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LIQUID-METAL LUBRICATED
JOURNAL BEARINGS

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LIST OF SYMBOLS

b	Local half-thickness of film
C_f	Friction coefficient
C_{xc}	Friction coefficient for Couette flow
D	Bearing diameter
F	Frictional force on journal surface
f	Ratio of friction force to load/also ratio of tangential to normal velocity
H	Dimensionless film thickness
h	Local film thickness
L	Bearing length
m	Velocity profile exponent
n	Velocity profile exponent
P	Dimensionless pressure
$\bar{P}$	Mean pressure in bearing
p	Pressure
Q	Dimensionless velocity $\sqrt{U^2 + W^2}$
q	Modulus of velocity vector $\sqrt{u^2 + w^2}$ /also velocity along stream line
R	Local Reynolds number, bu_b/ν
$\bar{R}$	Mean Reynolds number, $\bar{b}u_s/2\nu$
r	Bearing radius
U	u/u_{bc}
U^+	Dimensionless velocity, $u/\sqrt{\tau/\rho}$
u	Velocity in direction of turning
u'	Turbulent velocity component in direction of turning

LIST OF SYMBOLS (Cont'd)

V	Velocity vector modulus, $\sqrt{u^2 + w^2}$
W	w/u_{bc}
W_ℓ	Dimensional bearing load
$\bar{W}_\ell$	Dimensionless bearing load
w	Velocity in axial direction
X	Dimensionless coordinate in direction of sliding x/h
x	Coordinate in direction of turning
Y^+	Dimensionless wall distance, $y\sqrt{\tau/\rho}/\nu$
y	Coordinate normal to boundary
Z	Dimensionless axial coordinate
z	Axial coordinate
Γ	Surface tension
δ^*	Displacement thickness
ϵ	Eccentricity ratio
ζ	Dimensionless transport quantity
μ	Absolute viscosity
ν	Kinematic viscosity
ρ	Fluid density
ξ	Dimensionless distance
τ	Shear stress
T	Shear stress at wall
$()_b$	Refers to midchannel

LIST OF SYMBOLS (Cont'd)

- ()_{bc} Refers to Couette flow at midchannel
- ()_c Refers to Couette flow
- ()_o Refers to stationary wall
- ()_s Refers to surface velocity of journal

INTRODUCTION

This report covers the results of a 3-year program of analytical and experimental research. From the outset, the objective has been to provide a guidebook for the turbulent-film bearing designer. It was originally conceived that the experimental work would suggest refinements on previously tried analytical approaches and that the end result would be a series of design tables for bearings of a variety of geometries. As the program developed, the objective became more clearly defined as one of setting down an approach, quite broadly applicable to bearing films, and establishing confidence in the validity of this approach.

The final result, as reported here, presents the approach which was developed, showing it to be easily dealt with computationally. It is then applied to: (1) short bearings with predominant end leakage effects, (2) long tilted pads with approximately tangential flow, and (3) finite tilted pads. In addition, the problems of the turbulent squeeze film, when there is a relative velocity in the direction normal to the surfaces, is treated, and the problem of the free surface with surface-tension effects is dealt with.

The results of these calculations point up certain danger areas in the design of turbulent film bearings, particularly those associated with inertial effects. The results which are shown in tables and plots can be used directly by the designer. These results also serve as worked out examples of how the film equations may be dealt with, and as such should aid in the continued development of solutions for specific bearing configurations as the need arises.

LARGE-SCALE SIMULATION OF TURBULENT FLOWS

General

Despite the large amount of research on turbulent boundary layers and pipe flow, many questions have been left unanswered with regard to the flow conditions to be encountered in bearings. To answer these, a large-scale annular-flow apparatus was constructed with 6-ft journal diameter, 2-ft length, and 0.54-in. film thickness. Air was used as a working fluid, and operation with this in the large-scale configuration was felt to simulate liquid metals in typical smaller dimensions since the air was effectively incompressible over the pressure range encountered, and since Reynolds numbers and Taylor numbers could be made to correspond to those values encountered in bearings.

Pressure distributions were measured along with velocity and turbulence-intensity profiles. Wall shear stress is computed from the velocity profile measurements. The variations of these factors are here compared with available data for plane Couette flow, wall-law flow, and pressure flow in pipes, and the relationships among these are discussed. In addition, large pressure jumps at discontinuities are reported and are shown to provide a major influence on the overall pressure distributions.

A breakdown follows on the more important of such topics for investigation:

- (1) Shear stress and velocity profile relationships in vortex flow, and in fully turbulent flow.
- (2) Effects of strong tangential and axial pressure flows when superposed on a Couette flow.

- (3) Effects of a strong Couette flow on a weak axial pressure flow.
- (4) Effects of film thickness changes on shear stress and velocity profiles.
- (5) Circumferential transfer of turbulence which makes the state at a point in a film dependent upon the turbulent state elsewhere.
- (6) Effects of discontinuities in the flow, such as at the entry to a bearing pad or a step.
- (7) Pressure distribution data for use in the analysis of short full bearings, as well as long and short tilted pads and stepped pads.

Description of Apparatus

The measurements reported here were made on a large-scale (6-ft diameter, 20-in. length, 0.5-in. film thickness), apparatus, operating over a range of Reynolds numbers believed to be realistic for bearings. The working fluid was air, which lends itself to hot-wire anemometry. The pressure changes were so small that the air behaved as an incompressible fluid and was therefore analogous in its performance to liquid lubricants. The apparatus was operated as a short full bearing, as a tilted pad (with sections of the full bearing removed), and as a stepped pad made by modification of the tilting pad.

A schematic diagram of the apparatus is shown in Figure 1. The journal is supported on a shaft and bearings, and is mounted on a stand fixed to the floor. The journal runs at speeds up to 120 rpm, generating, in the 0.54-in. film thickness, Reynolds numbers which are comparable to those expected in the much smaller bearings to be used with liquid metals.

The bearing (or outer cylinder) is independently supported and, as shown in Figure 1, is mounted on a sled which can be moved back and forth in

FIGURE 1. SCHEMATIC DIAGRAM OF BEARING APPARATUS

the horizontal plane to allow variation of journal eccentricity. The bearing can also be rotated around the journal on ball bearing supports so that a pressure or velocity probe, fixed to the bearing, can be set at various attitude angles. A micrometer traversing-mechanism, mounted to the bearing, provides for radial as well as axial traversing of the air film with a velocity probe. This velocity probe is a hot-wire anemometer, with extra long filament-support needles to minimize the effect of the 1/8-in. diameter barrel of the probe on the flow.

Also shown in Figure 1 are removable sectors in the bearing, a feature which allows the simulation of a pad bearing. In the design of the apparatus, a film thickness had to be chosen large enough so that good resolution could be obtained in the velocity measurements, yet small enough so that a realistic b/r ratio could be maintained. The 0.54-in. film thickness was chosen as one which would provide an optimum balance between these two requirements, giving a dimensionless clearance of $b/r = 0.007$, which is comparable to 0.00436 used by Wilcock^{(1)*}.

Very low velocities are required to generate the desired range of Reynolds number in this film; therefore, low pressures are encountered. In this experiment, the problem of measuring low velocities was solved by using a hot-wire anemometer, which also allows the measurement of turbulence quantities. The problem of low pressures was solved by developing a pressure measuring device, which is described along with the discussion of pressure measurements.

*Superscript numbers in parentheses refer to List of References at end of report.

Velocity Profiles in Couette Flow

A typical family of time-mean velocity profiles is shown in Figure 2. These are for pure Couette flow, where Reynolds number was varied through changes of turning speed. They were made at midspan in the axial direction. For the very lowest Reynolds number, they approach a straight diagonal line. As either turbulence or vortex flow appears at higher Reynolds numbers, they show curvature and ultimately seem to approach an asymptotic limiting profile shape.

When half of one of these profiles is plotted, it has much the same appearance as the velocity profile in a boundary layer or in the half-channel of pipe flow. To illustrate this, we note that the Couette flows data of Robertson and Richardt have been fitted well with a power law equation based upon independent data for pipe flow⁽²⁾. The important observation here is that the profile shape in fully turbulent flow is not greatly sensitive to the details of the stress distribution. In the Couette flow, the shear stress is constant from the wall outward; in pipe flow (as in other parallel pressure flows), the shear stress varies linearly inward to zero at midchannel. Even with this extreme difference, velocity profiles for the two flows closely resemble one another.

The fact that these published profiles could so well be fitted by a power law suggests that this might provide a simple single-parameter means to characterize the velocity profile under any specified operating condition. The problem would be to determine the most appropriate exponent to use in an equation of the type:

FIGURE 2. VELOCITY PROFILES IN COUETTE FLOW

$$u/u_b = (y/b)^n \quad (1)$$

where n would vary with Reynolds number since the profiles in Figure 2 obviously span a broad range of curvatures. Such a fit at $R = 2500$ and 740 is illustrated by the linear and logarithmic plots in Figures 3 and 4. In the logarithmic plot, it is seen that the data for the higher Reynolds number do indeed conform to a straight line. From the slope of the line on the logarithmic plot, it is found that $n \approx 1/8$. The calculated sublayer thickness is indicated on the figure, showing that the data points were taken in the turbulent core.

For the lower Reynolds number, the sublayer fills much of the channel, and many of the data points fall within it. Outside this layer, the points tend toward the power law curve for the higher Reynolds number.

The sublayer thickness was calculated by the theoretical relation $(u/u_b)(y/b) = 100/R$, where y is the sublayer thickness and u is the velocity at y . This relation is derived from the wall-law equation which will be discussed in a later section in connection with friction coefficient measurements. One may conclude from the above observations that power law representation is, indeed, satisfactory when the sublayer is thin.

Displacement thickness serves as a more fundamental parameter than the power law exponent to characterize the velocity profile. It has the advantage of being free from arbitrary judgments in the processes of determination of its magnitude from profile data; and, being the result of an integration process, tends to represent the consensus of all of the data points in the profile.

FIGURE 3. NORMALIZED HALF-CHANNEL PROFILES SHOWING
LEAST-SQUARES POWER-LAW FIT FOR $R = 2500$

FIGURE 4. LOG-LOG PLOT OF NORMALIZED HALF-CHANNEL PROFILES, SHOWING POWER-LAW FIT AT $R = 2500$

Displacement thickness in the half channel is defined by

$$\delta^* = \int_0^b \frac{1}{b} \left(1 - \frac{u}{u_b} \right) dy \quad (2)$$

It is the measure of the velocity defect, or the discrepancy between the profile and one with constant velocity. For empirical profiles, the integration is accomplished by the use of planimeter with the profile-data plot. For laminar flow, the profile becomes linear, and $\delta^*/b = 0.5$.

The curves in Figure 5 serve to illustrate comparisons of displacement thickness variation for: Couette flow in the large-scale apparatus, Couette flow between parallel surfaces (called plane Couette flow), and a curve representing the "wall law," which will be discussed in a later section. Transition in the large-scale apparatus occurs at lower Reynolds number than that for the Couette flow. This may be attributed to the Taylor vortex transition. The plane Couette flow curve approaches the wall-law curve at higher Reynolds numbers, and in many ways resembles the curve for the present work, simply translated to a higher transition Reynolds number. It is of interest that the plane Couette flow data always lie above the wall-law curve, whereas the data for the present work lie below it.

The points in Figure 6 allow comparisons among data for different film thicknesses in the experimental apparatus, for eccentric displacement of the journal. This simulates a short full bearing, where h is local film thickness and is expressed in ratio to mean radial clearance $\bar{h}$. In all cases, the profiles were measured at midspan axially. The profiles were very near to those for

FIGURE 5. DISPLACEMENT-THICKNESS VARIATION
NEAR TRANSITION

FIGURE 6. EFFECT OF ECCENTRICITY ON DISPLACEMENT THICKNESS IN LARGE-SCALE TURBULENCE APPARATUS

Couette flow, since tangential pressure flow is very weak, as a consequence of the small length/diameter ratio of the apparatus. The Reynolds number $\bar{R}$ is based upon the mean clearance $\bar{h}$, such that $\bar{R} = \frac{u_s \bar{h}}{4\nu} = \frac{u_{bc} \bar{h}}{2\nu}$. With respect to the Reynolds number, $\bar{R}$, the thinner film thicknesses behave somewhat as though they had a higher transition than the simple Couette flow. This might suggest that local Reynolds number $R = u_{bc} h / 2\nu$, rather than $\bar{R}$, controls the transition. If so, this would require the 0.18-in. film thickness points to fall upon the broken line, which represents a transposition of the curve for concentric operation (solid line). Since the data points fall below this, it is clear that the film is controlled neither by the local R nor the over-all R but a combination of these. This behavior is apparently caused by circumferential transfer of turbulence, which makes the thin film become turbulent more easily than it would from local influences alone, but does not completely dominate the flow behavior.

Taylor⁽³⁾ has determined that the vortex-flow transition in Couette flow occurs at a fixed value of the number $T = 4R\sqrt{h/r}$. If it is assumed that local transition is determined by T/T_{CRIT} , then δ^*/b would be expected to be a function of T . When plotted against this, however, the correlation of the data is worse than in Figure 6.

Velocity Profiles in Blocked Flow

Very little has been published on extremely skewed flows, such as that in which a strong axial pressure flow interacts with a strong tangential Couette flow. To produce such flows experimentally, the journal was set concentric in the full bearing and a barrier placed in the film in the form of a 0.5-in.

thick dam which extended across the film, parallel to the axial direction. This blocked the flow in the tangential direction and forced the fluid to divert and pass out the ends of the bearing clearance.

Figure 7 shows a map of the local velocity vectors at the half-channel thickness as the flow approaches the blockage. Here, z/h is measured from the midspan of the bearing outward toward the edge, and x/h is measured from the blockage. Both measurements are in multiples of film thickness. The velocity at the midspan ($z = 0$) simply decelerates. As z increases, the velocity turns into the axial direction and increases in magnitude.

Figure 8 shows typical measurements of the x and z components of a skewed velocity profile. At the journal surface, the normalized circumferential component of velocity u/u_s is unity (corresponding to the journal surface speed), and the normalized axial component is zero. Near the bearing surface, both components of velocity go to zero as they should.

The displacement of the peak of the axial velocity curve toward the stationary wall indicates that the point of zero axial shear stress is located here rather than at $y = b$. The axial traction on the stationary wall must, therefore, be less than on the moving wall.

A strongly decelerated flow occurred at midspan when the blockage was placed across the channel. In effect, this blockage produced a pressure flow opposing the Couette flow. Figure 9 shows full channel velocity profiles measured at the axial midspan of the bearing, for $\bar{R} = 2400$. For two distances from the blockage (indicated by $x/h = 6$ and 8 , measured from the blockage), the profiles go very nearly normal to the stationary wall and show sharp

FIGURE 7. MAP OF MIDCHANNEL VELOCITY VECTORS
(at $y = b$), NEAR A BLOCKAGE IN THE FILM

FIGURE 8. COMPONENTS OF SKEWED VELOCITY PROFILE
IN FLOW BEHIND BLOCKAGE

FIGURE 9. VELOCITY PROFILES NEAR A BLOCKAGE
AT AXIAL MIDSPAN OF FILM

gradients at the moving wall. These full channel profiles are quite different from the full channel profiles for Couette flow alone, but can be shown to correspond closely to those in half-channel plots.

Measurements for Orthogonal Couette and Pressure Flows

It has been noted that the problem of pressure flow acting at right angles to a Couette flow (the case of axial leakage around a shaft) has not been dealt with adequately by experiments. This idealized situation does, however, make an excellent test case for how well any analytical approach can serve to treat the general problem of flow with shear stress varying in direction at different distances from a boundary wall. To provide the needed information, the experiments reported here were undertaken.

To generate the desired flows, the wheel or journal was set concentric in the housing. End leakage was blocked by a plastic curtain at one end, and a dam was extended across the clearance from this blockage, spanning three-fourths of the width of the bearing, and leaving only a very small clearance for the journal to pass beneath. This dam diverted the flow as shown in Figure 10 and produced combinations of end-leakage flow, Couette flow, and tangential-pressure flow. By probing various regions with a hot-wire anemometer, positions were found where only a very weak pressure flow acted tangentially, and a strong pressure flow acted in the axial direction. These positions were approximately 1 in. tangentially upstream and, respectively, 3, 2 and 1 in. outward from the dam tip.

Two techniques were possible for applying hot-wire anemometry to the measurement of the skewed velocity profiles. One was to select a fixed

FIGURE 10. ILLUSTRATION OF PLACEMENT OF DAM
RELATIVE TO MEASUREMENT POINTS

distance out from the wall for a given reading and to orient the wire so as to be normal to the local velocity vector. This was found to be very difficult since the precise angle of flow could not be determined accurately. A second technique was to orient the probe first in the axial direction and then in the tangential direction at each position out from a wall, and thereby to read the two velocity components separately. Unfortunately, there is some coupling effect between the readings. To determine the correction factor, pure Couette flow was generated in the apparatus with the dam removed, and the wire was placed at midchannel where the velocity was known to be half the sliding speed of the journal. The wire was then oriented at different known angles to the direction of flow. Readings for each of these orientations were plotted against the sine of the angle between the wire and the actual velocity direction, which would represent the velocity component normal to the wire. In this plot (Fig. 11), the normal velocity calculated from the hot-wire circuit was equal to the true normal velocity component, only when the hot wire was exactly normal to the total velocity vector. Indeed, the calibration of the wire was obtained by orienting it normal to the midstream velocity for the Couette flow and varying the wheel speed. When the wire was turned parallel to the total velocity vector, there was still a reading of approximately 15 percent of the total velocity magnitude. An equation which expresses the relationship for any angular orientation is as follows:

$$\frac{v_{n, \text{calc}}}{v} - 0.15 = 0.085 \frac{v_n}{v}, \text{ where } v = \sqrt{u^2 + w^2} \quad (3)$$

FIGURE 11. DIRECTIONAL SENSITIVITY OF
HOT-WIRE PROBE

(Note that one need not be concerned with a third velocity component since the flow was essentially parallel to the walls.) Applying Equation (1) to the flows where the velocity acts in some unknown direction, it is possible to take the measured axial velocity and measured tangential velocity and to compute the true axial, u , and tangential, w , from these using the following equations:

$$\begin{aligned} 0.85u &= u_{\text{calc}} - 0.15\sqrt{u^2 + w^2} \\ 0.85w &= w_{\text{calc}} - 0.15\sqrt{u^2 + w^2} \end{aligned} \tag{4}$$

Typical data are shown in Table 1 where U represents the velocity in the tangential direction expressed in a ratio to the midchannel Couette velocity, u_{bc} or $u_s/2$, and W represents the axial velocity similarly nondimensionalized. y/h is the wall distance measured from the stationary wall. A, B, B and C represent independent runs on different days with the letter indicating the measurement position in Figure 10. Looking at the profile of U , it is seen that the velocity is almost unity midway between the walls, in which case the profile would be nearly symmetrical, and one could assume that there was only very weak displacement flow in the direction of the turning. Similarly, the W profile is almost symmetrical, though slightly distorted by the presence of the weak displacement flow in the tangential direction. Averages of the runs are indicated at each wall-distance position. To draw upon as many points as possible, velocities at corresponding wall distances from both the moving and stationary walls were adjusted so as to be relative to the referenced wall in each case, and these were averaged. Thus, the corresponding velocities on the two sides of the channel were averaged with one

TABLE 1. PROFILE DATA FOR ORTHOGONAL
COUETTE AND PRESSURE FLOWS

<u>y/h</u>	<u>A</u>	<u>B</u>	<u>B</u>	<u>C</u>	<u>Avg</u>	<u>Symm</u>
Tangential velocity, U						
0	0	0	0	0	0	0
0.0625	0.4061	0.4796	0.4910	0.4287	0.451	0.539
0.1875	0.6772	0.6517	0.6530	0.5648	0.637	0.771
0.3125	0.7641	0.7449	0.8104	0.7054	0.757	0.885
0.4375	0.8255	0.8712	0.9426	0.7954	0.859	0.960
0.5625	0.9503	0.9562	0.9840	0.8636	0.938	1.040
0.6875	0.9812	1.0088	0.9840	0.9707	0.986	1.114
0.8125	1.0462	1.1020	1.1101	1.1209	1.095	1.228
0.9375	1.3227	1.3888	1.3903	1.3854	1.372	1.460
1.000	2.000	2.000	2.000	2.000	2.000	2.000
Axial velocity, W						
0	0	0	0	0	0	0
0.0625	0.3580	0.4594	0.4256	0.5265	0.442	0.421
0.1875	0.4462	0.6667	0.5571	0.6856	0.589	0.624
0.3125	0.5106	0.6794	0.6075	0.7589	0.639	0.668
0.4375	0.5150	0.6862	0.6075	0.7930	0.651	0.665
0.5625	0.5431	0.7183	0.6135	0.8425	0.679	0.665
0.6875	0.5406	0.7331	0.7045	0.8104	0.697	0.668
0.8125	0.4772	0.7223	0.6177	0.8236	0.660	0.624
0.9375	0.2885	0.3854	0.3734	0.5581	0.401	0.421
1.000	0	0	0	0	0	0

another to generate symmetrical profiles, eliminating, to some extent, the small effect of the tangential displacement flow and the small effect of asymmetry in the axial flow. This is not a large correction, and one would find that, if he plotted the flow in the half-channel normalized to have a value of unity at $y/h = 0.5$, the averaged and symmetrical profiles correspond very closely. The latter, however, has been used in subsequent calculations, and it is felt to be more accurate, drawing as it does on more data items for each point. Figures 12 and 13 illustrate the adjusted profiles of U and W .

Several fundamental questions can be answered through cross comparison and analysis of the profile data. These include: (1) verification of the eddy-viscosity concept for skewed flows, (2) determination of the relationship between midstream eddy viscosity in the skewed flow and that in wall-law flow of the same wall shear stress, and (3) determination of the friction coefficient and the direction of the friction force in the skewed flow. These will be discussed separately below.

The concept of the eddy viscosity requires that, at any wall distance in a turbulent flow, the direction of the velocity gradient be the same as the direction of the local shear stress (where velocities and stresses are taken parallel to the wall). It is of interest to test this concept in a flow where the shear stress direction varies significantly on moving outward from the wall to midstream. In the experimental situation, this direction rotates approximately 45° over the stated distance. At a given wall distance, irrespective of how eddy viscosity is determined physically, it should be possible to write, at a given value of y :

FIGURE 12. PROFILES OF TANGENTIAL VELOCITY COMPONENT FOR CROSSED SHEAR AND DISPLACEMENT FLOWS

FIGURE 13. PROFILES OF AXIAL COMPONENT OF CROSSED SHEAR AND DISPLACEMENT FLOWS

$$\tau_{xy} = \frac{\partial u}{\partial y} (\epsilon + \nu) \rho \quad (5)$$

$$\tau_{zy} = \frac{\partial w}{\partial z} (\epsilon + \nu) \rho$$

or, combining these and assuming that ϵ has the same magnitude in both the z- and x-directions, at any given value of y,

$$W = \int_0^y \frac{\tau_{zy}}{\tau_{xy}} \frac{\partial u}{\partial y} dy \quad (6)$$

For pure shear flow, where there is no pressure gradient along the flow, where streamlines are parallel, and where viscosity (eddy or otherwise) varies symmetrically with respect to the channel centerline, the shear stress will be constant across the channel, and it is possible to write

$$\tau_{xy} = T_{xy} \quad (7)$$

Then, similarly for displacement flow under corresponding conditions (but where there is no relative motion of the walls in the direction of the flow), it is possible to write

$$\tau_{zy} = T_{zy}(1 - 2y/h) \quad (8)$$

Using this information, the above integral can be written as

$$\frac{T_x}{T_z} W = \int_0^y (1 - 2y/h) \frac{\partial U}{\partial y} dy \quad (9)$$

Thus, it is possible to take data for the variation of U and use these to predict

the variation of W . This conversion is fairly accurate since it involves no differentiation of the data but, rather, integration. The results of such a calculation are illustrated in Table 2 where wall distance and the adjusted measured velocity are shown in the first two columns. The calculated value

TABLE 2. COMPARISON OF MEASURED W_{symm}
WITH THAT CALCULATED FROM V_{symm}

<u>y/h</u>	<u>W_{symm}</u>	<u>$\frac{T_x}{T_z} W_{\text{calc}}$</u>	<u>W_{calc}^*</u>
0	0	0	0
0.0625	0.421	0.524	0.448
0.1875	0.624	0.713	0.610
0.3125	0.668	0.774	0.662
0.4375	0.665	0.791	0.676
Mean	0.5945	0.6955	

*On basis of $\frac{T_x}{T_z} = \frac{0.6955}{0.5945} = 1.17$

of the integral of Equation (9) is shown in column 3. Averaging these quantities (and assuming $W_{\text{calc, mean}} = W_{\text{symm, mean}}$), it is possible to get a representative value of $T_x/T_z = 1.17$ as indicated. Inserting this into the values of column 3, the column of W_{calc} was produced. In Figure 13, these calculated points have been shown relative to the measured points, and it is seen that, within what is apparently experimental scatter, both serve to fit the same curve closely. This indicates that, to a good degree of accuracy, it is valid to assume that the eddy viscosity is not directional at a given wall distance y .

Friction Coefficient Measurements

In the reported experiments, the wall shear stress was very small (about 10^{-3} lb/ft² at the maximum) and could not be measured directly. Instead, a generally accepted alternative procedure was used, involving application of the "wall law" or "universal velocity profile" to a region in the vicinity of the solid boundary surface, to permit computation of the shear stress, as will be explained below. In a field of parallel flow in the vicinity of a single wall, where any other boundaries are very far distant, it may be assumed that, for purposes of dimensional analysis, only the following variables described the flow: y distance out from the wall; τ wall shear stress (which is also the shear stress throughout the flow); ρ fluid density; μ fluid viscosity; and u local velocity. These variables fully specify both the fluid properties and the kinematic and geometric properties of the flow, whether it be turbulent or laminar. They may be grouped into two dimensionless groups designated by U^+ and Y^+ , where $U^+ = u\sqrt{\rho/\tau}$ and $Y^+ = (y/\nu)(\sqrt{\tau/\rho})$, and also $U^+ = \sqrt{2/C_f}$ and $Y^+ = R\sqrt{C_f/2}$. When plotted, the representation of U^+ versus Y^+ is normally called the "universal velocity profile" or "wall-law profile" since its appearance corresponds geometrically to the shape of a velocity profile in the vicinity of a wall. Numerous experimenters have attempted to determine this fundamental function or profile from measurements in actual flows. It has been fairly well established by such experiments that, for pipe flow, a single universal velocity profile function will apply for at least a third of the pipe radius out from the wall and is a good first approximation right out to the center of the pipe. It has further

been shown, though less exhaustively, that plane Couette flow also follows the same universal velocity profile for a sizeable portion of the channel distance out from the wall. Boundary layer velocity profiles do the same.

This profile in the turbulent regime may be approximated to a fair degree by a simple power law and also by a logarithmic expression. In the laminar regime very near to the wall, the universal velocity profile approaches $U^+ = Y^+$. A simple expression which gives a very adequate fit of known data⁽⁴⁾ over the full range of interest is illustrated in Figure 14:

$$Y^+ = U^+ + \left(\frac{U^+}{8.74} \right)^7 \quad (10)$$

Using the above wall law, a computer program was written to determine the value of wall τ necessary to force the experimental data to satisfy Equation (10). Since $U^+ = u\sqrt{\rho/\tau}$ and $Y^+ = y\sqrt{\tau/\rho}/\nu$, Equation (10) can be rewritten as

$$y\sqrt{\tau/\rho}/\nu = U\sqrt{\rho/\tau} + (u\sqrt{\rho/\tau}/8.74)^7 \quad (11)$$

Here, ν and ρ are known constants, and y and u are obtained from the measured velocity profiles. Thus, τ may be determined and converted to C_f where $C_f = 2\tau/\rho u_b^2$. If the actual profile is of the same form as the universal velocity profile, the value of C_f computed for any pair of (u, y) coordinates will correspond to that for any other pair. If the values of computed C_f vary outward from the wall, this indicates that the velocity profile does not follow the universal velocity profile.

FIGURE 14. WALL-LAW PROFILE EQUATION COMPARED WITH EXPERIMENTAL DATA

Figure 15 shows a number of profiles of computed C_f for concentric flow, showing the value to remain nearly constant, but still dropping off toward midstream. This shows that the universal-velocity-profile shear-stress relationship underestimates the actual midstream shear stress, and indicates that an additional mode of momentum transfer must be at work, over and above the turbulent mode. This is undoubtedly Taylor vortex flow, still at work even when the flow is fully turbulent. By extrapolating the C_f profiles back to the wall, values are obtained which must be independent of this momentum transfer since the true function must merge toward the wall-law function as the laminar sublayer is approached. The results of such extrapolations are shown in Figure 16.

To show the effect of film thickness variation on C_f , velocity profiles were obtained at three different journal eccentricities. The resulting values of C_f are shown in Figure 16 where the points follow the same trend as those for the concentric journal.

When compared with the data of Couette⁽⁵⁾, as well as the thin film values of Smith and Fuller⁽⁶⁾, the magnitudes reported here are generally higher, tending toward Taylor's earlier results for even thicker films. Both this observation and the manner of variation of δ^*/b may be reconciled. Since, for a given midstream velocity, decreased δ^*/b implies increased velocity gradient at the wall, it calls for increased shear stress. Because δ^*/b is less than the wall-law value over the range of measurement, it would be expected that shear stress and C_f would exceed the wall-law values. On the other hand, Couette's data (for thin film and outer cylinder turning) as well as those of

FIGURE 15. WALL FRICTION COEFFICIENT CALCULATED FROM PROFILE DATA, AS AFFECTED BY REYNOLDS NUMBER AND WALL DISTANCE

FIGURE 16. FRICTION COEFFICIENT VARIATION
COMPARED WITH PRIOR WORK

Robertson and Reichardt were taken under conditions which do not give rise to early transition and vortex flow. Friction coefficient values of Robertson correlate closely with those of Couette, which are plotted, and both fall close to wall-law curve.

Blocked Flow. When the wall-law technique was applied to calculation of C_f in blocked flow, close agreement was found between the experimental profile and the wall-law profile as illustrated by Figure 17. In the figure, y is measured from the moving surface, and computed C_f is nearly constant, even though the midchannel profiles have zero velocity at the stationary wall and though the skewed profiles are strongly distorted. In the case of the skewed flow, the modulus of the velocity vector, rather than the x velocity component, was used in the calculation. The computed friction coefficient is comparable to that for wall law at the same Reynolds number and is below that for the Couette flow.

Friction Coefficient Calculation for Orthogonal Couette and Pressure Flows

Using the methods in the preceding paragraphs, the results obtained are shown in Figure 18. This figure displays values of C_f calculated from simple Couette flow, and from the full velocity vector, $V = \sqrt{W^2 + U^2}$, from the adjusted profiles under consideration here. Note that the Couette flow acting alone has a somewhat higher friction coefficient (wall intercept at $y/h = 0$) than if it followed the wall-law profile precisely. Indeed, the point at the extreme right would be the friction coefficient for wall-law Couette flow with the same midstream velocity. The added friction is believed to be attributable to the vortex flow which supplements turbulent mixing in

FIGURE 17. WALL FRICTION COEFFICIENT CALCULATED FROM PROFILE DATA NEAR A BLOCKAGE

FIGURE 18. VARIATION OF CALCULATED C_f WITH WALL DISTANCE FOR THREE PROFILE TYPES

transferring momentum in the flow. This is a consequence of the film thickness used here, and has been discussed elsewhere.

By way of confirmation of the technique by which all of the reported friction coefficients were calculated, wall shear stress has been determined separately by an x-probe technique at high Reynolds number. This employs two hot wires to measure Reynolds stress at midstream. At the highest Reynolds number, this stress at midstream should predominate over any viscous stresses, and thus should give a valid measure of wall stress for the Couette flow. Under such a condition, the measured stress was determined, and the corresponding C_f was found to agree with the reported measurements, again supporting the validity of the trends observed.

Turbulence Intensity

The hot-wire anemometer lends itself to measurement of the instantaneous magnitude of the velocity component u parallel to the direction of flow. The root-mean-square amplitude of the fluctuating component of this velocity, u' , provides a measure of turbulence intensity and can be determined as a function of position in the film, Reynolds number, and eccentricity. Such measurements shed light on circumferential transport of turbulence as well as other factors.

Figure 19 shows comparisons between measured data for Couette flow (concentric journal) and representative values for pressure flow from the literature. Near the wall, the two magnitudes correspond closely, as would be expected from the correspondence of wall friction coefficient between Couette flow and pipe flow. Near midstream, the Couette flow shows

FIGURE 19. TURBULENCE INTENSITY PROFILE IN COUETTE FLOW
COMPARED WITH PIPE FLOW DATA

consistently elevated magnitudes of u'/u_b , possibly as a result of the fact that the Couette flow has midstream shear stress while pressure flow does not.

Figure 20 shows half-channel journal side profiles of u'/u_s , for $\bar{R} = 1200$ and for four film thicknesses, in the full bearing with journal eccentricity. Two aspects of these profiles are of interest: (1) the variation near the wall, and (2) the variation in the bulk of the flow. Near the wall, $u'/u_s \doteq 0.06$ for all clearances; thus, the wall-turbulence is apparently responsive to local conditions. On the other hand, midstream turbulence is high for the thin film and low for the thick film. This indicates that turbulence is being swept into the thin film region from some more highly turbulent zone.

The group $\overline{(u'/u_s)}^2 h/\bar{h}$ is plotted against local R in Figure 21, and one must conclude that, except near transition, this is largely determined by local Reynolds number. Whether the transition, as indicated by this quantity, occurs at a given $\bar{R}$ cannot be verified by the points plotted since not enough extend into the lower range of $\overline{(u'/u_s)}^2 h/\bar{h}$.

Ross⁽⁷⁾, among others, has indicated that this quantity contributes an increment to the pressure gradient in the flow such that

$$\frac{1}{\rho u_{bc}^2} \frac{\partial p}{\partial x} = \int_0^h \frac{\partial}{\partial x} \frac{u'^2}{u_{bc}^2} dy \quad (12)$$

defining $p/\rho u_{bc}^2 = P$, $x/\bar{h} = X$

FIGURE 20. PROFILES OF TURBULENCE INTENSITY
FOR ECCENTRIC JOURNAL PLACEMENT

FIGURE 21. A MEASURE OF TURBULENCE LEVEL AS INFLUENCED BY FILM THICKNESS AND LOCAL R FOR ECCENTRIC JOURNAL PLACEMENT

$$\frac{\partial P}{\partial X} = \frac{\partial}{\partial X} \frac{h}{h} \left(\frac{u'}{u_{bc}} \right)^2 \quad (13)$$

For the eccentric full bearing, in which turbulence quantities were measured, the maximum $\partial P/\partial X$ was about 6×10^{-4} , whereas the corresponding value for the term on the right was about 3.6×10^{-5} . Hence, the term in question contributes less than 10% of the total pressure variation, and may be neglected in analytical approaches to the problem.

Figure 22 shows the variation of turbulence intensity profiles taken at axial midspan of the blocked flow, where $z = 0$. Here, the full channel profile for the experimental flow is compared with the half-channel profile for pipe flow so that the zero shear stress occurs at $y/h = 1$ in either case. Similarly, $u_{\max}$ is the centerline velocity in the pipe flow and is the velocity u_s in the blocked flow. R_h for this flow is $u_s h/\nu$. The close correspondence of the distributions indicates, as did the computed C_f variation, that the full channel profile for the blocked flow corresponds closely to the half-channel profile for pressure flow. This illustrates also that the turbulence intensity is much reduced at the stationary wall ($y/h = 1$), thereby explaining why the shear stress is low on this wall, for an axial flow is superposed on this flow as has previously been pointed out.

In other published data⁽⁸⁾, it is seen that $u'/u_{\max}$ is much lower than the chosen function. This may be explained by the typically high Reynolds numbers used, where the coefficient of friction is much reduced below the values represented here. From Reynolds equations for turbulent flow, it

FIGURE 22. PROFILE OF TURBULENCE INTENSITY
AT AXIAL MIDSPAN NEAR A BLOCKAGE

follows that $C_f = 2\overline{u'v'}/u_{\max}^2$. Hence, if C_f is small, u' tends to be reduced likewise.

Pressure Measurements

Pressures to be measured were typically 10^{-5} psi and lower. Although there are instruments available which will, at best, resolve pressures of approximately this order, difficulties associated with their use suggested the development of a simpler and more rugged device for use in measuring the static pressure distribution around the bearing.

The device which was developed employs a halogen detector with its sensing element placed in a tube, connected to a static pressure tap on the bearing. A halogen-containing gas is injected into the tube, a known distance upstream of the sensing element. This is forced along the tube toward ambient by the pressure to be measured, and the time required for the halogen to reach the sensing element is recorded. Using the Poiseuille flow equation, the unknown pressure can then be computed.

A schematic of the complete system is shown in Figure 23, where halogen-containing gas (Freon 12 in this case) is admitted into the sample chamber by a regulator. The pressure difference ($p_1 - p_2$) then causes a flow, in the direction shown, down the tube and across the sensing element of the halogen detector. The pressure p_2 can be either atmospheric, for determining gage pressure p_1 , or it may also be a pressure in the bearing, for measurement of the gradient between two points.

The procedure for operating the device is best understood by referring to Figure 23. First, the three-way valve A is turned so that it blocks the

FIGURE 23. ARRANGEMENT FOR MEASURING VERY SMALL PRESSURES

flow path from the bearing, valves B and C are shut, and valve D is opened so as to charge the sample chamber with Freon. Next, valve D is shut and C is opened so that the sample chamber exhausts the Freon to ambient pressure. Now, valve C is shut, and A and B are turned simultaneously so that the Freon sample begins to move in the tube toward the halogen detector. At this point, a stopwatch is started, to be stopped when a response is registered by the halogen detector.

