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SUBJECT: DEFLECTION EQUATIONS FOR VARIOUS LOADINGS
OF CIRCULAR-ARC CURVED BEAMS

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DEFLECTION EQUATIONS FOR VARIOUS LOADINGS OF
CIRCULAR-ARC/CURVED BEAMS

In analyzing stresses, deflections, and forces in piping and certain structural systems, it is useful to have a set of equations giving the deflections at any point of a curved beam in terms of the various loads acting on the beam. This paper presents the deflection equations for a curved circular-arc beam of variable length. The cases treated include in-plane and out-of-plane bending due to forces and moments applied to the end of the beam in three principal directions, and uniformly distributed loads applied along the beam in these three directions. Deflections were calculated by the strain energy method using Castigliano's theorem.

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NOMENCLATURE

P, Q, F	Forces acting on beam in tangential, radial, and direction normal to curve, respectively.
M_P, M_Q, M_F	Moments acting on beam in tangential, radial, and direction normal to curve, respectively.
R, T, N	Distributed loads acting on beam, in force per unit length of beam, in radial, tangential and direction normal to curve, respectively.
x, y, z	Rectangular coordinates.
X, Y, Z	Fictitious forces acting on beam in x-, y-, and z-directions, respectively.
M_i, M_j, M_k	Fictitious moments acting on beam in x-, y-, and z-directions, respectively.
$\delta x, \delta y, \delta z$	Deflections of beam in x-, y-, and z-directions, respectively.
r	Radius of curvature of beam.
U	Strain energy of flexure, and total strain energy of beam.
U_F	Strain energy of flexure of beam.
U_T	Strain energy of torsion of beam.
E	Young's modulus.
ν	Poisson's ratio.
G	Shear modulus, $G = \frac{E}{2(1+\nu)}$
I_z	Moment of inertia of section of beam about z-axis.
I_r	Moment of inertia of section of beam about axis radial to curve.
K	A function of the cross section of the beam, used to calculate the torsional rigidity.
$\alpha, \theta, \theta', \theta''$	Angular coordinates of beam.
$\delta\phi_x, \delta\phi_y, \delta\phi_z$	Angular deflections of beam with rotation vectors parallel to x-, y-, and z-axes, respectively.

Coordinates

The system of coordinates used is shown in Figures 1 and 2. The beam lies in the xy-plane with one end fixed at the origin, and is described by radius r and angular length α . Forces P , Q , and F , and moments M_P , M_Q , and M_F act on the end of the beam in three principal directions, respectively, tangential, radial, and normal to the plane of the curve. Distributed loads T , R , and N , per unit length of beam, act respectively, in the same principal directions. Deflections are given at a point on the beam described by angle θ . Linear deflections, δx , δy , and δz are given in the x , y , and z directions, and angular deflections, $\delta \phi_x$, $\delta \phi_y$, and $\delta \phi_z$ are given as positive when the angular rotation vectors are directed, according to the right-hand rule, along the positive x , y , and z directions.

Limitations on Solutions

The following conditions have been assumed for the calculations:

1. The beam has constant cross-sectional properties and is made of an isotropic and homogeneous material obeying Hook's law.
2. Plane sections remain plane and the neutral axis retains its original length after loading.
3. The planes of bending are the principal planes of the beam.
4. The radius of curvature of the beam is much greater than the depth of the beam (more than about 10 times).
5. The cross section of the beam does not deform upon loading.

These assumptions should be valid for many applications. Two situations occur in which the assumptions are not valid; however, with certain modifications the calculations and solutions may be used.

