.4
34	36.43	<10	<0.01
37	32.11	<10	3.4
39	33.12	40	0.02
48	35.14	<10	0.4
50	29.38	20	0.5
51	30.82	360	<0.01
53	30.67	<10	<0.01
54	32.83	150	<0.01
55	30.82	300	3.5
57	31.10	<10	0.07
58	32.11	40	<0.01
59	31.97	<10	<0.01
60	30.38	<10	0.5

Beam half-height = $1/8$ in.

= 3.18 mm

= 3180 μ

Average strength $\bar{\sigma}$ = 31.93

Table D-II. Transverse location of fracture.

Specimen no.	Location - mm from side
5	0
34	<0.01
51	<0.01
53	<0.01
54	<0.01
58	<0.01
59	<0.01
25	0.01
11	0.02
39	0.02
12	0.025
1	0.04
2	0.04
57	0.07
13	0.09
8	0.2
30	0.4
48	0.4
4	0.5
17	0.5
50	0.5
60	0.5
24	0.5
22	0.6
7	1.7
18	2
14	2.36
55	2.86
37	2.96
6	3.0

Note: Half width of beam is 3.18 mm.
Dimensions in excess of this in
Table D-I mean that the nearest
distance to the edge is 6.36 - D.

Cut-off (See Figure D-1)
Independent of transverse boundary

Fig. D-3. Transverse distribution of fracture-source location in Al_2O_3 beams subject to uniform bending.

In Table D-III these 16 observations are ranked according to the depth of the fracture-source and plotted as a cumulative distribution on Figure D-4. In view of the effect of the transverse surface it is expected that a substantial fraction of the remaining data will also exhibit the surface influence, this time being due to features on the tensile surface. The distribution of Figure D-4 does, in fact, demonstrate a surface effect which influences about half of these data. The remaining "surface-independent" observations comprising 8 data points are examined in Table D-IV. The simplest Weibull fit to the strength distribution yields a modulus between about 28 and 32 (as may be confirmed by testing some of the estimators of section C), while the fracture source distribution suggests a significantly lower value, about 17 as shown in Figure D-5. If the standardized form of equation B-7 is used a value of α may be estimated which will make consistent the strength and fracture-source distributions with $m = 17$. Using the method of reference (4) α is estimated to be about 0.5 and the plot is shown in Figure D-5. The suggestion is that the preferred strength-distribution model is that of equation B-7 based on the observed fracture-source distribution. A combination of strength and fracture source distributions may therefore be sufficient to identify which of the two statistical models is more appropriate.

It is evident that a great deal more data would be desired to reach firm conclusions about this lot of specimens, not to say drawing general conclusions about fundamental material characteristics.

It is possible nevertheless, to say with some assurance that the flaws governing failure in a brittle material will be drawn from two distinct distributions, those characterizing surface features, and those of the volumetric distribution (the latter being most frequently the explicit assumption in statistical strength analyses). The extent of the competition between the two modes of failure can be ascertained by fractographic methods and analysis such as described here, and should prove of interest in the endeavors to upgrade present-day brittle materials. In the present case it appears that the statistical flaw distributions of the surface and volume types are quite similar, yielding about the same average of strength and distribution.

One would expect surface features to be more severe since, for example, a round hole at the surface may reach a stress concentration of 3 while the spherical void achieves only 2. Such conclusions are altered by the effects of finite specimen size and details of shape.

Table D-III. Distribution of fracture-source depth of transverse edge-independent group of Table D-II.

Specimen	M-R (ksi)	Depth (microns)	Normalized depth*
30	33.67	<5	<0.00157
18	34.57	<5	<0.00157
48	35.14	<10	<0.00314
17	32.11	<10	<0.00314
37	32.11	<10	<0.00314
60	30.38	<10	<0.00314
22	30.38	<10	<0.00314
24	32.83	<10	<0.00314
8	32.26	12	0.00377
50	29.38	20	0.00628
4	30.67	30	0.00945
13	31.69	90	0.0283
7	30.1	165	0.0519
55	30.82	300	0.0945
14	32.26	340	0.1068
6	30.24	350	0.11

Surface independent

* Normalized depth is obtained by dividing 3180 into the observed value.