The equation for relating the time-to-response of the halogen detector to the static pressure in the bearing is derived as follows. The Poiseuille equation for laminar flow in a tube may be written

$$\Delta p = 8\mu \frac{\ell}{r_t^2} \bar{u} \quad (14)$$

Since the time for the fluid to traverse the distance ℓ is $t = \ell / \bar{u}$, it follows that

$$\Delta p = 8\mu \frac{\ell^2}{r_t^2} \frac{1}{t} \quad (15)$$

where

Δp - pressure drop across the tube

μ - viscosity of fluid

ℓ - length of tube

r_t - radius of tube

$\bar{u}$ - average flow velocity in the tube

That the assumptions are indeed valid has been checked experimentally, and this has been reported separately⁽⁹⁾.

Pressure in a Short Journal Bearing. In Figure 23, the dimensionless pressure P is plotted for two values of Reynolds number. The data were taken on the large-scale apparatus, with the journal eccentric in the bearing cylinder, in the same manner as for the previous studies of displacement thickness and turbulence intensity. The direction of journal turning is indicated in the figure, the position of minimum film thickness is taken as the origin for angular measurements, and $h/\bar{h}$ is 0.18 at this point. The maximum value of $h/\bar{h}$ is 1.67. The measurements appear to be well correlated by the dimensionless representation of pressure even though actual pressure varied over a ratio of nearly 2 to 1 for the two journal speeds.

Measurements on Tilted Pads. Segments of the bearing surface were removed to provide a single pad 36 in. long and 20 in. wide. This was tilted by the same means that produced varied eccentricity in the above measurements.

For the measurements reported in Figure 24, the pad was pivoted about the point $x/\ell = 0.734$ where the film thickness was maintained at the reference value $\bar{h} = 0.54$ inch. The film thickness was 1.0 in. at the leading edge, and 0.26 in. at the trailing edge. Pressures were measured at the axial midspan of the pad and along its length for two values of mean Reynolds number $\bar{R} = 2400$ and 1200. Dimensionless pressure, $P = p/\rho u_{bc}^2$, is plotted against the dimensionless position from the leading edge x/ℓ . The pressure is positive at all points and appears to peak at or near the pivot point.

End leakage was restricted by attaching sheet metal shields to restrict the axial flow and, for the case in Figure 24, did not completely prohibit this flow. Nevertheless, the figure illustrates that the dimensionless presentation still serves to draw the data together and that the leading edge pressure is of major significance in either case.

Figure 25 shows similar results for two angles of tilt and for more carefully restricted amounts of end leakage. Again, the predominant effect of leading edge "ram" pressure is illustrated. As indicated, this pressure decreased as tilt was decreased. When the tilt was reduced to zero (concentric operation), the leading edge pressure became negative, dropping to values of $P = -0.02$. This was true whether or not a sheet metal scoop was attached to the leading edge of the plate which served as the pad.

Velocity measurements were made near the leading edge, and it was found that the concept of a fixed dimensionless stagnation pressure at entry would serve to correlate the data, as illustrated in Figure 26, and the magnitude for best fit is $P_o = 0.34$, as indicated.

Pressure Measurements on a Pad Bearing with Blocked Flow. As has been discussed along with the velocity profile and turbulence measurements, a barrier was placed in the 0.54-in. thick film. This blocked the flow in the tangential direction and forced the air to divert and pass out the ends of the bearing clearance. To study the midstream pressure variations associated with this, the pressure measuring instrument was connected to a static pressure tap at fixed positions behind the leading edge of the pad, the barrier was set at various positions downstream of the points of pressure measurement

FIGURE 24. PRESSURE VARIATION FOR TILTED PADS FOR UNRESTRICTED AND PARTIALLY RESTRICTED END LEAKAGE

FIGURE 25. PRESSURE VARIATION FOR TWO TILT ANGLES, FOR A PAD WITHOUT END LEAKAGE

FIGURE 26. VARIATION OF PAD-ENTRY PRESSURE WITH INLET VELOCITY

and the pressure was obtained for each position of the barrier. This was done for eight different placements of the pressure measuring device and at a Reynolds number of $\bar{R} = 2400$. The results are shown in Figure 27, where the gage pressure p is nondimensionalized and plotted against the position of the barrier relative to the point of pressure measurement. All pressures are measured at the axial midspan of the bearing. The magnitude of the maximum pressure is not greatly dependent upon the location of the pressure tap with respect to the leading edge. Furthermore, when the barrier is very close to the leading edge, the maximum pressure is close to the 0.34 value of Figure 26.

Measurements on Stepped Pads. A stepped pad was created in the large-scale turbulent film apparatus by cementing a 1/4-in. thick sheet of rubber to the after half of the surface of the previously used tilting pad. When this was placed concentric with the rotating drum, it made a stepped pad for which the first half of the circumferential length had a film thickness almost double that of the second half. Pressure measurements were made at three points along the thicker film and at five points along the thinner film, these measurements being on the centerline in the axial direction. For the data shown in Figure 28, shields were not applied to the ends of the film, so this represents a short stepped-pad with end leakage. The pressure rise ahead of the step is analogous to that for blocked flow as shown in Figure 27.

Pressure is seen to drop to negative magnitudes at the step and rise back toward zero in the thin film portion under the pad. This is believed to be due almost entirely to inertial effects. One may ask (since inertial effects

FIGURE 27. VARIATION OF PRESSURE WITH POSITION OF BARRIER IN BLOCKED FLOW

FIGURE 28. PRESSURE VARIATION UNDER A SHORT STEPPED PAD

FIGURE 29. PRESSURE VARIATION UNDER A STEPPED PAD WITHOUT END LEAKAGE

are so important) why there was not a corresponding leading-edge pressure rise as encountered in the tilted pad. This may be explained by noting that the velocity u_b at entry was very near to u_{bc} , thus calling for a very small pressure rise.

When axial leakage was blocked by shields, the pressure distribution was as in Figure 29, showing a raised leading edge pressure and a correspondingly raised pressure maximum. In this case, the entry velocity was dropped considerably below u_{bc} .

Measurements in the Spiral-Groove Configuration

The spiral-groove configuration has been investigated as part of the general program of simulation of different bearing or seal configurations in a large-scale turbulence apparatus. Because of its many differences from the other configurations discussed, it is dealt with separately here. As will be shown, the measurements demonstrate the overwhelming influence of inertial effects. They also give indication of regions where the pressure falls significantly below ambient. When these negative pressures are sufficiently great in liquid-lubricated bearings or seals, the film will cavitate and disrupt, and the elements will cease to perform as intended. Whereas the numerical magnitudes of the results here are, to an extent, specific to the apparatus, they nevertheless point up design limitations additional to those previously applied to such configurations. The results further suggest that any analyses of turbulent screw-thread seals and spiral-groove bearings may be quite far removed from reality unless they take account of the inertial effects at discontinuities. In addition to dramatizing the effects of the steps,

the present work also gives some insights into how to deal with the changes which occur across these discontinuities.

Modification of Apparatus. The general configuration of the bearing in the large-scale turbulence apparatus has been described previously. This apparatus was based on a 6-ft diameter wheel, 20 in. long. Above this surface, which could be turned at speeds as high as 100 rpm, various stationary bearing configurations could be supported, including tilted pads, stepped pads, spiral grooves and the full bearing. End leakage could be restricted by plastic curtains, thus making possible the simulation of long or short bearings, depending upon the presence or absence of the curtains. Using air as the working fluid, Reynolds number in excess of $hu_s/2\nu = R = 2500$ was possible, where h is film thickness, u_s is journal surface speed and ν is kinematic viscosity. This is well within the turbulent regime.

For the present experiments, a cylindrical bearing was placed about the journal, providing an annular clearance of 0.54 inch. Spiral raised areas (made of 1/4-in. thick rubber sheets) were cemented to the stationary inner surface of the bearing to provide the lands of the spiral configuration. This is illustrated in Figure 30. Looking down on the developed surface of the bearing, one sees the rubber sections making an angle of 45° with the direction of turning. One end of the gas film is blocked against axial leakage by a seal made of a plastic sheet which rubs against the edge of the journal. The other end is open. Locations for pressure taps (or velocity probes) are illustrated in the figure. In the "groove" regions, the film thickness was 0.54 in.; in the regions over the "lands," it was approximately 0.27 in.; these dimensions

FIGURE 30. LAYOUT OF EXPERIMENTAL GROOVE AND LAND ARRANGEMENTS, SHOWING LOCATION OF PRESSURE TAPS

prevailed when the bearing was held concentric with the journal. Upon displacement of the bearing relative to the journal, the relative dimensions could be shifted locally. In the following discussion, the 0.54-in. clearance will be spoken of as 1/2 in., and 0.27 in. will be termed 1/4 inch.

Pressures were measured by the tracer technique already described. Both to generalize these and to convert them to a meaningful scale, these pressures had been nondimensionalized ($P = p / \rho u_{bc}^2$). Here, P is the dimensionless pressure, p is the actual measured pressure (psi), ρ is the fluid density, and u_{bc} is half the linear speed of the journal surface. The pressures actually measured were very small (sometimes as low as 10^{-5} psi). They were observed by attaching a capillary tube to the appropriate pressure tap and timing the rate of movement of a tracer gas (fluorinated hydrocarbons) down the length of the tube. This arrangement was fully calibrated and shown to be reproducible in its measurements. Because the pressure was low, density changes were negligible, and the gaseous fluid would be expected to simulate an incompressible liquid.

On the stated dimensionless basis, the distribution of P would be the same for different sizes of bearing provided dimensional similitude and Reynolds number were maintained constant. Thus, the same pressure distribution would be expected to prevail for small bearings geometrically similar to the large-scale bearing. Furthermore, it has been demonstrated that the dimensionless pressure distribution would not change greatly with changes of Reynolds number once in the turbulent regime⁽¹⁰⁾. This was true even

though the measured pressure, p , varied over a ratio of 4:1 in the reported experiments.

Dimensionless Pressure Variation. Figure 30 shows the distribution of pressure tap holes relative to the land and groove. Subsequent diagrams show magnitudes of pressure, velocities, and other variables so arrayed that the magnitude of the measured quantity appears over the location of the measuring point.

Figures 31, 32, and 33 show pressure distribution on the stated dimensionless basis, for three eccentricities. In Figure 31 (for concentric operation), one finds that pressures are predominantly negative in the thin-film regions and are of the order of 0.1. In Figure 32, the bearing has been displaced toward the journal, keeping the step height constant but thinning the local film throughout by approximately 1/8 inch. Again, pressures over the land are predominantly negative, going to even lower values than for concentric operation. In Figure 33, the bearing has been displaced away from the journal by approximately 1/8 in. from concentric operation, and the magnitudes of the negative pressures are reduced. The pressure distributions in all three drawings were measured at a Reynolds number $R = 2300$, where R is based upon the nominal 0.5-in. film thickness and a velocity equal to half the turning speed of the wheel. On the basis of prior experimentation, the results would be within experimental scatter, over a range of $1000 < R < 5000$.

The three pressure distributions shown differ considerably from those reported for laminar flow⁽¹¹⁾. In that case, pressure varies in nearly linear segments, rising in the direction of turning in the thick-film regions and

FIGURE 31. DIMENSIONLESS PRESSURE, P ,
FOR CONCENTRIC OPERATION

FIGURE 32. DIMENSIONLESS PRESSURE, P, FOR DISPLACEMENT TOWARD JOURNAL

FIGURE 33. DIMENSIONLESS PRESSURE, P , FOR
DISPLACEMENT FROM JOURNAL

falling in the direction of turning in the thin-film regions. In the present case, the effect is more nearly a "square wave" with sharp drops and rises at the discontinuities.

In Figure 34, the maximum negative pressure has been taken from each of the three distributions and plotted against film thickness ratio. To illustrate the significance of this, assume that the film thickness ratio is approximately 0.5 and P is approximately 0.09. If the fluid had a density equivalent to water (2 slugs/ft^3), then the velocity, u_s , at which the actual pressure would reach -15 psi would be 220 ft/sec. This is not a particularly high velocity in many machinery applications. Furthermore, if the configuration were used as a bearing and radial displacement took place under load, the negative pressure would be augmented at any given speed, and the permissible velocity would be decreased. One can visualize such a bearing operating without difficulty under a light load and then, under a disturbance, displacing and ultimately causing disruption of the film.

Load-Carrying Capacity. The pressure distribution in Figures 32, 33, and 34 can be used to provide an estimate of load-carrying capacity when each pressure is assumed to attribute to a parallelogram drawn about the point with boundaries equally spaced between adjacent rows of pressure taps. In this way, an area-weighted average has been obtained for the pressure, and this is plotted in Figures 35 and 36.

Figure 35 shows the dimensionless pressure as a function of ratio of minimum to maximum film thickness and indicates that there is a condition of maximum load support. Should the ratio $h_{\min}/h_{\max}$ be reduced below

FIGURE 34. EFFECT OF FILM THICKNESS RATIO ON
MAXIMUM NEGATIVE PRESSURE

FIGURE 35. EFFECT OF FILM THICKNESS RATIO ON MEAN PRESSURE, $\bar{P}$

FIGURE 36. EFFECT OF MINIMUM CLEARANCE ON
MEAN PRESSURE, P

approximately 0.5, load support decreases. Decrease of load support as the surfaces approach one another would represent the conditions of negative-spring or instability. This effect is more clearly demonstrated in Figure 36 where the minimum film thickness is expressed in ratio to journal radius, and again it is seen that, if this thickness becomes too small, the bearing becomes unstable (i. e., the net load support drops as the surfaces approach one another). This is because the negative pressure in the thin-film regions begins to predominate under such a circumstance. If the spiral grooves were made long enough in the axial direction, there is reason to believe that pressure-gradient effects might dominate and shift the peak to a lower magnitude. Put differently, note that the test bearing is relatively short and pressures are dominated by end conditions. Such end conditions may not be so significant in the overall load support in a long bearing. They would nevertheless remain important in giving rise to negative pressures in the film. In any event, the observations reported here must serve as a warning regarding the stability of the spiral-groove bearing under turbulent operating conditions since the high Reynolds numbers, which lead to turbulence, lead likewise to inertial effects.

Velocity Measurements. A hot-wire anemometer was inserted into each of the pressure holes and was used to measure the velocity at midchannel (i. e., midway between the bearing surfaces). It was also used to obtain a rough estimate of the direction of the velocity but could not resolve this better than within 10° . In this latter capacity, it is indicated that the flow was essentially tangential at all of the ports except possibly the one at the lower

right-hand corner of the array in Figure 30. Typical velocities are shown in Figure 37, expressed in ratio to u_{bc} , as Q . When $Q = 1$, pure Couette flow would prevail in the space. From the figures, it is seen that $Q > 1$ in the thin-film portions, thereby indicating a displacement flow in the direction of turning. This is as would have been anticipated from the theory of the stepped pad where a similar augmentation of flow occurs in the thin-film region. Diminution occurs in the thick-film region where $Q > 1$. In Figure 38, the quantity $Q^2/2$ is tabulated, showing the order of magnitude of the changes of this factor.

It is of interest to compute a stagnation pressure which, in the dimensionless terminology, would be $P_o = P + \zeta Q^2/2$. This would represent the mean condition of the flow below any pressure tap at any position in the

array. Here, ζ is defined as $\zeta = \left(\int_0^h u^2 dy \right) / (u_b^2 h)$. The curve of Figure 39

has been computed from data consistent with measured turbulent velocity profiles and different magnitudes of the dimensionless Q . Using such values, the data in Figures 40 and 41 were computed. There is some scatter in the values reported, but this does not obscure the important relationship. For example, at the central step from thick film to thin film, the mean change of P_o is 0.055 or approximately 10%. On the other hand, in going from the thin-film region (upper edge of data array) to one corresponding to the lowest edge of the thick-film region, the change of P_o is 0.149, or nearly three times the above value. This indicates that flow into a convergence is like that into a nozzle. Flow into a divergence is like flow into an inefficient diffuser.

FIGURE 37. DIMENSIONLESS VELOCITY, q/u_{bc}

FIGURE 38. DIMENSIONLESS $Q^2/2$

FIGURE 39. VARIATION OF MOMENTUM FLUX COEFFICIENT

FIGURE 40. DIMENSIONLESS MOMENTUM FLUX INTENSITY, $Q^2/2$

FIGURE 41. DIMENSIONLESS TOTAL PRESSURE, $P + \frac{1}{2}\rho Q^2/\rho$

Effect of Step Shape. The steps in the large-scale turbulence apparatus were almost true geometric discontinuities, while the tiny steps in an actual bearing must undoubtedly be rounded. Consequently, a separate set of measurements was called for to determine the effects of step-abruptness. These were carried out for steps set normal to the flow, and the results are illustrated in Figure 42 where the change across a vertical step and a 45° beveled step is indicated in the figure. For converging flow over the step, there was no measured change of pressure drop. For diverging flow, the pressure recovery was greater for the 45° step, thereby indicating some "diffuser" effect. On the other hand, neither divergence led to a recovery in excess of $1/3$ of the drop at a convergence. In this figure, the pressure magnitudes are shown to four significant figures, although they were not accurate to this degree.

Continuity Considerations. In discussing the measured velocities, it was noted that the flow was nearly tangential at each measuring point. Recognizing that the volume flow in any section of channel is approximately VH , it is seen that the velocity changes across the step cannot be reconciled with conservation-of-volume if the restriction of no axial movement is held rigid. For example, before the convergence, $QH \approx 0.75$; just after it, $QH \approx 0.5$. Further reflection provides a reconciliation of the observation with other factors. It requires that the condition of absolute-tangential flow be relaxed in the neighborhood of the step. If this drop occurred all along the step, it would necessitate a large tangential pressure gradient. Because the step is inclined to the axis of the cylinder, this would also necessitate a large axial pressure gradient and tend to give rise to an axial flow. On the other

$$P_1 - P_2 = 0.369$$

$$P_1 - P_2 = 0.369$$

$$P_2 - P_1 = 0.1015$$

$$P_2 - P_1 = 0.1122$$

FIGURE 42. EFFECT OF STEP ABRUPTNESS ON PRESSURE CHANGE

hand, if the velocity component tangential to the step is preserved, and velocity changes occur only normal to the step, there will be no pressure gradients along the step, and a self-consistent condition is predicted.

Figure 43 shows a case where a unit velocity to the left is reduced on entering a region of increased film thickness. The component of this velocity normal to the step is indicated as 0.707. If this is foreshortened by one-half in passing through, then the normal component becomes 0.35, and the tangential component remains 0.707. The overall velocity becomes 0.79. These velocities correspond roughly to those measured in the experimental arrangement.

In Figure 44, the opposite situation is shown where fluid moves from left to right going from thick film to thin film with an increase in velocity normal to the step. Here, the velocity normal to the step is 0.5 and increases to unity. The tangential component is 0.5 in either case, and the magnitude of the overall velocity vector is increased to 1.1. The angular change is not great and would be less noticeable for the actual case which appears to be between these two examples. After all, one would expect that outward fluid movement in one zone must be balanced by inward movement in the next. In essence, this means that there is a shift of the velocity pattern downward in one case and upward in the other, undoubtedly with standing vortices compensating for this shift in the regions near the end where the barrier is placed. At the open end, compensation is provided simply by movement of the fluid into or out of the clearance space.

FIGURE 43. EFFECT OF STEP UPON VELOCITIES, ON MOVING FROM THIN FILM TO THICK FILM

FIGURE 44. EFFECT OF STEP UPON VELOCITIES, ON MOVING FROM THICK FILM TO THIN FILM

Analysis of Pressure Jumps at Discontinuities. To obtain further insights into the pressure jump phenomena, let v_n be the velocity normal to a step and h be film thickness. Let the flow be from thin film, indicated by subscript ()_a, to a thick film, indicated by subscript ()_b as illustrated in Figure 45a. Continuity requires

$$v_{na}h_a = v_{nb}h_b \quad (16)$$

Momentum conservation applied to the mixing zone in the enlarged region requires

$$\rho v_{na}^2 h_a - \rho v_{nb}^2 h_b = (p_b - p_a)h_b \quad (17)$$

Introducing the nondimensionalization $V_{na} = v_{na}/u_{bc}$, $V_{nb} = v_{nb}/u_{bc}$, $(P_b - P_a)/\rho u_{bc}^2 = \Delta P$

$$V_{na}^2 \left(\frac{h_a}{h_b} \right) - V_{nb}^2 = \Delta P \quad (18)$$

Drawing upon the continuity equation,

$$V_{na}^2 \left(\frac{h_a}{h_b} \right) - V_{na}^2 \left(\frac{h_a}{h_b} \right)^2 = \Delta P \quad (19)$$

If, for example, $V_{na} = 0.707$ and $h_a/h_b = 1/2$, then $\Delta P = 0.125$. Had the process been one of a perfect diffuser, the ideal pressure rise, ΔP_i , would have been:

FIGURE 45a. ILLUSTRATION OF FLOW NEAR A DIVERGENCE OF FILM THICKNESS

$$\Delta P_i = \frac{V_{na}^2}{2} - \frac{V_{nb}^2}{2} = \frac{V_{na}^2}{2} \left[1 - \left(\frac{h_a}{h} \right) \right] \quad (20)$$

For the given conditions, then, $\Delta P_i = 0.375$.

An expression for the "loss" in mixing is derived by subtracting the actual pressure rise ΔP from the ideal ΔP_i , giving

$$\Delta P_{loss} = \frac{V_{na}^2}{2} \left[1 - \left(\frac{h_a}{h_b} \right) \right]^2 \quad (21)$$

or in terms of downstream velocity

$$\Delta P_{loss} = \frac{V_{na}^2}{2} \left(\frac{h_b}{h_a} - 1 \right)^2 \quad (22)$$

This provides a rough estimate of the loss of stagnation pressure in the experimental configuration.

The flow process at convergence is illustrated in Figure 45b. Here, it is seen that separation occurs at the convergence, and a vena contracta forms. Following this, the mixing process is similar to that above.

To use similar reasoning to the above in calculations would require knowledge of the contraction coefficient. Experience with orifices in pipes indicates this to be dependent upon the ratio of approach area to orifice area. Because of this uncertainty, predictions of loss will be widely variant. For the present, the measurements reported here can serve as a guide, however, as to what would be reasonable to expect in overall behavior at a convergence.

FIGURE 45b. ILLUSTRATION OF FLOW NEAR A
CONVERGENCE OF FILM THICKNESS

Conclusion

The topics treated here shed light on many different aspects of the turbulent film bearing problem, some serving to provide fundamental information to guide calculations and some giving sample data with which to compare calculated results.

With regard to velocity profile data, it is apparent that the thick-film flow (with early transition) has velocity profiles corresponding to plane Couette flows of much higher Reynolds number. Wall-law predictions fall between these two sets of data. Since typical bearing flows would be expected to reside between plane flow and that of the present experiment, it appears that δ^*/b predicted by the wall law does serve as a valid approximation.

Furthermore, the velocity profiles show that transition is not completely determined by local R or by the mean $\bar{R}$ of the film. Apparently, the thick-film portions of an eccentric flow reach transition early and transmit turbulence to the thin-film portions. It is possible that, with smaller $\bar{h}/r$ ratio, turbulence would be damped in transit and local conditions would become the more important. On the other hand, if analyses are restricted to regions far from transition, the data tend to merge toward the wall-law values, and become relatively insensitive to Reynolds number, thus eliminating the questions raised by the data.

The friction coefficient measurements show values higher than for wall-law flow or plane Couette flow. This is consistent with the smaller δ^*/b values which suggests that, at a given R , the experimental flow will have velocity gradients at the wall greater than in the plane flow. Withal,

the C_f values are not far from those for plane-Couette flow in the range where both flows are turbulent, and data for curved, but thinner films would be expected to lie between these.

The turbulence intensity measurements show interesting comparisons between the reported flows and pipe flow (pressure flow). They support the concept that turbulence is transported tangentially and verify that the quantity $(\overline{u'}/\overline{u})^2 h/\overline{h}$ is of secondary importance in determining the pressure gradients in the flow.

The pressure measurements provide data on both short bearings and simulated long bearings, where end leakage is blocked. In both cases, they provide useful data for checking theoretical analyses. An outstanding feature of the data is the large pressure discontinuity at a blockage, step, or entry. This discontinuity seems to be accounted for by a dimensionless stagnation pressure $P_0 = 0.34$, which is a consequence of the velocity-pressure of the flow carried by the journal prior to entry into the bearing. If the velocity of this flow were u_s (the journal surface speed), the magnitude of P_0 would be 2. The experimental value, however, corresponds to a velocity $u < u_{bc}$. This velocity is apparently the result of vortices peeling off from the journal surface and transferring momentum to the ambient fluid. The P_0 resulting from this effect would appear to provide a useful parameter in bearing analyses for the case of flooded entry.

The dimensionless pressure P has served to correlate the measurements, where $P = p/\rho u_{bc}^2$, and is significantly different from that used in laminar flow bearing calculations. Aside from the effectiveness it has shown

in bringing together the reported experimental results, its use may be justified by the fact that the surface tractions, as well as pressure rises due to velocity changes, both tend to be proportional to the square of velocity in the fully turbulent regime.

The pressure measurements show vividly the most important differences between turbulent-film operation and laminar-film operation. For the tilted and stepped pads, the pressure jumps at discontinuities dominate the entire pressure distribution. In the turbulent, short full bearing, the combined turbulence and inertial effects conspire to cause the pressure distribution to tend to increase shaft displacement rather than oppose it. In the experimental configuration, the lubricant film would be a negative spring, and the bearing would be unstable.

For a theory of turbulent film analysis to be valid, it must encompass these experimental cases and be able to predict the unusual pressure effects that have been measured.

In the study of orthogonal Couette and pressure flow, velocity profiles were generated for an axial pressure (or displacement) flow and a tangential Couette flow. Despite some scatter in the measurements, they clearly indicate:

- (1) The presence of the axial flow reduces the tangential wall shear stress.
- (2) The midstream slope of the tangential velocity profile is significantly smaller than that of a corresponding Couette flow acting alone.
- (3) The midstream eddy viscosity is reduced below that which would prevail for the wall-law eddy-viscosity distribution based upon the total wall shear stress.

The measurements on the spiral-groove configuration demonstrate clearly that the pressure pattern is dominated by the "inertial" jumps at the steps of film thickness. They show how load support in a film varies with changes in the spacing between journal and bearing. The results also give warning of film breakup due to excessive "negative" pressure, which may explain observed failure of herringbone or screw thrust seals when taken to high operating speeds.

DERIVATION OF EQUATIONS FOR THIN LUBRICANT FILMS
WITH END LEAKAGE

The Fundamental Equations of an Incompressible Turbulent Fluid

The momentum and continuity equations for an incompressible fluid under a steady-state condition are⁽¹²⁾:

$$\begin{aligned} \bar{u} \frac{\partial \bar{u}}{\partial x} + \bar{v} \frac{\partial \bar{u}}{\partial y} + \bar{w} \frac{\partial \bar{u}}{\partial z} = & -\frac{1}{\rho} \frac{\partial \bar{p}}{\partial x} + \nu \left(\frac{\partial^2 \bar{u}}{\partial x^2} + \frac{\partial^2 \bar{u}}{\partial y^2} + \frac{\partial^2 \bar{u}}{\partial z^2} \right) \\ & - \left(\overline{\frac{\partial u'^2}{\partial x}} + \overline{\frac{\partial u'v'}{\partial y}} + \overline{\frac{\partial u'w'}{\partial z}} \right) \end{aligned} \quad (23)$$

$$\begin{aligned} \bar{u} \frac{\partial \bar{v}}{\partial x} + \bar{v} \frac{\partial \bar{v}}{\partial y} + \bar{w} \frac{\partial \bar{v}}{\partial z} = & -\frac{1}{\rho} \frac{\partial \bar{p}}{\partial y} + \nu \left(\frac{\partial^2 \bar{v}}{\partial x^2} + \frac{\partial^2 \bar{v}}{\partial y^2} + \frac{\partial^2 \bar{v}}{\partial z^2} \right) \\ & - \left(\overline{\frac{\partial u'v'}{\partial x}} + \overline{\frac{\partial v'^2}{\partial y}} + \overline{\frac{\partial v'w'}{\partial z}} \right) \end{aligned} \quad (24)$$

$$\begin{aligned} \bar{u} \frac{\partial \bar{w}}{\partial x} + \bar{v} \frac{\partial \bar{w}}{\partial y} + \bar{w} \frac{\partial \bar{w}}{\partial z} = & -\frac{1}{\rho} \frac{\partial \bar{p}}{\partial z} + \nu \left(\frac{\partial^2 \bar{w}}{\partial x^2} + \frac{\partial^2 \bar{w}}{\partial y^2} + \frac{\partial^2 \bar{w}}{\partial z^2} \right) \\ & - \left(\overline{\frac{\partial u'w'}{\partial x}} + \overline{\frac{\partial v'w'}{\partial y}} + \overline{\frac{\partial w'^2}{\partial z}} \right) \end{aligned} \quad (25)$$

and

$$\frac{\partial \bar{u}}{\partial x} + \frac{\partial \bar{v}}{\partial y} + \frac{\partial \bar{w}}{\partial z} = 0 \quad (26)$$

In the above equations, the quantities with superscript bar and prime indicate time average and fluctuating components, respectively.

Let the bearing film thickness lie along the y-axis. For very thin film,

the simplifications in view of order of magnitude may be made as follows⁽¹³⁾:

$$\frac{\partial^2 \bar{u}}{\partial x^2}, \frac{\partial^2 \bar{u}}{\partial z^2}, \frac{\partial^2 \bar{w}}{\partial x^2}, \frac{\partial^2 \bar{w}}{\partial z^2} \ll \frac{\partial^2 \bar{u}}{\partial y^2}, \frac{\partial^2 \bar{w}}{\partial y^2} \quad (27)$$

$$\overline{\frac{\partial u'^2}{\partial x}}, \overline{\frac{\partial u'w'}{\partial z}}, \overline{\frac{\partial u'w'}{\partial x}}, \overline{\frac{\partial w'^2}{\partial z}}, \overline{\frac{\partial u'v'}{\partial x}}, \overline{\frac{\partial v'w'}{\partial z}} \ll \overline{\frac{\partial u'v'}{\partial y}}, \overline{\frac{\partial v'w'}{\partial y}}, \overline{\frac{\partial v'^2}{\partial y}} \quad (28)$$

and

$$\frac{\partial \bar{p}}{\partial y} = 0 \quad (29)$$

Here the frictional and inertial forces in the y-direction are neglected. By means of Equations (27) to (29) and omitting superscript bar over the quantities, Equations (23) to (26) then reduce to

$$\rho \left[u \frac{\partial u}{\partial x} + v \frac{\partial u}{\partial y} + w \frac{\partial u}{\partial z} \right] = - \frac{\partial p}{\partial x} + \frac{\partial \tau_x}{\partial y} \quad (30)$$

$$\rho \left[u \frac{\partial w}{\partial x} + v \frac{\partial w}{\partial y} + w \frac{\partial w}{\partial z} \right] = - \frac{\partial p}{\partial z} + \frac{\partial \tau_z}{\partial y} \quad (31)$$

and

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = 0 \quad (32)$$

where τ_x and τ_z are shearing stresses in the x- and z-directions, respectively.

Making use of Equation (32), the following identities may be obtained as

$$u \frac{\partial u}{\partial x} + v \frac{\partial u}{\partial z} + w \frac{\partial u}{\partial z} \equiv \frac{\partial u^2}{\partial x} + \frac{\partial(uw)}{\partial z} + \frac{\partial(uv)}{\partial y} \quad (33)$$

and

$$u \frac{\partial w}{\partial x} + v \frac{\partial v \partial w}{\partial y \partial z} + w \frac{\partial w}{\partial z} \equiv \frac{\partial w^2}{\partial z} + \frac{\partial(uw)}{\partial x} + \frac{\partial(wv)}{\partial y} \quad (34)$$

Substituting Equations (33) and (34) into Equations (30) and (31), respectively, and integrating them across the film thickness $h = h(x, z)$, the momentum equations may be simplified to

$$\rho \left[\int_0^h \frac{\partial u^2}{\partial x} dy + \int_0^h \frac{\partial(uv)}{\partial y} dy + \int_0^h \frac{\partial(uw)}{\partial z} dy \right] = - \frac{\partial p}{\partial x} h + \tau_{xh} - \tau_{xo} \quad (35)$$

$$\rho \left[\int_0^h \frac{\partial w^2}{\partial z} dy + \int_0^h \frac{\partial(vw)}{\partial y} dy + \int_0^h \frac{\partial(uw)}{\partial x} dy \right] = - \frac{\partial p}{\partial z} h + \tau_{zh} - \tau_{zo} \quad (36)$$

where

$$\tau_{zh} \equiv \mu \left. \frac{\partial u}{\partial y} \right|_{y=h} - \rho \overline{u'v'} \Big|_{y=h} \quad (37)$$

$$\tau_{zo} \equiv \mu \left. \frac{\partial u}{\partial y} \right|_{y=0} - \rho \overline{u'v'} \Big|_{y=0} \quad (38)$$

$$\tau_{zh} \equiv \mu \left. \frac{\partial w}{\partial y} \right|_{y=h} - \rho \overline{v'w'} \Big|_{y=h} \quad (39)$$

and

$$\tau_{zo} \equiv \mu \left. \frac{\partial w}{\partial y} \right|_{y=0} - \rho \overline{v'w'} \Big|_{y=0} \quad (40)$$

Similarly, integrating Equation (32) across the film thickness, the continuity equation is obtained:

$$\int_0^h \frac{\partial u}{\partial x} dy + \int_0^h \frac{\partial v}{\partial y} dy + \int_0^h \frac{\partial w}{\partial y} dy = 0 \quad (41)$$

Three equations [Eqs. (35), (36) and (41)] with seven dependent variables (u, v, w, p, u', v', and w') are impossible to solve. Consequently, further simplification of these three equations is necessary.

In terms of three dependent variables (Δu_b , w_b and p), three film equations can be handled either numerically or analytically.

Further Simplification of Equations (35) to (41)

Because of the emphasis upon friction coefficient (C_{xc}) and velocity profile parameters (n in x-direction and m in z-direction) in the

experimental program, it has been found desirable to carry through an analytical treatment of thin bearing film, in terms of these three quantities (n , m and C_{xc}). Introduce the definition

$$C_f = \frac{2|\tau|}{\rho u_b^2} \quad (42)$$

such that C_f is defined relative to the wall on which the shear stress is to be measured, and u_b is defined as midchannel velocity relative to that wall.

Let $h = h(x, z)$ and have continuous derivatives for $x_1 < x < x_2$ and $z_1 < z < z_2$. In addition, let $u(x, z)$, $w(x, z)$, $u^2(x, z)$, $w^2(x, z)$, and wu be continuous and have continuous derivatives $\frac{\partial u}{\partial x}$, $\frac{\partial u^2}{\partial z}$, $\frac{\partial w}{\partial z}$, $\frac{\partial w^2}{\partial z}$, $\frac{\partial uw}{\partial x}$, $\frac{\partial uw}{\partial z}$ in a domain of the x - z plane. Let the bearing problem be a case of $u_o = u_s$, $u_h = u_h$, $w_o = 0$ and $w_h = 0$; and $v_o = V$, $v_h = 0$. Then, by means of Leibnitz's rule, Equations (35), (36) and (41) may be written as

$$\begin{aligned} \rho \left[\left\{ \frac{\partial}{\partial x} \int_0^h u^2 dy - u_h^2 \frac{\partial h}{\partial x} \right\} + \left\{ \frac{\partial}{\partial z} \int_0^h w u dy - w_h u_h \frac{\partial h}{\partial z} \right\} + \left\{ u_h v_h - u_o v_o \right\} \right] \\ = - \frac{\partial p}{\partial x} h + \tau_{xh} - \tau_{xo} \end{aligned} \quad (43)$$

$$\begin{aligned} \rho \left[\left\{ \frac{\partial}{\partial z} \int_0^h w^2 dy - w_h^2 \frac{\partial h}{\partial z} \right\} + \left\{ \frac{\partial}{\partial x} \int_0^h u w dy - u_h w_h \frac{\partial h}{\partial x} \right\} + \left\{ v_h w_h - v_o w_o \right\} \right] \\ = - \frac{\partial p}{\partial z} h + \tau_{zh} - \tau_{zo} \end{aligned} \quad (44)$$

$$\frac{\partial}{\partial x} \int_0^h u dy - u_h \frac{\partial h}{\partial x} + \frac{\partial}{\partial z} \int_0^h w dy - w_h \frac{\partial h}{\partial z} = v_o - v_h \quad (45)$$

Because $w_o = v_h = w_h = 0$, Equations (43) and (44) with $u_h = 0$, are reduced to:

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$$\begin{aligned}
|Q_{bh}| &= \left[(u_{bh})^2 + (w_{bh})^2 \right]^{1/2} \\
&= u_{bc} \cdot \left[\left(1 + \frac{\Delta u_b}{u_{bc}} \right)^2 + \left(\frac{w_b}{u_{bc}} \right)^2 \right]^{1/2}
\end{aligned} \tag{55}$$

$$\begin{aligned}
|Q_{bo}| &= \left[(u_{bo})^2 + (w_{bo})^2 \right]^{1/2} \\
&= u_{bc} \cdot \left[\left(1 - \frac{\Delta u_b}{u_{bc}} \right)^2 + \left(\frac{w_b}{u_{bc}} \right)^2 \right]^{1/2}
\end{aligned} \tag{56}$$

In the discussion of the experimental results, it was shown that, over a large range of conditions the velocity profile in the half-channel is very close to the wall-law profile in shape. If one takes the midchannel velocity relative to a given wall, it will be approximately relative to a given wall shear stress which acts upon the fluid in opposition to this vector Q , and the stress magnitude can be expressed in terms of a friction coefficient C_f according to:

$$|\tau_h| = C_{fh} \rho \frac{(Q_{bh})^2}{2} \tag{57}$$

$$|\tau_o| = C_{fo} \rho \frac{(Q_{bo})^2}{2} \tag{58}$$

$$\begin{aligned}
C_{fh} &= C_{xc} \left(\frac{R_{bc}}{R_{bh}} \right)^k \\
&= C_{xc} \left(\frac{u_{bc}}{|Q_{bh}|} \right)^k
\end{aligned} \tag{59}$$

$$C_{fo} = C_{xc} \left(\frac{u_{bc}}{|Q_{bo}|} \right)^k \tag{60}$$

Now drawing upon the assumption that the total wall shear stress acts along the direction of Q ,

$$|\tau_{xh}| = C_{fh}\rho \cdot \frac{|Q_{bh}|^2}{2} \cdot \frac{|u_{bh}|}{|Q_{bh}|} \quad (61)$$

$$|\tau_{xo}| = C_{fo}\rho \cdot \frac{|Q_{bo}|^2}{2} \cdot \frac{|u_{bo}|}{|Q_{bo}|} \quad (62)$$

$$|\tau_{zh}| = C_{fh}\rho \cdot \frac{|Q_{bh}|^2}{2} \cdot \frac{|w_{bh}|}{|Q_{bh}|} \quad (63)$$

$$|\tau_{zo}| = C_{fo}\rho \cdot \frac{|Q_{bo}|^2}{2} \cdot \frac{|w_{bo}|}{|Q_{bo}|} \quad (64)$$

For convenience in the derivations which follow, define the following dimensionless quantities:

$$\zeta_1 = \frac{\int_0^h u^2 dy}{u_b^2 h} \quad (65)$$

$$\zeta_2 \equiv \zeta_4 = \frac{\int_0^h w u dy}{w_b u_b h} \quad (66)$$

$$\zeta_3 = \frac{\int_0^h w^2 dy}{w_b^2 h} \quad (67)$$

$$\zeta'_7 = \frac{\int_0^h u dy}{u_b h} \quad (68)$$

and

$$\zeta'_8 = \frac{\int_0^h w dy}{w_b h} \quad (69)$$

The Difference of Wall Shears, $\Delta\tau_x$, in the Tangential Direction for Codirectional Couette and Pressure Flows, with Wall Shear with Same Signs (Couette Flow Dominating)

Refer to Figure 46 where relative velocities u_{bh} and u_{bo} are defined.