Figure 1 - Coordinate System Showing End Forces and Moments (Right-hand rule)

Figure 2 - Coordinate System Showing Forces Acting Over Elemental Length of Beam due to Distributed Loads T, R and N

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The first situation occurs when the radius of the curve is not much greater than the thickness of the beam, and assumption (4) does not hold. Because of the initial curvature of the beam, the stress distribution over the cross section is not linear with distance from the neutral axis. This causes a shift in the neutral axis toward the center of the curvature. The equations for the deflections and the stresses must be modified to take this into account.¹

The second situation occurs in the bending of annular sections, such as thin-walled pipe and tubing. Because of the radial components of the fiber stresses, the cross section of the beam deforms, and assumption (5) is not valid. This deformation reduces both the strength and stiffness of the beam, therefore, the calculations should be modified for both deflections and stresses.²

Part 1 - In-Plane Bending

Referring to Figure 3, the strain energy of flexure of the section of beam between zero and θ is given by

$$U(\theta) = \int_0^\theta \frac{M_z^2 r d\theta'}{2 EI_z} \quad (1)$$

where M_z is the moment due to all loads acting on the beam, and is a function of the variable, θ' .

By Castigliano's theorem,³ the deflections at θ are given by the following equations, using the expression from Equation (1).

$$\delta x(\theta) = \frac{\partial U(\theta)}{\partial X} = \frac{r}{EI_z} \int_0^\theta M_z \frac{\partial M_z}{\partial X} d\theta' \quad (2)$$

$$\delta y(\theta) = \frac{\partial U(\theta)}{\partial Y} = \frac{r}{EI_z} \int_0^\theta M_z \frac{\partial M_z}{\partial Y} d\theta' \quad (3)$$

$$\delta \phi_z(\theta) = \frac{\partial U(\theta)}{\partial M_k} = \frac{r}{EI_z} \int_0^\theta M_z \frac{\partial M_z}{\partial M_k} d\theta' \quad (4)$$

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1. See Reference 1, pp. 362-366, and Reference 3, pp. 146-150.
2. See Reference 1, pp. 405-410, and Reference 4, pp. 52-60.
3. Reference 1, p. 378.

Figure 3 - Coordinate System Showing Loads Used to Calculate Deflections for In-Plane Bending

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In Equations (2), (3), and (4), X , Y , and M_k are loads which are applied at θ in order to determine the partial derivatives as a function of θ' , but they are not used in calculating the moment, M_z . Since the loads themselves do not enter into the expressions for the deflections, they are sometimes referred to as fictitious loads. The moment at θ' due to these loads is given by

$$M_z(\theta') = -X r (\sin\theta - \sin\theta') + Y r (\cos\theta' - \cos\theta) + M_k \quad (5)$$

The partial derivatives are given by Equations (6), (7) and (8).

$$\frac{\partial M_z(\theta')}{\partial X} = -r (\sin\theta - \sin\theta') \quad (6)$$

$$\frac{\partial M_z(\theta')}{\partial Y} = r (\cos\theta' - \cos\theta) \quad (7)$$

$$\frac{\partial M_z(\theta')}{\partial M_k} = 1 \quad (8)$$

Substituting Equations (6), (7), and (8) into Equations (2), (3), and (4),

$$\delta x(\theta) = -\frac{r^2}{EI_z} \int_0^\theta M_z (\sin\theta - \sin\theta') d\theta' \quad (9)$$

$$\delta y(\theta) = \frac{r^2}{EI_z} \int_0^\theta M_z (\cos\theta' - \cos\theta) d\theta' \quad (10)$$

$$\delta \phi_z(\theta) = \frac{r}{EI_z} \int_0^\theta M_z d\theta' \quad (11)$$

Equations (9), (10), and (11) may now be used to determine the deflections for the various cases of in-plane bending.

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Case I - Tangential Load P at α

The moment at θ' due to P is given by

$$M_z(\theta') = -P r \left[1 - \cos(\alpha - \theta') \right] \quad (12)$$

Substituting Equation (12) into Equations (9), (10) and (11), and carrying out the integrations,

$$\begin{aligned} \delta x(\theta) = & \frac{Pr^3}{EI_z} \left[\sin\alpha \left(\frac{\theta}{2} - \sin\theta + 1/2 \sin\theta \cos\theta \right) \right. \\ & \left. - 1/2 \cos\alpha \sin^2\theta + \theta \sin\theta + \cos\theta - 1 \right] \end{aligned} \quad (13)$$

$$\begin{aligned} \delta y(\theta) = & -\frac{Pr^3}{EI_z} \left[\sin\alpha (1/2 \sin^2\theta + \cos\theta - 1) \right. \\ & \left. + 1/2 \cos\alpha (\sin\theta \cos\theta - \theta) + \sin\theta - \theta \cos\theta \right] \end{aligned} \quad (14)$$

$$\delta \phi_z(\theta) = -\frac{Pr^2}{EI_z} \left[\theta - \sin\alpha (1 - \cos\theta) - \cos\alpha \sin\theta \right] \quad (15)$$