Table D-IV. The strength and depth distribution of the surface-independent observations (estimated from Figure D-2).

Strength				Depth			
Spec No.	Rank	Strength ksi	Normalized strength	Spec No.	Rank	Strength ksi	Normalized depth
50	1	29.38	0.945	8	2	12	0.00377
7	2	30.10	0.97	50	3	20	0.00628
6	3	30.24	0.975	4	4	30	0.00945
4	4	30.67	0.99	13	5	90	0.0283
55	5	30.82	0.995	7	6	165	0.0519
13	6	31.69	1.02	55	7	300	0.0945
8	7	32.26	1.04	14	8	340	0.1068
14	8	32.26	1.04	6	9	350	0.11

Avg. = 31.0

The strength is normalized to the average value of the sample = 31.

The depth is normalized to the beam half-height, i. e., $(1/8)^{in.} = 3180\mu$.

Fig. D-4. Fracture depth distribution showing effect of tensile surface.

Fig. D-5. Depth distribution of edge-independent observations.

SECTION E

THE PARALLEL MODEL

The parallel model for strength of materials differs in essential respects from the series model considered thus far. It should, in principle, exhibit a size-effect only with length and not with the number of parallel elements; it exhibits a certain amount of fracture tolerance; and it offers a means of achieving higher strength than is possible in a monolithic structure, especially in a highly variable material. Since the behavior of a composite structure is governed mainly by the properties of the reinforcing elements, it is necessary to have adequate information on the properties of the fiber or other reinforcing element. Because fiber properties turn out to be highly variable, it is necessary to use statistical analysis to describe them, and again the "worst flaw" hypothesis seems to yield consistent and useable predictions.

The standardized Weibull Distribution will be applied here to the idealized parallel model to exhibit some important features of the parallel model and the utility of the standardized form.

We have as the starting point, the Weibull form

$$S = \exp (-\beta^m \Gamma^m) \quad (E-1)$$

and the ideal strength of a parallel structure

$$\beta_{||} = \beta_* S(\beta_*) \quad (E-2)$$

where β_* maximizes the righthand side of equation E-2 and $\beta_{||}$ is the apparent strength of the parallel structure relative to the average strength of the elements. It readily follows that β_* , the maximum stress achieved in the fibers, is

$$\beta_* = \left(\frac{1}{m}\right)^{1/m} \frac{1}{\Gamma} \quad (E-3)$$

and

$$\beta_{||} = \left(\frac{1}{m\epsilon}\right)^{1/m} \frac{1}{\Gamma} \quad (E-4)$$

gives the measured strength of the parallel structure relative to the element average strength. We also note that the fraction of elements which still support load at the onset of instability is

$$S_* = \exp -\frac{1}{m}$$

As a means of assessing the efficiency of fiber-reinforcement we compare the strength of the parallel structure with that of the same size monolithic structure characterized by the same Weibull Modulus. The average strength of such a monolithic structure relative to the fiber average is, of course, $\left(\frac{1}{n}\right)^{1/m}$ where n is the number of fibers as expected from the conventional size effect, and this gives,

$$\frac{\sigma_{\parallel} \text{ (Parallel Strength)}}{\sigma_S \text{ (Series Strength)}} = \left(\frac{n}{me}\right)^{1/m} \frac{1}{\bar{F}} \quad (\text{E-5})$$

Equation E-5 is plotted on Figure E-1 and equations E-1 and E-2 on Figure E-2. Figure E-1 shows that for any m a minimum number of fibers exist below which there is no strength improvement over a series model. The low values of m are seen to respond more rapidly to the relative improvements of parallel construction. Figure E-2 gives an immediate estimate of the composite strength attainable with known fiber properties as well as the expected maximum fiber stress. In most cases of real interest, over 75% of the fibers are still active at the maximum stress so that composite behavior is yet dictated by some of the weakest elements. This suggests that a most direct means of upgrading a composite is to proof-test the fibers used in its fabrication. Such proof test need not exceed the maximum fiber stress which would be expected in the unmodified composite, in order to produce enhancement of strength. The elimination from the structure of the 15 to 20% weakest elements will produce a similar increase in composite strength although now the composite will exhibit less crack tolerance. It will fail, like the series model, at first fracture, if the proof stress equals or exceeds the maximum fiber stress in an unmodified system.