By definition, the quantity $\Delta\tau_x$ may be written as

$$\Delta\tau_x = -|\tau_{xh}| - (-1)|\tau_{xo}| \quad (70)$$

Substituting Equations (52), (53), (55), (56), (59) to (69) into (70), $\Delta\tau_x$

is found to be

$$\Delta\tau_x = -2\rho\zeta_5 C_{xc} u_{bc} \Delta u_b \quad (71)$$

where

$$\zeta_5 = \frac{A - B}{4U} \quad (0 \leq |U| \leq 1) \quad (72)$$

$$A \equiv \left| 1 + \frac{\Delta u_b}{u_{bc}} \right| \cdot \left| \left[\left(1 + \frac{\Delta u_b}{u_{bc}} \right)^2 + \left(\frac{w_b}{u_{bc}} \right)^2 \right]^{\frac{1-k}{2}} \right| \quad (73)$$

and

$$B \equiv \left| -1 + \frac{\Delta u_b}{u_{bc}} \right| \cdot \left| \left[\left(1 - \frac{\Delta u_b}{u_{bc}} \right)^2 + \left(\frac{w_b}{u_{bc}} \right)^2 \right]^{\frac{1-k}{2}} \right| \quad (74)$$

When $U \equiv \frac{\Delta u_b}{u_{bc}} \rightarrow 0$, the limit of ζ_5 in Equation (72) is:

FIGURE 46. WALL SHEAR WITH SAME SIGNS
(COUETTE EFFECT DOMINATING)

$$\lim_{U \rightarrow 0} \zeta_5 = \left(1 - \frac{k}{2}\right) \cdot \left| (1 + W^2)^{\frac{1-k}{2}} \right| - \frac{(1-k)W^2}{2} \cdot \left| (1 + W^2)^{\frac{-(1+k)}{2}} \right| \quad (75)$$

The Difference of Wall Shears, $\Delta\tau_x$, in the Tangential Direction for Codirectional Couette and Pressure Flows, with Wall Shear with Opposite Signs (Pressure Flow Dominating)

Lower Surface Positive Shear, Upper Surface Negative Shear. As

shown in Figure 47, wall shears appear to be opposite in sign.

The definition of u_{bh} , u_{bo} remains the same as in Figure 46 (a). Simi-

larly, $\Delta\tau_x$ is

$$\begin{aligned} \Delta\tau_x &= -|\tau_{xh}| - |\tau_{xo}| \\ &= -2\rho\zeta_5 C_{xc} u_{bc} \Delta u_b \end{aligned} \quad (76)$$

where

$$\zeta_5 = \frac{A + B}{4U} \quad (1 \leq U) \quad (77)$$

such that A and B in Equation (77) are in the same form as in Equations (73) and (74)

Lower Surface Negative Shear, Upper Surface Positive Shear. Using the

definition of $\Delta\tau_x$ and taking the sign of τ_{xh} and τ_{xo} into account, $\Delta\tau_x$

is (see Figure 48)

$$\begin{aligned} \Delta\tau_x &= |\tau_{xh}| - (-1)|\tau_{xo}| \\ &= -2\rho\zeta_5 C_{xc} u_{bc} \Delta u_b \end{aligned} \quad (78)$$

where

$$\zeta_5 = -\frac{(A + B)}{4U} \quad (U \leq -1) \quad (79)$$

FIGURE 47. COUETTE AND PRESSURE FLOW EFFECTS IN THE SAME DIRECTION FOR THE CASE OF U SUCH THAT $1 \leq U$

FIGURE 48. COUETTE AND PRESSURE FLOW EFFECTS IN THE OPPOSITE DIRECTION FOR THE CASE OF U SUCH THAT $U \leq -1$

such that A and B in Equation (79) are in the same form as in Equations (73) and (74).

Remarks. For the above three different cases, ζ_5 is obtained by Equations (72), (77) and (79). Each form for ζ_5 in Equations (72), (77) and (79) will be chosen in a corresponding bearing problem. Although $\Delta\tau_x$ in Equations (71), (76) and (78) is in the same form, the value of ζ_5 in Equations (72), (77) and (79) is different. Consequently, $\Delta\tau_x$ corresponding to each case is different.

The Difference of Wall Shears, $\Delta\tau_z$, in the Axial Direction for Either Positive z-Direction or Negative z-Direction Flow

As a first approximation, the velocity profile of w can be assumed as a turbulent power law velocity profile symmetrical about $y = b$ as shown in Figure 49.

In the positive z-direction, $\Delta\tau_z$,

$$\Delta\tau_z = -|\tau_{zh}| - |\tau_{zo}| \quad (80)$$

Substituting Equations (52) to (56), (59), (60), (63) and (64) into Equation (80), $\Delta\tau_z$ is obtained as

$$\Delta\tau_z = -\zeta_6 C_{xc} \rho u_{bc} w_b \quad (81)$$

where

$$\zeta_6 = (E + F)/2 \quad (82)$$

$$E = \left| \left[\left(1 + \frac{\Delta u_b}{u_{bc}} \right)^2 + \left(\frac{w_b}{u_{bc}} \right)^2 \right]^{\frac{1-k}{2}} \right| \quad (83)$$

FIGURE 49. TURBULENT VELOCITY PROFILE IN AXIAL DIRECTION

and

$$F = \left| \left[\left(1 - \frac{\Delta u_b}{u_{bc}} \right)^2 + \left(\frac{w_b}{u_{bc}} \right)^2 \right]^{\frac{1-k}{2}} \right| \quad (84)$$

In the negative z-direction, $\Delta\tau_z$ is

$$\begin{aligned} \Delta\tau_z &= \left| \tau_{zh} \right| - (-1) \left| \tau_{zo} \right| \\ &= -\zeta_6 C_{xc} \rho u_{bc} w_b \end{aligned} \quad (85)$$

where

$$\zeta_6 = (E + F)/2 \quad (86)$$

Comparing Equations (81) and (82) with (85) and (86), ζ_6 remains the same in either the positive or negative z-direction.

The Simplified Fundamental Thin Film Equations of Incompressible Turbulent Fluid

Considering Equations (70), (76), (78) and (81), and neglecting small quantities of $\frac{\partial \zeta_1}{\partial x}$, $\frac{\partial \zeta_2}{\partial z}$, $\frac{\partial \zeta_3}{\partial z}$, $\frac{\partial \zeta_4}{\partial x}$, $\frac{\partial(\zeta_7')}{\partial x}$, and $\frac{\partial(\zeta_8')}{\partial z}$, Equations (49) to (51) may be simplified to

$$\begin{aligned} & -\frac{U_o V_o}{\zeta_2 H} + \frac{\zeta_1}{\zeta_2} \left[2(1+U) \frac{\partial U}{\partial X} + \frac{(dH/dX)}{H} (1+U)^2 \right] + W \frac{\partial U}{\partial Z} + (1+U) \frac{\partial W}{\partial Z} \\ & = -\frac{1}{\zeta_2} \frac{\partial P}{\partial X} - \frac{2\zeta_5}{H\zeta_2} C_{xc} U \end{aligned} \quad (87)$$

$$\begin{aligned} & \frac{2\zeta_3}{\zeta_2} W \frac{\partial W}{\partial Z} + (1+U) \frac{\partial W}{\partial X} + W \frac{\partial U}{\partial X} + \frac{(dH/dX)}{H} W (1+U) \\ & = -\frac{1}{\zeta_2} \frac{\partial P}{\partial Z} - \frac{\zeta_6}{\zeta_2} C_{xc} \frac{W}{H} \end{aligned} \quad (88)$$

$$\frac{(dH/dX)}{H} (1+U) + \frac{\partial U}{\partial X} + \zeta_7 \frac{\partial W}{\partial Z} = \frac{V_o}{H\zeta_7} \quad (89)$$

where

$$\frac{p}{\rho u_{bc}^2} = P, \quad \frac{\Delta u_b}{u_{bc}} = U, \quad \frac{w_b}{u_{bc}} = W, \quad \frac{h}{\bar{h}} = H, \quad \frac{u_o}{u_{bc}} = U_o, \quad \frac{v_o}{u_{bc}} = V_o,$$

$$\frac{x}{\bar{h}} = X, \quad \frac{z}{\bar{h}} = Z \text{ and } \bar{h} = \text{the reference film thickness} \quad (90)$$

and

$$\zeta_7 = \frac{\zeta_8}{\zeta_7} \quad (91)$$

Given sets of $\zeta_1, \zeta_2, \zeta_3, \zeta_5, \zeta_6, \zeta_7$ and C_{xc} , Equations (87) to (89) constitute the starting point for the mathematical treatment of thin turbulent bearing fluid film subjected to the given boundary values.

Evaluation of the Dimensionless Coefficients

The dimensionless coefficients ($\zeta_1, \zeta_2, \zeta_3$ and ζ_7) are evaluated to account for the variations of U, n and m . Irrespective of n and m , ζ_5 and ζ_6 are obtained to take the quantities U, k and W into account. It is of interest to determine the general magnitude of all the dimensionless coefficients and their sensitivity to the quantities U, W, n, m and k .

Velocity Profiles. The velocity profiles for u as shown in Figures 46 to 48 are represented by the turbulent power law as follows:

u - profile:

$$u = \pm 2u_{bc} + u_h \mp (u_{bc} \mp \Delta u_b) \left(\frac{y_o}{b} \right)^n \quad \left(0 \leq \frac{y_o}{b} \leq 1, \begin{matrix} u_o > u_h \\ u_o < u_h \end{matrix} \right) \quad (92)$$

$$u = \pm (u_{bc} \pm \Delta u_b) \left(\frac{y_h}{b} \right)^n \pm u_h \quad \left(0 \leq \frac{y_h}{b} \leq 1, \begin{matrix} u_o > u_h \\ u_o < u_h \end{matrix} \right) \quad (93)$$

The velocity profile for both $u_o > u_h$ and $u_o < u_h$ as shown in

Figure 49 is the w - profile:

$$w = w_b \left(\frac{y_o}{b} \right)^m \quad \left(0 \leq \frac{y_o}{b} \leq 1 \right) \quad (94)$$

$$w = w_b \left(\frac{y_h}{b} \right)^m \quad \left(0 \leq \frac{y_h}{b} \leq 1 \right) \quad (95)$$

Let y_o and y_h denote the reference position across channel with respect to lower and upper surface of the wall, respectively. The values of n and m refer to the velocity profiles in the tangential and axial directions, respectively. In the axial direction, the w - profile is assumed to be symmetrical about $y = b$.

Evaluation of ζ_1 , ζ_2 , ζ_3 , ζ_6 and ζ_7 . Substituting Equations (92) to (95)

into Equations (65) to (69) and (91), the general magnitude of the dimensionless coefficients and their sensitivity to the shape of the turbulent velocity profile in the tangential and axial directions, respectively, are obtained as

$$\begin{aligned} \zeta_1 &= \frac{b \left[\int_0^1 (u_{bc} + \Delta u_b)^2 \left(\frac{y_h}{b} \right)^{2n} d \left(\frac{y_h}{b} \right) + \int_0^1 \left\{ 2u_{bc} - (u_{bc} - \Delta u_b) \left(\frac{y_o}{b} \right)^n \right\}^2 d \left(\frac{y_o}{b} \right) \right]}{2b(u_{bc} + \Delta u_b)^2} \\ &= \frac{(1 + U^2)/(1 + 2n) + 2(n + U)/(1 + n)}{(1 + U)^2} \end{aligned} \quad (96)$$

$$\begin{aligned} \zeta_2 \equiv \zeta_4 &= \frac{b \left[\int_0^1 w_b (u_{bc} + \Delta u_b) \left(\frac{y_h}{b} \right)^{n+m} d \left(\frac{y_h}{b} \right) + \int_0^1 \left\{ 2u_{bc} - (u_{bc} - \Delta u_b) \left(\frac{y_o}{b} \right)^n \right\} w_b \left(\frac{y_o}{b} \right)^m d \left(\frac{y_o}{b} \right) \right]}{2bw_b u_b} \\ &= \frac{\frac{1}{1+m} + \frac{U}{1+n+m}}{1+U} \end{aligned} \quad (97)$$

$$\lim_{U \rightarrow -1} \frac{\zeta_1}{\zeta_2} = \frac{2(1+m)(1+n+m)}{(1+n)(1+2n)} \quad (98)$$

$$\zeta_3 = \frac{b \left[\int_0^1 w_b \left(\frac{y_o}{b} \right)^{2m} d \left(\frac{y_o}{b} \right) + \int_0^1 w_b \left(\frac{y_h}{b} \right)^{2m} d \left(\frac{y_h}{b} \right) \right]}{2b(w_b)^2}$$

$$= \frac{1}{1 + 2m} \quad (99)$$

$$\zeta_7' = \frac{b \left[\int_0^1 (u_{bc} + \Delta u_b) \left(\frac{y_h}{b} \right)^n d \left(\frac{y_h}{b} \right) + \int_0^1 \left\{ 2u_{bc} - (u_{bc} - \Delta u_b) \left(\frac{y_o}{b} \right)^n \right\} d \left(\frac{y_o}{b} \right) \right]}{2bu_b}$$

$$= \frac{1 + U/(1 + n)}{1 + U} \quad (100)$$

$$\zeta_8' = \frac{b \left[\int_0^1 w_b \left(\frac{y_h}{b} \right)^m d \left(\frac{y_h}{b} \right) + \int_0^1 w_b \left(\frac{y_o}{b} \right)^m d \left(\frac{y_o}{b} \right) \right]}{2bw_b}$$

$$= \frac{1}{1 + m} \quad (101)$$

and

$$\zeta_7 \equiv \zeta_8' / \zeta_7'$$

$$= \frac{1 + U}{(1 + m) \left(1 + \frac{U}{1 + n} \right)} \quad (102)$$

The value of ζ_6 in Equation (82), used for evaluating its general magnitude, is rewritten as

$$\zeta_6 = \frac{\left| \left[(1 + U)^2 + W^2 \right]^{\frac{1-k}{2}} \right|}{2} + \frac{\left| \left[(-1 + U)^2 + W^2 \right]^{\frac{1-k}{2}} \right|}{2} \quad (103)$$

For both cases of Couette and pressure flows effect dominating, the equations of ζ_1 , ζ_2 , ζ_3 , ζ_6 and ζ_7 remain unchanged because of identical equations for velocity profiles in the x and z directions, respectively. However, the equation of ζ_5 will be different for each case.

Evaluation of ζ_5 for Wall Shear with Same Sign (Couette Flow Dominating).

Equation (72) of ζ_5 will be used

$$\zeta_5 = \frac{A - B}{4U} \quad (0 \leq |U| \leq 1) \quad (104)$$

where A and B are the same as Equation (73) and Equation (74).

Evaluation of ζ_5 for Wall Shear with Opposite Sign (Pressure Flow Dominating)

Lower Surface Positive Shear, Upper Surface Negative Shear. The value of ζ_5 in Equation (77) will be used

$$\zeta_5 = \frac{A + B}{4U} \quad (1 \leq U) \quad (105)$$

Lower Surface Negative Shear, Upper Surface Positive Shear. The value of ζ_5 in Equation (79) will be used

$$\zeta_5 = - \frac{(A + B)}{4U} \quad (U \leq -1) \quad (106)$$

Results. The corresponding symbols in the analysis and computer programs are listed as follows:

TABLE 3. CORRESPONDING SYMBOLS IN ANALYSIS
AND COMPUTER PROGRAMS

<u>Symbols</u>	
<u>In Analysis</u>	<u>In Computer Programs</u>
U	U
W	W
n	N
m	M
k	K
ζ_1	ZETA 1
ζ_2	ZETA 2
ζ_3	ZETA 3
ζ_5	ZETA 5
ζ_6	ZETA 6
ζ_7	ZETA 7

All the dimensionless coefficients expressed by Equations (96) to (106) are calculated for the range of U such that $-6 \leq U \leq 6$ in the following Tables 4 to 22. For example, in Table 4 corresponding to $U = -6$, the general magnitudes of ζ_1 , ζ_2 , ζ_3 and ζ_7 , and of ζ_5 and ζ_6 are listed in Table 4(a) and (b), respectively. Correspondingly, Tables 5 to 22 refer to $U = -5$ to 6, respectively. Analyzing Tables 4 to 22, the effect of U on the extreme values of ζ_1 , ζ_2 , ζ_3 and ζ_7 for $n \left(\frac{1}{10} \leq n \leq \frac{1}{3} \right)$ and $m \left(\frac{1}{10} \leq m \leq \frac{1}{3} \right)$ are shown in Table 23. Similarly, Table 24 shows the effect of U on the extreme values of ζ_5 and ζ_6 (independent of n and m) for W ($0 \leq W \leq 2$) and $k \left(\frac{1}{4} \leq k \leq \frac{1}{2} \right)$.

TABLE 4. THE GENERAL MAGNITUDES OF THE
DIMENSIONLESS COEFFICIENTS FOR U = -6

(a) Effect of the Values of n and m Such That

$$\frac{1}{10} \leq [n, m] \leq \frac{1}{3} \text{ on the Values of } \zeta_1, \zeta_2,$$

ζ_3 and ζ_7

(b) Effect of the Values of k

$$\left(\frac{1}{4} \leq k \leq \frac{1}{2}\right) \text{ and } W(0 \leq W \leq 2)$$

on the Values of ζ_5 and ζ_6

N	M	ZETA 1	ZETA 2	ZETA 3	ZETA 7				
0.33333	0.33333	0.54800	0.57000	0.60000	1.07143				
	0.25000	0.54800	0.59789	0.66667	1.14286				
	0.20000	0.54800	0.61594	0.71429	1.19048				
	0.16667	0.54800	0.62857	0.75000	1.22449				
	0.14286	0.54800	0.63790	0.77778	1.25000				
	0.12500	0.54800	0.64500	0.80000	1.26984				
0.25000	0.11111	0.54800	0.65077	0.81818	1.28571				
	0.10000	0.54800	0.65539	0.83333	1.29870				
	0.33333	0.61867	0.60789	0.60000	0.98684				
	0.25000	0.61867	0.64000	0.66667	1.05263				
	0.20000	0.61867	0.66092	0.71429	1.09649				
	0.16667	0.61867	0.67563	0.75000	1.12782				
0.20000	0.14286	0.61867	0.68654	0.77778	1.15132				
	0.12500	0.61867	0.69495	0.80000	1.16959				
	0.11111	0.61867	0.70163	0.81818	1.18421				
	0.10000	0.61867	0.70707	0.83333	1.19617				
	0.33333	0.67048	0.63261	0.60000	0.93750				
	0.25000	0.67048	0.66759	0.66667	1.00000				
0.16667	0.20000	0.67048	0.69048	0.71429	1.04167				
	0.16667	0.67048	0.70662	0.75000	1.07143				
	0.14286	0.67048	0.71862	0.77778	1.09375				
	0.12500	0.67048	0.72788	0.80000	1.11111				
	0.11111	0.67048	0.73525	0.81818	1.12500				
	0.10000	0.67048	0.74126	0.83333	1.13636				
0.14286	0.33333	0.71000	0.65000	0.60000	0.90517				
	0.25000	0.71000	0.68706	0.66667	0.96552				
	0.20000	0.71000	0.71138	0.71429	1.00575				
	0.16667	0.71000	0.72857	0.75000	1.03448				
	0.14286	0.71000	0.74136	0.77778	1.05603				
	0.12500	0.71000	0.75125	0.80000	1.07280				
0.12500	0.11111	0.71000	0.75913	0.81818	1.08621				
	0.10000	0.71000	0.76555	0.83333	1.09718				
	0.33333	0.74111	0.66290	0.60000	0.88235				
	0.25000	0.74111	0.70154	0.66667	0.94118				
	0.20000	0.74111	0.72695	0.71429	0.98039				
	0.16667	0.74111	0.74494	0.75000	1.00840				
0.11111	0.14286	0.74111	0.75833	0.77778	1.02941				
	0.12500	0.74111	0.76870	0.80000	1.04575				
	0.11111	0.74111	0.77696	0.81818	1.05882				
	0.10000	0.74111	0.78370	0.83333	1.06952				
	0.33333	0.76622	0.67286	0.60000	0.86538				
	0.25000	0.76622	0.71273	0.66667	0.92308				
0.10000	0.20000	0.76622	0.73899	0.71429	0.96154				
	0.16667	0.76622	0.75760	0.75000	0.98901				
	0.14286	0.76622	0.77148	0.77778	1.00962				
	0.12500	0.76622	0.78222	0.80000	1.02564				
	0.11111	0.76622	0.79079	0.81818	1.03846				
	0.10000	0.76622	0.79777	0.83333	1.04895				
0.08333	0.33333	0.78691	0.68077	0.60000	0.85227				
	0.25000	0.78691	0.72163	0.66667	0.90909				
	0.20000	0.78691	0.74859	0.71429	0.94697				
	0.16667	0.78691	0.76770	0.75000	0.97403				
	0.14286	0.78691	0.78196	0.77778	0.99432				
	0.12500	0.78691	0.79301	0.80000	1.01010				
0.06667	0.11111	0.78691	0.80182	0.81818	1.02273				
	0.10000	0.78691	0.80901	0.83333	1.03306				
	0.33333	0.80424	0.68721	0.60000	0.84184				
	0.25000	0.80424	0.72889	0.66667	0.89796				
	0.20000	0.80424	0.75641	0.71429	0.93537				
	0.16667	0.80424	0.77594	0.75000	0.96210				
0.05000	0.14286	0.80424	0.79052	0.77778	0.98214				
	0.12500	0.80424	0.80181	0.80000	0.99773				
	0.11111	0.80424	0.81083	0.81818	1.01020				
	0.10000	0.80424	0.81818	0.83333	1.02041				
	0.33333	0.84224	0.68721	0.60000	0.84184				
	0.25000	0.84224	0.72889	0.66667	0.89796				
0.04167	0.20000	0.84224	0.75641	0.71429	0.93537				
	0.16667	0.84224	0.77594	0.75000	0.96210				
	0.14286	0.84224	0.79052	0.77778	0.98214				
	0.12500	0.84224	0.80181	0.80000	0.99773				
	0.11111	0.84224	0.81083	0.81818	1.01020				
	0.10000	0.84224	0.81818	0.83333	1.02041				

TABLE 5. THE GENERAL MAGNITUDES OF THE
DIMENSIONLESS COEFFICIENTS FOR $U = -5$

(a) Effect of the Values of n and m Such That

$\frac{1}{10} \leq [n, m] \leq \frac{1}{3}$ on the Values of $\zeta_1, \zeta_2,$
 ζ_3 and ζ_7

(b) Effect of the Values of k

$\left(\frac{1}{4} \leq k \leq \frac{1}{2}\right)$ and $W(0 \leq W \leq 2)$
on the Values of ζ_5 and ζ_6

N	M	ZETA 1	ZETA 2	ZETA 3	ZETA 7	K	W	ZETA 5	ZETA 6
0.55333	0.55333	0.53750	0.56250	0.60000	1.09091	0.50000	0.00000	1.13485	2.22474
	0.25000	0.53750	0.58947	0.66667	1.16364		0.25000	1.13556	2.22625
	0.20000	0.53750	0.60688	0.71429	1.21212		0.66667	1.13985	2.23538
	0.16667	0.53750	0.61905	0.75000	1.24675		1.00000	1.14601	2.24843
	0.14286	0.53750	0.62802	0.77778	1.27273		1.25000	1.15212	2.26139
	0.12500	0.53750	0.63492	0.80000	1.29293		1.33333	1.15443	2.26628
	0.11111	0.53750	0.64038	0.81818	1.30909		1.66667	1.16496	2.28855
	0.10000	0.53750	0.64482	0.83333	1.32231		2.00000	1.17741	2.31480
	0.33333	0.60833	0.60197	0.60000	1.00000		0.00000	1.49455	2.91088
	0.25000	0.60833	0.63333	0.66667	1.06667	0.33333	0.25000	1.49577	2.91348
0.25000	0.20000	0.60833	0.65374	0.71429	1.11111		0.66667	1.50323	2.92921
	0.16667	0.60833	0.66807	0.75000	1.14286		1.00000	1.51392	2.95175
	0.14286	0.60833	0.67869	0.77778	1.16667		1.25000	1.52457	2.97418
	0.12500	0.60833	0.68687	0.80000	1.18519		1.33333	1.52861	2.98266
	0.11111	0.60833	0.69337	0.81818	1.20000		1.66667	1.54702	3.02136
	0.10000	0.60833	0.69865	0.83333	1.21212		2.00000	1.56887	3.06718
	0.33333	0.66071	0.62772	0.60000	0.94737		0.00000	1.71578	3.33104
	0.25000	0.66071	0.66207	0.66667	1.01053		0.25000	1.71736	3.33436
	0.20000	0.66071	0.68452	0.71429	1.05263		0.66667	1.72693	3.35449
	0.16667	0.66071	0.70035	0.75000	1.08271	0.25000	1.00000	1.74067	3.38336
0.16667	0.14286	0.66071	0.71210	0.77778	1.10526		1.25000	1.75437	3.41212
	0.12500	0.66071	0.72117	0.80000	1.12281		1.33333	1.75956	3.42300
	0.11111	0.66071	0.72839	0.81818	1.13684		1.66667	1.78329	3.47272
	0.10000	0.66071	0.73427	0.83333	1.14833		2.00000	1.81151	3.53173
	0.33333	0.70089	0.64583	0.60000	0.91304				
	0.25000	0.70089	0.68235	0.66667	0.97391				
	0.20000	0.70089	0.70630	0.71429	1.01449				
	0.16667	0.70089	0.72321	0.75000	1.04348				
	0.14286	0.70089	0.73580	0.77778	1.06522				
	0.12500	0.70089	0.74552	0.80000	1.08213				
0.14286	0.11111	0.70089	0.75326	0.81818	1.09565	0.12500	0.00000	1.04516	
	0.10000	0.70089	0.75957	0.83333	1.10672		0.25000	1.05572	
	0.33333	0.73264	0.65927	0.60000	0.88889		0.66667	0.94815	
	0.25000	0.73264	0.69744	0.66667	0.94815		1.00000	0.98765	
	0.20000	0.73264	0.72222	0.71429	1.01587		1.25000	1.03704	
	0.16667	0.73264	0.74026	0.75000	1.03704		1.33333	1.05350	
	0.14286	0.73264	0.75347	0.77778	1.05350		1.66667	1.06667	
	0.12500	0.73264	0.76369	0.80000	1.07744		2.00000	1.08213	
	0.11111	0.73264	0.77184	0.81818	1.09565				
	0.10000	0.73264	0.77847	0.83333	1.10672				
0.12500	0.33333	0.75833	0.66964	0.60000	0.87097	0.11111	0.00000	0.85714	
	0.25000	0.75833	0.70909	0.66667	0.92903		0.25000	0.91429	
	0.20000	0.75833	0.73506	0.71429	0.96774		0.66667	0.95238	
	0.16667	0.75833	0.75346	0.75000	0.99539		1.00000	0.97959	
	0.14286	0.75833	0.76717	0.77778	1.01613		1.25000	1.00000	
	0.12500	0.75833	0.77778	0.80000	1.03226		1.33333	1.01587	
	0.11111	0.75833	0.78624	0.81818	1.04516		1.66667	1.02857	
	0.10000	0.75833	0.79314	0.83333	1.05572		2.00000	1.03896	
	0.33333	0.77955	0.67788	0.60000	0.85714				
	0.25000	0.77955	0.71837	0.66667	0.91429				
0.11111	0.20000	0.77955	0.74506	0.71429	0.95238	0.10000	0.00000	0.84615	
	0.16667	0.77955	0.76398	0.75000	0.97959		0.25000	0.90256	
	0.14286	0.77955	0.77809	0.77778	1.00000		0.66667	0.94017	
	0.12500	0.77955	0.78901	0.80000	1.01587		1.00000	0.96703	
	0.11111	0.77955	0.79773	0.81818	1.02857		1.25000	0.98718	
	0.10000	0.77955	0.80484	0.83333	1.03896		1.33333	1.00285	
	0.33333	0.79735	0.68459	0.60000	0.84615		1.66667	1.01538	
	0.25000	0.79735	0.72593	0.66667	0.90256		2.00000	1.02564	
	0.20000	0.79735	0.75321	0.71429	0.94017				
	0.16667	0.79735	0.77256	0.75000	0.96703				
0.10000	0.14286	0.79735	0.78700	0.77778	0.98718				
	0.12500	0.79735	0.79819	0.80000	1.00285				
	0.11111	0.79735	0.80711	0.81818	1.01538				
	0.10000	0.79735	0.81439	0.83333	1.02564				

TABLE 6. THE GENERAL MAGNITUDES OF THE
DIMENSIONLESS COEFFICIENTS FOR $U = -4$

(a) Effect of the Values of n and m Such That
 $\frac{1}{10} \leq [n, m] \leq \frac{1}{3}$ on the Values of $\zeta_1, \zeta_2,$
 ζ_3 and ζ_7

(b) Effect of the Values of k
 $\left(\frac{1}{4} \leq k \leq \frac{1}{2}\right)$ and $W(0 \leq W \leq 2)$
on the Values of ζ_5 and ζ_6

I	M	ZETA							
		1	2	3	7	K	W	5	6
0.3333	0.3333	0.52222	0.55000	0.60000	1.12500				
	0.25000	0.52222	0.57544	0.66667	1.20000				
	0.20000	0.52222	0.59179	0.71429	1.25000				
	0.16667	0.52222	0.60317	0.75000	1.28571				
	0.14286	0.52222	0.61156	0.77778	1.31250				
	0.12500	0.52222	0.61799	0.80000	1.33333				
	0.11111	0.52222	0.62308	0.81818	1.35000				
	0.10000	0.52222	0.62720	0.83333	1.36364				
	0.33333	0.59259	0.59211	0.60000	1.02273				
	0.25000	0.59259	0.62222	0.66667	1.09091				
0.25000	0.20000	0.59259	0.64176	0.71429	1.13636				
	0.16667	0.59259	0.65546	0.75000	1.16883				
	0.14286	0.59259	0.66560	0.77778	1.19318				
	0.12500	0.59259	0.67340	0.80000	1.21212				
	0.11111	0.59259	0.67959	0.81818	1.22727				
	0.10000	0.59259	0.68462	0.83333	1.23967				
	0.33333	0.64550	0.61957	0.60000	0.96429				
	0.25000	0.64550	0.65287	0.66667	1.02857				
	0.20000	0.64550	0.67460	0.71429	1.07143				
	0.16667	0.64550	0.68990	0.75000	1.10204				
0.20000	0.14286	0.64550	0.70124	0.77778	1.12500				
	0.12500	0.64550	0.70999	0.80000	1.14286				
	0.11111	0.64550	0.71695	0.81818	1.15714				
	0.10000	0.64550	0.72261	0.83333	1.16883				
	0.33333	0.68651	0.63889	0.60000	0.92647				
	0.25000	0.68651	0.67451	0.66667	0.98824				
	0.20000	0.68651	0.69783	0.71429	1.02941				
	0.16667	0.68651	0.71429	0.75000	1.05882				
	0.14286	0.68651	0.72652	0.77778	1.08088				
	0.12500	0.68651	0.73596	0.80000	1.09804				
0.16667	0.11111	0.68651	0.74348	0.81818	1.11176				
	0.10000	0.68651	0.74960	0.83333	1.12299				
	0.33333	0.71914	0.65323	0.60000	0.90000				
	0.25000	0.71914	0.69060	0.66667	0.96000				
	0.20000	0.71914	0.71513	0.71429	1.00000				
	0.16667	0.71914	0.73247	0.75000	1.02857				
	0.14286	0.71914	0.74537	0.77778	1.05000				
	0.12500	0.71914	0.75535	0.80000	1.06667				
	0.11111	0.71914	0.76329	0.81818	1.08000				
	0.10000	0.71914	0.76977	0.83333	1.09091				
0.12500	0.33333	0.74568	0.66429	0.60000	0.88043				
	0.25000	0.74568	0.70303	0.66667	0.93913				
	0.20000	0.74568	0.72851	0.71429	0.97826				
	0.16667	0.74568	0.74654	0.75000	1.00621				
	0.14286	0.74568	0.75998	0.77778	1.02717				
	0.12500	0.74568	0.77037	0.80000	1.04348				
	0.11111	0.74568	0.77865	0.81818	1.05652				
	0.10000	0.74568	0.78541	0.83333	1.06719				
	0.33333	0.76768	0.67308	0.60000	0.86538				
	0.25000	0.76768	0.71293	0.66667	0.92308				
0.11111	0.20000	0.76768	0.73917	0.71429	0.96154				
	0.16667	0.76768	0.75776	0.75000	0.98901				
	0.14286	0.76768	0.77162	0.77778	1.00962				
	0.12500	0.76768	0.78236	0.80000	1.02564				
	0.11111	0.76768	0.79091	0.81818	1.03846				
	0.10000	0.76768	0.79789	0.83333	1.04895				
	0.33333	0.78620	0.68023	0.60000	0.85345				
	0.25000	0.78620	0.72099	0.66667	0.91034				
	0.20000	0.78620	0.74786	0.71429	0.94828				
	0.16667	0.78620	0.76692	0.75000	0.97537				
0.10000	0.14286	0.78620	0.78113	0.77778	0.99569				
	0.12500	0.78620	0.79214	0.80000	1.01149				
	0.11111	0.78620	0.80092	0.81818	1.02414				
	0.10000	0.78620	0.80808	0.83333	1.03448				

TABLE 7. THE GENERAL MAGNITUDES OF THE
DIMENSIONLESS COEFFICIENTS FOR $U = -3$

(a) Effect of the Values of n and m Such That

$$\frac{1}{10} \leq [n, m] \leq \frac{1}{3} \text{ on the Values of } \zeta_1, \zeta_2,$$