For the special case $\alpha = \theta = \frac{\pi}{2}$,

$$\delta x = \frac{Pr^3}{EI_z} \left(\frac{3\pi}{4} - 2 \right) = 0.35619 \frac{Pr^3}{EI_z} \quad (16)$$

$$\delta y = -1/2 \frac{Pr^3}{EI_z} \quad (17)$$

$$\delta \phi_z = -\frac{Pr^2}{EI_z} \left(\frac{\pi}{2} - 1 \right) = -0.57080 \frac{Pr^2}{EI_z} \quad (18)$$

10Case II - Radial Load Q at α

The moment at θ' due to Q is given by

$$M_z(\theta') = Q r \sin(\alpha - \theta') \quad (19)$$

Substituting Equation (19) into Equations (9), (10), and (11), and carrying out the integrations,

$$\delta x(\theta) = - \frac{Qr^3}{EI_z} \left[\cos\alpha \left(\frac{1}{2} \sin\theta \cos\theta + \frac{\theta}{2} - \sin\theta \right) + \frac{1}{2} \sin\alpha \sin^2\theta \right] \quad (20)$$

$$\delta y(\theta) = \frac{Qr^3}{EI_z} \left[\frac{1}{2} \sin\alpha (\theta - \sin\theta \cos\theta) - \cos\alpha \left(\frac{1}{2} + \frac{1}{2} \cos^2\theta - \cos\theta \right) \right] \quad (21)$$

$$\delta \phi_z(\theta) = \frac{Qr^2}{EI_z} \left[\sin\alpha \sin\theta - \cos\alpha (1 - \cos\theta) \right] \quad (22)$$

For the special case $\alpha = \theta = \frac{\pi}{2}$,

$$\delta x = - \frac{1}{2} \frac{Qr^3}{EI_z} \quad (23)$$

$$\delta y = \frac{\pi}{4} \frac{Qr^3}{EI_z} = 0.78540 \frac{Qr^3}{EI_z} \quad (24)$$

$$\delta \phi_z = \frac{Qr^2}{EI_z} \quad (25)$$

Case III - Moment M_F at α

For this case,

$$M_z(\theta') = M_F \quad (26)$$

Substituting Equation (26) into Equations (9), (10), and (11),

$$\delta x(\theta) = - \frac{M_F r^2}{EI_z} (\theta \sin \theta + \cos \theta - 1) \quad (27)$$

$$\delta y(\theta) = \frac{M_F r^2}{EI_z} (\sin \theta - \theta \cos \theta) \quad (28)$$

$$\delta \phi_z(\theta) = \frac{M_F r}{EI_z} \theta \quad (29)$$

For the special case $\alpha = \theta = \frac{\pi}{2}$,

$$\delta x = - \frac{M_F r^2}{EI_z} \left(\frac{\pi}{2} - 1 \right) = - 0.57080 \frac{M_F r^2}{EI_z} \quad (30)$$

$$\delta y = \frac{M_F r^2}{EI_z} \quad (31)$$

$$\delta \phi_z = \frac{\pi}{2} \frac{M_F r}{EI_z} \quad (32)$$

Case IV - Uniformly Distributed Tangential Load T

Referring to Figure 2, the incremental moment at θ' due to the distributed load T acting over the incremental length of beam, $r d\theta''$, at θ'' is given by

$$dM_z(\theta') = - (Tr d\theta'') r [1 - \cos(\theta'' - \theta')] \quad (33)$$

Integrating Equation (33) from θ' to α ,

$$\begin{aligned} M_z(\theta') &= - Tr^2 \int_{\theta'}^{\alpha} [1 - \cos(\theta'' - \theta')] d\theta'' \\ &= - Tr^2 [\alpha - \theta' - \sin(\alpha - \theta')] \end{aligned} \quad (34)$$