The equations used above are appropriate for fibers which are not coupled by the shearing action of a matrix and which support the applied load equally. If the matrix can adhere to the fibers without causing degradation the localized fracture of a fiber does not entirely relieve it of load transmission and hence a greater number of fiber fractures must occur before collapse than is the case for the idealized model. While this mechanism serves to increase the effective strength there may be other factors reducing it such as kinks, matrix voids, chemical effects, etc. In reality, then, there are many conditions which would effect deviation from the ideal situation, nevertheless, the use of the function shown in Figure E-2 to predict composite strength has been remarkably successful. For proper application it is necessary to have the fiber average strength and size, its Weibull Modulus, the composite strength and size, and the proportion of fiber content. Frequently it is found in the literature that not all this information is available and guesses have to be made regarding the fiber sizes or the Weibull Modulus. When the extreme values and/or standard deviations are given the graphs included in section C of this report may be used to estimate m. For thumb-rule purposes, Figure E-2 shows that the parallel model develops about 70% $\pm$ 5% of the average strength in the region of m from 5 to 10 and if correction is made for fiber quantity and size,

2

Fig. E-1. Relative strengths of a parallel structure and equal volume monolithic structure (series model).

Fig. E-2. Parallel strength and max. fiber stress relative to fiber average.

reasonable estimates of composite strengths will result. Table E-1 summarizes some fiber data and Weibull Moduli to demonstrate the difference in the data reported by various observers of the "same" material. It is risky to take fiber data from one source and apply it to composite data from another. When composite experiments accompany such fiber studies it is found that predictions made on the basis of the ideal model are reasonable and generally underestimate the observed composite strength. It is also seen that the conventional size effect estimates of the form $\left(\frac{V_2}{V_1}\right)^{1/m}$ are slightly too strong and the decrease of strength with increasing length or diameter is not so great. Thus, in several respects, the model is a reasonable and nearly always conservative estimator of scale-up and composite strength.

For a specific example we may cite the SiC fiber study of Huang (5). These fibers exhibited $m = 5$ in strength distribution with an average of 330 ksi and about $m = 5.5$ in length effect. The fibers were used in uniaxial reinforcement of a nickel matrix which, from stress vs strain studies, achieved about 60 ksi at composite failure. Since the composite gage length was about the same as that of the fiber gage length no size correction is necessary. From Figure E-2 we find that at $m = 5$ about 65% of the average may be developed, thus the "expected" composite strength in ksi will be given by

$$\sigma_{\text{comp}} = 215(F) + 60(1 - F)$$

where F is fiber fraction, so

$$\sigma_{\text{comp}} = 60 + 155(F)$$

Figure E-3 is a plot of this data, the line derived from the law of mixtures by Huang, and the equation above. A region which requires some comment is that of the transition between the non-reinforced matrix, $F = 0$, and the first observed values at $F = 16\%$. The addition of a small amount of fibers causes a degradation of strength. Such an effect is observed with a variety of materials and is due in part to the dynamic properties of the matrix, and the fact that fiber fracture occurs at strains well below those at which ultimate matrix properties are developed. It is therefore not to be expected that the linear relation above should be capable of extrapolation to the ordinate at $F = 0$.

In respect to the size effect in fibers it is almost invariably observed that fibers of the same lot may be caused to exhibit reduced strength and increased variability by the damage resulting from casual handling or experimental control. When damage has occurred such that strength is degraded this may be attributed to an increase in the number of high intensity stress concentrations and therefore a decrease in the distribution exponent m . Such a change in m also implies an increased coefficient of variation as is, in fact, observed. The results of one study (6) in which E-glass fibers from a common source were subjected to a variety of handling environments bear out the corresponding increase of coefficient of variation with decrease of average strength. The observed relation between the strength and the Weibull Modulus is shown in Figure E-4.

Fig. E-3. Tensile strength of SiC fiber reinforced Ni after Huang.

Fig. E-4. Degradation of strength and statistics of E-glass due to handling damage.