ζ_3 and ζ_7

(b) Effect of the Values of k

$$\left(\frac{1}{4} \leq k \leq \frac{1}{2}\right) \text{ and } W(0 \leq W \leq 2)$$

on the Values of ζ_5 and ζ_6

N	M	ZETA 1	ZETA 2	ZETA 3	ZETA 7	K	W	ZETA 5	ZETA 6
0.43333	0.43333	0.50000	0.52500	0.60000	1.20000	1.50000	0.00000	0.90237	1.70711
	0.25000	0.50000	0.54737	0.66667	1.28000		0.25000	0.90393	1.71083
	0.20000	0.50000	0.56159	0.71429	1.33333		0.66667	0.91324	1.73285
	0.16667	0.50000	0.57143	0.75000	1.37143		1.00000	0.92607	1.76295
	0.14286	0.50000	0.57863	0.77778	1.40000		1.25000	0.93834	1.79144
	0.12500	0.50000	0.58413	0.80000	1.42222		1.33333	0.94286	1.80188
	0.11111	0.50000	0.58846	0.81818	1.44000		1.66667	0.96281	1.84759
	0.10000	0.50000	0.59197	0.83333	1.45455		2.00000	0.98521	1.89827
	0.33333	0.56667	0.57237	0.60000	1.07143		0.00000	1.10451	2.05362
	0.25000	0.56667	0.60000	0.66667	1.14286	0.43333	0.25000	1.10698	2.05937
0.25000	0.20000	0.56667	0.61782	0.71429	1.19048		0.66667	1.12168	2.09355
	0.16667	0.56667	0.63025	0.75000	1.22449		1.00000	1.14209	2.14063
	0.14286	0.56667	0.63942	0.77778	1.25000		1.25000	1.16175	2.18558
	0.12500	0.56667	0.64646	0.80000	1.26984		1.33333	1.16904	2.20216
	0.11111	0.56667	0.65204	0.81818	1.28571		1.66667	1.20140	2.27521
	0.10000	0.56667	0.65657	0.83333	1.29870		2.00000	1.23814	2.35721
	0.33333	0.61905	0.60326	0.60000	1.00000		0.00000	1.22311	2.25511
	0.25000	0.61905	0.63448	0.66667	1.06667		0.25000	1.22612	2.26208
	0.20000	0.61905	0.65476	0.71429	1.11111		0.66667	1.24414	2.30360
	0.16667	0.61905	0.66899	0.75000	1.14286	0.25000	1.00000	1.26925	2.36102
0.16667	0.14286	0.61905	0.67952	0.77778	1.16667		1.25000	1.29353	2.41608
	0.12500	0.61905	0.68763	0.80000	1.18519		1.33333	1.30255	2.43644
	0.11111	0.61905	0.69407	0.81818	1.20000		1.66667	1.34273	2.52648
	0.10000	0.61905	0.69930	0.83333	1.21212		2.00000	1.38860	2.62815
	0.33333	0.66071	0.62500	0.60000	0.95455				
	0.25000	0.66071	0.65882	0.66667	1.01818				
	0.20000	0.66071	0.68089	0.71429	1.06061				
	0.16667	0.66071	0.69643	0.75000	1.09091				
	0.14286	0.66071	0.70795	0.77778	1.11364				
	0.12500	0.66071	0.71685	0.80000	1.13131				
0.14286	0.11111	0.66071	0.72391	0.81818	1.14545	0.12500	0.00000	0.90237	1.70711
	0.10000	0.66071	0.72967	0.83333	1.15702		0.25000	0.90393	1.71083
	0.33333	0.69444	0.64113	0.60000	0.92308		0.66667	0.91324	1.73285
	0.25000	0.69444	0.67692	0.66667	0.98462		1.00000	0.92607	1.76295
	0.20000	0.69444	0.70035	0.71429	1.02564		1.25000	0.93834	1.79144
	0.16667	0.69444	0.71688	0.75000	1.06494		1.33333	0.94286	1.80188
	0.14286	0.69444	0.72917	0.77778	1.07692		1.66667	0.96281	1.84759
	0.12500	0.69444	0.73865	0.80000	1.09402		2.00000	0.98521	1.89827
	0.11111	0.69444	0.74620	0.81818	1.10769		0.00000	1.10451	2.05362
	0.10000	0.69444	0.75235	0.83333	1.11888	0.16667	0.25000	1.10698	2.05937
0.12500	0.33333	0.72222	0.65357	0.60000	0.90000		0.66667	1.12168	2.09355
	0.25000	0.72222	0.69091	0.66667	0.96000		1.00000	1.14209	2.14063
	0.20000	0.72222	0.71541	0.71429	1.00000		1.25000	1.16175	2.18558
	0.16667	0.72222	0.73272	0.75000	1.02857		1.33333	1.16904	2.20216
	0.14286	0.72222	0.74560	0.77778	1.05000		1.66667	1.20140	2.27521
	0.12500	0.72222	0.75556	0.80000	1.06667		2.00000	1.23814	2.35721
	0.11111	0.72222	0.76348	0.81818	1.08000		0.00000	1.22311	2.25511
	0.10000	0.72222	0.76994	0.83333	1.09091		0.25000	1.22612	2.26208
	0.33333	0.74545	0.66346	0.60000	0.88235		0.66667	1.24414	2.30360
	0.25000	0.74545	0.70204	0.66667	0.94118	0.14286	1.00000	1.26925	2.36102
0.11111	0.20000	0.74545	0.72740	0.71429	0.98039		1.25000	1.29353	2.41608
	0.16667	0.74545	0.74534	0.75000	1.00840		1.33333	1.30255	2.43644
	0.14286	0.74545	0.75870	0.77778	1.02941		1.66667	1.34273	2.52648
	0.12500	0.74545	0.76904	0.80000	1.04575		2.00000	1.38860	2.62815
	0.11111	0.74545	0.77727	0.81818	1.05882				
	0.10000	0.74545	0.78399	0.83333	1.06952				
	0.33333	0.76515	0.67151	0.60000	0.86842				
	0.25000	0.76515	0.71111	0.66667	0.92632				
	0.20000	0.76515	0.73718	0.71429	0.96491				
	0.16667	0.76515	0.75564	0.75000	0.99248	0.10000	0.00000	0.90237	1.70711
0.10000	0.14286	0.76515	0.76940	0.77778	1.01316		0.25000	0.90393	1.71083
	0.12500	0.76515	0.78005	0.80000	1.02924		0.66667	0.91324	1.73285
	0.11111	0.76515	0.78853	0.81818	1.04211		1.00000	0.92607	1.76295
	0.10000	0.76515	0.79545	0.83333	1.05263		1.25000	0.93834	1.79144
							1.33333	0.94286	1.80188
							1.66667	0.96281	1.84759
							2.00000	0.98521	1.89827
							0.00000	1.10451	2.05362
							0.25000	1.10698	2.05937
							0.66667	1.12168	2.09355

TABLE 8. THE GENERAL MAGNITUDES OF THE
DIMENSIONLESS COEFFICIENTS FOR $U = -2.5$

(a) Effect of the Values of n and m Such That

$$\frac{1}{10} \leq [n, m] \leq \frac{1}{3} \text{ on the Values of } \zeta_1, \zeta_2,$$

ζ_3 and ζ_7

(b) Effect of the Values of k

$$\left(\frac{1}{4} \leq k \leq \frac{1}{2}\right) \text{ and } W(0 \leq W \leq 2)$$

on the Values of ζ_5 and ζ_6

n	m	ZETA 1	ZETA 2	ZETA 3	ZETA 7	k	W	ZETA 5	ZETA 6
0.33333	0.33333	0.48889	0.50000	0.60000	1.28571				
	0.25000	0.48889	0.51930	0.66667	1.37143				
	0.20000	0.48889	0.53140	0.71429	1.42857				
	0.16667	0.48889	0.53968	0.75000	1.46939				
	0.14286	0.48889	0.54570	0.77778	1.50000				
	0.12500	0.48889	0.55026	0.80000	1.52381				
	0.11111	0.48889	0.55385	0.81818	1.54286				
	0.10000	0.48889	0.55673	0.83333	1.55844				
	0.33333	0.54815	0.55263	0.60000	1.12500				
	0.25000	0.54815	0.57778	0.66667	1.20000				
0.25000	0.20000	0.54815	0.59387	0.71429	1.25000				
	0.16667	0.54815	0.60504	0.75000	1.28571				
	0.14286	0.54815	0.61325	0.77778	1.31250				
	0.12500	0.54815	0.61953	0.80000	1.33333				
	0.11111	0.54815	0.62449	0.81818	1.35000				
	0.10000	0.54815	0.62851	0.83333	1.36364				
	0.33333	0.59788	0.58696	0.60000	1.03846				
	0.25000	0.59788	0.61609	0.66667	1.10769				
	0.20000	0.59788	0.63492	0.71429	1.15385				
	0.16667	0.59788	0.64808	0.75000	1.18681				
0.20000	0.14286	0.59788	0.65780	0.77778	1.21154				
	0.12500	0.59788	0.66527	0.80000	1.23077				
	0.11111	0.59788	0.67119	0.81818	1.24615				
	0.10000	0.59788	0.67599	0.83333	1.25874				
	0.33333	0.63889	0.61111	0.60000	0.98438				
	0.25000	0.63889	0.64314	0.66667	1.05000				
	0.20000	0.63889	0.66396	0.71429	1.09375				
	0.16667	0.63889	0.67857	0.75000	1.12500				
	0.14286	0.63889	0.68939	0.77778	1.14844				
	0.12500	0.63889	0.69773	0.80000	1.16667				
0.16667	0.11111	0.63889	0.70435	0.81818	1.18125				
	0.10000	0.63889	0.70973	0.83333	1.19318				
	0.33333	0.67284	0.62903	0.60000	0.94737				
	0.25000	0.67284	0.66325	0.66667	1.01053				
	0.20000	0.67284	0.68558	0.71429	1.05263				
	0.16667	0.67284	0.70130	0.75000	1.08271				
	0.14286	0.67284	0.71296	0.77778	1.10526				
	0.12500	0.67284	0.72196	0.80000	1.12281				
	0.11111	0.67284	0.72911	0.81818	1.13684				
	0.10000	0.67284	0.73494	0.83333	1.14833				
0.12500	0.33333	0.70123	0.64286	0.60000	0.92045				
	0.25000	0.70123	0.67879	0.66667	0.98182				
	0.20000	0.70123	0.70231	0.71429	1.02273				
	0.16667	0.70123	0.71889	0.75000	1.05195				
	0.14286	0.70123	0.73122	0.77778	1.07386				
	0.12500	0.70123	0.74074	0.80000	1.09091				
	0.11111	0.70123	0.74831	0.81818	1.10455				
	0.10000	0.70123	0.75448	0.83333	1.11570				
	0.33333	0.72525	0.65385	0.60000	0.90000				
	0.25000	0.72525	0.69116	0.66667	0.96000				
0.11111	0.20000	0.72525	0.71563	0.71429	1.00000				
	0.16667	0.72525	0.73292	0.75000	1.02857				
	0.14286	0.72525	0.74578	0.77778	1.05000				
	0.12500	0.72525	0.75572	0.80000	1.06667				
	0.11111	0.72525	0.76364	0.81818	1.08000				
	0.10000	0.72525	0.77009	0.83333	1.09091				
	0.33333	0.74579	0.66279	0.60000	0.88393				
	0.25000	0.74579	0.70123	0.66667	0.94286				
	0.20000	0.74579	0.72650	0.71429	0.98214				
	0.16667	0.74579	0.74436	0.75000	1.01020				
0.10000	0.14286	0.74579	0.75766	0.77778	1.03125				
	0.12500	0.74579	0.76795	0.80000	1.04762				
	0.11111	0.74579	0.77615	0.81818	1.06071				
	0.10000	0.74579	0.78283	0.83333	1.07143				

TABLE 9. THE GENERAL MAGNITUDES OF THE
DIMENSIONLESS COEFFICIENTS FOR $U = -2$

(a) Effect of the Values of n and m Such That

$$\frac{1}{10} \leq [n, m] \leq \frac{1}{3} \text{ on the Values of } \zeta_1, \zeta_2,$$

ζ_3 and ζ_7

(b) Effect of the Values of k

$$\left(\frac{1}{4} \leq k \leq \frac{1}{2}\right) \text{ and } W(0 \leq W \leq 2)$$

on the Values of ζ_5 and ζ_6

N	M	ZETA 1	ZETA 2	ZETA 3	ZETA 7	K	W	ZETA 5	ZETA 6
0.55533	0.33333	0.50000	0.45000	0.60000	1.50000	0.50000	0.00000	0.77452	1.36603
	0.25000	0.50000	0.46316	0.66667	1.60000		0.25000	0.77755	1.37516
	0.20000	0.50000	0.47101	0.71429	1.66667		0.66667	0.79443	1.42467
	0.16667	0.50000	0.47619	0.75000	1.71429		1.00000	0.81551	1.48374
	0.14286	0.50000	0.47984	0.77778	1.75000		1.25000	0.83419	1.53400
	0.12500	0.50000	0.48254	0.80000	1.77778		1.33333	0.84083	1.55144
	0.11111	0.50000	0.48462	0.81818	1.80000		1.66667	0.86897	1.62334
0.25000	0.10000	0.50000	0.48626	0.83333	1.81818	0.33333	2.00000	0.89898	1.69709
	0.33333	0.53333	0.51316	0.60000	1.25000		0.00000	0.90503	1.54004
	0.25000	0.53333	0.53333	0.66667	1.33333		0.25000	0.90938	1.55265
	0.20000	0.53333	0.54598	0.71429	1.38889		0.66667	0.93397	1.62209
	0.16667	0.53333	0.55462	0.75000	1.42857		1.00000	0.96540	1.70718
	0.14286	0.53333	0.56090	0.77778	1.45833		1.25000	0.99384	1.78126
	0.12500	0.53333	0.56566	0.80000	1.48148		1.33333	1.00405	1.80731
0.20000	0.11111	0.53333	0.56939	0.81818	1.50000	0.25000	1.66667	1.04788	1.91633
	0.10000	0.53333	0.57239	0.83333	1.51515		2.00000	1.09550	2.03066
	0.33333	0.57143	0.55435	0.60000	1.12500		0.00000	0.97982	1.63975
	0.25000	0.57143	0.57931	0.66667	1.20000		0.25000	0.98491	1.65421
	0.20000	0.57143	0.59524	0.71429	1.25000		0.66667	1.01389	1.73447
	0.16667	0.57143	0.60627	0.75000	1.28571		1.00000	1.05137	1.83411
	0.14286	0.57143	0.61436	0.77778	1.31250		1.25000	1.08560	1.92184
0.16667	0.12500	0.57143	0.62055	0.80000	1.33333	0.16667	1.33333	1.09795	1.95289
	0.11111	0.57143	0.62542	0.81818	1.35000		1.66667	1.15131	2.08378
	0.10000	0.57143	0.62937	0.83333	1.36364		2.00000	1.20978	2.22256
	0.33333	0.60714	0.58333	0.60000	1.05000				
	0.25000	0.60714	0.61176	0.66667	1.12000				
	0.20000	0.60714	0.63008	0.71429	1.16667				
	0.16667	0.60714	0.64286	0.75000	1.20000				
0.14286	0.14286	0.60714	0.65227	0.77778	1.22500	0.12500			
	0.12500	0.60714	0.65950	0.80000	1.24444				
	0.11111	0.60714	0.66522	0.81818	1.26000				
	0.10000	0.60714	0.66986	0.83333	1.27273				
	0.33333	0.63889	0.60484	0.60000	1.00000				
	0.25000	0.63889	0.63590	0.66667	1.06667				
	0.20000	0.63889	0.65603	0.71429	1.11111				
0.12500	0.16667	0.63889	0.67013	0.75000	1.14286	0.11111			
	0.14286	0.63889	0.68056	0.77778	1.16667				
	0.12500	0.63889	0.68858	0.80000	1.18519				
	0.11111	0.63889	0.69494	0.81818	1.20000				
	0.10000	0.63889	0.70010	0.83333	1.21212				
	0.33333	0.66667	0.62143	0.60000	0.96429				
	0.25000	0.66667	0.65455	0.66667	1.02857				
0.11111	0.20000	0.66667	0.67610	0.71429	1.07143	0.10000			
	0.16667	0.66667	0.69124	0.75000	1.10204				
	0.14286	0.66667	0.70246	0.77778	1.12500				
	0.12500	0.66667	0.71111	0.80000	1.14286				
	0.11111	0.66667	0.71798	0.81818	1.15714				
	0.10000	0.66667	0.72356	0.83333	1.16883				
	0.33333	0.69091	0.63462	0.60000	0.93750				
0.10000	0.25000	0.69091	0.66939	0.66667	1.00000	0.08333			
	0.20000	0.69091	0.69209	0.71429	1.04167				
	0.16667	0.69091	0.70807	0.75000	1.07143				
	0.14286	0.69091	0.71994	0.77778	1.09375				
	0.12500	0.69091	0.72909	0.80000	1.11111				
	0.11111	0.69091	0.73636	0.81818	1.12500				
	0.10000	0.69091	0.74229	0.83333	1.13636				
0.08333	0.33333	0.71212	0.64535	0.60000	0.91667	0.07500			
	0.25000	0.71212	0.68148	0.66667	0.97778				
	0.20000	0.71212	0.70513	0.71429	1.01852				
	0.16667	0.71212	0.72180	0.75000	1.04762				
	0.14286	0.71212	0.73420	0.77778	1.06944				
	0.12500	0.71212	0.74376	0.80000	1.08642				
	0.11111	0.71212	0.75138	0.81818	1.10000				
0.07500	0.10000	0.71212	0.75758	0.83333	1.11111				

TABLE 10. THE GENERAL MAGNITUDES OF THE
DIMENSIONLESS COEFFICIENTS FOR $U = -1.5$

(a) Effect of the Values of n and m Such That
 $\frac{1}{10} \leq [n, m] \leq \frac{1}{3}$ on the Values of $\zeta_1, \zeta_2,$
 ζ_3 and ζ_7

(b) Effect of the Values of k
 $\left(\frac{1}{4} \leq k \leq \frac{1}{2}\right)$ and $W(0 \leq W \leq 2)$
on the Values of ζ_5 and ζ_6

N	M	ZETA 1	ZETA 2	ZETA 3	ZETA 7	K	W	ZETA 5	ZETA 6
0.3333	0.33333	0.80000	0.30000	0.60000	3.00000	0.50000	0.00000	0.71773	1.14412
	0.25000	0.80000	0.29474	0.66667	3.20000		0.25000	0.72275	1.16638
	0.20000	0.80000	0.28986	0.71429	3.33333		0.66667	0.74629	1.26070
	0.16667	0.80000	0.28571	0.75000	3.42857		1.00000	0.77183	1.34914
	0.14286	0.80000	0.28226	0.77778	3.50000		1.25000	0.79330	1.41607
	0.12500	0.80000	0.27937	0.80000	3.55556		1.33333	0.80080	1.43828
	0.11111	0.80000	0.27692	0.81818	3.60000		1.66667	0.83217	1.52625
	0.10000	0.80000	0.27484	0.83333	3.63636		2.00000	0.86519	1.61255
	0.33333	0.66667	0.39474	0.60000	1.87500		0.00000	0.82000	1.23599
	0.25000	0.66667	0.40000	0.66667	2.00000	0.33333	0.25000	0.82661	1.26337
0.25000	0.20000	0.66667	0.40230	0.71429	2.08333		0.66667	0.85908	1.38512
	0.16667	0.66667	0.40336	0.75000	2.14286		1.00000	0.89620	1.50633
	0.14286	0.66667	0.40385	0.77778	2.18750		1.25000	0.92838	1.60175
	0.12500	0.66667	0.40404	0.80000	2.22222		1.33333	0.93977	1.63402
	0.11111	0.66667	0.40408	0.81818	2.25000		1.66667	0.98815	1.76445
	0.10000	0.66667	0.40404	0.83333	2.27273		2.00000	1.04008	1.89602
	0.33333	0.61905	0.45652	0.60000	1.50000		0.00000	0.87796	1.29139
	0.25000	0.61905	0.46897	0.66667	1.60000		0.25000	0.88538	1.32105
	0.20000	0.61905	0.47619	0.71429	1.66667		0.66667	0.92271	1.45613
	0.16667	0.61905	0.48084	0.75000	1.71429	0.25000	1.00000	0.96643	1.59463
0.16667	0.14286	0.61905	0.48404	0.77778	1.75000		1.25000	1.00486	1.70577
	0.12500	0.61905	0.48637	0.80000	1.77778		1.33333	1.01857	1.74371
	0.11111	0.61905	0.48814	0.81818	1.80000		1.66667	1.07715	1.89859
	0.10000	0.61905	0.48951	0.83333	1.81818		2.00000	1.14064	2.05695
	0.33333	0.60714	0.50000	0.60000	1.31250				
	0.25000	0.60714	0.51765	0.66667	1.40000				
	0.20000	0.60714	0.52846	0.71429	1.45833				
	0.16667	0.60714	0.53571	0.75000	1.50000				
	0.14286	0.60714	0.54091	0.77778	1.53125				
	0.12500	0.60714	0.54480	0.80000	1.55556				
0.14286	0.11111	0.60714	0.54783	0.81818	1.57500	0.12500	0.00000	0.87796	1.29139
	0.10000	0.60714	0.55024	0.83333	1.59091		0.25000	0.88538	1.32105
	0.33333	0.61111	0.53226	0.60000	1.20000		0.66667	0.92271	1.45613
	0.25000	0.61111	0.55385	0.66667	1.28000		1.00000	0.96643	1.59463
	0.20000	0.61111	0.56738	0.71429	1.33333		1.25000	1.00486	1.70577
	0.16667	0.61111	0.57662	0.75000	1.37143		1.33333	1.01857	1.74371
	0.14286	0.61111	0.58333	0.77778	1.40000		1.66667	1.07715	1.89859
	0.12500	0.61111	0.58842	0.80000	1.42222		2.00000	1.14064	2.05695
	0.11111	0.61111	0.59241	0.81818	1.44000				
	0.10000	0.61111	0.59561	0.83333	1.45455				
0.12500	0.33333	0.62222	0.55714	0.60000	1.12500	0.11111	0.00000	0.87796	1.29139
	0.25000	0.62222	0.58182	0.66667	1.20000		0.25000	0.88538	1.32105
	0.20000	0.62222	0.59748	0.71429	1.25000		0.66667	0.92271	1.45613
	0.16667	0.62222	0.60829	0.75000	1.28571		1.00000	0.96643	1.59463
	0.14286	0.62222	0.61620	0.77778	1.31250		1.25000	1.00486	1.70577
	0.12500	0.62222	0.62222	0.80000	1.33333		1.33333	1.01857	1.74371
	0.11111	0.62222	0.62697	0.81818	1.35000		1.66667	1.07715	1.89859
	0.10000	0.62222	0.63080	0.83333	1.36364		2.00000	1.14064	2.05695
	0.33333	0.63636	0.57692	0.60000	1.07143				
	0.25000	0.63636	0.60408	0.66667	1.14286	0.10000	0.00000	0.87796	1.29139
0.11111	0.20000	0.63636	0.62147	0.71429	1.19048		0.25000	0.88538	1.32105
	0.16667	0.63636	0.63354	0.75000	1.22449		0.66667	0.92271	1.45613
	0.14286	0.63636	0.64241	0.77778	1.25000		1.00000	0.96643	1.59463
	0.12500	0.63636	0.64919	0.80000	1.26984		1.25000	1.00486	1.70577
	0.11111	0.63636	0.65455	0.81818	1.28571		1.33333	1.01857	1.74371
	0.10000	0.63636	0.65888	0.83333	1.29870		1.66667	1.07715	1.89859
	0.33333	0.65152	0.59302	0.60000	1.03125		2.00000	1.14064	2.05695
	0.25000	0.65152	0.62222	0.66667	1.10000				
	0.20000	0.65152	0.64103	0.71429	1.14583				
	0.16667	0.65152	0.65414	0.75000	1.17857	0.10000	0.00000	0.87796	1.29139
0.10000	0.14286	0.65152	0.66379	0.77778	1.20313		0.25000	0.88538	1.32105
	0.12500	0.65152	0.67120	0.80000	1.22222		0.66667	0.92271	1.45613
	0.11111	0.65152	0.67706	0.81818	1.23750		1.00000	0.96643	1.59463
	0.10000	0.65152	0.68182	0.83333	1.25000		1.25000	1.00486	1.70577
							1.33333	1.01857	1.74371
							1.66667	1.07715	1.89859
							2.00000	1.14064	2.05695

TABLE 11. THE GENERAL MAGNITUDES OF THE
DIMENSIONLESS COEFFICIENTS FOR $U = -1$

(a) Effect of the Values of n and m Such That

$\frac{1}{10} \leq [n, m] \leq \frac{1}{3}$ on the Values of $\zeta_1, \zeta_2,$
 ζ_3 and ζ_7

(b) Effect of the Values of k

$\left(\frac{1}{4} \leq k \leq \frac{1}{2}\right)$ and $W(0 \leq W \leq 2)$
on the Values of ζ_5 and ζ_6

N	M	ZETA 1	ZETA 2	ZETA 3	ZETA 7	ZETA1/ZETA2	K	W	ZETA 5	ZETA 6
0.55555	0.55533	INFINITE	INFINITE	0.60000	0.00000	2.00000	0.50000	0.00000	0.70711	0.70711
	0.25000	INFINITE	INFINITE	0.66667	0.00000	1.78125		0.25000	0.70985	0.95985
	0.20000	INFINITE	INFINITE	0.71429	0.00000	1.65600		0.66667	0.72598	1.13423
	0.16667	INFINITE	INFINITE	0.75000	0.00000	1.57500		1.00000	0.74767	1.24767
	0.14286	INFINITE	INFINITE	0.77778	0.00000	1.51837		1.25000	0.76787	1.32689
	0.12500	INFINITE	INFINITE	0.80000	0.00000	1.47656		1.33333	0.77519	1.35254
	0.11111	INFINITE	INFINITE	0.81818	0.00000	1.44444		1.66667	0.80676	1.45225
	0.10000	INFINITE	INFINITE	0.83333	0.00000	1.41900		2.00000	0.84090	1.54800
	0.25000	INFINITE	INFINITE	0.60000	0.00000	2.25185		0.33333	0.79370	0.79370
	0.25000	INFINITE	INFINITE	0.66667	0.00000	2.00000		0.25000	0.79781	0.99624
0.25000	0.20000	INFINITE	INFINITE	0.71429	0.00000	1.85600	0.25000	0.00000	0.82207	1.20364
	0.16667	INFINITE	INFINITE	0.75000	0.00000	1.76296		1.00000	0.85499	1.35499
	0.14286	INFINITE	INFINITE	0.77778	0.00000	1.69796		1.25000	0.88592	1.46612
	0.12500	INFINITE	INFINITE	0.80000	0.00000	1.65000		1.33333	0.89720	1.50291
	0.11111	INFINITE	INFINITE	0.81818	0.00000	1.61317		1.66667	0.94624	1.64910
	0.10000	INFINITE	INFINITE	0.83333	0.00000	1.58400		2.00000	1.00000	1.79370
	0.33333	INFINITE	INFINITE	0.60000	0.00000	2.43386		0.00000	0.84090	0.84090
	0.25000	INFINITE	INFINITE	0.66667	0.00000	2.15774		0.25000	0.84580	1.02258
	0.20000	INFINITE	INFINITE	0.71429	0.00000	2.00000		0.66667	0.87479	1.24368
	0.16667	INFINITE	INFINITE	0.75000	0.00000	1.89815		1.00000	0.91429	1.41429
0.16667	0.14286	INFINITE	INFINITE	0.77778	0.00000	1.82702	0.25000	1.25000	0.95158	1.54267
	0.12500	INFINITE	INFINITE	0.80000	0.00000	1.77455		1.33333	0.96523	1.58563
	0.11111	INFINITE	INFINITE	0.81818	0.00000	1.73427		1.66667	1.02477	1.75820
	0.10000	INFINITE	INFINITE	0.83333	0.00000	1.70238		2.00000	1.09051	1.93140
	0.33333	INFINITE	INFINITE	0.60000	0.00000	2.57143		0.00000		
	0.25000	INFINITE	INFINITE	0.66667	0.00000	2.27679				
	0.20000	INFINITE	INFINITE	0.71429	0.00000	2.10857				
	0.16667	INFINITE	INFINITE	0.75000	0.00000	2.00000				
	0.14286	INFINITE	INFINITE	0.77778	0.00000	1.92420				
	0.12500	INFINITE	INFINITE	0.80000	0.00000	1.86830				
0.14286	0.11111	INFINITE	INFINITE	0.81818	0.00000	1.82540	0.25000			
	0.10000	INFINITE	INFINITE	0.83333	0.00000	1.79143				
	0.33333	INFINITE	INFINITE	0.60000	0.00000	2.67901				
	0.25000	INFINITE	INFINITE	0.66667	0.00000	2.36979				
	0.20000	INFINITE	INFINITE	0.71429	0.00000	2.19333				
	0.16667	INFINITE	INFINITE	0.75000	0.00000	2.07948				
	0.14286	INFINITE	INFINITE	0.77778	0.00000	2.00000				
	0.12500	INFINITE	INFINITE	0.80000	0.00000	1.94141				
	0.11111	INFINITE	INFINITE	0.81818	0.00000	1.89643				
	0.10000	INFINITE	INFINITE	0.83333	0.00000	1.86083				
0.12500	0.33333	INFINITE	INFINITE	0.60000	0.00000	2.76543	0.25000			
	0.25000	INFINITE	INFINITE	0.66667	0.00000	2.44444				
	0.20000	INFINITE	INFINITE	0.71429	0.00000	2.26133				
	0.16667	INFINITE	INFINITE	0.75000	0.00000	2.14321				
	0.14286	INFINITE	INFINITE	0.77778	0.00000	2.06077				
	0.12500	INFINITE	INFINITE	0.80000	0.00000	2.00000				
	0.11111	INFINITE	INFINITE	0.81818	0.00000	1.95336				
	0.10000	INFINITE	INFINITE	0.83333	0.00000	1.91644				
	0.33333	INFINITE	INFINITE	0.60000	0.00000	2.83636				
	0.25000	INFINITE	INFINITE	0.66667	0.00000	2.50568				
0.11111	0.20000	INFINITE	INFINITE	0.71429	0.00000	2.31709	0.25000			
	0.16667	INFINITE	INFINITE	0.75000	0.00000	2.19545				
	0.14286	INFINITE	INFINITE	0.77778	0.00000	2.11057				
	0.12500	INFINITE	INFINITE	0.80000	0.00000	2.04801				
	0.11111	INFINITE	INFINITE	0.81818	0.00000	2.00000				
	0.10000	INFINITE	INFINITE	0.83333	0.00000	1.96200				
	0.33333	INFINITE	INFINITE	0.60000	0.00000	2.89562				
	0.25000	INFINITE	INFINITE	0.66667	0.00000	2.55682				
	0.20000	INFINITE	INFINITE	0.71429	0.00000	2.36364				
	0.16667	INFINITE	INFINITE	0.75000	0.00000	2.23906				
0.10000	0.14286	INFINITE	INFINITE	0.77778	0.00000	2.15213	0.25000			
	0.12500	INFINITE	INFINITE	0.80000	0.00000	2.08807				
	0.11111	INFINITE	INFINITE	0.81818	0.00000	2.03891				
	0.10000	INFINITE	INFINITE	0.83333	0.00000	2.00000				

TABLE 12. THE GENERAL MAGNITUDES OF THE
DIMENSIONLESS COEFFICIENTS FOR $U = -0.5$

(a) Effect of the Values of n and m Such That
 $\frac{1}{10} \leq [n, m] \leq \frac{1}{3}$ on the Values of $\zeta_1, \zeta_2,$
 ζ_3 and ζ_7

(b) Effect of the Values of k
 $\left(\frac{1}{4} \leq k \leq \frac{1}{2}\right)$ and $W(0 \leq W \leq 2)$
on the Values of ζ_5 and ζ_6

I									
	M	ZETA 1	ZETA 2	ZETA 3	ZETA 7	K	W	ZETA 5	ZETA 6
0.55555	0.33333	2.00000	0.90000	0.60000	0.60000	0.50000	0.00000	0.74178	0.96593
	0.25000	2.00000	0.96842	0.66667	0.64000		0.25000	0.73795	0.99042
	0.20000	2.00000	1.01449	0.71429	0.66667		0.66667	0.73268	1.09704
	0.16667	2.00000	1.04762	0.75000	0.68571		1.00000	0.74266	1.20002
	0.14286	2.00000	1.07258	0.77778	0.70000		1.25000	0.75793	1.27882
	0.12500	2.00000	1.09206	0.80000	0.71111		1.33333	0.76417	1.30499
	0.11111	2.00000	1.10769	0.81818	0.72000		1.66667	0.79329	1.40827
	0.10000	2.00000	1.12051	0.83333	0.72727		2.00000	0.82690	1.50847
	0.33333	1.73333	0.86842	0.60000	0.62500		0.00000	0.82529	0.97017
	0.25000	1.73333	0.93333	0.66667	0.66667	0.33333	0.25000	0.82214	1.00050
0.25000	0.20000	1.73333	0.97701	0.71429	0.69444		0.66667	0.82225	1.13853
	0.16667	1.73333	1.00840	0.75000	0.71429		1.00000	0.84163	1.27923
	0.14286	1.73333	1.03205	0.77778	0.72917		1.25000	0.86684	1.39072
	0.12500	1.73333	1.05051	0.80000	0.74074		1.33333	0.87687	1.42840
	0.11111	1.73333	1.06531	0.81818	0.75000		1.66667	0.92318	1.57991
	0.10000	1.73333	1.07744	0.83333	0.75758		2.00000	0.97656	1.73091
	0.33333	1.57143	0.84783	0.60000	0.64286		0.00000	0.86790	0.97500
	0.25000	1.57143	0.91034	0.66667	0.68571		0.25000	0.86543	1.00795
	0.20000	1.57143	0.95238	0.71429	0.71429		0.66667	0.86960	1.16119
	0.16667	1.57143	0.98258	0.75000	0.73469	0.25000	1.00000	0.89504	1.32154
0.16667	0.14286	1.57143	1.00532	0.77778	0.75000		1.25000	0.92638	1.45081
	0.12500	1.57143	1.02306	0.80000	0.76190		1.33333	0.93873	1.49486
	0.11111	1.57143	1.03729	0.81818	0.77143		1.66667	0.99553	1.67371
	0.10000	1.57143	1.04895	0.83333	0.77922		2.00000	1.06102	1.85432
	0.33333	1.46429	0.83333	0.60000	0.65625				
	0.25000	1.46429	0.89412	0.66667	0.70000				
	0.20000	1.46429	0.93496	0.71429	0.72917				
	0.16667	1.46429	0.96429	0.75000	0.75000				
	0.14286	1.46429	0.98636	0.77778	0.76563				
	0.12500	1.46429	1.00358	0.80000	0.77778				
0.14286	0.11111	1.46429	1.01739	0.81818	0.78750	0.12500	0.00000	0.74178	0.96593
	0.10000	1.46429	1.02871	0.83333	0.79545		0.25000	0.73795	0.99042
	0.33333	1.38889	0.82258	0.60000	0.66667		0.66667	0.73268	1.09704
	0.25000	1.38889	0.88205	0.66667	0.71111		1.00000	0.74266	1.20002
	0.20000	1.38889	0.92199	0.71429	0.74074		1.25000	0.75793	1.27882
	0.16667	1.38889	0.95065	0.75000	0.76190		1.33333	0.76417	1.30499
	0.14286	1.38889	0.97222	0.77778	0.77778		1.66667	0.79329	1.40827
	0.12500	1.38889	0.98905	0.80000	0.79012		2.00000	0.82690	1.50847
	0.11111	1.38889	1.00253	0.81818	0.80000				
	0.10000	1.38889	1.01358	0.83333	0.80808				
0.12500	0.33333	1.33333	0.81429	0.60000	0.67500	0.11111	0.00000	0.74178	0.96593
	0.25000	1.33333	0.87273	0.66667	0.72000		0.25000	0.73795	0.99042
	0.20000	1.33333	0.91195	0.71429	0.75000		0.66667	0.73268	1.09704
	0.16667	1.33333	0.94009	0.75000	0.77143		1.00000	0.74266	1.20002
	0.14286	1.33333	0.96127	0.77778	0.78750		1.25000	0.75793	1.27882
	0.12500	1.33333	0.97778	0.80000	0.80000		1.33333	0.76417	1.30499
	0.11111	1.33333	0.99101	0.81818	0.81000		1.66667	0.79329	1.40827
	0.10000	1.33333	1.00186	0.83333	0.81818		2.00000	0.82690	1.50847
	0.33333	1.29091	0.80769	0.60000	0.68182				
	0.25000	1.29091	0.86531	0.66667	0.72727				
0.11111	0.20000	1.29091	0.90395	0.71429	0.75758	0.10000	0.00000	0.74178	0.96593
	0.16667	1.29091	0.93168	0.75000	0.77922		0.25000	0.73795	0.99042
	0.14286	1.29091	0.95253	0.77778	0.79545		0.66667	0.73268	1.09704
	0.12500	1.29091	0.96879	0.80000	0.80808		1.00000	0.74266	1.20002
	0.11111	1.29091	0.98182	0.81818	0.81818		1.25000	0.75793	1.27882
	0.10000	1.29091	0.99249	0.83333	0.82645		1.33333	0.76417	1.30499
	0.33333	1.25758	0.80233	0.60000	0.68750		1.66667	0.79329	1.40827
	0.25000	1.25758	0.85926	0.66667	0.73333		2.00000	0.82690	1.50847
	0.20000	1.25758	0.89744	0.71429	0.76389				
	0.16667	1.25758	0.92481	0.75000	0.78571				
0.10000	0.14286	1.25758	0.94540	0.77778	0.80208	0.09091	0.00000	0.74178	0.96593
	0.12500	1.25758	0.96145	0.80000	0.81481		0.25000	0.73795	0.99042
	0.11111	1.25758	0.97431	0.81818	0.82500		0.66667	0.73268	1.09704
	0.10000	1.25758	0.98485	0.83333	0.83333		1.00000	0.74266	1.20002
							1.25000	0.75793	1.27882

TABLE 13. THE GENERAL MAGNITUDES OF THE
DIMENSIONLESS COEFFICIENTS FOR $U = 0$

(a) Effect of the Values of n and m Such That

$$\frac{1}{10} \leq [n, m] \leq \frac{1}{3} \text{ on the Values of } \zeta_1, \zeta_2,$$