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Substituting Equation (34) into Equations (9), (10), and (11), and carrying out the integrations

$$\delta x(\theta) = \frac{Tr^4}{EI_z} \left[\left(\alpha\theta - \frac{\theta^2}{2} + 1 + \cos\alpha \right) \sin\theta + (\alpha - \theta - 1/2 \cos\alpha \sin\theta) \cos\theta - 1/2 \sin\alpha \sin^2\theta - \frac{\theta}{2} \cos\alpha - \alpha \right] \quad (35)$$

$$\begin{aligned} \delta y(\theta) = & - \frac{Tr^4}{EI_z} \left[(\alpha - \theta) \sin\theta + \left(\frac{\theta^2}{2} - \alpha\theta - 1 - \cos\alpha \right) \cos\theta \right. \\ & + 1/2 \sin\alpha \sin\theta \cos\theta + 1/2 \cos\alpha \cos^2\theta \\ & \left. + 1/2 \cos\alpha - \frac{\theta}{2} \sin\alpha + 1 \right] \quad (36) \end{aligned}$$

$$\delta \phi_z(\theta) = - \frac{Tr^3}{EI_z} \left[\theta(\alpha - \frac{\theta}{2}) + \cos\alpha (1 - \cos\theta) - \sin\alpha \sin\theta \right] \quad (37)$$

For the special case $\alpha = \theta = \frac{\pi}{2}$,

$$\delta x = \frac{Tr^4}{EI_z} \left(\frac{\pi^2}{8} - \frac{\pi}{2} + 1/2 \right) = 0.16290 \frac{Tr^4}{EI_z} \quad (38)$$

$$\delta y = - \frac{Tr^4}{EI_z} \left(1 - \frac{\pi}{4} \right) = - 0.21460 \frac{Tr^4}{EI_z} \quad (39)$$

$$\delta \phi_z = - \frac{Tr^3}{EI_z} \left(\frac{\pi^2}{8} - 1 \right) = - 0.23370 \frac{Tr^3}{EI_z} \quad (40)$$

Case V - Uniformly Distributed Radial Load R

Referring to Figure 2, the incremental moment at θ' due to the distributed load R acting over the incremental length of beam, $rd\theta''$, at θ'' is given by

$$dM_z(\theta') = (Rrd\theta'') r \sin(\theta'' - \theta') = Rr^2 \sin(\theta'' - \theta') d\theta'' \quad (41)$$

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Integrating Equation (41) from θ' to α ,

$$M_z(\theta') = Rr^2 \int_{\theta'}^{\alpha} \sin(\theta'' - \theta') d\theta'' = Rr^2 [1 - \cos(\alpha - \theta')] \quad (42)$$

Substituting Equation (42) into Equations (9), (10), and (11), and carrying out the integration

$$\begin{aligned} \delta x(\theta) = & -\frac{Rr^4}{EI_z} \left[(\theta - \sin\alpha)\sin\theta + \cos\theta + 1/2 \sin\alpha \sin\theta \cos\theta \right. \\ & \left. - 1/2 \cos\alpha \sin^2\theta + \frac{\theta}{2} \sin\alpha - 1 \right] \end{aligned} \quad (43)$$

$$\begin{aligned} \delta y(\theta) = & \frac{Rr^4}{EI_z} \left[\sin\alpha (1/2 \sin^2\theta + \cos\theta - 1) \right. \\ & \left. + 1/2 \cos\alpha (\sin\theta \cos\theta - \theta) + \sin\theta - \theta \cos\theta \right] \end{aligned} \quad (44)$$

$$\delta \phi_z(\theta) = \frac{Rr^3}{EI_z} \left[\theta + \sin(\alpha - \theta) - \sin\alpha \right] \quad (45)$$