Table E-I

Source*	Fiber material	Diam - mils gage - in.	Fiber Weibull Modulus	Avg strength (ksi)	Remarks
5	SiC	4.8×1	5	330	Gage length 0.5 to 10 size effect predicted.
6	"E" glass	$0.37 \times 3/8$	9	483	Treatment of fibers produced values from 4 to 14.
7	"E" glass	0.5×1	70	530	Tested immediately after fabrication. No size effect, very low variance.
8	"E" glass	3.5×1	12	180	Size effect estimate, from 0.2 to 5 in. gage.
9	Boron	1.8×1	8	366	Gage length varied to 120 in. Conventional size effect too strong.
10	Boron	4×1	5,6	460	
11	Boron	4×1	12	460	Size effect estimate.
12	Tungsten moly	$1 \times 1(?)$	3	600	
13	Graphite (thornel)	?	15-20	184	Other graphite fibers shown reach 360 ksi with the same high modulus.

*References in Bibliography.

As pointed out in another section, when the distribution of flaw (or stress concentration) intensity and frequency is undisturbed the effect of size causes a change in variability such that the coefficient of variation remains constant.

In this case the formulation in terms of the stress concentration and constant "true" failure stress described in section A allows an estimate to be made of the limiting strength, equation A-7c.

The data of reference (6) on "E" glass fibers may be used with this equation to estimate a limiting value of strength σ_L . The information given is the average strength of fibers about 11 inches long, $\sigma_A = 423,000$ psi, and fibers about 1-3/8 inches long, $\sigma_B = 483,000$ psi. From the variance of the short fiber data the Weibull Modulus is estimated roughly as $m = 7.4$, and all this is put into equation A-7c. Such a computation leads to the value

$$\sigma_L = 860,000 \text{ psi}$$

for the "E" glass fibers described. The prediction of theoretical strength is still an uncertain process, with various models producing differences of five-fold. Condon (14), gives a brief discussion of the matter and produces rough theoretical strength estimates for glass ranging from about 810,000 psi to over $4(10^6)$ psi. It seems that σ_L should not be identified with the theoretical strength of a material although it may give conservative estimates of it.

As fracture propagates through a composite and the fiber stress rises, the shear coupling length to adjacent broken fibers requires an ever increasing length so that the longest slivers of a broken composite are representative of the last stages of fracture and hence of the strongest fibers—a criterion which may aid fractographic interpretation.

SECTION F

DESIGN APPLICATION

The Graphite Expander Development Group at the Lawrence Radiation Laboratory has been concerned with the construction and study of conversion machinery suitable for use with very high temperature inert-gas cooled nuclear reactors. Initial studies (15) have shown some performance advantages of graphite rotating machinery over MHD ducts and that present state-of-the-art of graphite materials technology is sufficient to provide the necessary strength in the large sizes required (ca. 60 in. diam). Many studies are still in progress and only a preliminary review of current results is possible at this time. The operation of a small-scale expander has been reported (16), describing the operation of graphite rotating machinery in argon at about 2700°F and speeds to 14,000 rpm.

One of the problems associated with construction of a true small-scale prototype is the attachment of shafting through flexible couplings in the high-temperature regions. The method selected for this function was the crowned-spline coupling, and the material a high strength very fine grained graphite designated EP1924 (currently it is called AXF) by the POCO Graphite Company. The distribution of strength for this material is shown in Figure B-3. A simplified approximation to this distribution may be made using $m = 9$ to demonstrate prediction of size effects in the splined shafts. The details of this study are reported in an internal memorandum (17). Briefly, spline-tooth load carrying capacity was measured by single-tooth mockups and torsional capability was estimated from tests of 0.25 inch diameter specimens. These exhibited a mean torque capability of 23.5 in.-lb and a maximum shear stress of 7650 psi assuming linear elastic behavior. There is of course some non-linearity but the error in stress for the present purpose is neglected. The distribution was consistent with the Weibull approximation $m = 9$. The prototype shafts were 1-5/8 inch diameter and represented a scale up of 100 over the 0.25 inch diameter test specimens. Involving the conventional size effect led to a prediction of 3850 in.-lb for the large shafts. Two large shafts were tested and in both cases failure was in the cylindrical portion, indicating adequate spline-tooth design. The observed torques were 4130 in.-lb and 3430 in.-lb. The weaker of the two exhibited a pronounced void near the surface, at the site of fracture initiation. The average, 3780 in.-lb is in reasonable agreement with the prediction.