ζ_3 and ζ_7

(b) Effect of the Values of k

$$\left(\frac{1}{4} \leq k \leq \frac{1}{2}\right) \text{ and } W(0 \leq W \leq 2)$$

on the Values of ζ_5 and ζ_6

N	M	ZETA 1	ZETA 2	ZETA 3	ZETA 7	K	W	ZETA 5	ZETA 6
0.33333	0.33333	1.10000	0.75000	0.60000	0.75000	0.50000	0.00000	0.75000	1.00000
	0.25000	1.10000	0.80000	0.66667	0.80000		0.25000	0.74652	1.01527
	0.20000	1.10000	0.83333	0.71429	0.83333		0.66667	0.73789	1.09629
	0.16667	1.10000	0.85714	0.75000	0.85714		1.00000	0.74325	1.18921
	0.14286	1.10000	0.87500	0.77778	0.87500		1.25000	0.75605	1.26522
	0.12500	1.10000	0.88889	0.80000	0.88889		1.33333	0.76169	1.29099
	0.11111	1.10000	0.90000	0.81818	0.90000		1.66667	0.78933	1.39415
	0.10000	1.10000	0.90909	0.83333	0.90909		2.00000	0.82244	1.49535
	0.33333	1.06667	0.75000	0.60000	0.75000		0.33333	0.83333	1.00000
	0.25000	1.06667	0.80000	0.66667	0.80000	0.25000	0.00000	0.83034	1.02041
0.25000	0.20000	1.06667	0.83333	0.71429	0.83333		0.66667	0.82606	1.13040
	0.16667	1.06667	0.85714	0.75000	0.85714		1.00000	0.83995	1.25992
	0.14286	1.06667	0.87500	0.77778	0.87500		1.25000	0.86222	1.36843
	0.12500	1.06667	0.88889	0.80000	0.88889		1.33333	0.87155	1.40572
	0.11111	1.06667	0.90000	0.81818	0.90000		1.66667	0.91614	1.55744
	0.10000	1.06667	0.90909	0.83333	0.90909		2.00000	0.96899	1.70998
	0.33333	1.04762	0.75000	0.60000	0.75000		0.00000	0.87500	1.00000
	0.25000	1.04762	0.80000	0.66667	0.80000		0.25000	0.87255	1.02299
	0.20000	1.04762	0.83333	0.71429	0.83333		0.66667	0.87193	1.14786
	0.16667	1.04762	0.85714	0.75000	0.85714		1.00000	0.89158	1.29684
0.16667	0.14286	1.04762	0.87500	0.77778	0.87500		1.25000	0.91984	1.42314
	0.12500	1.04762	0.88889	0.80000	0.88889	0.12500	1.33333	0.93145	1.46685
	0.11111	1.04762	0.90000	0.81818	0.90000		1.66667	0.98647	1.64613
	0.10000	1.04762	0.90909	0.83333	0.90909		2.00000	1.05143	1.82858
	0.33333	1.03571	0.75000	0.60000	0.75000				
	0.25000	1.03571	0.80000	0.66667	0.80000				
	0.20000	1.03571	0.83333	0.71429	0.83333				
	0.16667	1.03571	0.85714	0.75000	0.85714				
	0.14286	1.03571	0.87500	0.77778	0.87500				
	0.12500	1.03571	0.88889	0.80000	0.88889				
	0.11111	1.03571	0.90000	0.81818	0.90000				
0.14286	0.10000	1.03571	0.90909	0.83333	0.90909	0.10000			
	0.33333	1.02778	0.75000	0.60000	0.75000				
	0.25000	1.02778	0.80000	0.66667	0.80000				
	0.20000	1.02778	0.83333	0.71429	0.83333				
	0.16667	1.02778	0.85714	0.75000	0.85714				
	0.14286	1.02778	0.87500	0.77778	0.87500				
	0.12500	1.02778	0.88889	0.80000	0.88889				
	0.11111	1.02778	0.90000	0.81818	0.90000				
	0.10000	1.02778	0.90909	0.83333	0.90909				
	0.33333	1.02222	0.75000	0.60000	0.75000	0.07500			
0.12500	0.25000	1.02222	0.80000	0.66667	0.80000				
	0.20000	1.02222	0.83333	0.71429	0.83333				
	0.16667	1.02222	0.85714	0.75000	0.85714				
	0.14286	1.02222	0.87500	0.77778	0.87500				
	0.12500	1.02222	0.88889	0.80000	0.88889				
	0.11111	1.02222	0.90000	0.81818	0.90000				
	0.10000	1.02222	0.90909	0.83333	0.90909				
	0.33333	1.01818	0.75000	0.60000	0.75000				
	0.25000	1.01818	0.80000	0.66667	0.80000				
	0.20000	1.01818	0.83333	0.71429	0.83333	0.05000			
0.11111	0.16667	1.01818	0.85714	0.75000	0.85714				
	0.14286	1.01818	0.87500	0.77778	0.87500				
	0.12500	1.01818	0.88889	0.80000	0.88889				
	0.11111	1.01818	0.90000	0.81818	0.90000				
	0.10000	1.01818	0.90909	0.83333	0.90909				
	0.33333	1.01515	0.75000	0.60000	0.75000				
	0.25000	1.01515	0.80000	0.66667	0.80000				
	0.20000	1.01515	0.83333	0.71429	0.83333				
	0.16667	1.01515	0.85714	0.75000	0.85714				
	0.14286	1.01515	0.87500	0.77778	0.87500	0.02500			
0.10000	0.12500	1.01515	0.88889	0.80000	0.88889				
	0.11111	1.01515	0.90000	0.81818	0.90000				
	0.10000	1.01515	0.90909	0.83333	0.90909				
	0.33333	1.01515	0.75000	0.60000	0.75000				
	0.25000	1.01515	0.80000	0.66667	0.80000				
	0.20000	1.01515	0.83333	0.71429	0.83333				
	0.16667	1.01515	0.85714	0.75000	0.85714				
	0.14286	1.01515	0.87500	0.77778	0.87500				
	0.12500	1.01515	0.88889	0.80000	0.88889				
	0.11111	1.01515	0.90000	0.81818	0.90000				
	0.10000	1.01515	0.90909	0.83333	0.90909				

TABLE 14. THE GENERAL MAGNITUDES OF THE
DIMENSIONLESS COEFFICIENTS FOR $U = 0.5$

(a) Effect of the Values of n and m Such That

$\frac{1}{10} \leq n, m \leq \frac{1}{3}$ on the Values of $\zeta_1, \zeta_2,$
 ζ_3 and ζ_7

(b) Effect of the Values of k

$\left(\frac{1}{4} \leq k \leq \frac{1}{2}\right)$ and $W(0 \leq W \leq 2)$
on the Values of ζ_5 and ζ_6

N	M	ZETA 1	ZETA 2	ZETA 3	ZETA 7				
0.55533	0.33333	0.88889	0.70000	0.60000	0.81818				
	0.25000	0.88889	0.74386	0.66667	0.87273				
	0.20000	0.88889	0.77295	0.71429	0.90909				
0.25000	0.16667	0.88889	0.79365	0.75000	0.93506				
	0.14286	0.88889	0.80914	0.77778	0.95455				
	0.12500	0.88889	0.82116	0.80000	0.96970				
	0.11111	0.88889	0.83077	0.81818	0.98182				
	0.10000	0.88889	0.83862	0.83333	0.99174				
	0.33333	0.90370	0.71053	0.60000	0.80357				
	0.25000	0.90370	0.75556	0.66667	0.85714				
	0.20000	0.90370	0.78544	0.71429	0.89286				
	0.16667	0.90370	0.80672	0.75000	0.91837				
	0.14286	0.90370	0.82265	0.77778	0.93750				
0.20000	0.12500	0.90370	0.83502	0.80000	0.95238				
	0.11111	0.90370	0.84490	0.81818	0.96429				
	0.10000	0.90370	0.85297	0.83333	0.97403				
	0.33333	0.91534	0.71739	0.60000	0.79412				
	0.25000	0.91534	0.76322	0.66667	0.84706				
	0.20000	0.91534	0.79365	0.71429	0.88235				
	0.16667	0.91534	0.81533	0.75000	0.90756				
	0.14286	0.91534	0.83156	0.77778	0.92647				
	0.12500	0.91534	0.84416	0.80000	0.94118				
	0.11111	0.91534	0.85424	0.81818	0.95294				
0.16667	0.10000	0.91534	0.86247	0.83333	0.96257				
	0.33333	0.92460	0.72222	0.60000	0.78750				
	0.25000	0.92460	0.76863	0.66667	0.84000				
	0.20000	0.92460	0.79946	0.71429	0.87500				
	0.16667	0.92460	0.82143	0.75000	0.90000				
	0.14286	0.92460	0.83788	0.77778	0.91875				
	0.12500	0.92460	0.85066	0.80000	0.93333				
	0.11111	0.92460	0.86087	0.81818	0.94500				
	0.10000	0.92460	0.86922	0.83333	0.95455				
	0.33333	0.93210	0.72581	0.60000	0.78261				
0.14286	0.25000	0.93210	0.77265	0.66667	0.83478				
	0.20000	0.93210	0.80378	0.71429	0.86957				
	0.16667	0.93210	0.82597	0.75000	0.89441				
	0.14286	0.93210	0.84259	0.77778	0.91304				
	0.12500	0.93210	0.85550	0.80000	0.92754				
	0.11111	0.93210	0.86582	0.81818	0.93913				
	0.10000	0.93210	0.87426	0.83333	0.94862				
	0.33333	0.93827	0.72857	0.60000	0.77885				
	0.25000	0.93827	0.77576	0.66667	0.83077				
	0.20000	0.93827	0.80713	0.71429	0.86538				
0.12500	0.16667	0.93827	0.82949	0.75000	0.89011				
	0.14286	0.93827	0.84624	0.77778	0.90865				
	0.12500	0.93827	0.85926	0.80000	0.92308				
	0.11111	0.93827	0.86966	0.81818	0.93462				
	0.10000	0.93827	0.87817	0.83333	0.94406				
	0.33333	0.94343	0.73077	0.60000	0.77586				
	0.25000	0.94343	0.77823	0.66667	0.82759				
	0.20000	0.94343	0.80979	0.71429	0.86207				
	0.16667	0.94343	0.83230	0.75000	0.88670				
	0.14286	0.94343	0.84916	0.77778	0.90517				
0.11111	0.12500	0.94343	0.86226	0.80000	0.91954				
	0.11111	0.94343	0.87273	0.81818	0.93103				
	0.10000	0.94343	0.88129	0.83333	0.94044				
	0.33333	0.94781	0.73256	0.60000	0.77344				
	0.25000	0.94781	0.78025	0.66667	0.82500				
	0.20000	0.94781	0.81197	0.71429	0.85938				
	0.16667	0.94781	0.83459	0.75000	0.88393				
	0.14286	0.94781	0.85153	0.77778	0.90234				
	0.12500	0.94781	0.86470	0.80000	0.91667				
	0.11111	0.94781	0.87523	0.81818	0.92813				
0.10000	0.10000	0.94781	0.88384	0.83333	0.93750				

TABLE 15. THE GENERAL MAGNITUDES OF THE
DIMENSIONLESS COEFFICIENTS FOR $U = 1$

(a) Effect of the Values of n and m Such That

$$\frac{1}{10} \leq [n, m] \leq \frac{1}{3} \text{ on the Values of } \zeta_1, \zeta_2,$$

ζ_3 and ζ_7

(b) Effect of the Values of k

$$\left(\frac{1}{4} \leq k \leq \frac{1}{2}\right) \text{ and } W(0 \leq W \leq 2)$$

on the Values of ζ_5 and ζ_6

N	M	ZETA 1	ZETA 2	ZETA 3	ZETA 7	K	W	ZETA 5	ZETA 6
0.33333	0.33333	0.80000	0.67500	0.60000	0.85714				
	0.25000	0.80000	0.71579	0.66667	0.91429				
	0.20000	0.80000	0.74275	0.71429	0.95238				
	0.16667	0.80000	0.76190	0.75000	0.97959				
	0.14286	0.80000	0.77621	0.77778	1.00000				
	0.12500	0.80000	0.78730	0.80000	1.01587				
	0.11111	0.80000	0.79615	0.81818	1.02857				
	0.10000	0.80000	0.80338	0.83333	1.03896				
	0.33333	0.83333	0.69079	0.60000	0.83333				
	0.25000	0.83333	0.73333	0.66667	0.88889				
0.25000	0.20000	0.83333	0.76149	0.71429	0.92593				
	0.16667	0.83333	0.78151	0.75000	0.95238				
	0.14286	0.83333	0.79647	0.77778	0.97222				
	0.12500	0.83333	0.80808	0.80000	0.98765				
	0.11111	0.83333	0.81735	0.81818	1.00000				
	0.10000	0.83333	0.82492	0.83333	1.01010				
	0.33333	0.85714	0.70100	0.60000	0.81818				
	0.25000	0.85714	0.74483	0.66667	0.87273				
	0.20000	0.85714	0.77381	0.71429	0.90909				
	0.16667	0.85714	0.79443	0.75000	0.93506				
0.20000	0.14286	0.85714	0.80984	0.77778	0.95455				
	0.12500	0.85714	0.82180	0.80000	0.96970				
	0.11111	0.85714	0.83136	0.81818	0.98182				
	0.10000	0.85714	0.83916	0.83333	0.99174				
	0.33333	0.87500	0.70833	0.60000	0.80769				
	0.25000	0.87500	0.75294	0.66667	0.86154				
	0.20000	0.87500	0.78252	0.71429	0.89744				
	0.16667	0.87500	0.80357	0.75000	0.92308				
	0.14286	0.87500	0.81932	0.77778	0.94231				
	0.12500	0.87500	0.83154	0.80000	0.95726				
0.16667	0.11111	0.87500	0.84130	0.81818	0.96923				
	0.10000	0.87500	0.84928	0.83333	0.97902				
	0.33333	0.88889	0.71371	0.60000	0.80000				
	0.25000	0.88889	0.75897	0.66667	0.85333				
	0.20000	0.88889	0.78901	0.71429	0.88889				
	0.16667	0.88889	0.81039	0.75000	0.91429				
	0.14286	0.88889	0.82639	0.77778	0.93333				
	0.12500	0.88889	0.83881	0.80000	0.94815				
	0.11111	0.88889	0.84873	0.81818	0.96000				
	0.10000	0.88889	0.85684	0.83333	0.96970				
0.12500	0.33333	0.90000	0.71786	0.60000	0.79412				
	0.25000	0.90000	0.76364	0.66667	0.84706				
	0.20000	0.90000	0.79403	0.71429	0.88235				
	0.16667	0.90000	0.81567	0.75000	0.90756				
	0.14286	0.90000	0.83187	0.77778	0.92647				
	0.12500	0.90000	0.84444	0.80000	0.94118				
	0.11111	0.90000	0.85449	0.81818	0.95294				
	0.10000	0.90000	0.86271	0.83333	0.96257				
	0.33333	0.90909	0.72115	0.60000	0.78947				
	0.25000	0.90909	0.76735	0.66667	0.84211				
0.11111	0.20000	0.90909	0.79802	0.71429	0.87719				
	0.16667	0.90909	0.81988	0.75000	0.90226				
	0.14286	0.90909	0.83623	0.77778	0.92165				
	0.12500	0.90909	0.84894	0.80000	0.93567				
	0.11111	0.90909	0.85909	0.81818	0.94737				
	0.10000	0.90909	0.86739	0.83333	0.95694				
	0.33333	0.91667	0.72384	0.60000	0.78571				
	0.25000	0.91667	0.77037	0.66667	0.83810				
	0.20000	0.91667	0.80128	0.71429	0.87302				
	0.16667	0.91667	0.82331	0.75000	0.89796				
0.10000	0.14286	0.91667	0.83980	0.77778	0.91667				
	0.12500	0.91667	0.85261	0.80000	0.93122				
	0.11111	0.91667	0.86284	0.81818	0.94286				
	0.10000	0.91667	0.87121	0.83333	0.95238				

TABLE 16. THE GENERAL MAGNITUDES OF THE
DIMENSIONLESS COEFFICIENTS FOR $U = 1.5$

(a) Effect of the Values of n and m Such That

$\frac{1}{10} \leq [n, m] \leq \frac{1}{3}$ on the Values of $\zeta_1, \zeta_2,$
 ζ_3 and ζ_7

(b) Effect of the Values of k

$\left(\frac{1}{4} \leq k \leq \frac{1}{2}\right)$ and $W(0 \leq W \leq 2)$
on the Values of ζ_5 and ζ_6

N	M	ZETA 1	ZETA 2	ZETA 3	ZETA 7	K	W	ZETA 5	ZETA 6
0.33333	0.33333	0.75200	0.66000	0.60000	0.88235	0.50000	0.00000	0.71773	1.14412
	0.25000	0.75200	0.69895	0.66667	0.94118		0.25000	0.72275	1.16638
	0.20000	0.75200	0.72464	0.71429	0.98039		0.66667	0.74629	1.26070
	0.16667	0.75200	0.74286	0.75000	1.00840		1.00000	0.77183	1.34914
	0.14286	0.75200	0.75645	0.77778	1.02941		1.25000	0.79330	1.41607
0.25000	0.12500	0.75200	0.76698	0.80000	1.04575	0.33333	0.00000	0.86519	1.61255
	0.11111	0.75200	0.77538	0.81818	1.05882		0.25000	0.82000	1.23599
	0.10000	0.75200	0.78224	0.83333	1.06952		0.66667	0.85908	1.38512
	0.33333	0.79467	0.67895	0.60000	0.85227		1.00000	0.89620	1.50633
	0.25000	0.79467	0.72000	0.66667	0.90909		1.25000	0.92838	1.60175
0.20000	0.20000	0.79467	0.74713	0.71429	0.94697	0.25000	0.00000	0.87796	1.29139
	0.16667	0.79467	0.76639	0.75000	0.97403		0.25000	0.88538	1.32105
	0.14286	0.79467	0.78077	0.77778	0.99432		0.66667	0.92271	1.45613
	0.12500	0.79467	0.79192	0.80000	1.01010		1.00000	0.96643	1.59463
	0.11111	0.79467	0.80082	0.81818	1.02273		1.25000	1.00486	1.70577
0.16667	0.10000	0.79467	0.80808	0.83333	1.03306	0.16667	0.00000	0.82661	1.26337
	0.33333	0.82476	0.69130	0.60000	0.83333		0.25000	0.82661	1.26337
	0.25000	0.82476	0.73379	0.66667	0.88889		0.66667	0.85908	1.38512
	0.20000	0.82476	0.76190	0.71429	0.92593		1.00000	0.89620	1.50633
	0.16667	0.82476	0.78188	0.75000	0.95238		1.25000	0.92838	1.60175
0.14286	0.14286	0.82476	0.79681	0.77778	0.97222	0.12500	0.00000	0.86519	1.61255
	0.12500	0.82476	0.80839	0.80000	0.98765		0.25000	0.82000	1.23599
	0.11111	0.82476	0.81763	0.81818	1.00000		0.66667	0.85908	1.38512
	0.10000	0.82476	0.82517	0.83333	1.01010		1.00000	0.89620	1.50633
	0.33333	0.84714	0.70000	0.60000	0.82031		1.25000	0.92838	1.60175
0.12500	0.25000	0.84714	0.74353	0.66667	0.87500	0.10000	0.00000	0.87796	1.29139
	0.20000	0.84714	0.77236	0.71429	0.91146		0.25000	0.88538	1.32105
	0.16667	0.84714	0.79286	0.75000	0.93750		0.66667	0.92271	1.45613
	0.14286	0.84714	0.80818	0.77778	0.95703		1.00000	0.96643	1.59463
	0.12500	0.84714	0.82007	0.80000	0.97222		1.25000	1.00486	1.70577
0.11111	0.11111	0.84714	0.82957	0.81818	0.98438	0.08333	0.00000	0.82661	1.26337
	0.10000	0.84714	0.83732	0.83333	0.99432		0.25000	0.82661	1.26337
	0.33333	0.86444	0.70645	0.60000	0.81081		0.66667	0.85908	1.38512
	0.25000	0.86444	0.75077	0.66667	0.86486		1.00000	0.89620	1.50633
	0.20000	0.86444	0.78014	0.71429	0.90090		1.25000	0.92838	1.60175
0.10000	0.16667	0.86444	0.80104	0.75000	0.92664	0.06667	0.00000	0.87796	1.29139
	0.14286	0.86444	0.81667	0.77778	0.94595		0.25000	0.88538	1.32105
	0.12500	0.86444	0.82880	0.80000	0.96096		0.66667	0.92271	1.45613
	0.11111	0.86444	0.83848	0.81818	0.97297		1.00000	0.96643	1.59463
	0.10000	0.86444	0.84640	0.83333	0.98280		1.25000	1.00486	1.70577
0.08333	0.33333	0.87822	0.71143	0.60000	0.80357	0.05000	0.00000	0.82661	1.26337
	0.25000	0.87822	0.75636	0.66667	0.85714		0.25000	0.82661	1.26337
	0.20000	0.87822	0.78616	0.71429	0.89286		0.66667	0.85908	1.38512
	0.16667	0.87822	0.80737	0.75000	0.91837		1.00000	0.89620	1.50633
	0.14286	0.87822	0.82324	0.77778	0.93750		1.25000	0.92838	1.60175
0.06667	0.12500	0.87822	0.83556	0.80000	0.95238	0.04167	0.00000	0.86519	1.61255
	0.11111	0.87822	0.84539	0.81818	0.96429		0.25000	0.82000	1.23599
	0.10000	0.87822	0.85343	0.83333	0.97403		0.66667	0.85908	1.38512
	0.33333	0.88945	0.71538	0.60000	0.79787		1.00000	0.89620	1.50633
	0.25000	0.88945	0.76082	0.66667	0.85106		1.25000	0.92838	1.60175
0.05000	0.20000	0.88945	0.79096	0.71429	0.88652	0.03333	0.00000	0.87796	1.29139
	0.16667	0.88945	0.81242	0.75000	0.91185		0.25000	0.88538	1.32105
	0.14286	0.88945	0.82848	0.77778	0.93085		0.66667	0.92271	1.45613
	0.12500	0.88945	0.84095	0.80000	0.94563		1.00000	0.96643	1.59463
	0.11111	0.88945	0.85091	0.81818	0.95745		1.25000	1.00486	1.70577
0.04167	0.10000	0.88945	0.85905	0.83333	0.96712	0.02500	0.00000	0.82661	1.26337
	0.33333	0.89879	0.71860	0.60000	0.79327		0.25000	0.82661	1.26337
	0.25000	0.89879	0.76444	0.66667	0.84615		0.66667	0.85908	1.38512
	0.20000	0.89879	0.79487	0.71429	0.88141		1.00000	0.89620	1.50633
	0.16667	0.89879	0.81654	0.75000	0.90659		1.25000	0.92838	1.60175
0.03333	0.14286	0.89879	0.83276	0.77778	0.92548	0.01667	0.00000	0.87796	1.29139
	0.12500	0.89879	0.84535	0.80000	0.94017		0.25000	0.88538	1.32105
	0.11111	0.89879	0.85541	0.81818	0.95192		0.66667	0.92271	1.45613
	0.10000	0.89879	0.86364	0.83333	0.96154		1.00000	0.96643	1.59463
							1.25000	1.00486	1.70577

TABLE 17. THE GENERAL MAGNITUDES OF THE
DIMENSIONLESS COEFFICIENTS FOR $U = 2$

(a) Effect of the Values of n and m Such That

$$\frac{1}{10} \leq [n, m] \leq \frac{1}{3} \text{ on the Values of } \zeta_1, \zeta_2,$$

ζ_3 and ζ_7

(b) Effect of the Values of k

$$\left(\frac{1}{4} \leq k \leq \frac{1}{2}\right) \text{ and } W(0 \leq W \leq 2)$$

on the Values of ζ_5 and ζ_6

N	M	ZETA 1	ZETA 2	ZETA 3	ZETA 7	K	W	ZETA 5	ZETA 6
0.53333	0.33333	0.72222	0.65000	0.60000	0.90000				
	0.25000	0.72222	0.68772	0.66667	0.96000				
	0.20000	0.72222	0.71256	0.71429	1.00000				
	0.16667	0.72222	0.73016	0.75000	1.02857				
0.25000	0.14286	0.72222	0.74328	0.77778	1.05000				
	0.12500	0.72222	0.75344	0.80000	1.06667				
	0.11111	0.72222	0.76154	0.81818	1.08000				
	0.10000	0.72222	0.76815	0.83333	1.09091				
0.20000	0.33333	0.77037	0.67105	0.60000	0.86538				
	0.25000	0.77037	0.71111	0.66667	0.92308				
	0.20000	0.77037	0.73755	0.71429	0.96154				
	0.16667	0.77037	0.75630	0.75000	0.98901				
0.16667	0.14286	0.77037	0.77030	0.77778	1.00962				
	0.12500	0.77037	0.78114	0.80000	1.02564				
	0.11111	0.77037	0.78980	0.81818	1.03846				
	0.10000	0.77037	0.79686	0.83333	1.04895				
0.20000	0.33333	0.80423	0.68478	0.60000	0.84375				
	0.25000	0.80423	0.72644	0.66667	0.90000				
	0.20000	0.80423	0.75397	0.71429	0.93750				
	0.16667	0.80423	0.77352	0.75000	0.96429				
0.16667	0.14286	0.80423	0.78812	0.77778	0.98438				
	0.12500	0.80423	0.79944	0.80000	1.00000				
	0.11111	0.80423	0.80847	0.81818	1.01250				
	0.10000	0.80423	0.81585	0.83333	1.02273				
0.12500	0.33333	0.82937	0.69444	0.60000	0.82895				
	0.25000	0.82937	0.73725	0.66667	0.88421				
	0.20000	0.82937	0.76558	0.71429	0.92105				
	0.16667	0.82937	0.78571	0.75000	0.94737				
0.14286	0.14286	0.82937	0.80076	0.77778	0.96711				
	0.12500	0.82937	0.81243	0.80000	0.98246				
	0.11111	0.82937	0.82174	0.81818	0.99474				
	0.10000	0.82937	0.82935	0.83333	1.00478				
0.12500	0.33333	0.84877	0.70161	0.60000	0.81818				
	0.25000	0.84877	0.74530	0.66667	0.87273				
	0.20000	0.84877	0.77423	0.71429	0.90909				
	0.16667	0.84877	0.79481	0.75000	0.93506				
0.11111	0.14286	0.84877	0.81019	0.77778	0.95455				
	0.12500	0.84877	0.82212	0.80000	0.96970				
	0.11111	0.84877	0.83165	0.81818	0.98182				
	0.10000	0.84877	0.83943	0.83333	0.99174				
0.10000	0.33333	0.86420	0.70714	0.60000	0.81000				
	0.25000	0.86420	0.75152	0.66667	0.86400				
	0.20000	0.86420	0.78092	0.71429	0.90000				
	0.16667	0.86420	0.80184	0.75000	0.92571				
0.11111	0.14286	0.86420	0.81749	0.77778	0.94500				
	0.12500	0.86420	0.82963	0.80000	0.96000				
	0.11111	0.86420	0.83933	0.81818	0.97200				
	0.10000	0.86420	0.84725	0.83333	0.98182				
0.10000	0.33333	0.87677	0.71154	0.60000	0.80357				
	0.25000	0.87677	0.75646	0.66667	0.85714				
	0.20000	0.87677	0.78625	0.71429	0.89286				
	0.16667	0.87677	0.80745	0.75000	0.91837				
0.10000	0.14286	0.87677	0.82331	0.77778	0.93750				
	0.12500	0.87677	0.83562	0.80000	0.95238				
	0.11111	0.87677	0.84545	0.81818	0.96429				
	0.10000	0.87677	0.85349	0.83333	0.97403				
0.10000	0.33333	0.88721	0.71512	0.60000	0.79839				
	0.25000	0.88721	0.76049	0.66667	0.85161				
	0.20000	0.88721	0.79060	0.71429	0.88710				
	0.16667	0.88721	0.81203	0.75000	0.91244				
0.10000	0.14286	0.88721	0.82807	0.77778	0.93145				
	0.12500	0.88721	0.84051	0.80000	0.94624				
	0.11111	0.88721	0.85046	0.81818	0.95806				
	0.10000	0.88721	0.85859	0.83333	0.96774				

TABLE 18. THE GENERAL MAGNITUDES OF THE
DIMENSIONLESS COEFFICIENTS FOR $U = 2.5$

(a) Effect of the Values of n and m Such That
 $\frac{1}{10} \leq [n, m] \leq \frac{1}{3}$ on the Values of $\zeta_1, \zeta_2,$
 ζ_3 and ζ_7

(b) Effect of the Values of k
 $\left(\frac{1}{4} \leq k \leq \frac{1}{2}\right)$ and $W(0 \leq W \leq 2)$
on the Values of ζ_5 and ζ_6

N	M	ZETA 1	ZETA 2	ZETA 3	ZETA 7	K	W	ZETA 5	ZETA 6
0.33333	0.33333	0.70204	0.64286	0.60000	0.91304	0.50000	0.00000	0.83850	1.54779
	0.25000	0.70204	0.67970	0.66667	0.97391		0.25000	0.84060	1.55319
	0.20000	0.70204	0.70393	0.71429	1.01449		0.66667	0.85283	1.58439
	0.16667	0.70204	0.72109	0.75000	1.04348		1.00000	0.86916	1.62528
	0.14286	0.70204	0.73387	0.77778	1.06522		1.25000	0.88434	1.66258
	0.12500	0.70204	0.74376	0.80000	1.08213		1.33333	0.88985	1.67598
	0.11111	0.70204	0.75165	0.81818	1.09565		1.66667	0.91373	1.73316
	0.10000	0.70204	0.75808	0.83333	1.10672		2.00000	0.93989	1.79445
	0.33333	0.75374	0.66541	0.60000	0.87500		0.00000	1.00338	1.80779
	0.25000	0.75374	0.70476	0.66667	0.93333	0.33333	0.25000	1.00656	1.81576
0.25000	0.20000	0.75374	0.73071	0.71429	0.97222		0.66667	1.02520	1.86214
	0.16667	0.75374	0.74910	0.75000	1.00000		1.00000	1.05040	1.92378
	0.14286	0.75374	0.76282	0.77778	1.02083		1.25000	1.07410	1.98077
	0.12500	0.75374	0.77345	0.80000	1.03704		1.33333	1.08277	2.00140
	0.11111	0.75374	0.78192	0.81818	1.05000		1.66667	1.12068	2.09044
	0.10000	0.75374	0.78884	0.83333	1.06061		2.00000	1.16282	2.18746
	0.33333	0.79009	0.68012	0.60000	0.85135		0.00000	1.09892	1.95714
	0.25000	0.79009	0.72118	0.66667	0.90811		0.25000	1.10273	1.96659
	0.20000	0.79009	0.74830	0.71429	0.94595		0.66667	1.12519	2.02175
	0.16667	0.79009	0.76755	0.75000	0.97297	0.25000	1.00000	1.15573	2.09556
0.16667	0.14286	0.79009	0.78191	0.77778	0.99324		1.25000	1.18462	2.16425
	0.12500	0.79009	0.79305	0.80000	1.00901		1.33333	1.19522	2.18922
	0.11111	0.79009	0.80194	0.81818	1.02162		1.66667	1.24181	2.29755
	0.10000	0.79009	0.80919	0.83333	1.03194		2.00000	1.29395	2.41655
	0.33333	0.81706	0.69048	0.60000	0.83523		0.00000	1.09892	1.95714
	0.25000	0.81706	0.73277	0.66667	0.89091		0.25000	1.10273	1.96659
	0.20000	0.81706	0.76074	0.71429	0.92803		0.66667	1.12519	2.02175
	0.16667	0.81706	0.78061	0.75000	0.95455		1.00000	1.15573	2.09556
	0.14286	0.81706	0.79545	0.77778	0.97443		1.25000	1.18462	2.16425
	0.12500	0.81706	0.80696	0.80000	0.98990	0.12500	1.33333	1.19522	2.18922
0.14286	0.11111	0.81706	0.81615	0.81818	1.00227		1.66667	1.24181	2.29755
	0.10000	0.81706	0.82365	0.83333	1.01240		2.00000	1.29395	2.41655
	0.33333	0.83787	0.69816	0.60000	0.82353		0.00000	1.09892	1.95714
	0.25000	0.83787	0.74139	0.66667	0.87843		0.25000	1.10273	1.96659
	0.20000	0.83787	0.77001	0.71429	0.91503		0.66667	1.12519	2.02175
	0.16667	0.83787	0.79035	0.75000	0.94118		1.00000	1.15573	2.09556
	0.14286	0.83787	0.80556	0.77778	0.96078		1.25000	1.18462	2.16425
	0.12500	0.83787	0.81735	0.80000	0.97603		1.33333	1.19522	2.18922
	0.11111	0.83787	0.82676	0.81818	0.98824		1.66667	1.24181	2.29755
	0.10000	0.83787	0.83445	0.83333	0.99822	0.10000	2.00000	1.29395	2.41655
0.12500	0.33333	0.85442	0.70408	0.60000	0.81466		0.00000	1.09892	1.95714
	0.25000	0.85442	0.74805	0.66667	0.86897		0.25000	1.10273	1.96659
	0.20000	0.85442	0.77718	0.71429	0.90517		0.66667	1.12519	2.02175
	0.16667	0.85442	0.79789	0.75000	0.93103		1.00000	1.15573	2.09556
	0.14286	0.85442	0.81338	0.77778	0.95043		1.25000	1.18462	2.16425
	0.12500	0.85442	0.82540	0.80000	0.96552		1.33333	1.19522	2.18922
	0.11111	0.85442	0.83499	0.81818	0.97759		1.66667	1.24181	2.29755
	0.10000	0.85442	0.84283	0.83333	0.98746		2.00000	1.29395	2.41655
	0.33333	0.86790	0.70879	0.60000	0.80769		0.00000	1.09892	1.95714
	0.25000	0.86790	0.75335	0.66667	0.86154	0.08333	0.25000	1.10273	1.96659
0.11111	0.20000	0.86790	0.78289	0.71429	0.89744		0.66667	1.12519	2.02175
	0.16667	0.86790	0.80390	0.75000	0.92308		1.00000	1.15573	2.09556
	0.14286	0.86790	0.81962	0.77778	0.94231		1.25000	1.18462	2.16425
	0.12500	0.86790	0.83182	0.80000	0.95726		1.33333	1.19522	2.18922
	0.11111	0.86790	0.84156	0.81818	0.96923		1.66667	1.24181	2.29755
	0.10000	0.86790	0.84952	0.83333	0.97902		2.00000	1.29395	2.41655
	0.33333	0.87910	0.71262	0.60000	0.80208		0.00000	1.09892	1.95714
	0.25000	0.87910	0.75767	0.66667	0.85556		0.25000	1.10273	1.96659
	0.20000	0.87910	0.78755	0.71429	0.89120		0.66667	1.12519	2.02175
	0.16667	0.87910	0.80881	0.75000	0.91667	0.06667	1.00000	1.15573	2.09556
0.10000	0.14286	0.87910	0.82471	0.77778	0.93576		1.25000	1.18462	2.16425
	0.12500	0.87910	0.83706	0.80000	0.95062		1.33333	1.19522	2.18922
	0.11111	0.87910	0.84692	0.81818	0.96250		1.66667	1.24181	2.29755
0.10000	0.10000	0.87910	0.85498	0.83333	0.97222	0.05000	2.00000	1.29395	2.41655
	0.08333	0.87910	0.86498	0.85000	0.98000		0.00000	1.09892	1.95714

TABLE 19. THE GENERAL MAGNITUDES OF THE
DIMENSIONLESS COEFFICIENTS FOR $U = 3$

(a) Effect of the Values of n and m Such That

$$\frac{1}{10} \leq [n, m] \leq \frac{1}{3} \text{ on the Values of } \zeta_1, \zeta_2,$$

ζ_3 and ζ_7

(b) Effect of the Values of k

$$\left(\frac{1}{4} \leq k \leq \frac{1}{2}\right) \text{ and } W(0 \leq W \leq 2)$$

on the Values of ζ_5 and ζ_6

N	M	ZETA 1	ZETA 2	ZETA 3	ZETA 7	K	W	ZETA 5	ZETA 6
0.33333	0.33333	0.68750	0.63750	0.60000	0.92308				
	0.25000	0.68750	0.67368	0.66667	0.98462				
	0.20000	0.68750	0.69746	0.71429	1.02564				
	0.16667	0.68750	0.71429	0.75000	1.05495				
	0.14286	0.68750	0.72681	0.77778	1.07692				
0.25000	0.12500	0.68750	0.73651	0.80000	1.09402				
	0.11111	0.68750	0.74423	0.81818	1.10769				
	0.10000	0.68750	0.75053	0.83333	1.11888				
	0.33333	0.74167	0.66118	0.60000	0.88235				
	0.25000	0.74167	0.70000	0.66667	0.94118				
0.20000	0.20000	0.74167	0.72557	0.71429	0.98039				
	0.16667	0.74167	0.74370	0.75000	1.00840				
	0.14286	0.74167	0.75721	0.77778	1.02941				
	0.12500	0.74167	0.76768	0.80000	1.04575				
	0.11111	0.74167	0.77602	0.81818	1.05882				
0.16667	0.10000	0.74167	0.78283	0.83333	1.06952				
	0.33333	0.77976	0.67663	0.60000	0.85714				
	0.25000	0.77976	0.71724	0.66667	0.91429				
	0.20000	0.77976	0.74405	0.71429	0.95238				
	0.16667	0.77976	0.76307	0.75000	0.97959				
0.14286	0.12500	0.77976	0.77726	0.77778	1.00000				
	0.11111	0.77976	0.78826	0.80000	1.01587				
	0.10000	0.77976	0.79703	0.81818	1.02857				
	0.33333	0.80804	0.80420	0.83333	1.03896				
	0.25000	0.80804	0.80804	0.80000	0.84000				
0.12500	0.20000	0.80804	0.72941	0.66667	0.89660				
	0.16667	0.80804	0.75711	0.71429	0.93333				
	0.14286	0.80804	0.77679	0.75000	0.96000				
	0.12500	0.80804	0.79148	0.77778	0.98000				
	0.11111	0.80804	0.80287	0.80000	0.99556				
0.11111	0.10000	0.80804	0.81196	0.81818	1.00800				
	0.33333	0.82986	0.81938	0.83333	1.01818				
	0.25000	0.82986	0.69556	0.60000	0.82759				
	0.20000	0.82986	0.73846	0.66667	0.88276				
	0.16667	0.82986	0.76684	0.71429	0.91954				
0.10000	0.14286	0.82986	0.78701	0.75000	0.94581				
	0.12500	0.82986	0.80208	0.77778	0.96552				
	0.11111	0.82986	0.81377	0.80000	0.98084				
	0.10000	0.82986	0.82310	0.81818	0.99310				
	0.33333	0.84722	0.83072	0.83333	1.00313				
0.08333	0.25000	0.84722	0.70179	0.60000	0.81818				
	0.20000	0.84722	0.74545	0.66667	0.87273				
	0.16667	0.84722	0.77437	0.71429	0.90909				
	0.14286	0.84722	0.79493	0.75000	0.93506				
	0.12500	0.84722	0.81030	0.77778	0.95455				
0.07692	0.11111	0.84722	0.82222	0.80000	0.96970				
	0.10000	0.84722	0.83174	0.81818	0.98182				
	0.33333	0.86136	0.83952	0.83333	0.99174				
	0.25000	0.86136	0.70673	0.60000	0.81081				
	0.20000	0.86136	0.75102	0.66667	0.86486				
0.07143	0.16667	0.86136	0.78037	0.71429	0.90090				
	0.14286	0.86136	0.80124	0.75000	0.92664				
	0.12500	0.86136	0.81685	0.77778	0.94595				
	0.11111	0.86136	0.82896	0.80000	0.96096				
	0.10000	0.86136	0.83804	0.81818	0.97297				
0.06774	0.33333	0.87311	0.84654	0.83333	0.98280				
	0.25000	0.87311	0.71076	0.60000	0.80488				
	0.20000	0.87311	0.75556	0.66667	0.85854				
	0.16667	0.87311	0.78526	0.71429	0.89431				
	0.14286	0.87311	0.80639	0.75000	0.91986				
0.06452	0.12500	0.87311	0.82220	0.77778	0.93902				
	0.11111	0.87311	0.83447	0.80000	0.95393				
	0.10000	0.87311	0.84427	0.81818	0.96585				
	0.33333	0.87311	0.85227	0.83333	0.97561				
	0.25000	0.87311	0.85227	0.83333	0.97561				