For the special case $\alpha = \theta = \frac{\pi}{2}$,

$$\delta x = -\frac{Rr^4}{EI_z} \left(\frac{3\pi}{4} - 2 \right) = -0.35619 \frac{Rr^4}{EI_z} \quad (46)$$

$$\delta y = 1/2 \frac{Rr^4}{EI_z} \quad (47)$$

$$\delta \phi_z = \frac{Rr^3}{EI_z} \left(\frac{\pi}{2} - 1 \right) = 0.57080 \frac{Rr^3}{EI_z} \quad (48)$$

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Part 2 - Out-of Plane Bending

For out-of-plane bending the beam deforms by flexure and torsion. Referring to Figure 4, the strain energy of flexure is given by

$$U_F(\theta) = \int_0^\theta \frac{M_r^2 r d\theta'}{2EI_r} \quad (49)$$

where M_r is the moment with axis along the radius at θ' due to all loads acting on the beam, and I_r is the moment of inertia of the cross-section of the beam about the same axis. The strain energy of torsion is given by

$$U_T(\theta) = \int_0^\theta \frac{M_T^2 r d\theta'}{2GK} \quad (50)$$

where M_T is the twisting moment with axis along the tangent at θ' and GK is the torsional rigidity of the beam. For beams of circular and concentric-hollow circular cross-sections, K is the polar moment of inertia of the cross-section. For noncircular beams, various methods of calculating K are available.⁴

The total strain energy is the sum of Equations (49) and (50).

$$U(\theta) = U_F(\theta) + U_T(\theta) = \int_0^\theta \left(\frac{M_r^2}{2EI_r} + \frac{M_T^2}{2GK} \right) r d\theta' \quad (51)$$

By Castigliano's theorem, as in Part 1, the deflections at θ are given by the following equations:

$$\delta_z(\theta) = \frac{\partial U(\theta)}{\partial Z} = \int_0^\theta \left(\frac{M_r}{EI_r} \frac{\partial M_r}{\partial Z} + \frac{M_T}{GK} \frac{\partial M_T}{\partial Z} \right) r d\theta' \quad (52)$$

$$\delta\phi_x(\theta) = \frac{\partial U(\theta)}{\partial M_1} = \int_0^\theta \left(\frac{M_r}{EI_r} \frac{\partial M_r}{\partial M_1} + \frac{M_T}{GK} \frac{\partial M_T}{\partial M_1} \right) r d\theta' \quad (53)$$

$$\delta\phi_y(\theta) = \frac{\partial U(\theta)}{\partial M_j} = \int_0^\theta \left(\frac{M_r}{EI_r} \frac{\partial M_r}{\partial M_j} + \frac{M_T}{GK} \frac{\partial M_T}{\partial M_j} \right) r d\theta' \quad (54)$$

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4. See Reference 3, pp. 170-186, and Reference 2, pp. 258-313.

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Figure 4 - Coordinate System Showing Loads Used to Calculate Deflections for Out-of-Plane Bending

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In Equations (52), (53), and (54), Z , M_i , and M_j are the loads applied to the beam at θ in order to calculate the corresponding deflections. The moments M_r and M_T at θ' due to these loads at θ are given by Equations (55) and (56).

$$M_r(\theta') = -Z r \sin(\theta - \theta') - M_i \cos\theta' + M_j \sin\theta' \quad (55)$$

$$M_T(\theta') = -Z r [1 - \cos(\theta - \theta')] + M_i \sin\theta' + M_j \cos\theta' \quad (56)$$

The partial derivatives are given by Equations (57) through (62).

$$\frac{\partial M_r(\theta')}{\partial Z} = -r \sin(\theta - \theta') \quad (57)$$

$$\frac{\partial M_r(\theta')}{\partial M_i} = -\cos\theta' \quad (58)$$

$$\frac{\partial M_r(\theta')}{\partial M_j} = \sin\theta' \quad (59)$$

$$\frac{\partial M_T(\theta')}{\partial Z} = -r [1 - \cos(\theta - \theta')] \quad (60)$$

$$\frac{\partial M_T(\theta')}{\partial M_i} = \sin\theta' \quad (61)$$

$$\frac{\partial M_T(\theta')}{\partial M_j} = \cos\theta' \quad (62)$$

Substituting Equations (57) through (62) into Equations (52), (53), and (54),

$$\delta z(\theta) = -r^2 \int_0^\theta \left\{ \frac{M_r}{EI_r} \sin(\theta - \theta') + \frac{M_T}{GK} [1 - \cos(\theta - \theta')] \right\} d\theta' \quad (63)$$