In design estimates of graphite rotors for full scale application the scale-up estimates cover over five orders of magnitude and questions regarding the reliability of the scale-up estimate, the proper treatment of stress distribution, and stress concentration are of primary importance. An instructive example is that of the specific geometry of the graphite helical rotors, shown in Figure F-1.

The male rotor has a small-radius fillet where the lobe is faired into the root, as shown more explicitly in Figure F-2. Photoelastic analysis of such a rotor yields a stress concentration of about 1.7 in this fillet. It appeared that this was the critical

Fig. F-1. Graphite helical rotor pair.

Fig. F-2. Stress concentration regions in male rotor.

region and was the limiting factor in operation, however, a number of other considerations imply that some mitigation of the maximum stresses and failure probability occurs. An analytical study of the strength reduction in a brittle plate with a circular hole was undertaken to assess the general effects of a stress concentration when the Weibull statistical analysis is invoked. Using the equations for the stresses around a hole in an infinite plate (18), the model shown in Figure F-3 was examined. By equating the risks of rupture in the region of the hole, using the true and the nominal stress distribution, an estimate was obtained for the expected strength reduction. The risk of rupture with the true stress distribution was computed on the basis of the maximum principal tension. The results, shown in Figure F-4, demonstrate the dramatic differences between the expected strength reduction and the theoretical stress concentration of 3 which exists in these plates. A quick and limited test was performed on graphite rods, the results of which are consistent with Figure F-4. The graphite, grade ATJ, exhibits a Weibull Modulus between 10 and 20. Two rods 0.375 inch diameter were drilled transversely with a hole about 0.04 inch diameter, and were pulled quickly to failure. Four rods having no hole were also tested. All tests were run at 1800°C. When correction was made for the net area supporting the load in the case of the drilled specimens the observed strength reductions were 1.52 and 1.37 which, on Figure F-4, correspond to $m = 13$ and $m = 10$ respectively, values consistent with the statistical behavior of the graphite. Thus, invocation of the proper theoretical treatment has brought into accord with prediction some experimental data which otherwise, in view of the maximum stress concentration of 3, is apparently anomalous. The effect of stress concentration in reducing the strength of a brittle structure will depend on the relative size of the concentration to the component and the appropriate statistical parameters. The strength reduction is likely to be substantially less than the stress concentration. An additional factor relevant to the operation of graphite at elevated temperatures is the effect of creep relaxation in smoothing out sharp stress gradients. Some preliminary data on this matter was also obtained during the testing described above. Two drilled rods were loaded very slowly to failure at 1800°C. The strength reduction exhibited by these drilled rods was 1.06 and 1.21, values which are less than the case of rapid loading because of creep relaxation in the regions of high stress.

In proceeding to make design estimates for the very large graphite rotors contemplated in central-station application, some account must be made of the stress concentration at the male lobe fillet discussed previously. This was done in a conservative fashion by taking the maximum stress, given by the factor 1.7, to act uniformly over a local cylindrical region of diameter equal to the fillet radius. On this basis it was concluded that the strength of a full size graphite rotor characterized by $m = 15$ would be approximately 1/3 that of tensile specimens 1/4 inch diameter by 1.5 inch long. Such a strength reduction is equivalent to a volume scale up of $1.4(10^7)$, well beyond any experimental data. Although conventional size effects computations seem to invariably underestimate the strength of large structures, some experimental observation of such enormous scale-ups

Fig. F-3. Model for estimating strength reduction in a brittle plate with circular hole.

- For infinite plate approximation to hold, $\frac{b}{a} \geq 6$.
- The risk of rupture under maximum principal tension in the region of radius b is compared between the case with a hole using the true stress distribution, and the case of a uniform stress S acting on the same region.