TABLE 20. THE GENERAL MAGNITUDES OF THE
DIMENSIONLESS COEFFICIENTS FOR $U = 4$

(a) Effect of the Values of n and m Such That

$$\frac{1}{10} \leq [n, m] \leq \frac{1}{3} \text{ on the Values of } \zeta_1, \zeta_2,$$

ζ_3 and ζ_7

(b) Effect of the Values of k

$$\left(\frac{1}{4} \leq k \leq \frac{1}{2}\right) \text{ and } W(0 \leq W \leq 2)$$

on the Values of ζ_5 and ζ_6

N	M	ZETA 1	ZETA 2	ZETA 3	ZETA 7	K	W	ZETA 5	ZETA 6
0.33333	0.33333	0.66800	0.63000	0.60000	0.93750	0.50000	0.00000	1.02353	1.98406
	0.25000	0.66800	0.66526	0.66667	1.00000		0.25000	1.02453	1.98626
	0.20000	0.66800	0.68841	0.71429	1.04167		0.66667	1.03055	1.99949
	0.16667	0.66800	0.70476	0.75000	1.07143		1.00000	1.03908	2.01819
	0.14286	0.66800	0.71694	0.77778	1.09375		1.25000	1.04746	2.03650
	0.12500	0.66800	0.72635	0.80000	1.11111		1.33333	1.05061	2.04335
0.25000	0.11111	0.66800	0.73385	0.81818	1.12500	0.33333	1.66667	1.06477	2.07414
	0.10000	0.66800	0.73996	0.83333	1.13636		2.00000	1.08122	2.10971
	0.33333	0.72533	0.65526	0.60000	0.89286		0.00000	1.30377	2.50205
	0.25000	0.72533	0.69333	0.66667	0.95238		0.25000	1.30543	2.50567
	0.20000	0.72533	0.71839	0.71429	0.99206		0.66667	1.31547	2.52751
	0.16667	0.72533	0.73613	0.75000	1.02041		1.00000	1.32974	2.55847
0.20000	0.14286	0.72533	0.74936	0.77778	1.04167	0.25000	1.25000	1.34380	2.58890
	0.12500	0.72533	0.75960	0.80000	1.05820		1.33333	1.34909	2.60032
	0.11111	0.72533	0.76776	0.81818	1.07143		1.66667	1.37302	2.65187
	0.10000	0.72533	0.77441	0.83333	1.08225		2.00000	1.40097	2.71183
	0.33333	0.76571	0.67174	0.60000	0.86538		0.00000	1.47231	2.81160
	0.25000	0.76571	0.71172	0.66667	0.92308		0.25000	1.47440	2.81613
0.16667	0.20000	0.76571	0.73810	0.71429	0.96154	0.16667	0.66667	1.48704	2.84348
	0.16667	0.76571	0.75679	0.75000	0.98901		1.00000	1.50502	2.88231
	0.14286	0.76571	0.77074	0.77778	1.00962		1.25000	1.52279	2.92057
	0.12500	0.76571	0.78155	0.80000	1.02564		1.33333	1.52947	2.93494
	0.11111	0.76571	0.79017	0.81818	1.03846		1.66667	1.55979	2.99995
	0.10000	0.76571	0.79720	0.83333	1.04895		2.00000	1.59532	3.07581
0.14286	0.33333	0.79571	0.68333	0.60000	0.84677	0.12500	0.00000	1.02353	1.98406
	0.25000	0.79571	0.72471	0.66667	0.90323		0.25000	1.02453	1.98626
	0.20000	0.79571	0.75203	0.71429	0.94086		0.66667	1.03055	1.99949
	0.16667	0.79571	0.77143	0.75000	0.96774		1.00000	1.03908	2.01819
	0.14286	0.79571	0.78591	0.77778	0.98790		1.25000	1.04746	2.03650
	0.12500	0.79571	0.79713	0.80000	1.00358		1.33333	1.05061	2.04335
0.12500	0.11111	0.79571	0.80609	0.81818	1.01613	0.10000	1.66667	1.06477	2.07414
	0.10000	0.79571	0.81340	0.83333	1.02639		2.00000	1.08122	2.10971
	0.33333	0.81889	0.69194	0.60000	0.83333		0.00000	1.30377	2.50205
	0.25000	0.81889	0.73436	0.66667	0.88889		0.25000	1.30543	2.50567
	0.20000	0.81889	0.76241	0.71429	0.92593		0.66667	1.31547	2.52751
	0.16667	0.81889	0.78234	0.75000	0.95238		1.00000	1.32974	2.55847
0.11111	0.14286	0.81889	0.79722	0.77778	0.97222	0.08333	1.25000	1.34380	2.58890
	0.12500	0.81889	0.80876	0.80000	0.98765		1.33333	1.34909	2.60032
	0.11111	0.81889	0.81797	0.81818	1.00000		1.66667	1.37302	2.65187
	0.10000	0.81889	0.82550	0.83333	1.01010		2.00000	1.40097	2.71183
	0.33333	0.83733	0.69857	0.60000	0.82317		0.00000	1.47231	2.81160
	0.25000	0.83733	0.74182	0.66667	0.87805		0.25000	1.47440	2.81613
0.10000	0.20000	0.83733	0.77044	0.71429	0.91463	0.06667	0.66667	1.48704	2.84348
	0.16667	0.83733	0.79078	0.75000	0.94077		1.00000	1.50502	2.88231
	0.14286	0.83733	0.80599	0.77778	0.96037		1.25000	1.52279	2.92057
	0.12500	0.83733	0.81778	0.80000	0.97561		1.33333	1.52947	2.93494
	0.11111	0.83733	0.82719	0.81818	0.98780		1.66667	1.55979	2.99995
	0.10000	0.83733	0.83488	0.83333	0.99778		2.00000	1.59532	3.07581
0.08333	0.33333	0.85236	0.70385	0.60000	0.81522	0.05000	0.00000	1.02353	1.98406
	0.25000	0.85236	0.74776	0.66667	0.86957		0.25000	1.02453	1.98626
	0.20000	0.85236	0.77684	0.71429	0.90580		0.66667	1.03055	1.99949
	0.16667	0.85236	0.79752	0.75000	0.93168		1.00000	1.03908	2.01819
	0.14286	0.85236	0.81297	0.77778	0.95109		1.25000	1.04746	2.03650
	0.12500	0.85236	0.82497	0.80000	0.96618		1.33333	1.05061	2.04335
0.06667	0.11111	0.85236	0.83455	0.81818	0.97826	0.04000	1.66667	1.06477	2.07414
	0.10000	0.85236	0.84237	0.83333	0.98814		2.00000	1.08122	2.10971
	0.33333	0.86485	0.70814	0.60000	0.80882		0.00000	1.30377	2.50205
	0.25000	0.86485	0.75259	0.66667	0.86275		0.25000	1.30543	2.50567
	0.20000	0.86485	0.78205	0.71429	0.89869		0.66667	1.31547	2.52751
	0.16667	0.86485	0.80301	0.75000	0.92437		1.00000	1.32974	2.55847
0.05000	0.14286	0.86485	0.81868	0.77778	0.94363	0.03000	1.25000	1.34380	2.58890
	0.12500	0.86485	0.83084	0.80000	0.95861		1.33333	1.34909	2.60032
	0.11111	0.86485	0.84055	0.81818	0.97059		1.66667	1.37302	2.65187
	0.10000	0.86485	0.84848	0.83333	0.98039		2.00000	1.40097	2.71183
	0.33333	0.88000	0.71429	0.60000	0.81522		0.00000	1.47231	2.81160
	0.25000	0.88000	0.75259	0.66667	0.86275		0.25000	1.47440	2.81613

TABLE 21. THE GENERAL MAGNITUDES OF THE
DIMENSIONLESS COEFFICIENTS FOR $U = 5$

(a) Effect of the Values of n and m Such That

$$\frac{1}{10} \leq [n, m] \leq \frac{1}{3} \text{ on the Values of } \zeta_1, \zeta_2,$$

ζ_3 and ζ_7

(b) Effect of the Values of k

$$\left(\frac{1}{4} \leq k \leq \frac{1}{2}\right) \text{ and } W(0 \leq W \leq 2)$$

on the Values of ζ_5 and ζ_6

N	M	ZETA 1	ZETA 2	ZETA 3	ZETA 7				
0.55555	0.33333	0.65556	0.62500	0.60000	0.94737				
	0.25000	0.65556	0.65965	0.66667	1.01053				
	0.20000	0.65556	0.68237	0.71429	1.05263				
	0.16667	0.65556	0.69841	0.75000	1.08271				
	0.14286	0.65556	0.71035	0.77778	1.10526				
0.25000	0.12500	0.65556	0.71958	0.80000	1.12281				
	0.11111	0.65556	0.72692	0.81818	1.13684				
	0.10000	0.65556	0.73291	0.83333	1.14833				
	0.33333	0.71481	0.65132	0.60000	0.90000				
	0.25000	0.71481	0.68889	0.66667	0.96000				
0.20000	0.20000	0.71481	0.71360	0.71429	1.00000				
	0.16667	0.71481	0.73109	0.75000	1.02857				
	0.14286	0.71481	0.74412	0.77778	1.05000				
	0.12500	0.71481	0.75421	0.80000	1.06667				
	0.11111	0.71481	0.76224	0.81818	1.08000				
0.16667	0.10000	0.71481	0.76880	0.83333	1.09091				
	0.33333	0.75661	0.66848	0.60000	0.87097				
	0.25000	0.75661	0.70805	0.66667	0.92903				
	0.20000	0.75661	0.73413	0.71429	0.96774				
	0.16667	0.75661	0.75261	0.75000	0.99539				
0.14286	0.14286	0.75661	0.76640	0.77778	1.01613				
	0.12500	0.75661	0.77708	0.80000	1.03226				
	0.11111	0.75661	0.78559	0.81818	1.04516				
	0.10000	0.75661	0.79254	0.83333	1.05572				
	0.33333	0.78770	0.68056	0.60000	0.85135				
0.12500	0.25000	0.78770	0.72157	0.66667	0.90811				
	0.20000	0.78770	0.74864	0.71429	0.94595				
	0.16667	0.78770	0.76786	0.75000	0.97297				
	0.14286	0.78770	0.78220	0.77778	0.99324				
	0.12500	0.78770	0.79331	0.80000	1.00901				
0.11111	0.11111	0.78770	0.80217	0.81818	1.02162				
	0.10000	0.78770	0.80941	0.83333	1.03194				
	0.33333	0.81173	0.68952	0.60000	0.83721				
	0.25000	0.81173	0.73162	0.66667	0.89302				
	0.20000	0.81173	0.75946	0.71429	0.93023				
0.10000	0.16667	0.81173	0.77922	0.75000	0.95681				
	0.14286	0.81173	0.79398	0.77778	0.97674				
	0.12500	0.81173	0.80543	0.80000	0.99225				
	0.11111	0.81173	0.81456	0.81818	1.00465				
	0.10000	0.81173	0.82201	0.83333	1.01480				
0.08594	0.33333	0.83086	0.69643	0.60000	0.82653				
	0.25000	0.83086	0.73939	0.66667	0.88163				
	0.20000	0.83086	0.76782	0.71429	0.91837				
	0.16667	0.83086	0.78802	0.75000	0.94461				
	0.14286	0.83086	0.80311	0.77778	0.96429				
0.07795	0.12500	0.83086	0.81481	0.80000	0.97959				
	0.11111	0.83086	0.82416	0.81818	0.99184				
	0.10000	0.83086	0.83179	0.83333	1.00186				
	0.33333	0.84646	0.70192	0.60000	0.81818				
	0.25000	0.84646	0.74558	0.66667	0.87273				
0.06990	0.20000	0.84646	0.77448	0.71429	0.90909				
	0.16667	0.84646	0.79503	0.75000	0.93506				
	0.14286	0.84646	0.81039	0.77778	0.95455				
	0.12500	0.84646	0.82231	0.80000	0.96970				
	0.11111	0.84646	0.83182	0.81818	0.98182				
0.06175	0.10000	0.84646	0.83959	0.83333	0.99174				
	0.33333	0.85943	0.70640	0.60000	0.81148				
	0.25000	0.85943	0.75062	0.66667	0.86557				
	0.20000	0.85943	0.77991	0.71429	0.90164				
	0.16667	0.85943	0.80075	0.75000	0.92740				
0.05462	0.14286	0.85943	0.81633	0.77778	0.94672				
	0.12500	0.85943	0.82842	0.80000	0.96175				
	0.11111	0.85943	0.83807	0.81818	0.97377				
	0.10000	0.85943	0.84596	0.83333	0.98361				

TABLE 22. THE GENERAL MAGNITUDES OF THE
DIMENSIONLESS COEFFICIENTS FOR $U = 6$

(a) Effect of the Values of n and m Such That

$$\frac{1}{10} \leq [n, m] \leq \frac{1}{3} \text{ on the Values of } \zeta_1, \zeta_2,$$

ζ_3 and ζ_7

(b) Effect of the Values of k

$$\left(\frac{1}{4} \leq k \leq \frac{1}{2}\right) \text{ and } W(0 \leq W \leq 2)$$

on the Values of ζ_5 and ζ_6

N	M	ZETA 1	ZETA 2	ZETA 3	ZETA 7	K	W	ZETA 5	ZETA 6
0.33333	0.33333	0.64694	0.62143	0.60000	0.95455				
	0.25000	0.64694	0.65564	0.66667	1.01818				
	0.20000	0.64694	0.67805	0.71429	1.06061				
	0.16667	0.64694	0.69388	0.75000	1.09091				
	0.14286	0.64694	0.70565	0.77778	1.11364				
0.25000	0.12500	0.64694	0.71474	0.80000	1.13131				
	0.11111	0.64694	0.72198	0.81818	1.14545				
	0.10000	0.64694	0.72788	0.83333	1.15702				
	0.33333	0.70748	0.64850	0.60000	0.90517				
	0.25000	0.70748	0.68571	0.66667	0.96552				
0.20000	0.20000	0.70748	0.71018	0.71429	1.00575				
	0.16667	0.70748	0.72749	0.75000	1.03448				
	0.14286	0.70748	0.74038	0.77778	1.05603				
	0.12500	0.70748	0.75036	0.80000	1.07280				
	0.11111	0.70748	0.75831	0.81818	1.08621				
0.16667	0.10000	0.70748	0.76479	0.83333	1.09718				
	0.33333	0.75024	0.66615	0.60000	0.87500				
	0.25000	0.75024	0.70542	0.66667	0.93333				
	0.20000	0.75024	0.73129	0.71429	0.97222				
	0.16667	0.75024	0.74963	0.75000	1.00000				
0.14286	0.14286	0.75024	0.76330	0.77778	1.02083				
	0.12500	0.75024	0.77388	0.80000	1.03704				
	0.11111	0.75024	0.78232	0.81818	1.05000				
	0.10000	0.75024	0.78921	0.83333	1.06061				
	0.33333	0.78207	0.67857	0.60000	0.85465				
0.12500	0.25000	0.78207	0.71933	0.66667	0.91163				
	0.20000	0.78207	0.74623	0.71429	0.94961				
	0.16667	0.78207	0.76531	0.75000	0.97674				
	0.14286	0.78207	0.77955	0.77778	0.99709				
	0.12500	0.78207	0.79058	0.80000	1.01292				
0.11111	0.11111	0.78207	0.79938	0.81818	1.02558				
	0.10000	0.78207	0.80656	0.83333	1.03594				
	0.33333	0.80669	0.68779	0.60000	0.84000				
	0.25000	0.80669	0.72967	0.66667	0.89600				
	0.20000	0.80669	0.75735	0.71429	0.93333				
0.10000	0.16667	0.80669	0.77699	0.75000	0.96000				
	0.14286	0.80669	0.79167	0.77778	0.98000				
	0.12500	0.80669	0.80304	0.80000	0.99556				
	0.11111	0.80669	0.81212	0.81818	1.00800				
	0.10000	0.80669	0.81953	0.83333	1.01818				
0.08333	0.33333	0.82630	0.69490	0.60000	0.82695				
	0.25000	0.82630	0.73766	0.66667	0.88421				
	0.20000	0.82630	0.76595	0.71429	0.92105				
	0.16667	0.82630	0.78604	0.75000	0.94737				
	0.14286	0.82630	0.80106	0.77778	0.96711				
0.07143	0.12500	0.82630	0.81270	0.80000	0.98246				
	0.11111	0.82630	0.82199	0.81818	0.99474				
	0.10000	0.82630	0.82958	0.83333	1.00478				
	0.33333	0.84230	0.70055	0.60000	0.82031				
	0.25000	0.84230	0.74402	0.66667	0.87500				
0.06250	0.20000	0.84230	0.77280	0.71429	0.91146				
	0.16667	0.84230	0.79326	0.75000	0.93750				
	0.14286	0.84230	0.80854	0.77778	0.95703				
	0.12500	0.84230	0.82040	0.80000	0.97222				
	0.11111	0.84230	0.82987	0.81818	0.98437				
0.05556	0.10000	0.84230	0.83760	0.83333	0.99432				
	0.33333	0.85560	0.70515	0.60000	0.81338				
	0.25000	0.85560	0.74921	0.66667	0.86761				
	0.20000	0.85560	0.77839	0.71429	0.90376				
	0.16667	0.85560	0.79914	0.75000	0.92958				
0.05000	0.14286	0.85560	0.81466	0.77778	0.94894				
	0.12500	0.85560	0.82669	0.80000	0.96401				
	0.11111	0.85560	0.83630	0.81818	0.97606				
	0.10000	0.85560	0.84416	0.83333	0.98592				

TABLE 23. EFFECT OF U ON THE EXTREME VALUES OF ξ_1 , ξ_2 , ξ_3 , AND ξ_7 ,
 FOR $n \left(\frac{1}{10} \leq n \leq \frac{3}{1} \right)$ AND $m \left(\frac{1}{10} \leq m \leq \frac{3}{1} \right)$

U		ξ_1		ξ_2		ξ_3		ξ_7		$\lim_{U \rightarrow -1} \frac{\xi_1}{\xi_2}$	
Max	Min	Max	Min	Max	Min	Max	Min	Max	Min	Max	Min
$-6 \leq U < -1$											
0.80424	0.48889	0.81818	0.27484	∞	∞	0.83333	0.6	3.63636	0.84615	0	0
$U = -1$											
2	1.25758	1.12051	0.80769	∞	∞	0.83333	0.6	0	0	2.89562	1.419
$U = -\frac{1}{2}$											
1.1	1.01515	0.90909	0.75	0.83333	0.6	0.90909	0.75	0.99174	0.81818	0.8	0.79327
$U = 0$											
0.94781	0.88889	0.88384	0.7	0.99174	0.81818	1.03896	0.8	1.15702	0.79327		
$U = \frac{1}{2}$											
0.91667	0.8	0.87121	0.675								
$U = 1$											
0.89879	0.64694	0.86364	0.62143								
$1 < U \leq 6$											

TABLE 24. EFFECT OF U ON THE EXTREME VALUES OF
 ζ_5 AND ζ_6 FOR $W(0 \leq W \leq 2)$, $k\left(\frac{1}{4} \leq k \leq \frac{1}{2}\right)$ AND ANY
 VALUE OF n AND m SUCH THAT $\frac{1}{10} \leq (n, m) \leq \frac{1}{3}$

U	W	k	ζ_5		ζ_6	
			Max	Min	Max	Min
$-6 \leq U < -1$		1/2	1.27042	0.71773	2.50938	1.14412
		1/3	1.73565	0.82	3.4143	1.23599
		1/4	2.02915	0.87796	3.98356	1.29139
U = -1		1/2	0.8409	0.7	1.548	0.70711
		1/3	1	0.7937	1.7937	0.7937
		1/4	1.09051	0.8409	1.9314	0.8409
U = - $\frac{1}{2}$		1/2	0.8269	0.73268	1.50847	0.96593
		1/3	0.97656	0.82214	1.73091	0.97017
		1/4	1.06102	0.86543	1.85432	0.975
U = 0		1/2	0.82244	0.73789	1.49535	1
		1/3	0.96899	0.82606	1.70998	1
		1/4	1.05143	0.87193	1.82858	1
U = $\frac{1}{2}$		1/2	0.8269	0.73268	1.50847	0.96593
		1/3	0.97656	0.82214	1.73091	0.97017
		1/4	1.06102	0.86543	1.85432	0.975
U = 1		1/2	0.8409	0.70711	1.548	0.70711
		1/3	1	0.7937	1.7937	0.7937
		1/4	1.09051	0.8409	1.9314	0.8409
$1 < U \leq 6$		1/2	1.27042	0.71773	2.50938	1.14412
		1/3	1.73565	0.82	3.4143	1.23599
		1/4	2.02915	0.87796	3.98356	1.29139

ANALYSIS OF SHORT JOURNAL BEARINGS

To provide a concrete illustration of the application of the equations of the preceding section, the limiting case of the short journal bearing will be dealt with here. In this case, the pressure distribution is dominated by end leakage effects. Hence, a comparison of the results of this analysis with experimental results serves as a critical test of the ability of the chosen system for analysis to deal with such flows.

The short bearing limiting case may be characterized by a tangential Couette flow and an axial pressure flow. This combined flow may be obtained from the flow in the finite bearing Equations (87) to (89), when the tangential pressure-flow component U approaches zero

$$\frac{2\zeta_3}{\zeta_2} W \frac{\partial W}{\partial Z} + \frac{\partial W}{\partial X} + \frac{1}{H} \frac{dH}{dX} W = - \frac{1}{\zeta_2} \frac{\partial P}{\partial Z} - \frac{\zeta_6}{\zeta_2} C_{xc} \frac{W}{H} \quad (107)$$

and

$$\frac{1}{H} \frac{dH}{dX} + \zeta_7 \frac{\partial W}{\partial Z} = 0 \quad (108)$$

By using the condition $W = 0$ at $Z = 0$, Equation (108) can be integrated as

$$W = - \frac{1}{\zeta_7} \frac{\frac{dH}{dX}}{H} Z \quad (109)$$

Differentiating W with respect X and Z , respectively:

$$\frac{\partial W}{\partial X} = - \frac{H \frac{d^2 H}{dX^2} - \left(\frac{dH}{dX} \right)^2}{\zeta_7 H^2} \cdot Z \quad (110)$$

and

$$\frac{\partial W}{\partial Z} = - \frac{1}{\zeta_7} \frac{\frac{dH}{dX}}{H} \quad (111)$$

Substituting Equations (109) to (111) into Equation (107),

$$\frac{\partial P}{\partial Z} = 2 A(X) Z \quad (112)$$

where

$$2A(X) = \frac{\zeta_6 C_{xc}}{\zeta_7} \cdot \frac{\frac{dH}{dX}}{H^2} - \frac{2\zeta_3}{(\zeta_7)^2} \left(\frac{\frac{dH}{dX}}{H} \right)^2 + \frac{\zeta_2}{\zeta_7} \cdot \frac{\left\{ H \frac{d^2 H}{dX^2} - \left(\frac{dH}{dX} \right)^2 \right\}}{H^2} + \frac{\zeta_2}{\zeta_7} \left(\frac{\frac{dH}{dX}}{H} \right)^2 \quad (113)$$

Integrating $\frac{\partial P}{\partial Z}$ in Equation (112) with respect to Z, and using the boundary condition $P = 0$ at $Z = Z_{\max} = \frac{L/2}{h}$, $P(X, Z)$ is

$$P(X, Z) = \left\{ A(X) \right\} \cdot \left\{ Z^2 - Z_{\max}^2 \right\} \quad (114)$$

The pressure distribution in Equation (114) is parabolic in Z.

Short Bearing Equations for Full Journal Bearing

Using the notation shown in Figure 50, the dimensionless film equation for H is

$$H = 1 + \epsilon \cos \phi = 1 + \epsilon \cos \beta X \quad (115)$$

FIGURE 50. NOTATION FOR SHORT FULL JOURNAL BEARING

Substituting Equation (115) into Equation (114), P is

$$P(\phi, z) = \left\{ 1 - \left(\frac{2z}{L} \right)^2 \right\} \cdot \{ P_f + P_i \} \quad (116)$$

where

$$P_f = P_{\text{friction}}$$

$$= \frac{\zeta_6 C_{xc} \epsilon \left(\frac{L}{D} \right)}{4 \left(\frac{h}{L} \right) \zeta_7} \frac{\sin \phi}{(1 + \epsilon \cos \phi)^2} \quad (117)$$

and

$$P_i = P_{\text{inertia}}$$

$$= \frac{\zeta_2}{2\zeta_7} \frac{\epsilon \left(\frac{L}{D} \right)^2 \cos \phi}{(1 + \epsilon \cos \phi)} + \frac{\zeta_3}{(\zeta_7)^2} \frac{\left(\epsilon \frac{L}{D} \right)^2 \sin^2 \phi}{(1 + \epsilon \cos \phi)^2} \quad (118)$$

In the right-hand side of Equation (116), the first and second terms denote the pressure due to friction and inertia, respectively. If the inertia effect upon the pressure is neglected, $P(\phi, z)$ is reduced to

$$P(\phi, z) = P_f(\phi, z) \quad (119)$$

where $P_f(\phi, z)$ is in the same form as in Equation (117).

The pressure distribution in Equations (116) and (119) is parabolic in z . For the case of $P = P_f$ in Equation (119), $P(\phi, z)$ is antisymmetrical about $\phi = \pi$ and $P = 0$; that is, the region of negative pressures is identical to that of positive pressures.

At the central line, P is obtained by putting $z = 0$ in Equation (116)

$$P(\phi, 0) = P_f + P_i \quad (120)$$

where P_f and P_i are in the same form as in Equations (117) and (118), respectively.

Estimation of the Pressure Produced by the Centrifugal Force in the Fluid Film. As shown in Figure 51, the centrifugal force of fluid element (adr) at radius r , F_c , is

$$F_c = (\rho a dr) (\omega^2 r) \quad (121)$$

Then, the centrifugal force at $r = r_m$, F_{cm} , is obtained by integrating Equation (121) with respect to r

$$\begin{aligned} F_{cm} &= \int_{r_m}^{r_o} \rho a \omega^2 r dr \\ &= \rho a u_{bc}^2 \frac{h}{2r_m} \left(1 + \frac{h}{4r_m} \right) \end{aligned} \quad (122)$$

where

$$\frac{h}{2} \equiv r_o - r_m \quad (123)$$

Consequently, the centrifugal stress, σ_{cm} , at $r = r_m$, is

$$\sigma_{cm} = -\frac{F_{cm}}{a} \quad (124)$$

Substituting Equations (122) into (124), one obtains

$$\sigma_{cm} = \rho u_{bc}^2 \frac{h}{2r_m} \left(1 + \frac{h}{4r_m} \right) \quad (125)$$

FIGURE 51. SCHEMATIC DIAGRAM OF FLUID ELEMENT

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and any value of n in Table 13(a), ζ_2 , ζ_3 and ζ_7 can be found as 0.8889, 0.8, and 0.8889. Because of very strong Couette flow in the short bearing, we may assume $W \ll 1$ (i.e., $W \rightarrow 0$). In Table 13(b), no matter which value of k is chosen, $\zeta_6 = 1$ for $W \rightarrow 0$. As for the evaluation of C_{xc} , we may use $\bar{R} = 2100$ and Figure 16 to obtain C_{xc} equal to the value of 0.0075.

Substituting the numerical values of the dimensionless coefficients ($\zeta_2 = \zeta_7 = 0.0889$, $\zeta_3 = 0.8$, $\zeta_6 = 1$, and $C_{xc} = 0.0075$), and geometrical parameters ($\epsilon = 0.5$, $\bar{h} = 0.045$ ft, $D = 6$ ft, $L = \frac{10}{6}$ ft and $\beta = 0.015$) into Equation (120), $P(\phi, 0)$ is calculated as shown in Table 25 and plotted in Figure 52. A small correction for centrifugal force effects is included, the calculation of which is outlined in the section which follows.

TABLE 25. THEORETICAL PRESSURES FOR THE EXPERIMENTAL SHORT JOURNAL BEARING CONFIGURATION

ϕ (degree)	$P_f + P_i$	$\Delta P \equiv \Delta P_{\text{centrifugal}}$	$(P_f + P_i + \Delta P)$
0	0.0130	0.0056	0.019
30	0.0177	0.0054	0.023
60	0.0256	0.0047	0.030
90	0.0348	0.0037	0.038
120	0.0565	0.0021	0.058
150	0.0867	0.0028	0.015
180	-0.0391	0.0018	-0.037
210	-0.0355	0.0021	-0.033
240	-0.0684	0.0028	-0.040
270	0.0264	0.0037	0.030
300	0.0101	0.0047	0.015
330	0.0109	0.0053	0.016
360	0.0130	0.0056	0.019

$P(\phi, 0)$, predicted by theory and measured by experiment, are seen to correspond well, particularly in regions of positive pressure, and throughout

FIGURE 52. THEORETICAL (SOLID LINE) AND EXPERIMENTAL (O AND Δ) RESULTS

they agree in general magnitude. This fact supports the arguments that the equations for the short bearing model $\left(\frac{L}{D} \doteq \frac{1}{3}\right)$ are valid in such regions. In the regions of negative pressure, the discrepancy may result from the components of momentum of the fluid being drawn from the ambient into the film.

Analytical Estimates of Short Journal Bearing Performance

In the right-hand side of Equation (116), the first and second terms denote the pressure components due to friction and inertia, respectively.

If the inertia effect upon the pressure is neglected, P is reduced to

$$P = P_f \quad (130)$$

where P_f is in the same form as in Equation (117).

From the pressure solution, bearing performance estimates for total load, attitude angle, frictional force and frictional coefficient may be obtained for the following cases:

- (1) Pressure distributions for $0 \leq \phi \leq 360^\circ$, both with and without accounting of inertial effects.
- (2) Positive pressure distribution only, without inertia effect, for $0 \leq \phi \leq 180^\circ$.

The general equations of the components of the total load and the circumferential frictional force at shaft are

$$W_l \sin \psi = \frac{D}{2} \int_{-L/2}^{L/2} \int_{\phi = \phi_1}^{\phi = \phi_2} p(\phi, z) \sin \phi d\phi dz \quad (131)$$

$$W_l \cos \psi = - \frac{D}{2} \int_{-L/2}^{L/2} \int_{\phi = \phi_1}^{\phi = \phi_2} p(\phi, z) \cos \phi d\phi dz \quad (132)$$

where

- ψ - attitude angle, degree
- ϕ - angular coordinates measured from lines of centers in direction of journal motion, degree
- L - bearing length
- D - journal diameter
- W_l - bearing load

and

$p = \rho u_{bc}^2 P$, where P is the dimensionless pressure.

$$F = \frac{LD}{2} \int_{\phi = \phi_1}^{\phi = \phi_2} \tau_{xo} d\phi \quad (133)$$

where

$$\tau_{xo} = \frac{\rho C_{xc}}{2} (u_{bc})^2 \quad (134)$$

F - circumferential frictional force at shaft

Short Full 360° Journal Bearing, without Inertial Effects and with Admissible

Negative Pressure Regions. Using $\phi_1 = 0^\circ$, $\phi_2 = 360^\circ$ and substituting

Equation (130) into Equations (131) and (132), the components of the total

dimensionless load, obtained by integration, are

$$\bar{W}_l \sin \psi = \frac{\pi \zeta_6 C_{xc} (L/D)}{6 \zeta_7 (\bar{h}/L) \epsilon} [(1 - \epsilon^2)^{-1/2} - 1] \quad (135)$$

$$\bar{W}_l \cos \psi = 0 \quad (136)$$

where

$$\bar{W}_\ell = \frac{W_\ell}{\rho(u_{bc})^2 DL} \quad (137)$$

From Equations (135) and (136), the quantities of $\bar{W}_\ell$ and ψ are

$$\bar{W}_\ell = \frac{\pi \zeta_6 C_{xc}(L/D)}{6 \zeta_7 (\bar{h}/L) \epsilon} [(1 - \epsilon^2)^{-1/2} - 1] \quad (138)$$

and

$$\psi = 90^\circ \quad (139)$$

Using $\phi_1 = 0^\circ$, $\phi_2 = 360^\circ$ and Equations (133) and (134), the frictional force, F , is given by

$$F = \frac{\pi}{2} \rho C_{xc} L D (u_{bc})^2 \quad (140)$$

Using the definition of f and Equations (138) and (140), the frictional coefficient at the shaft is found to be

$$\begin{aligned} f &= F/W_\ell \\ &= \frac{3 \zeta_7 (\bar{h}/L) \epsilon}{\zeta_6 (L/D) [(1 - \epsilon^2)^{-1/2} - 1]} \end{aligned} \quad (141)$$

Short Full 360° Journal Bearing, with Inertial Effects and Admissible Negative Pressure Regions. Using $\phi_1 = 0^\circ$, $\phi_2 = 360^\circ$ and substituting Equation (116) into Equations (131) and (132), the components of the total dimensionless load, obtained by integration, are:

$$\psi = \lim_{\epsilon \rightarrow 0} \psi = \tan^{-1} \left[\frac{\zeta_6 C_{xc}}{-2\zeta_2(\bar{h}/L)(L/D)} \right] \quad (147)$$

When $\epsilon \rightarrow 1$, the limit of ψ in Equation (146) is

$$\psi = \lim_{\epsilon \rightarrow 1} \psi = \tan^{-1} \left\{ \frac{(\zeta_6 C_{xc}) / [(2L/D)(\bar{h}/L)]}{-\zeta_2 + 2\zeta_3/\zeta_7} \right\} \quad (148)$$

The frictional force, F , is in the same general form as Equation (140).

Using Equations (140), (144), and (145),

$$\begin{aligned} f &= F/W_\ell \\ &= \frac{3C_{xc}\epsilon(1 - \epsilon^2)^{1/2}}{4(L/D)^2 G} \end{aligned} \quad (149)$$

where G is identical in form with Equation (145).

When $\zeta_2 = \zeta_3/\zeta_7 = 0$, the quantities of $\bar{W}_\ell$, ψ , and f in Equations (144), (146), and (149) can be reduced to those without inertia effect in Equations (138), (139), and (141), respectively.

Discussion of Performance Estimates for Short Full Journal Bearings. The parameters corresponding to $\bar{R} = 1200$ and $\bar{R} = 3400$ are listed in Table 26.

TABLE 26. VALUES OF PARAMETERS FOR
 $\bar{R} = 1200$ AND $\bar{R} = 3400$

$\bar{R}$	C_{xc}	U	W	ζ_2	ζ_3	ζ_6	ζ_7
1200	0.011	0	$W \doteq 0$	0.8571	0.75	1	0.85714
3400	0.0068	0	$W \doteq 0$	0.9	0.81818	1	0.9

The quantities of W_ℓ in Equations (138) and (144), ψ in Equations (139) and (146), and f in Equations (141) and (149) are plotted in Figures 53 to 61 for

$$\bar{W}_\ell \sin \psi = \frac{\pi \zeta_6 C_{xc} (L/D) [(1 - \epsilon^2)^{-1/2} - 1]}{6 \zeta_7 (\bar{h}/L) \epsilon} \quad (142)$$

$$\begin{aligned} \bar{W}_\ell \cos \psi = & \frac{\pi \zeta_2 (L/D)^2 [1 - (1 - \epsilon^2)^{-1/2}]}{3 \epsilon \zeta_7} \\ & - \frac{2 \pi \zeta_3 (L/D)^2}{3 \epsilon (\zeta_7)^2} \left\{ 2 - (2 - \epsilon^2)(1 - \epsilon^2)^{-1/2} \right\} \end{aligned} \quad (143)$$

From Equations (142) and (143), $\bar{W}_\ell$ is

$$\begin{aligned} \bar{W}_\ell &= [(\bar{W}_\ell \sin \psi)^2 + (\bar{W}_\ell \cos \psi)^2]^{1/2} \\ &= \frac{2 \pi (L/D)^2}{3 \epsilon (1 - \epsilon^2)^{1/2}} \cdot G \end{aligned} \quad (144)$$

where

$$\begin{aligned} G = \frac{1}{2 \zeta_7} \left\{ \left[\left(\frac{\zeta_6 C_{xc}}{2 \cdot \frac{L}{D} \cdot \frac{\bar{h}}{L}} \right)^2 + (\zeta_2)^2 \right] [(2 - \epsilon^2) - 2(1 - \epsilon^2)^{1/2}] \right. \\ + 4(\zeta_3/\zeta_7)^2 [8(1 - 3\epsilon^2)(1 - \epsilon^2)^{-2} - (8 - 12\epsilon^2 + 4\epsilon^4)(1 - \epsilon^2)^{-1/2} \\ + \epsilon^4(5 - \epsilon^2)^2(1 - \epsilon^2)^{-2}] - 4(\zeta_2 \zeta_3/\zeta_7) [(4 - 3\epsilon^2) \\ \left. - (4 - \epsilon^2)(1 - \epsilon^2)^{1/2}] \right\}^{1/2} \end{aligned} \quad (145)$$

Dividing Equation (142) by (143), ψ is

$$\psi = \tan^{-1} \left[\frac{\zeta_6 C_{xc} (1 - \epsilon^2) [1 - (1 - \epsilon^2)^{1/2}]}{2 \left(\frac{L}{D} \right) \left(\frac{\bar{h}}{L} \right) \left\{ -\zeta_2 (1 - \epsilon^2) [1 - (1 - \epsilon^2)^{1/2}] + \frac{2 \zeta_3}{\zeta_7} [(2 - 3\epsilon^2 + \epsilon^4) - 2(1 - \epsilon^2)^{3/2}] \right\}} \right] \quad (146)$$

When $\epsilon \rightarrow 0$, the limit of ψ in Equation (146) is:

FIGURE 53. VARIATION OF LOAD SUPPORT WITH ECCENTRICITY FOR A FULL BEARING WITH $L/D = 1/7$

FIGURE 54. VARIATION OF LOAD SUPPORT WITH ECCENTRICITY FOR A FULL BEARING WITH $L/D = 1/4$

FIGURE 55. VARIATION OF LOAD SUPPORT WITH ECCENTRICITY FOR A FULL BEARING WITH $L/D = 1/3$

FIGURE 56. VARIATION OF ATTITUDE ANGLE WITH ECCENTRICITY RATIO
FOR A FULL BEARING FOR $L/D = 1/7$

FIGURE 57. VARIATION OF ATTITUDE ANGLE WITH ECCENTRICITY RATIO FOR A FULL BEARING FOR $L/D = 1/4$

FIGURE 58. VARIATION OF ATTITUDE ANGLE WITH ECCENTRICITY RATIO FOR A FULL BEARING FOR $L/D = 1/3$

FIGURE 59. FRICTION COEFFICIENT VERSUS ECCENTRICITY FOR $L/D = 1/7$

FIGURE 60. FRICTION COEFFICIENT VERSUS ECCENTRICITY FOR $L/D = 1/4$

FIGURE 61. FRICTION COEFFICIENT VERSUS ECCENTRICITY FOR $L/D = 1/3$

various values of L/D , $\bar{h}/L$ and $\bar{R}$ for the cases without and with inertia effect, respectively. Plots of load versus eccentricity are shown in Figures 53 to 55, for different values of L/D and $\bar{h}/L$ for two Reynolds numbers $\bar{R} = 1200$ and $\bar{R} = 3400$. The lower the Reynolds number, the higher the dimensionless load capacity, with or without accounting of inertia effect. For a corresponding film-thickness/journal-diameter ratio, $\bar{h}/D$ and $\bar{R}$, it appears that the load capacity increases as eccentricity ratio increases. The inertia effect on the load capacity becomes most significant for the relatively large values of $\bar{h}/D$ and $\bar{R}$.