$$\delta \phi_x(\theta) = -r \int_0^\theta \left[\frac{M_r}{EI_r} \cos\theta' - \frac{M_T}{GK} \sin\theta' \right] d\theta' \quad (64)$$

$$\delta \phi_y(\theta) = r \int_0^\theta \left[\frac{M_r}{EI_r} \sin\theta' + \frac{M_T}{GK} \cos\theta' \right] d\theta' \quad (65)$$

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Equations (63), (64), and (65), may now be used to determine the deflections for the various cases of out-of-plane bending.

Case VI - Normal Load F at α

The moments at θ' due to F are given by

$$M_r = -F r \sin(\alpha - \theta') \quad (66)$$

$$M_T = -F r [1 - \cos(\alpha - \theta')] \quad (67)$$

Substituting Equations (66) and (67) into Equations (63), (64), and (65), and carrying out the integrations,

$$\begin{aligned} \delta z(\theta) = Fr^3 \left\{ \frac{1}{2EI_r} [\theta \cos(\alpha - \theta) - \cos\alpha \sin\theta] \right. \\ \left. + \frac{1}{GK} [\theta + \sin(\alpha - \theta) - \sin\alpha - \sin\theta + \frac{\theta}{2} \cos(\alpha - \theta) \right. \\ \left. + 1/2 \cos\alpha \sin\theta] \right\} \quad (68) \end{aligned}$$

$$\begin{aligned} \delta \phi_x(\theta) = \frac{Fr^2}{2} \left\{ \frac{1}{EI_r} [\sin\theta \sin(\alpha - \theta) + \theta \sin\alpha] \right. \\ \left. - \frac{1}{GK} [2(1 - \cos\theta) + \sin\theta \sin(\alpha - \theta) - \theta \sin\alpha] \right\} \quad (69) \end{aligned}$$

$$\begin{aligned} \delta \phi_y(\theta) = -\frac{Fr^2}{2} \left(\frac{1}{EI_r} [\sin\theta \cos(\alpha - \theta) - \theta \cos\alpha] \right. \\ \left. + \frac{1}{GK} \left\{ \sin\theta [2 - \cos(\alpha - \theta)] - \theta \cos\alpha \right\} \right) \quad (70) \end{aligned}$$

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For the special case $\alpha = \theta = \frac{\pi}{2}$,

$$\delta z = \frac{Fr^3}{2} \left[\frac{1}{EI_r} \left(\frac{\pi}{2} \right) + \frac{1}{GK} \left(\frac{3\pi}{2} - 4 \right) \right] = Fr^3 \left(\frac{0.78540}{EI_r} + \frac{0.35619}{GK} \right) \quad (71)$$

$$\delta \phi_x = \frac{Fr^2}{2} \left[\frac{1}{EI_r} \left(\frac{\pi}{2} \right) - \frac{1}{GK} \left(2 - \frac{\pi}{2} \right) \right] = Fr^2 \left(\frac{0.78540}{EI_r} - \frac{0.21460}{GK} \right) \quad (72)$$

$$\delta \phi_y = - \frac{Fr^2}{2} \left(\frac{1}{EI_r} + \frac{1}{GK} \right) \quad (73)$$

Case VII - Moment M_Q at α

The moments at θ' due to M_Q are given by

$$M_r = M_Q \cos(\alpha - \theta') \quad (74)$$

$$M_T = M_Q \sin(\alpha - \theta') \quad (75)$$

Substituting Equations (74) and (75) into Equations (63), (64), and (65) and carrying out the integrations,

$$\delta z(\theta) = - \frac{M_Q r^2}{2} \left\{ \frac{1}{EI_r} \left[\sin \alpha \sin \theta - \theta \sin(\alpha - \theta) \right] + \frac{1}{GK} \left[2 \cos(\alpha - \theta) - 2 \cos \alpha - \sin^2 \theta \right] \right\} \quad (76)$$