Fig. F-4. Strength reduction in a brittle plate with small hole.

is highly desirable. To this end a test was performed on a massive piece of RVA grade graphite (Union Carbide Corporation). The test item was a graphite duct 40.25 inches inside diameter, 44.75 inches outside diameter and 41.875 inches tall. This duct, was subject to internal pressure at room temperature and failed at 260 psi. At this writing small-specimen data has not yet been collected from the same billet. Referring to other published data on this material, but from an earlier time in the manufacturing program (19), produced the following data on small RVA specimens:

Specimen	Number	Measured strength (psi)			
		Least	Mean	Most	Std. dev.
Uniform flexure 1.25 × 1.25 × 5 1.67 center span	136	2245	3159	3950	300
Uniform flexure 0.5 × 0.5 × 5 1.67 center span	143	2550	3348	4120	368
Tension 0.25 diameter 1.5 long	130	1855	2632	3340	282
Other tensile data 0.25 diameter 1.5 long	30		2975		423
	30		2975		423

But the test cylinder, with a volume of 12,500 cubic inches, represents a scale-up from the tabulated tensile specimens of $1.7(10^5)$ and exhibits an apparent strength of about 2600 psi. Such size effect as actually occurred depends on the behavior of small specimens from the same billet. It seems very likely, however, that once again conventional predictions of size effect will underestimate the true strength of the large structure. Possible causes for depression of the theoretical size effect are discussed elsewhere and it is well to note that the "correct" determination of thresholds, bounds and other modifications of Weibull's Distribution require very large samples for precise description.

Whenever possible the operational stress in a brittle material should be compressive. Figure F-5 shows a conceptual gas cooled reactor containing no metal components to resist aerodynamic and structural forces. Instead, at the hot discharge end of the core a graphite arch or dome resists the thrust forces while itself subject only to compression. The load is eventually transferred to a cool region where metallic structure can be incorporated with active tensile stresses. A similar philosophy can be brought to bear in a variety of applications with the realization that thermal-stress induced fracture in such a structure does not usually result in collapse and the redundant support afforded by the conical seat introduces a design which inherently exhibits a great deal of fracture tolerance. The concept of redundant support and the accompanying benefit of fracture tolerance should be employed in design with brittle material if possible.

As a final comment, the use of fractography as a diagnostic tool for brittle materials should be recognized. Frequently, in the presence of sharp stress gradients, relaxation may occur and, following temperature cycling, residual stresses will appear. These may

Fig. F-5. Conceptual graphite ultrahigh-temperature, gas-cooled reactor.

cause fractures and failures which are likely to be attributed to the computed elastic stresses in operation. An excellent example of this occurred with small silicon-carbide rocket nozzles wherein diametrically opposite fractures were attributed to operational thermal stress. The case is illustrated in Figure F-6 where proper fractographic analysis serves to identify the mechanism causing failure. This nozzle failed after burnout and not during operation.

Fig. F-6 Fracture of rocket nozzle.

REFERENCES

1. B. Epstein, private communication.
2. J. V. J. Ravennis, *Electro-Technology*, March, 1964.
3. L. G. Johnson, *Industrial Mathematics* 2 (1951), pp. 1-9.
4. E. T. Robinson, UCRL report 7729.
5. K. C. Huang, USAF Dept. AFML-TR-66-330, Wright Patterson AFB, Ohio, October 1966.
6. R. A. Wallhaus, U. S. Dept. of Commerce Rept. AD276494, Office of Technical Services Washington, D. C., May 1962.
7. W. F. Thomas, *Physics and Chemistry of Glasses*, 1 (1960).
8. Fiber Composite Materials, ASM, October 1964.
9. H. W. Herring, *Selected Mechanical and Physical Properties at Boron Filaments*, NASA TN D-3202, January 1966.
10. Advanced Fibrous Reinforced Composites, 10th National Symposium, Society of Aerospace Materials and Process Engineers, November 1966, p. D-6.
11. Ibid p. G-3.
12. Ibid p. F-25.
13. Ibid p. A-22.
14. E. U. Condon, *American Journal of Physics* 22, 224 (1954).
15. W. M. Wells, *On Central Station Application of Graphite Helical-Rotor Expanders*, UCRL-70258 Rev. I.
16. W. M. Wells et al., UCRL-14660 Rev. II.
17. H. Leideter, *Graphite Engine Memo No. 1*, Feb. 1967 (Internal Memorandum - Research Programs Engineering Division, Lawrence Radiation Laboratory).
18. S. Timoshenko, *Theory of Elasticity*, McGraw Hill.
19. AFML-TR-65-183.

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