The eccentricity ratio is plotted against attitude angle in Figures 56 to 58 for different values of L/D and $\bar{h}/L$ for two Reynolds numbers $\bar{R} = 1200$ and $\bar{R} = 3400$. For the relatively large values of $\bar{h}/D$ and $\bar{R}$, the contribution to the attitude angle from the inertial effect is dominant. Particularly significant is the fact that displacement opposes load up to some limiting eccentricity, then behaves in the more expected way for further eccentricity. For the case without inertia effect, the attitude angle must be 90° . In other words, the load, W_l , acts perpendicular to the centerline. This means that the journal center will (under load and without considering inertia effect) move over along a line at right angles to the load line. For the very thin film, i.e., smaller values of $\bar{h}/D$, the attitude angle for different value of eccentricity with inertia effect is in the neighborhood of $\psi = 90^\circ$. But, for those cases where it is greater than 90° , the bearing is unstable statically. It would appear that the journal would orbit outward until out of this region of operation. Hence, with rotating load alone, a sizeable orbit would be anticipated. This is quite

different from half-speed whirl and is different from resonant synchronous whirl, since it is only slightly speed sensitive. Plots of frictional coefficient versus eccentricity ratio are shown in Figures 59 to 61 for different values of L/D and $\bar{h}/L$ for two Reynolds numbers $\bar{R} = 1200$ and $\bar{R} = 3400$. For the relatively large values of $\bar{h}/D$ and $\bar{R}$, the inertial effect on the frictional coefficient becomes important in the range of ϵ such that $0.4 \leq \epsilon \leq 0.6$. For a corresponding value of $\bar{h}/D$ and $\bar{R}$, the lower the frictional coefficient, the higher the eccentricity ratio. The smaller the value of $\bar{h}/D$, the smaller the frictional coefficient.

It is concluded from these data that a thin turbulent film behavior causes a reduction in frictional coefficient and an increase in load capacity as compared with a thick turbulent film, for the same values of $\bar{R}$ and ϵ .

180° Sector, Short Journal Bearing, Neglecting Inertia. In Equation (116), the value of $P(\phi, z)$ is positive in the converging film such that $0 \leq \phi \leq 180^\circ$. Substituting Equation (116) into Equations (131) and (132), the components of the total dimensionless load are

$$\bar{W}_\ell \sin \psi = \frac{\pi \zeta_6 C_{xc} (L/D) [(1 - \epsilon^2)^{-1/2} - 1]}{12 \zeta_7 \epsilon (\bar{h}/L)} \quad (150)$$

$$\bar{W}_\ell \cos \psi = \frac{\zeta_6 C_{xc} (L/D)}{12 \zeta_7 (\bar{h}/L)} \left[\frac{2}{(1 - \epsilon^2)} - \frac{1}{\epsilon} \ln \frac{(1 + \epsilon)}{(1 - \epsilon)} \right] \quad (151)$$

where

$$\bar{W}_\ell = \frac{W_\ell}{\rho (u_{bc})^2 DL} \quad (152)$$

From Equations (150) and (151), $\bar{W}_\ell$ is:

$$\bar{W}_\ell = \frac{\zeta_6 C_{xc} (L/D) M}{6\epsilon (\bar{h}/L)} \quad (153)$$

where

$$M = \frac{1}{2\zeta_7} \left\{ \frac{\pi^2}{(1-\epsilon^2)} [(2-\epsilon^2) - 2(1-\epsilon^2)^{1/2}] + \left[\ln \frac{1+\epsilon}{1-\epsilon} \right]^2 - \frac{4\epsilon}{(1-\epsilon^2)} \ln \frac{(1+\epsilon)}{(1-\epsilon)} + \frac{4\epsilon^2}{(1-\epsilon^2)^2} \right\}^{1/2} \quad (154)$$

Dividing Equation (150) over (151), ψ is

$$\psi = \tan^{-1} \left\{ \frac{\pi(1-\epsilon^2)^{1/2} [1 - (1-\epsilon^2)^{1/2}]}{2\epsilon - (1-\epsilon^2) \ln [(1+\epsilon)/(1-\epsilon)]} \right\} \quad (155)$$

When $\epsilon \rightarrow 0$, the limit of ψ in Equation (155) is

$$\psi = \lim_{\epsilon \rightarrow 0} \psi = 90^\circ \quad (156)$$

When $\epsilon \rightarrow 1$, the limit of ψ in Equation (155) is

$$\psi = 0^\circ \quad (157)$$

Using $\phi_1 = 0^\circ$, $\phi_2 = 180^\circ$, and Equations (133) and (134), F is

$$F = \pi \rho C_{xc} (LD) (u_{bc})^2 \quad (158)$$

Using the quantities of F and W_ℓ from Equations (158) and (153), the value of f is found to be

$$\begin{aligned} f &= F/W_\ell \\ &= \frac{6\pi\epsilon(\bar{h}/L)}{\zeta_6 M(L/D)} \end{aligned} \quad (159)$$

where M is identical to that in Equation (154).

For the convergent laminar film without inertial effect⁽¹⁴⁾

$(0^\circ \leq \phi \leq 180^\circ)$, ψ is

$$\psi = \frac{\pi(1 - \epsilon^2)^{1/2}}{4\epsilon} \quad (160)$$

The values of ψ for the laminar and turbulent films in Equations (160) and (155) are plotted in Figure 62.

As values of ϵ near zero, the line of centers becomes nearly perpendicular to the load line, and it is parallel to the load line at $\epsilon = 1$ for the laminar and turbulent flows, respectively. The locus of the journal for the case of laminar flow lies within a semicircle; but, for the case of turbulent flow, it lies outside a semicircle.

Numerical Solution of the Short Full Journal and Short Centrally Loaded Half-Journal Bearings, with Inertia and Turbulence Effects and without Negative Pressure Distribution

Substituting Equations (116) into (131) and (132), one obtains

$$\bar{W}_\ell \sin \psi = \frac{1}{3} \int_{\phi_1}^{\phi_2} (P_f + P_i) \sin \phi d\phi = A \quad (161)$$

$$\bar{W}_\ell \cos \psi = -\frac{1}{3} \int_{\phi_1}^{\phi_2} (P_f + P_i) \cos \phi d\phi = B \quad (162)$$

Here, the terms of P_f and P_i are identical to ones in Equations (117) and (118), respectively. From Equations (161) and (162), the values of ψ and $\bar{W}_\ell$ are, respectively:

FIGURE 62. ATTITUDE PLOTS FOR 180°, SHORT, JOURNAL-BEARING

$$\psi = \tan^{-1} (A/B) \quad (163)$$

and

$$\bar{W}_\ell = \frac{1}{3} [(A^2 + B^2)^{1/2}] \quad (164)$$

Full Journal Bearing. Using $\phi_1 = 0$ and $\phi_2 = 360^\circ$ and neglecting the negative pressure distribution, the numerical solutions of ψ and $\bar{W}_\ell$ are calculated from Equations (163) and (164), respectively. The limiting value of ψ for $\epsilon = 0$ from Equation (163) is

$$\lim_{\epsilon \rightarrow 0} \psi = \tan^{-1} \left\{ \frac{\int_0^{2\pi} E \sin^2 \phi d\phi}{-\int_0^{2\pi} E \sin \phi \cos \phi d\phi} \right\} \quad (165)$$

where

$$E = 1 + \frac{2 \left(\frac{\bar{h}}{L} \right) \left(\frac{L}{D} \right) \zeta_2}{\zeta_6 C_{xc} \tan \phi} \quad (166)$$

For $L/D = 1/4$, the quantities of $\bar{W}_\ell$ and ψ are plotted in Figures 63 to 66 for various values of $\bar{h}/L$ and $\bar{R}$. The lower the Reynolds number, the higher the dimensionless load capacity. For a set of values of $\bar{h}/D$ and $\bar{R}$, it seems that the load capacity increases as eccentricity ratio increases. Of practical interest is the condition of static stability where the displacement of the journal center opposes the fluid pressure load at the higher eccentricity ratio. For instance, the journal is stable for the case of $\bar{R} = 3400$ and $\bar{h}/L = 0.01$ in Figure 64, as the attitude angle of ($0 \leq \psi \leq 90^\circ$) is bound within the range of ϵ such that $0.5 \leq \epsilon \leq 0.98$.

FIGURE 63. VARIATION OF LOAD SUPPORT WITH ECCENTRICITY RATIO FOR A CENTRALLY LOADED HALF-JOURNAL BEARING FOR $L/D = 1/4$ AND $\bar{h}/L = 0.01, 0.005$, AND 0.001

FIGURE 64. VARIATION OF ATTITUDE ANGLE
 WITH ECCENTRICITY RATIO FOR
 $L/D = 1/4$, AND $\bar{h}/L = 0.01$

FIGURE 65. VARIATION OF ATTITUDE ANGLE
 WITH ECCENTRICITY RATIO FOR
 $L/D = 1/4$ AND $\bar{h}/L = 0.005$

FIGURE 66. VARIATION OF ATTITUDE ANGLE
WITH ECCENTRICITY RATIO FOR
 $L/D = 1/4$ AND $\bar{h}/L = 0.001$

Centrally Loaded Half-Journal Bearing. Denote ϕ_{cut} as the included angle between the leading edge of half-journal bearing and centers $O_b O_j$. Here, ϕ_{cut} is in the range such that $0^\circ \leq \phi_{\text{cut}} \leq 360^\circ$. For the loaded half-journal bearing, the general equation of α , as shown in Figure 67, is

$$\alpha = 180^\circ - (\phi_{\text{cut}} + \psi) \quad (167)$$

Here, ψ is the attitude angle, and α is the load angle with respect to the leading edge. The limiting value of ψ for $\epsilon = 0$ from Equation (163) is

$$\lim_{\epsilon=0} \psi = \tan^{-1} \left\{ \frac{\int_{\phi_{\text{cut}}}^{\phi_{\text{cut}} + 180^\circ} E \sin^2 \phi d\phi}{-\int_{\phi_{\text{cut}}}^{\phi_{\text{cut}} + 180^\circ} E \sin \phi \cos \phi d\phi} \right\} \quad (168)$$

where the value of E is identical to that in Equation (166).

Neglecting the negative pressure distribution, the values of ψ , $\bar{W}_l$ and α are calculated from Equations (163), (164) and (167) for various values of ϕ_{cut} and ϵ . Plot the curves of ϕ_{cut} , ψ and $\bar{W}_l$ versus α in the neighborhood of $\alpha = 90^\circ$, for various values of $\bar{h}/D$, $\bar{R}$ and ϵ . The corresponding values of ϕ_{cut} , ψ and $\bar{W}_l$ at $\alpha = 90^\circ$ are then determined and are plotted against ϵ in Figures 68 to 76. For a given set of values of ϵ , L/D and $\bar{h}/L$, the proper value of ϕ_{cut} for the centrally loaded half-journal bearing, i.e., $\alpha = 90^\circ$, can be obtained from Figures 68 to 70. As shown in Figures 71 to 73, the load capacity increases as eccentricity ratio increases, for a given set of values of L/D , $\bar{h}/L$ and $\bar{R}$. For the very thin film in Figure 76, the attitude angle,

FIGURE 67. SCHEMATIC DIAGRAM OF CENTRALLY LOADED HALF-JOURNAL BEARING

FIGURE 68. VARIATION OF ϕ_{cut} WITH ϵ
FOR $L/D = 1/4$ AND $\bar{h}/L = 0.01$

FIGURE 69. VARIATION OF ϕ_{cut} WITH ϵ
FOR $L/D = 1/4$ AND $\bar{h}/L = 0.005$

FIGURE 70. VARIATION OF ϕ_{cut} WITH ϵ FOR
 $L/D = 1/4$ AND $\bar{h}/L = 0.001$

FIGURE 71. VARIATION OF $\bar{w}_\ell$ WITH ϵ FOR $L/D = 1/4$ AND $\bar{h}/L = 0.01$

FIGURE 72. VARIATION OF $\bar{w}_\ell$ WITH ϵ FOR
 $L/D = 1/4$ AND $\bar{h}/L = 0.005$

FIGURE 73. VARIATION OF $\bar{w}_l$ WITH ϵ FOR $L/D = 1/4$ AND $\bar{h}/L = 0.001$

FIGURE 74. VARIATION OF ψ WITH ϵ FOR $L/D = 1/4$ AND $\bar{h}/L = 0.01$

FIGURE 75. VARIATION OF ψ WITH ϵ FOR $L/D = 1/4$ AND $\bar{h}/L = 0.005$

FIGURE 76. VARIATION OF ψ WITH ϵ FOR
 $L/D = 1/4$ AND $\bar{h}/L = 0.001$

ψ , is approximately in the range of ψ such that $0 < \psi \leq 90^\circ$ except in the neighborhood of $\epsilon = 0$. Within that range of ψ , the displacement of the journal center opposes the fluid pressure load, and consequently the journal is stable.

ANALYSIS OF A BEARING WITHOUT END LEAKAGE

General

In the preceding chapter, the basic equations have been shown to apply to the case of a bearing when end leakage effects predominate. It is of interest here to examine the other extreme, a tilted pad without end leakage, where velocity components in the tangential direction predominate in determining the pressure distribution. Here, it is shown that measurements in the large-scale apparatus can be predicted quite closely by the chosen analytical approach.

As a second topic, the equations are applied to a parametric study of a tilted pad without end leakage and without leading edge ram pressure to show the effects of inclusion or omission of the inertial terms in the calculation.

Experimental Results

In the large-scale turbulence apparatus, a tilted pad was simulated by removing all but one 3-ft section of the external cylindrical surface and tilting this at selected angles to the rotating drum. End leakage was restricted by stiff plastic curtains which spanned the gap between the stationary pad surface and the rotating drum surface, rubbing at the latter contact. Pressure was so small that it did not tend to force these curtains away from the drum. For such a pad, velocity profile traverses and pressure data were obtained at different angles of pivoting, holding the film thickness constant at a distance, x , back from the leading edge, equal to $x/l = 0.7$. The thickness at this point was held to 0.54 inch. The film thickness at leading and trailing edges was measured, and, in the analytical approach to the problem, a best parabola

was fitted to these dimensions for each operating condition to give an approximation of the actual thickness variation.

Pressure was measured at midspan in the axial direction, at the leading edge and at $x/l = 0.7$. Table 27 summarizes the dimensional data and pressure data. The results of these measurements are shown on the data plots of Figure 77, compared with theoretical curves for the same dimensional variation of the film thickness and Reynolds number for which the data were taken. In viewing these data, it is of interest that a significant amount of the pressure rise occurred at the leading edge due to what has been termed an "impact pressure" effect. It must be remembered that this effect is somewhat exaggerated due to the emphasis of inertial contributions, as a consequence of the relative dimensions chosen for the large-scale turbulent-film apparatus. Such inertial contributions would be expected to be minimized somewhat in actual bearing clearances, just as inertial effects were minimized in the thinner clearances in the preceding chapter on short bearings. Nevertheless, the leading edge pressure rise (which is not calculable from film considerations) forms a significant part of the problem of estimating tilted pad performance.

Analytical Approach to the Problem

It is of interest to determine how closely the simplified film equations which have been developed in previous progress reports fit the measured data for the tilted pad without end leakage. As has been pointed out, the analytical approach to the short bearing fit experimental data very well. The opposite extreme test would be the case of no end leakage where the

FIGURE 77. THEORETICAL AND EXPERIMENTAL PRESSURE DISTRIBUTIONS FOR A TILTED PAD

x-velocity components rather than the z-velocity components dominate. Consequently, application of the simplified analysis to the no-end leakage problem provides a second test of this analysis under a limiting extreme case. As will be shown, satisfactory agreement was obtained insofar as it could be checked, and confidence in the analytical approach is heightened.

From Equations (87) to (89), a solution for dimensionless pressure without end leakage was shown to be

$$-\frac{dP}{dX} = \frac{-\zeta_1}{\chi^2} \frac{dH/dX}{H^3} - \frac{\zeta_1(\chi - 1) dH}{H^2 \chi^2 dX} + 2 \left(\frac{1 - H}{H^2} \right) C_{xc} \frac{\zeta_5}{\chi} \quad (169)$$

where

$$\frac{x}{h_c} = X, \quad \frac{\Delta u_b}{u_{bc}} = U, \quad \frac{p}{\rho u_{bc}^2} = P \quad (170)$$

$$H = \frac{h}{h_c} = \frac{a}{h_c} + \frac{b}{(r + h_c)} x + \frac{c}{(r + h_c)^2} x^2 \quad (171)$$

$$\zeta_1 = \frac{(1 + U^2)/(1 + 2n) + 2(n + U)/(1 + n)}{(1 + u)^2} \quad (172)$$

$$\zeta_5 = \frac{(1 + U)^{2-k} - (1 - U)^{2-k}}{4U} \quad (173)$$

$$\chi = \frac{1}{1 + n} = \frac{1 + U}{U} \zeta_7 - \frac{1}{U} \quad (174)$$

The general magnitudes of ζ_1 , ζ_5 , and χ in Equations (172) to (174) are valid for any kind of bearing films without end leakage. The geometric constants a , b , and c of Equation (171), measured film thickness, and measured pressures are listed in Table 27.

TABLE 27. EXPERIMENTAL DATA FOR TILTING PADS

Case	Film Thickness, h, inches			Parameters, Equation (171)			Pressure, P	
	$x/l = 0$	$x/l = 0.7$	$x/l = 1$	a	b	c	$x/l = 0$	$x/l = 0.7$
1	1.0	0.54	0.265	0.28	0.428	1.0	0.28	0.428
2	0.85	0.54	0.357	0.257	0.264	0.85	0.257	0.264
3	0.7	0.54	0.449	0.149	0.099	0.7	0.149	0.099

Equation (169) may be solved directly for pressure gradient when the variation of H is inserted and the remaining parameters are assigned numerical values. Given values of U , n , and k , the quantities ζ_1 , ζ_5 , and χ were evaluated by means of Equations (172) to (174). The required quantities so calculated, or set by the experimental conditions of Figure 77, are listed in Table 28.

TABLE 28. REQUIRED QUANTITIES FOR EQUATION (169)

$\bar{R}$	U	k	n	ζ_1	ζ_5	χ	C_{xc}
2400	-0.31	0.25	0.125	1.155	0.621	0.889	0.0075

Once the gradient was calculated by Equation (169) for points along the pad, numerical integration provided magnitudes of P . To evaluate the constants of integration, initial pressure was taken as the experimental value, the trailing edge pressure was taken as zero.

Referring back to Figure 77, it is seen that for the two extreme cases, the fit between data and predictions is exceptional. For the intermediate case, the fit is less satisfactory. The error in this one data point may possibly be due to errors in measurement of pressure or film thickness.

Leading Edge Pressure Rise

It is of interest to compare cases on the basis of stagnation pressure, $p/\rho u_b^2 = P + (\xi_1/2)(1 + U)^2 = P_o$. The velocity U was calculated from the computed curves in Figure 77, and results are summarized in Table 29.

TABLE 29. LEADING EDGE PRESSURE DATA

$h_{L.E.},$ <u>in.</u>	<u>U</u>	<u>ξ_1</u>	<u>$(\xi_1/2)(1 + U)^2$</u>	<u>P</u>	<u>P_o</u>
1	-0.9665	1.155	0.000649	0.28	0.280
0.85	-0.5939	1.155	0.095244	0.257	0.352
0.7	-0.528	1.155	0.12829	0.149	0.277

The value of P_o is not far from $P_o = \text{const.} = 0.34$ which was discussed earlier.

Inertial Effects in Tilted Pads without End Leakage

The equations of the preceding paragraphs may be applied directly to the problem of estimating the importance of inertial effects in continuous portions of the bearing pad. For this purpose, a flat slide was assumed by letting the parameters in Equation (171) be as follows:

$$a = h_c, \quad b = -a \left(\frac{r}{h_c} + 1 \right), \quad c = 0 \quad (175)$$

Other quantities for this parametric study are as shown in Table 30. In interpreting the above, the choice of a sets the coordinate origin at the pressure maximum. The quantity α is the slope or inclination of the pad.

TABLE 30. RANGE OF VARIABLES USED IN
TILTED PAD COMPUTATIONS

ζ_1/χ	C_{xc}	α	$X_o \equiv x_o/h_c$		
0	0.005	0.001	-200	-400	-800
0	0.010	0.001	-200	-400	-800
0	0.015	0.001	-200	-400	-800
1	0.005	0.001	-200	-400	-800
1	0.010	0.001	-200	-400	-800
1	0.015	0.001	-200	-400	-800

The choice of $\zeta_1/\chi = 0$ simply eliminates the inertial term from Equation (169). The quantity C_{xc} represents a plausible magnitude from the previously shown charts. The quantity X_o measures the position of the pad's leading edge from an arbitrary point, where H is assigned a value of unity. This arbitrary point corresponds approximately to the mean film thickness. For these comparisons, the leading edge pressure is allowed to be zero.

Pressure Profiles. Figure 78 shows a set of pressure profiles, showing the effect of dimensionless pad length L (i. e., $L = \ell/h_c$) and ζ_1/χ (the inertial factor). Because L was not one of the preset parameters in the program, exactly comparable values are not available for all of the magnitudes of ζ_1/χ . On the other hand, they are sufficiently close to permit useful comparisons. The principal feature to note in these plots is that the higher the inertial factor ζ_1/χ , the lower the profile. Figures 79 and 80 show similar relations for different friction coefficients C_{xc} .

FIGURE 78. PRESSURE DISTRIBUTION UNDER TILTED PADS
SHOWING EFFECT OF INERTIA FOR $C_{xc} = 0.05$

FIGURE 79. PRESSURE DISTRIBUTION UNDER TILTED PADS SHOWING EFFECT OF INERTIA FOR $C_{xc} = 0.01$

FIGURE 80. PRESSURE DISTRIBUTION UNDER TILTED PADS SHOWING EFFECT OF INERTIA FOR $C_{xc} = 0.015$

Mean Dimensionless Pressure. Pressures may be integrated and divided by length to provide the dimensionless mean pressure. Typical values are shown in Figures 81, 82, and 83. In this case, as in the others, it is seen that inertia has a significant though not first-order effect. Neglect of it in the continuous portions of the bearing would not seriously distort calculated results. On the other hand, the leading edge ram effect is very significant and may not be neglected.

Location of Center of Pressure. In Table 31, the symbol C_p/L represents the dimensionless distance back from the leading edge to the center of pressure, expressed as a fraction of dimensionless pad length. The center of pressure is behind the pad center, as is true in laminar flow. Again, it is apparent that the inertial effect, while significant, is of secondary importance.

TABLE 31. EFFECT OF BEARING PARAMETERS ON
COMPUTED CENTER-OF-PRESSURE LOCATION

<u>a</u>	<u>C_{xc}</u>	<u>ζ_1/χ</u>	<u>L</u>	<u>C_p/L</u>
0.001 ↓	0.005	0	358	0.523
	0.005	0	661	0.543
	0.005	0	1188	0.571
	0.005	1	204	0.510
	0.005	1	530	0.533
	0.005	1	1082	0.562
	0.01	0	358	0.523
	0.01	0	661	0.542
	0.01	0	1188	0.570
	0.01	1	279	0.520
	0.01	1	594	0.537
	0.01	1	1134	0.565
	0.015	0	358	0.524
	0.015	0	661	0.541
	0.015	0	1188	0.571
	0.015	1	305	0.521
	0.015	1	616	0.571
	0.015	1	1151	0.567

FIGURE 81. MEAN PRESSURE AS A FUNCTION OF LENGTH TO CLEARANCE RATIO, SHOWING EFFECT OF INERTIA FOR $C_{xc} = 0.005$

FIGURE 82. MEAN PRESSURE AS A FUNCTION OF LENGTH TO CLEARANCE RATIO, SHOWING EFFECT OF INERTIA
FOR $C_{xc} = 0.01$

FIGURE 83. MEAN PRESSURE AS A FUNCTION OF LENGTH TO CLEARANCE RATIO, SHOWING EFFECT OF INERTIA
FOR $C_{xc} = 0.015$

INFINITELY LONG TURBULENT SQUEEZE FILMS

Basic Equations

As shown in Figure 84, the upper surface will be held stationary. Because of the infinite length of bearing, the velocity component in length or z-direction is neglected. Based on these assumptions and the conventions of Figure 84, Equation (87) may be made to apply by changing the minus sign in front of $\zeta_5 C_{xc}$ to plus, whereupon it and Equation (88) are reduced to

$$\zeta_1 \left\{ 2(1 + U) \frac{\partial U}{\partial X} + \frac{dH/dX}{H} (1 + U)^2 \right\} = - \frac{\partial P}{\partial X} + \frac{2}{H} \zeta_5 C_{xc} U + \frac{u_s v_s}{H} \quad (176)$$

$$\zeta_7' (1 + U) \frac{dH}{dX} + \zeta_7' H \frac{\partial U}{\partial X} = v_s \quad (177)$$

Here, u_s and v_s , instead of U_0 and V_0 , denote the lower surface velocities. Equations (176) and (177) are equations of motion in X-direction and of continuity, respectively. Equation (177) may be written as

$$\int_0^h u dy = v_s x - u_{bc} h_c \quad (178)$$

Here the term of $u_{bc} h_c$ is a constant.

The velocity profiles for u are assumed to be represented by the power law as follows:

$$u = -2u_{bc} + (u_{bc} - \Delta u_b) \left(\frac{y_0}{b} \right)^n \quad \left(0 \leq \frac{y_0}{b} \leq 1, h = 2b \right) \quad (179)$$

FIGURE 84. NOTATION OF INFINITELY LONG
SQUEEZE TURBULENT BEARING

$$u = - (u_{bc} + \Delta u_b) \left(\frac{y_h}{b} \right)^n \quad \left(0 \leq \frac{y_h}{b} \leq 1, h = 2b \right) \quad (180)$$

Here y_o and y_h denote the coordinate across film with respect to the lower and upper surfaces, respectively. Substituting Equations (179) and (180) into the left side of Equation (178) and performing integration, the term of Δu_b , which represents the intensity of co-directional pressure flow with turbulent shear flow, becomes

$$\Delta u_b = (1 + n) \left\{ \left(\frac{h_c}{h} - 1 \right) u_{bc} - \frac{v_s x}{h} \right\} \quad (181)$$

Equation (176) may be written in the form such that

$$\begin{aligned} \rho \zeta_1 \left\{ 2(u_{bc} + \Delta u_b) \frac{\partial(\Delta u_b)}{\partial x} + (u_{bc} + \Delta u_b)^2 \frac{h'}{h} \right\} \\ = - \frac{\partial p}{\partial x} + 2\zeta_5 \rho C_{xc} u_{bc} \frac{\Delta u_b}{h} + \frac{\rho u_s v_s}{h} \end{aligned} \quad (182)$$

where

$$h' = \frac{dh}{dx} \quad (183)$$

Putting $u_{bc} = -u_s/2$ and Equation (181) into (182), the resulting equation of dp/dx is:

$$\begin{aligned}
\frac{dp}{dx} = & -\rho\zeta_1 \left(\frac{u_s}{2}\right)^2 \frac{h'}{h} \left[\left\{ 1 + (1+n) \left(\frac{h_c}{h} - 1\right) \right\} \left\{ 1 - (1+n) \left(1 + \frac{h_c}{h}\right) \right\} \right] \\
& - \frac{\rho\zeta_1 u_s v_s (1+n)}{h} \left[\frac{2(1+n)v_s x}{u_s h} - n + (1+n) \frac{h_c}{h} - \frac{x(1+n)h'h_c}{h^2} \right. \\
& \left. - \frac{v_s}{u_s} h' \left(\frac{x}{h}\right)^2 (1+n) \right] + \frac{\rho u_s v_s}{h} \\
& + 2 \left(\frac{u_s}{2}\right)^2 (1+n) \rho\zeta_5 C_{xc} \frac{1}{h} \left[\left(\frac{h_c}{h} - 1\right) + \frac{2v_s x}{u_s h} \right] \quad (184)
\end{aligned}$$

The velocity boundary conditions are

$$y = 0, \quad u = u_s = -2u_{bc} \quad (185)$$

$$y = h, \quad u = u_h = 0 \quad (186)$$

Considering the general case, the pressure boundary conditions are

$$x = 0, \quad p = p_o \quad (187)$$

$$x = \ell, \quad p = p_\ell \quad (188)$$

Equations (179) and (180) do satisfy conditions in Equations (185) and (186).

Once the shape of bearing, h , is defined, the problem is to solve Equation (184) for p subjected to the conditions in Equations (187) and (188).

Application of Equations (184), (187) and (188) to Plane Wedge Pad

The wedge film thickness is given by

$$h = h_m + \alpha x \quad (189)$$

Here h_m and α denote the minimum film thickness and the slope of the bearing,

respectively. Introduce the following dimensionless quantities: $X = x/\ell$, $P = p/\rho(u_s/2)^2$, $H = h/h_m$, $H_c = h_c/h_m$ and $\beta = a\ell/h_m$. Therefore, Equations (189), (184), (187) and (188) can be written in dimensionless form such that

$$H = 1 + \beta X \quad (190)$$

$$\frac{dP}{dH} = \frac{A}{H} - \frac{B}{H^2} + \frac{E}{H^3} \quad (191)$$

$$X = 0 \quad (\text{i.e., } H = 1), \quad P = P_0 \quad (192)$$

$$X = 1 \quad (\text{i.e., } H = 1 + \beta), \quad P = P_1 \quad (193)$$

where

$$A = \frac{4v_s}{au_s} - \frac{4\zeta_1(1+n)}{a} \frac{v_s}{u_s} \left\{ \frac{v_s(1+n)}{au_s} - n \right\} - n^2\zeta_1 - \frac{2\zeta_5 C_{xc}(1+n)}{a} \left(1 - \frac{2v_s}{au_s} \right) \quad (194)$$

$$B = \frac{4n(1+n)\zeta_1 v_s H_c}{au_s} - \frac{2(1+n)C_{xc} \left(H_c - \frac{2v_s}{au_s} \right) \zeta_5}{a} \quad (195)$$

$$E = \zeta_1(1+n)^2 \left\{ (H_c)^2 + \frac{4v_s}{au_s} \left(\frac{v_s}{au_s} - \frac{H_c}{1+n} \right) \right\} \quad (196)$$

Integrating Equation (191) with respect to H , the resulting equation for P becomes:

$$P = A \ln H + \frac{B}{H} - \frac{E}{2H^2} + D \quad (197)$$

Here, D and H_c are the integration constants which will be determined by Equations (192) and (193). Satisfied by Equations (192) and (193), the solution for P obtained from Equation (197) is

$$P = A \ln H + B \left(\frac{1}{H} - 1 \right) + \frac{E}{2} \left(1 - \frac{1}{H^2} \right) + P_o \quad (198)$$

where

$$H_c = - \frac{J}{2G} \left[1 \pm \left(\frac{1 - 4GK}{J^2} \right)^{1/2} \right] \quad (199)$$

$$G = \frac{(1+n)(2+\beta)\alpha\zeta_1}{2(1+\beta)} \quad (200)$$

$$J = 2\zeta_5 C_{xc} - \frac{2f\zeta_1 \left\{ (2+\beta) + 2n(1+\beta) \right\}}{(1+\beta)} \quad (201)$$

$$\begin{aligned} M = & \frac{(1+\beta)\alpha(P_o - P_1)}{(1+n)\beta} - \frac{2(1+\beta) \ln(1+\beta)}{\beta} \left[- \frac{2f}{1+n} \right. \\ & + 2f\zeta_1 \left\{ \frac{(1+n)f}{a} - n \right\} + \frac{\zeta_1 \alpha n^2}{2(1+n)} + \zeta_5 C_{xc} \left(1 - \frac{2f}{a} \right) \Big] \\ & + 4f \left[\frac{\zeta_1(1+n)(2+\beta)f}{2\alpha(1+\beta)} - \frac{\zeta_5 C_{xc}}{a} \right] \end{aligned} \quad (202)$$

$$f = \frac{v_s}{u_s} = -\sigma^* R_c \frac{(h_m/\ell)^2}{H_c} \quad (203)$$

$$\sigma^* - \text{squeeze-film bearing no.} = \frac{4\mu\ell^2 v_s}{\rho(u_s)^2 h_m^3} \quad (204)$$

$$R_c - \text{Reynolds number based on local shear-flow film thickness} = \frac{h_c u_{bc}}{2\nu} \quad (205)$$

In Equation (198), the terms of A, B and E are in the same forms as in Equations (194) to (196), respectively. Here, the term of H_c is the solution of the following equation

$$GH_c^2 + JH_c + M = 0 \quad (206)$$

For the real positive value of H_c , the "plus" sign in front of $(1 - 4GM/J^2)^{1/2}$ in Equation (199) must be chosen. In the range of parameters considered, $4GM/J^2 < 1$ and $-\frac{\zeta_1^f}{\zeta_5 C_{xc}} \left(\frac{2 + \beta}{1 + \beta} + 2n \right) < 1$, Equation (199) can be simplified to be

$$H_c \cong -M/J \quad (207)$$

Substitute Equations (201) and (202) into (207) and expand H_c in terms of H_c^0 and H_c^f . Here, H_c^0 and H_c^f denote the expressions without and with f , respectively. Therefore, Equation (207) becomes

$$H_c = H_c^0 + H_c^f \quad (208)$$

where

$$H_c^0 = \frac{a(P_1 - P_0)(1 + \beta)}{2\zeta_5(1 + n)\beta C_{xc}} + \frac{(1 + \beta)\ln(1 + \beta)}{\beta} \left\{ 1 + \frac{\zeta_1 n^2 a}{2\zeta_5 C_{xc}(1 + n)} \right\} \quad (209)$$

$$\begin{aligned}
H_C^f = & - \frac{1}{2\zeta_5 C_{xc}} \left[1 + \frac{\zeta_1 f}{\zeta_5 C_{xc}} \left\{ 2n + \frac{2 + \beta}{1 + \beta} \right\} \right] \times \\
& \times \left[\frac{4(1 + \beta) \ln(1 + \beta)}{\beta} \left\{ \frac{1}{(1 + n)} + \frac{\zeta_5 C_{xc}}{a} + \zeta_1 \left(n - \frac{1 + n}{a} f \right) \right\} \right. \\
& + \frac{4}{a} \left\{ - \zeta_5 C_{xc} + \frac{\zeta_1 f(1 + n)(2 + \beta)}{2(1 + \beta)} \right\} f \\
& + \frac{\zeta_1(1 + \beta) \left\{ \frac{(2 + \beta)}{(1 + \beta)} + 2n \right\}}{2(\zeta_5 C_{xc})^2(1 + n)\beta} \left[a(P_1 - P_0) + \left\{ \zeta_1 a n^2 \right. \right. \\
& \left. \left. + 2(1 + n)\zeta_5 C_{xc} \right\} \ln(1 + \beta) \right] \quad (210)
\end{aligned}$$

The dimensionless load-carrying capacity, $\overline{W}_\ell$, is

$$\overline{W}_\ell = \int_0^1 P dX \quad (211)$$

where

$$\overline{W}_\ell = \frac{4W_\ell}{\ell \rho(u_s)^2} \quad (212)$$

W_ℓ = dimensional quantity of load-carrying capacity

Substituting Equations (198) with (208) into (211) and performing integration, the dimensionless load-carrying capacity, $\overline{W}_\ell$, is

$$\overline{W}_\ell = \overline{W}_{\ell o} + \overline{W}_{\ell f} \quad (213)$$

where

$$\begin{aligned}
\overline{W}_{\ell o} = & P_o + \frac{\zeta_1 \beta (1+n)^2}{2(1+\beta)} (H_c^o)^2 \\
& + \frac{2\zeta_5 C_{xc} (1+n) H_c^o}{a} \left\{ 1 - \frac{\ln(1+\beta)}{\beta} \right\} \\
& + \left\{ n^2 \zeta_1 + \frac{2\zeta_5 C_{xc} (1+n)}{a} \right\} \left\{ 1 - \frac{(1+\beta) \ln(1+\beta)}{\beta} \right\}
\end{aligned} \tag{214}$$

$$\begin{aligned}
\overline{W}_{\ell f} = & \frac{4(1+n)f}{a} \left[\left\{ 1 - \frac{(1+\beta) \ln(1+\beta)}{\beta} \right\} \left\{ \frac{\zeta_1 (1+n)^f}{a} - \zeta_1 n \right. \right. \\
& - \left. \frac{\zeta_5 C_{xc}}{a} - \frac{1}{1+n} \right\} - \left\{ 1 - \frac{\ln(1+\beta)}{\beta} \right\} \left\{ \zeta_1 n H_c + \frac{\zeta_5 C_{xc}}{a} \right\} \Big] \\
& + \frac{2\zeta_1 (1+n)^2 \beta f}{a(1+\beta)} \left[\frac{f}{a} - \frac{H_c}{1+n} \right] + \frac{2(1+n) \zeta_5 C_{xc} H_c^f}{a} \times \\
& \times \left[1 - \frac{\ln(1+\beta)}{\beta} \right] + \frac{\zeta_1 \beta (1+n)^2 H_c^f}{(1+\beta)} \left[H_c^o + \frac{H_c^f}{2} \right]
\end{aligned} \tag{215}$$

Here, the terms $\overline{W}_{\ell o}$ and $\overline{W}_{\ell f}$ are ones without and with f , respectively.