$$\delta \phi_x(\theta) = - \frac{M_Q r}{2} \left\{ \frac{1}{EI_r} \left[\sin \theta \cos(\alpha - \theta) + \theta \cos \alpha \right] - \frac{1}{GK} \left[\sin \theta \cos(\alpha - \theta) - \theta \cos \alpha \right] \right\} \quad (77)$$

$$\delta\phi_y(\theta) = \frac{M_Q r}{2} \left\{ \frac{1}{EI_r} \left[\theta \sin\alpha - \sin\theta \sin(\alpha - \theta) \right] + \frac{1}{GK} \left[\sin\theta \sin(\alpha - \theta) + \theta \sin\alpha \right] \right\} \quad (78)$$

For the special case $\alpha = \theta = \frac{\pi}{2}$,

$$\delta z = - \frac{M_Q r^2}{2} \left(\frac{1}{EI_r} + \frac{1}{GK} \right) \quad (79)$$

$$\delta\phi_x = - \frac{M_Q r}{2} \left(\frac{1}{EI_r} - \frac{1}{GK} \right) \quad (80)$$

$$\delta\phi_y = \frac{\pi}{4} M_Q r \left(\frac{1}{EI_r} + \frac{1}{GK} \right) = 0.78540 M_Q r \left(\frac{1}{EI_r} + \frac{1}{GK} \right) \quad (81)$$

Case VIII - Moment M_P at α

The moments at θ' due to M_P are given by

$$M_r(\theta') = - M_P \sin(\alpha - \theta') \quad (82)$$

$$M_t(\theta') = M_P \cos(\alpha - \theta') \quad (83)$$

Substituting Equations (82) and (83) into Equations (63), (64), and (65), and carrying out the integrations,

$$\delta z(\theta) = \frac{M_P r^2}{2} \left\{ \frac{1}{EI_r} \left[\theta \cos(\alpha - \theta) - \sin\theta \cos\alpha \right] + \frac{1}{GK} \left[2 \sin(\alpha - \theta) - 2 \sin\alpha + \theta \cos(\alpha - \theta) + \sin\theta \cos\alpha \right] \right\} \quad (84)$$

$$\delta\phi_x(\theta) = \frac{M_{Pr}}{2} \left\{ \frac{1}{EI_r} \left[\sin\theta \sin(\alpha - \theta) + \theta \sin\alpha \right] + \frac{1}{GK} \left[\theta \sin\alpha - \sin\theta \sin(\alpha - \theta) \right] \right\} \quad (85)$$

$$\delta\phi_y(\theta) = - \frac{M_{Pr}}{2} \left\{ \frac{1}{EI_r} \left[\sin\theta \cos(\alpha - \theta) - \theta \cos\alpha \right] - \frac{1}{GK} \left[\sin\theta \cos(\alpha - \theta) + \theta \cos\alpha \right] \right\} \quad (86)$$

For the special case $\alpha = \theta = \frac{\pi}{2}$,

$$\delta z = \frac{M_{Pr}^2}{2} \left[\frac{1}{EI_r} \left(\frac{\pi}{2} \right) + \frac{1}{GK} \left(\frac{\pi}{2} - 2 \right) \right] = M_{Pr}^2 \left(\frac{0.78540}{EI_r} - \frac{0.21460}{GK} \right) \quad (87)$$

$$\delta\phi_x = \frac{\pi}{4} M_{Pr} \left(\frac{1}{EI_r} + \frac{1}{GK} \right) = 0.78540 M_{Pr} \left(\frac{1}{EI_r} + \frac{1}{GK} \right) \quad (88)$$

$$\delta\phi_y = - \frac{M_{Pr}}{2} \left(\frac{1}{EI_r} - \frac{1}{GK} \right) \quad (89)$$

Case IX - Uniformly Distributed Normal Load N

Referring to Figure 2, the incremental moments at θ' due to the distributed load N acting over the incremental length of beam, $r d\theta''$, at θ'' , are given by

$$d M_r(\theta') = - (N r d\theta'') r \sin(\theta'' - \theta') \quad (90)$$

$$d M_T(\theta') = - (N r d\theta'') r [1 - \cos(\theta'' - \theta')] \quad (91)$$

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Integrating Equations (90) and (91) from θ' to α ,

$$\begin{aligned} M_r(\theta') &= -Nr^2 \int_{\theta'}^{\alpha} \sin(\theta'' - \theta') d\theta'' \\ &= -Nr^2 [1 - \cos(\alpha - \theta')] \end{aligned} \quad (92)$$

$$\begin{aligned} M_T(\theta') &= -Nr^2 \int_{\theta'}^{\alpha} [1 - \cos(\theta'' - \theta')] d\theta'' \\ &= -Nr^2 [\alpha - \theta' - \sin(\alpha - \theta')] \end{aligned} \quad (93)$$

Substituting Equations (92) and (93) into Equations (63), (64), and (65), and carrying out the integrations,

$$\begin{aligned} \delta z(\theta) &= Nr^4 \left\{ \frac{1}{EI_r} \left[1 - \cos\theta - 1/2 \sin\alpha \sin\theta + \frac{\theta}{2} \sin(\alpha - \theta) \right] \right. \\ &\quad \left. + \frac{1}{GK} \left[\alpha\theta - \frac{\theta^2}{2} - (\alpha + 1/2 \sin\alpha) \sin\theta - (1 + \cos\alpha) \cos\theta \right. \right. \\ &\quad \left. \left. + \cos\alpha + 1 + \frac{\theta}{2} \sin(\alpha - \theta) \right] \right\} \end{aligned} \quad (94)$$

$$\begin{aligned} \delta \phi_x(\theta) &= Nr^3 \left\{ \frac{1}{EI_r} (\sin\theta - 1/2 \sin\alpha \sin^2\theta - \frac{\theta}{2} \cos\alpha \right. \\ &\quad \left. - 1/2 \cos\alpha \sin\theta \cos\theta) - \frac{1}{GK} [\alpha - (\alpha - \theta) \cos\theta \right. \\ &\quad \left. - \sin\theta - 1/2 \sin\alpha \sin^2\theta + 1/2 \cos\alpha(\theta - \sin\theta \cos\theta)] \right\} \end{aligned} \quad (95)$$

$$\begin{aligned} \delta \phi_y(\theta) &= -Nr^3 \left\{ \frac{1}{EI_r} [1 - \cos\theta - 1/2 \sin\alpha (\theta - \sin\theta \cos\theta) \right. \\ &\quad \left. - 1/2 \cos\alpha \sin^2\theta] + \frac{1}{GK} [(\alpha - \theta) \sin\theta + 1 - \cos\theta \right. \\ &\quad \left. - 1/2 \sin\alpha (\theta + \sin\theta \cos\theta) + 1/2 \cos\alpha \sin^2\theta] \right\} \end{aligned} \quad (96)$$

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For the special case $\alpha = \theta = \frac{\pi}{2}$,

$$\begin{aligned}\delta z &= Nr^4 \left[\frac{1}{2} \frac{1}{EI_r} + \frac{1}{GK} \left(\frac{\pi^2}{8} - \frac{\pi}{2} + \frac{1}{2} \right) \right] \\ &= Nr^4 \left(\frac{1}{2} \frac{1}{EI_r} + \frac{0.16290}{GK} \right)\end{aligned}\quad (97)$$

$$\begin{aligned}\delta \phi_x &= Nr^3 \left[\frac{1}{2} \frac{1}{EI_r} - \frac{1}{GK} \left(\frac{\pi}{2} - \frac{3}{2} \right) \right] \\ &= Nr^3 \left(\frac{1}{2} \frac{1}{EI_r} - \frac{0.07080}{GK} \right)\end{aligned}\quad (98)$$

$$\begin{aligned}\delta \phi_y &= - \left(1 - \frac{\pi}{4} \right) Nr^3 \left(\frac{1}{EI_r} + \frac{1}{GK} \right) \\ &= - 0.21460 Nr^3 \left(\frac{1}{EI_r} + \frac{1}{GK} \right)\end{aligned}\quad (99)$$

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