The general closed-form solutions are obtained for the calculation of pressure distribution [in Eq. (198)] and load-carrying capacity [in Eq. (213)] for the infinitely long plane wedge squeeze bearing.

The solutions of P and $\overline{W}_{\ell}$ reduce to ones without squeeze effect as $f = 0$. As $f = 0$, $\overline{W}_{\ell}$ becomes, with $W_{\ell f} = 0$,

$$\begin{aligned}
\overline{W}_{\ell} & \equiv W_{\ell o} \\
& = W_{\ell o}^P + W_{\ell o}^F + W_{\ell o}^I
\end{aligned} \tag{216}$$

where

$$\overline{W}_{\ell o}^P = P_o + (1 + \gamma)(P_1 - P_o) \left[1 - \gamma \ln \left(\frac{1 + \gamma}{\gamma} \right) \right] \quad (217)$$

$$\overline{W}_{\ell o}^F = \frac{2\zeta_5 C_{xc}(1 + n)}{a} \left[1 - \gamma(1 + \gamma) \left\{ \ln \left(\frac{1 + \gamma}{\gamma} \right) \right\}^2 \right] \quad (218)$$

and

$$\overline{W}_{\ell o}^I = \zeta_1 \left[\left\{ \frac{(1 + n)H_c^o}{2(1 + \gamma)} \right\}^2 + n^2 - n^2(1 + \gamma)\gamma \left\{ \ln \left(\frac{1 + \gamma}{\gamma} \right) \right\}^2 \right] \quad (219)$$

Here, the terms $\overline{W}_{\ell o}^P$, $\overline{W}_{\ell o}^F$ and $\overline{W}_{\ell o}^I$ denote the dimensionless load-carrying capacity, without squeeze effect, due to pressure boundary conditions, friction and inertia, respectively.

Spring and Damping Coefficients for Nontilting Plane Wedge Pad

The dimensional spring coefficient for untilting pad, k_s , is defined as

$$k_s = - \frac{dW_{\ell o}}{dh_m} \quad (220)$$

For a given value of $\alpha\ell$, the spring coefficient, k_s , in Equation (220) for untilting pad, becomes

$$k_s = - \frac{1}{\alpha\ell} \frac{dW_{\ell o}}{d\left(\frac{1}{\beta}\right)} \quad (221)$$

Introduce

$$\gamma = \frac{1}{\beta} \quad (222)$$

Substituting Equation (222) into Equation (221) and performing the differentiation of $W_{\ell o}$ with respect to γ , K_s becomes

$$K_s = - \frac{1}{a} \frac{d\bar{W}_{\ell o}}{d\gamma} \quad (223)$$

where

$$\bar{W}_{\ell o} = \frac{4W_{\ell o}}{\rho \ell (u_s)^2} \quad (224)$$

$$K_s = \frac{4k_s}{\rho (u_s)^2} \quad (225)$$

$$\begin{aligned} \frac{d\bar{W}_{\ell o}}{d\gamma} = & \frac{\zeta_1(1+n)^2}{2} \left[\frac{2H_c^o}{1+\gamma} \frac{dH_c^o}{d\gamma} - \left(\frac{H_c^o}{1+\gamma} \right)^2 \right] \\ & + \frac{2\zeta_5 C_{xc}(1+n)}{a} \left[\left\{ 1 - \gamma \ln \left(\frac{1+\gamma}{\gamma} \right) \right\} \frac{dH_c^o}{d\gamma} \right. \\ & \left. + H_c^o \left\{ \frac{1}{1+\gamma} - \ln \left(\frac{1+\gamma}{\gamma} \right) \right\} \right] + \left[\frac{1}{\gamma} - \ln \left(\frac{1+\gamma}{\gamma} \right) \right] \\ & \left[\zeta_1 n^2 + \frac{2\zeta_5 C_{xc}(1+n)}{a} \right] \end{aligned} \quad (226)$$

$$\begin{aligned} \frac{dH_c^o}{d\gamma} = & \frac{1}{2\zeta_5 C_{xc}} \left[\frac{a(P_1 - P_o)}{1+n} - \left\{ \frac{1}{\gamma} - \ln \left(\frac{1+\gamma}{\gamma} \right) \right\} \times \right. \\ & \left. \left\{ 2\zeta_5 C_{xc} + \frac{\zeta_1 n^2 a}{(1+n)} \right\} \right] \end{aligned} \quad (227)$$

and

$$\begin{aligned} H_c^o = & \frac{a(1+\gamma)(P_1 - P_o)}{2\zeta_5 C_{xc}(1+n)} + \left\{ 1 + \frac{\zeta_1 a n^2}{2\zeta_5 C_{xc}(1+n)} \right\} \times \\ & \left\{ (1+\gamma) \ln \left(\frac{1+\gamma}{\gamma} \right) \right\} \end{aligned} \quad (228)$$

In fact, the dimensionless spring coefficient K_s in Equation (223) may be written to be the sum of three terms because of the pressure boundary condition, friction and inertia, respectively.

$$K_s = K_s^P + K_s^F + K_s^I \quad (229)$$

where

$$K_s^P = \frac{(P_1 - P_0)}{a} \left\{ (1 + 2\gamma) \ln \left(\frac{1 + \gamma}{\gamma} \right) - 2 \right\} \quad (230)$$

$$K_s^F = \frac{2\zeta_5 C_{xc}(1 + n)}{a^2} \left\{ (1 + 2\gamma) \ln \left(\frac{1 + \gamma}{\gamma} \right) - 2 \right\} \ln \left(\frac{1 + \gamma}{\gamma} \right) \quad (231)$$

$$K_s^I = \frac{\zeta_1(1 + n)^2}{2a} \left\{ \left(\frac{H_c^0}{1 + \gamma} \right)^2 - \frac{2H_c^0}{(1 + \gamma)} \frac{dH_c^0}{d\gamma} \right\} + \frac{\zeta_1 n^2}{a} \left\{ (1 + 2\gamma) \ln \left(\frac{1 + \gamma}{\gamma} \right) - 2 \right\} \ln \left(\frac{1 + \gamma}{\gamma} \right) \quad (232)$$

The dimensional damping coefficient for untilting pad, k_d , is defined as

$$k_d = - \frac{\overline{W}_{lf}}{v_s} \quad (233)$$

where

$$v_s = fu_s \quad (234)$$

and

$$\overline{W}_{\ell f} = \frac{4W_{\ell f}}{\zeta \ell (u_s)^2} \quad (235)$$

As shown in Figure 84, the term u_s is always negative. As the lower surface of the bearing approaches the fixed upper surface, the term of v_s is positive. Therefore, the value of f chosen for this case must be negative. Substituting Equations (234) and (235) into Equation (233), one obtains

$$K_d = - \frac{\overline{W}_{\ell f}}{f} \quad (236)$$

where

$$K_d = \frac{4}{\rho \ell u_s} k_d \quad (237)$$

and

$$\begin{aligned} \overline{W}_{\ell f} = & \frac{4(1+n)f}{a} \left[\left\{ 1 - (1+\gamma) \ln \left(\frac{1+\gamma}{\gamma} \right) \right\} \left\{ \frac{1}{1+n} \right. \right. \\ & - \frac{\zeta_5 C_{xc}}{a} + \frac{\zeta_1 (1+n)f}{a} - \zeta_1 n \left. \right\} \\ & + \left\{ \gamma \ln \left(\frac{1+\gamma}{\gamma} \right) - 1 \right\} \left\{ \frac{\zeta_5 C_{xc}}{a} + \zeta_1 n H_c \right\} \left. \right] \\ & + \frac{2\zeta_5 C_{xc} (1+n) H_c^f}{a} \left[1 - \gamma \ln \left(\frac{1+\gamma}{\gamma} \right) \right] \\ & + \frac{\zeta_1 (1+n)^2}{(1+\gamma)} \left[H_c^f \left\{ H_c^o + \frac{H_c^f}{2} \right\} + \frac{2f}{a} \left\{ \frac{f}{a} - \frac{H_c}{1+n} \right\} \right] \end{aligned} \quad (238)$$

The dimensionless forms of spring and damping coefficients are expressed in Equations (229) and (236), respectively.

Discussion

Equations (217) to (219) for $\overline{W}_{\ell o}^P$, $\overline{W}_{\ell o}^F$ and $\overline{W}_{\ell o}^I$ are evaluated over a range of $h_m/\alpha\ell$. The results are plotted in Figure 85 which shows the variation of $\overline{W}_{\ell o}^P$, $\overline{W}_{\ell o}^F$ and $\overline{W}_{\ell o}^I$ with the value of $h_m/\alpha\ell$. The principal feature is that the thinner the minimum film thickness, the higher the load capacity. It is interesting to note how $\overline{W}_{\ell o}^F$ and $\overline{W}_{\ell o}^I$ build up as $h_m/\alpha\ell$ decreases. The values of $\overline{W}_{\ell o}^F$ and $\overline{W}_{\ell o}^I$, near $h_m/\alpha\ell = 0.1$, become significantly sensitive to the value of $h_m/\alpha\ell$. The contribution to $\overline{W}_{\ell o}$ due to the pressure boundary conditions, $\overline{W}_{\ell o}^P$, is almost independent of $h_m/\alpha\ell$ in the range such that $2 \leq h_m/\alpha\ell$.

Equations (213) to (215) for $\overline{W}_\ell$, $\overline{W}_{\ell o}$ and $\overline{W}_{\ell f}$ give the values of the dimensionless load capacity, respectively. The results are plotted in Figure 86. In view of Figure 86, the value of $\overline{W}_{\ell o}$ is higher than that of $\overline{W}_{\ell f}$. For small values of $h_m/\alpha\ell$, the value of $\overline{W}_\ell$ is seen to be strongly affected by $h_m/\alpha\ell$. Given the values of five parameters (n , C_{xc} , ζ_1 , ζ_5 and f) and two pressure boundary conditions (P_1 and P_o), Equation (213) can be used as a general design equation to predict $\overline{W}_\ell$ for any value of $h_m/\alpha\ell$.

Equations (223) and (236) for K_s and K_d are evaluated with $h_m/\alpha\ell$, respectively. Figure 87 shows the variation of dimensionless spring and damping coefficients with the value of $h_m/\alpha\ell$ for $\alpha = 0.001$. The smaller the values of $h_m/\alpha\ell$, the larger the values of K_s and K_d , as would be expected.

FIGURE 85. DIMENSIONLESS LOAD CAPACITY WITHOUT SQUEEZE EFFECT UNDER TILTED PAD FOR $\alpha = 0.001$

FIGURE 86. DIMENSIONLESS LOAD CAPACITY UNDER
TILTED PAD FOR $\alpha = 0.001$

FIGURE 87. DIMENSIONLESS SPRING AND DAMPING COEFFICIENTS UNDER TILTED PAD FOR $\alpha = 0.001$

ANALYSES OF LAMINAR AND TURBULENT FLOWS UNDER TILTED PADS

Comparisons between Laminar and Turbulent Films

In earlier sections, turbulent film equations have been derived in the following form:

$$\frac{\partial P}{\partial X} = - \frac{2\zeta_5}{H} C_{xc} U \quad (239)$$

$$\frac{\partial P}{\partial Z} = - \frac{\zeta_6 C_{xc}}{H} W \quad (240)$$

$$\frac{\partial \left\{ H(1+U) \right\}}{\partial X} + \frac{\zeta_8}{\zeta_7} \frac{\partial (WH)}{\partial Z} = 0 \quad (241)$$

where C_{xc} , ζ_5 , ζ_6 and ζ_8/ζ_7 can be treated as constants to a good first approximation.

For laminar flow with quadratic velocity profiles, under the same assumptions that go into the Reynolds equation, a similar set may be derived

$$\frac{\partial P}{\partial X} = - 8 \frac{U}{H^2} \quad (242)$$

$$\frac{\partial P}{\partial Z} = - 8 \frac{W}{H^2} \quad (243)$$

$$\frac{\partial \left\{ H \left(\frac{3}{4} + U \right) \right\}}{\partial X} + \frac{\partial (WH)}{\partial Z} = 0 \quad (244)$$

When these are combined, they do indeed become Reynolds equation, but, when left separate as they are, they may be used in a computation procedure similar to that which has been developed for Equations (239), (240), and (241).

The similarities between the two sets of equations make possible easy comparison between the two types of flow. For example, the continuity equations, Equations (241) and (244), are seen to be very nearly the same when it is recognized that ζ_7 is near unity. To make possible additional comparison, let $P' = P/8$ and $P'' = P/2\zeta_5 C_{xc}$ in which the remaining equations become

$$\text{Turbulent} \quad \left\{ \begin{array}{l} \frac{\partial P''}{\partial X} = -\frac{U}{H} \\ \frac{\partial P''}{\partial Z} = -\frac{W}{2H} \frac{\zeta_6}{\zeta_5} \end{array} \right. \quad \begin{array}{l} (245) \\ (246) \end{array}$$

$$\text{Laminar} \quad \left\{ \begin{array}{l} \frac{\partial P'}{\partial X} = -\frac{U}{H^2} \\ \frac{\partial P'}{\partial Z} = -\frac{W}{H^2} \end{array} \right. \quad \begin{array}{l} (247) \\ (248) \end{array}$$

Here, it is seen that the equations differ in their sensitivity to H and in the presence of $\zeta_6/2\zeta_5$ in Equation (246). For very small amounts of tilt, $H \rightarrow 1$; if also $\zeta_6/\zeta_5 \rightarrow 1$, then the turbulent film bearing shows greater leakage in the Z -direction than does the laminar film, on the selected dimensionless basis. Of course, the real implications of these differences can best be demonstrated by complete calculations of pressure for the given bearing configurations.

Iterative Procedure

Linearity of Equation (241) is achieved by dropping account of U in this equation in the first iteration and then supplying the approximate value of this variable for the second iteration. Letting $2\zeta_5 C_{XC}$ be defined as A, $\zeta_6 C_{XC}$ be defined as B, and letting $(1 + U) \frac{(dH/dX)}{H}$ be defined as F, Equations (239), (240), and (241) may be written in reverse order to become

$$-F = \frac{\partial U}{\partial X} + \zeta_7 \frac{\partial W}{\partial Z} \quad (249)$$

$$\frac{\partial P}{\partial X} = -\frac{AU}{H} \quad (250)$$

$$\frac{\partial P}{\partial Z} = -\frac{BW}{H} \quad (251)$$

Solutions may be obtained that satisfy Equation (249) and that may or may not satisfy Equations (250) and (251); however, if a systematic group of solutions can be obtained, it is permissible to sum these to achieve a combined solution which satisfies all three equations. The problem is to devise a system of generation and summing such solutions which is economical of computer time. The method chosen here is to begin with a primitive solution of Equation (249) and allow the discrepancy between the pressures computed from Equations (250) and (251) from this solution to serve as a guide in generation of successive solutions. For example, the following represents a valid solution for Equation (249):

$$\frac{\partial U}{\partial X} = - \frac{F}{2} \quad (252)$$

with the complementary statement

$$\zeta_7 \frac{\partial W}{\partial Z} = - \frac{F}{2} \quad (253)$$

It follows from (252) that

$$U = U_o - \frac{1}{2} \int_0^X F dX \quad (254)$$

where U_o is an initial value at $X = 0$. It also follows from (250) that

$$P_x = P_o - \int_0^X \frac{A}{H} U dX \quad (255)$$

where P_o is an initial value, and P_x is P as computed by the X integration.

Combining Equations (254) and (255),

$$P_x = P_o - \int_0^X \frac{A}{H} \left[U_o - \frac{1}{2} \int_0^X F dX \right] dX \quad (256)$$

but U_o is determined by the requirement that the trailing edge pressure be zero; hence,

$$U_o = \left\{ P_o + \int_0^L \left[\frac{A}{2H} \int_0^X F dX \right] dX \right\} / \left\{ \int_0^L \frac{A}{H} dX \right\} \quad (257)$$

From Equation (253),

$$\frac{\partial W}{\partial Z} = - \frac{F}{2\zeta_7} \quad (258)$$

or

$$W = - \frac{1}{2} \int_0^Z \frac{F}{\zeta_7} dZ \quad (259)$$

because $W = 0$ at $Z = 0$.

Since $P = 0$ at $Z_{\max}$ and P_z is computed by integration in the Z direction,

$$P_z = - \frac{1}{2} \int_{Z_{\max}}^Z \frac{B}{H} \int_0^Z \frac{F}{\zeta_7} dZ dZ \quad (260)$$

it would only be fortuitous that P_z equal P_x throughout the entire field. A solution can be obtained, however, which, when added to the prior solution, tends to produce values of P_z closer to those of P_x than in the primitive solution. This is obtained by letting, at each point,

$$\frac{\partial U}{\partial X} = \alpha (P_z - P_x) \quad (261)$$

and noting that, to satisfy Equation (249), there must be a complementary solution

$$\zeta_7 \frac{\partial W}{\partial Z} = - \alpha (P_z - P_x) \quad (262)$$

In Equations (261) and (262), $\alpha \ll 1$, say 0.001, and is called the relaxation coefficient.

For the next approximate solution, the boundary conditions become

$$W = 0 \text{ at } Z = 0$$

$$P = 0 \text{ at } Z = Z_{\max}$$

$$P = 0 \text{ at } X = 0$$

$$P = 0 \text{ at } X = L$$

A solution which satisfies these boundary conditions may be added to the previous solution, and thus the sum will satisfy all of the previous boundary conditions as stated. For the incremental solution, this is equivalent to substituting $-\alpha(P_z - P_x)$ for $F/2$, in Equations (252), (254), (256) and $\alpha(P_z - P_x)$ for $F/2$ in Equations (258), (259), (260), and letting $P_0 = 0$. By this means, the new P_z and P_x magnitudes may be produced for all (X, Z) points and may be added to the previously obtained matrices of P_x and P_z . The new matrices of P_x and P_z may be used in Equations (261) and (262) for additional repeats by the process of successive approximation.

Pressures under a Finite Tilted Pad

The laminar film Equations (242), (243), and (244) are close enough mathematically to the turbulent ones to serve as a test case for the iterative procedure which has been developed for turbulent films. This is of interest since exact solutions exist for finite tilted pads with laminar flow⁽¹⁵⁾, and these may be compared with corresponding computed solutions. To this end, calculations were carried out for $h_1/\ell_x = 0.001$, $h_1/h_2 = 2$, $L/W = 1$. A grid size of fifteen spaces in the X-direction and nine spaces in the Z-direction was selected, and 100 iterations were carried out, though this was considerably

more than was necessary. The computer time per case on a CDC 3600 digital computer was 4 sec, printing out a complete solution for dimensionless pressures and velocities at 135 grid points. The relatively coarse grid was justified for the integration techniques used since these tend to smooth irregularities.

Table 32 illustrates the convergence of the mean pressure $\bar{P}$ for successive groups of iterations. Primes will be dropped from the P quantities

TABLE 32. CONVERGENCE OF COMPUTATION

<u>Iteration</u>	<u>$\bar{P}$</u>	<u>$\bar{X}$</u>
5	306.8	0.577
10	275.4	0.586
15	269.8	0.587
20	269.8	0.586

in the remaining discussion. Figure 88 shows the pressure distribution for length/width ratio of unity as computed, this being compared with the pressure distributions obtained by Hays⁽¹⁵⁾ in an exact solution. In Figure 89, the axial distribution near the pressure peak is plotted out and is also compared with that calculated by Hays. In both plots, the calculated pressures are slightly high. However, it can be shown that, with coarser grids, pressures would indeed tend to be high, converging toward the exact solution as grid number was increased. Though increased number of grid points could have been used here to obtain a more precise fit, this was not done since it was

FIGURE 88. VARIATION OF PRESSURE IN DIRECTION OF SLIDING (AT MIDSPAN) SHOWING EXACT SOLUTIONS (CURVES) AND COMPUTED DATA (POINTS) FOR $L/W = 1$

FIGURE 89. VARIATION OF PRESSURE NORMAL TO DIRECTION OF SLIDING, AT $X/B = 0.7$, SHOWING EXACT SOLUTIONS (CURVES) AND COMPUTED DATA (POINTS) FOR $L/W = 1$

felt that, with the practical grid size used, the agreement was quite good enough for the type of comparisons to be made in subsequent calculations and that the economies resulting from the small grid size would be justified.

Computed Performance of a Turbulent Film Bearing of Comparable Dimensions to the Above

To illustrate the difference between turbulent-film and laminar-film behavior, even when the leading-edge pressure jump is ignored, a calculation was carried out for the same bearing geometry, but using Equations (245), (246), and (241). Here, the two quantities ξ were allowed to be unity, the friction coefficient was taken as 0.0075, and the dimensionless pressure as used in Equations (245) and (246) serves to permit the plotting of the turbulent film results on the laminar-flow dimensionless plots. Figures 90 and 91 show that the turbulent-film bearing behaves rather like a much shorter laminar-film bearing on this basis. Though the length to width ratio is unity, the turbulent-film bearing behaved more nearly like a laminar one of width 0.6 times the length. Otherwise, the general distribution of pressure is not significantly different from that for laminar flow. This would have been anticipated from our earlier reported work where it was shown that the relation resistance to axial and tangential pressure flows was unity for laminar operation but was in the ratio 1/2.59 for turbulent flow.

Computed Results for Tilted Pads with Leading-Edge Stagnation

To see how computed results would be affected by the assumed leading-edge pressure relationship, previously, this was used to set leading-edge pressure distribution of bearings. This has made possible comparisons of effects of film thickness variation for the same length/width ratio, and

FIGURE 90. VARIATION OF PRESSURE IN DIRECTION OF SLIDING (AT MIDSPAN) COMPARING LAMINAR SOLUTIONS WITH COMPUTED RESULTS FOR A TURBULENT FILM, WHERE $R = 1$ AND LEADING-EDGE PRESSURE IS ZERO

FIGURE 91. VARIATION OF PRESSURE NORMAL TO DIRECTION OF SLIDING, AT $X/B = 0.7$, COMPARING LAMINAR SOLUTIONS WITH COMPUTED RESULTS FOR A TURBULENT FILM, WHERE $L/W = 1$, AND LEADING-EDGE PRESSURE IS ZERO

length/width variations for the same film thickness, all accompanied by tilt angle variation. Figure 92 shows the sensitivity of the several parameters of the bearing performance to leading-edge pressure. Calculations were made in this case for assumed dimensionless leading-edge pressure of 0, 0.1, 0.2 and 0.5. The circled points on the diagram indicate the solution which corresponds to the combination of centerline leading-edge pressure and velocity given previously.

Figures 93, 94 and 95 show computed results for a set of pads for which the dimensionless length L is 750, and for which L/W is assigned the values of $1/2$, 1 and $3/2$. The length is expressed in ratio to the leading-edge film thickness, and this tilt angle is measured in radians. Figure 93 shows that, at small tilt angles, all three bearings may have negative leading-edge pressure. This is in accord with the experimental findings.

Figure 94 shows the corresponding variation of the mean pressure which is a measure of load support. In the range plotted, this is positive, but it goes negative near zero tilt.

Figure 95 shows the movement of the center-of-pressure under variation of tilt angle. Each curve shows negative slope in the region where P_1 is negative. If such pads were mounted on pivots and if the tilt angle were allowed to become small, the toe of the pad would tend to drop down and contact the shaft. The possible consequences of this condition are sufficiently serious as to suggest this as a prime cause of failure of tilted pads operating in the turbulent regime.

FIGURE 92. SENSITIVITY OF MEAN PRESSURE, $\bar{P}$, AND CENTER-OF-PRESSURE POSITION $\bar{X}$ TO VARIATION OF LEADING-EDGE PRESSURE P_1

FIGURE 93. VARIATION OF LEADING-EDGE PRESSURE WITH TILT ANGLES FOR TILTED PADS, AS INFLUENCED BY LENGTH/WIDTH RATIO

FIGURE 94. VARIATION OF MEAN PRESSURE WITH TILT ANGLES FOR TILTED PADS, AS INFLUENCED BY LENGTH/WIDTH RATIO

FIGURE 95. VARIATION OF CENTER-OF-PRESSURE LOCATION FOR TILTED PADS, AS INFLUENCED BY LENGTH/WIDTH RATIO AND TILT ANGLE

To explore further the effects of design parameters on bearing performance, the data reported in Figures 96, 97 and 98 were computed. In this series, the length/width ratio is fixed, but two values of film thickness are used so as to make the dimensionless length take on the magnitudes of 750 and 500. Since the thicker film can tolerate a greater tilt without contact, it has been desirable to set the tilt angle in ratio to the maximum allowable value, and maximum in order to facilitate comparisons.

Figure 96 shows that leading-edge pressure is not greatly influenced by film thickness for a corresponding ratio of leading edge/trailing edge clearance. [Note that $\alpha/\alpha_{\max} = (h_1 - h_2)/h_1$]

Figure 97 shows that the load support curves do not as closely follow one another, there being a definite penalty for the thicker film ($L = 500$).

In Figure 98, the variation for center-of-pressure location is plotted. Though the two curves diverge toward the right, they lie upon one another at the left. Both show the negative slope for a sizeable part of the expected operating range.

A corresponding set of curves to the above is shown in Figures 99, 100 and 101 for a relatively wide bearing. Though the magnitudes plotted differ for those of the prior set, the general behavior of the curve is the same.

FIGURE 96. EFFECT OF LAND
TILT ANGLE ON LEADING-
EDGE PRESSURE

FIGURE 97. EFFECT OF LAND
TILT ANGLE ON MEAN
PRESSURE

FIGURE 98. EFFECT OF LAND
TILT ANGLE ON CENTER-OF-
PRESSURE LOCATION

FIGURE 99. LEADING-EDGE
PRESSURE VARIATION FOR
A LONG BEARING

FIGURE 100. MEAN PRESSURE
VARIATION FOR A LONG
BEARING

FIGURE 101. VARIATION OF CENTER-
OF-PRESSURE LOCATION FOR A
LONG BEARING

THEORY OF A TURBULENT THIN LIQUID FILM IN THE PRESENCE OF SURFACE TENSION

On the basis of experimental observations, it has become clear that thin film flow with free surfaces occurs in bearings and that surface tension has a strong effect upon such flow.

Coyne and Elrod⁽¹⁶⁾ have carried out an analytical study of the effects of surface tension on the free surface contour of a liquid film in laminar flow and have shown that their solution relates to the boundary conditions in bearing lubrication problems. In the work reported here, a corresponding solution is carried out for turbulent flow.

Basic Equations

The Cartesian (x-y) coordinates are, locally, tangent and normal to the free, smoothly-separating film boundary as shown in Figure 102. The difference between the normal fluid stress, p_l , and the cavity pressure, p_a , is balanced by surface tension, Γ .

$$p_a - p_l = \frac{\Gamma}{r} \quad (263)$$

Here,

$$\frac{1}{r} = \frac{d\theta}{ds} \quad (264)$$

Substituting Equation (264) into (263), Equation (263) leads to

$$p_a - p_l = \Gamma \frac{d\theta}{ds} \quad (265)$$

FIGURE 102 DIAGRAM OF PHYSICAL MODEL

where $\frac{d\theta}{ds}$ is the local curvature of the free boundary. The Newtonian stress-strain rate relation gives p_ℓ as

$$-p_\ell = -p + 2\mu \left(\frac{\partial v}{\partial y} \right)_{y=0} \quad (266)$$

Two-dimensional continuity equation in (x-y) plane gives

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0 \quad (267)$$

Here, u and v are the velocity components along the x and y coordinates, respectively. At the free boundary, Equation (267) reads

$$\left(\frac{\partial v}{\partial y} \right)_{y=0} = - \left(\frac{\partial u}{\partial x} \right)_{y=0} \quad (268)$$

Substituting Equation (268) into Equations (265) and (266), the term $\Gamma \frac{d\theta}{ds}$ is

$$\Gamma \frac{d\theta}{ds} = p_a - p - 2\mu \left(\frac{\partial u}{\partial x} \right)_{y=0} \quad (269)$$

The x-component of the Navier-Stokes equation for the incompressible Newtonian flow is

$$\frac{\partial p}{\partial x} = \mu \left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} \right) - \rho \left(u \frac{\partial u}{\partial x} + v \frac{\partial u}{\partial y} \right) \quad (270)$$

The shear stress, τ_{yx} , in the fluid is

$$\tau_{yx} = \mu \left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} \right) \quad (271)$$

At the free boundary, Equation (271) reads

$$\left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} \right)_{y=0} \equiv 0 \quad (272)$$

For turbulent flow, the following quantities are introduced as

$$p = \bar{p} + p' \quad (273)$$

$$p_a = \bar{p}_a + p'_a \quad (274)$$

$$u = \bar{u} + u' \quad (275)$$

$$v = \bar{v} + v' \quad (276)$$

Here, the quantities with super $\bar{}$ and superscript $'$ are referred to as time average value and fluctuation, respectively.

General Theory

Based upon the basic equations in the previous section, the general formulation follows. Substitute Equations (273) to (276) into Equations (267), (269), (270) and (272). Then, the averaging procedure is carried out to eliminate the quantities which fluctuate with time. Following this, Equations (267), (269), (270) and (272) become, respectively,

$$\frac{\partial \bar{u}}{\partial x} + \frac{\partial \bar{v}}{\partial y} = 0 \quad (277)$$

$$\Gamma \frac{d\theta}{ds} = \bar{p}_a - \bar{p} - 2\mu \left(\frac{\partial \bar{u}}{\partial x} \right)_{y=0} \quad (278)$$

$$\frac{\partial \bar{p}}{\partial x} = -\rho \left(\bar{u} \frac{\partial \bar{u}}{\partial x} + \bar{v} \frac{\partial \bar{u}}{\partial y} \right) + \frac{\partial \tau_{yx}}{\partial y} \quad (279)$$

$$\left(\frac{\partial \bar{u}}{\partial y} + \frac{\partial \bar{v}}{\partial x} \right)_{y=0} = 0 \quad (280)$$

where τ_{yx} represents the total shear stress consisting of the laminar and turbulent shear stresses.

Rewrite Equation (279) as follows:

$$\frac{\partial \bar{p}}{\partial x} = -\rho \left[\bar{u} \frac{\partial \bar{u}}{\partial x} + \bar{v} \frac{\partial \bar{u}}{\partial y} + \frac{\partial}{\partial y} (\bar{u}\bar{v}) - \bar{u} \frac{\partial \bar{v}}{\partial y} - \bar{v} \frac{\partial \bar{u}}{\partial y} \right] + \frac{\partial \tau_{yx}}{\partial y}$$

and substituting Equation (277) into the previous form, the term of $\frac{\partial \bar{p}}{\partial x}$ is

$$\frac{\partial \bar{p}}{\partial x} = -\rho \left[\frac{\partial \bar{u}^2}{\partial x} + \frac{\partial}{\partial y} (\bar{u}\bar{v}) \right] + \frac{\partial \tau_{yx}}{\partial y} \quad (281)$$

For convenience, all the quantities with super "-" are omitted in the following equations.

Introduce the new $(x - \eta)$ coordinates as

$$x = x \quad (282)$$

$$\eta = y + \ell \quad (283)$$

Rewrite Equations (277), (278), (280) and (281) in $(x - \eta)$ coordinates, respectively,

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial \eta} = 0 \quad (284)$$

$$\Gamma \frac{d\theta}{ds} = p_a - p - 2\mu \left(\frac{\partial u}{\partial x} \right)_{\eta=\ell} \quad (285)$$

$$\left(\frac{\partial u}{\partial \eta} + \frac{\partial v}{\partial x} \right)_{\eta=\ell} = 0 \quad (286)$$

$$\frac{\partial p}{\partial x} = -\rho \left[\frac{\partial u^2}{\partial x} + \frac{\partial}{\partial \eta} (uv) \right] + \frac{\partial \tau_{yx}}{\partial \eta} \quad (287)$$

Due to the thinness of the film, the term of $\frac{\partial p}{\partial x}$ is assumed to be constant across the film. Thus, integrating Equation (287) with respect to η gives

$$\ell \frac{\partial p}{\partial x} = -\rho \left[\int_0^\ell \frac{\partial u^2}{\partial x} d\eta + (uv)_\ell - (uv)_0 \right] + (\tau_{\eta x})_\ell - (\tau_{\eta x})_0 \quad (288)$$

Because the terms of $(uv)_\ell$, $(uv)_0$, τ_ℓ and τ_0 are independent with η , Equation (288) at $\eta = \ell$ becomes

$$\begin{aligned} \ell \left(\frac{\partial p}{\partial x} \right)_{\eta=\ell} = & -\rho \left[\frac{\partial}{\partial x} \int_0^\ell u^2 d\eta \right]_{\eta=\ell} - \rho [(uv)_\ell - (uv)_0] \\ & + (\tau_{\eta x})_\ell - (\tau_{\eta x})_0 \end{aligned} \quad (289)$$

In general, at the free boundary,

$$v_\ell = 0 \quad (290)$$

$$(\tau_{\eta x})_\ell = 0 \quad (291)$$

Substituting Equations (290) and (291) into (289), $\left(\frac{\partial p}{\partial x} \right)_{\eta=\ell}$ is:

$$\ell \left(\frac{\partial p}{\partial x} \right)_{\eta=\ell} = -\rho \left[\frac{\partial}{\partial x} \int_0^\ell u^2 d\eta \right]_{\eta=\ell} + \left[\rho(uv)_o - (\tau_{\eta x})_o \right]_{\eta=\ell} \quad (292)$$

where

$$(\tau_{\eta x})_o = \frac{C_{xc}\rho(-u_o + u_\ell)^2}{2} \quad (293)$$

By means of substitution of Equations (A-13) into (A-16), Equations (285), (286) and (292) become, respectively,

$$\Gamma \frac{d\theta}{ds} = p_a - p - 2\mu \frac{dq}{ds} \quad (294)$$

$$\frac{d\theta}{ds} = -\frac{1}{q} \left(\frac{\partial u}{\partial \eta} \right)_{\eta=\ell} \quad (295)$$

$$\frac{dp}{ds} = -\frac{\rho}{\ell} \left[\frac{d}{ds} \left(\int_0^\ell u^2 d\eta \right) \right] + \left[\frac{\rho(uv)_o}{\ell} - \frac{C_{xc}\rho(-u_o + u_\ell)^2}{2\ell} \right]_{\eta=\ell} \quad (296)$$

Integrate Equation (296) with respect to s from $s = -\infty$ to $s = s$.

$$p - p_a = -\rho \int_0^\theta d \left[\int_0^\ell u^2 d \left(\frac{\eta}{\ell} \right) \right] + p \left[\int_{-\infty}^s \frac{(uv)_o}{\ell} ds - \frac{\rho C_{xc}}{2} \int_{-\infty}^s \frac{(-u_o + u_\ell)^2}{\ell} ds \right]_{\eta=\ell} \quad (297)$$

Here,

$$\ell = \frac{h}{\cos \theta} \quad (298)$$

$$\frac{dh}{ds} = \sin \theta \quad (299)$$

$$\frac{dx'}{ds} = \cos \theta \quad (300)$$

$$u_o = -U \cos \theta \quad (301)$$

$$v_o = U \sin \theta \quad (302)$$

From Equation (298), the term of $\frac{d\ell}{ds}$ is

$$\frac{d\ell}{ds} = h \frac{\tan \theta}{\cos \theta} \frac{d\theta}{ds} + \frac{l}{\cos \theta} \frac{dh}{ds} \quad (303)$$

Substituting Equation (299) into Equation (303), the term of ds is

$$ds = \frac{d\ell}{\tan \theta} - l d\theta \quad (304)$$

Substituting Equations (201), (202) and (204) into Equation (297), the term of $(p - p_a)$ is

$$\begin{aligned} p - p_a = & -\rho \left[\left\{ \int_0^l u^2 d\left(\frac{\eta}{l}\right) \right\}_{\theta=\theta} - \left\{ \int_0^l u^2 d\left(\frac{\eta}{l}\right) \right\}_{\theta=0} \right] \\ & + \rho U^2 \left[-h^2 \int_{h-\infty}^{h/\cos \theta} \frac{d\ell}{l^3} + \frac{1}{2} \int_0^\theta \sin 2\theta d\theta \right] \\ & - \frac{\rho C_{xc}}{2} \left[\int_{h-\infty}^{h/\cos \theta} \frac{(U \cos \theta + u_\ell)}{l \tan \theta} d\ell - \int_0^\theta (U \cos \theta + u_\ell)^2 d\theta \right] \quad (305) \end{aligned}$$

The flow swept by the plate is constant and is

$$\int_0^{\ell} u d\eta \equiv \text{constant} \equiv -Uh_{-\infty} \quad (306)$$

The boundary value of u_{ℓ} can be obtained from Equation (306) provided that the proper velocity profile of u along η -axis is specified.

The six quantities of p , q , θ , u_{ℓ} , x' and h in six equations of Equations (294), (295), (299), (300), (305) and (306) can be solved numerically. At present, the analysis to the problem is quite general. In this general theory, the velocity profile of u relating to the turbulent flow in the thin film must be, properly, assumed in order to have six equations with six unknowns. In the next section, a realistic velocity profile of u is chosen in order that the terms of $\int_0^1 u^2 d\left(\frac{\eta}{\ell}\right)$, $\frac{\partial u}{\partial \eta}$ and u_{ℓ} can be evaluated.

Power-Law Velocity Profile for u along η .

Governing Differential Equations. The velocity profile along the η -axis is represented by a power-law form

$$u = -U \cos \theta + (U \cos \theta + u_{\ell}) \left(\frac{\eta}{\ell}\right)^n \quad (307)$$

Here, the term n is the power-law parameter. Equation (307) does satisfy the boundary condition, Equation (307), at $\eta = 0$ and reduces to u_{ℓ} at $\eta = \ell$. Substituting Equation (307) into Equation (306) and integrating it with respect to η , the ratio of $\frac{u_{\ell}}{U}$ is:

$$\frac{u_\ell}{-U} = \left[\frac{1+n}{H} - n \right] \cos \theta \quad (308)$$

where

$$H \equiv \frac{h}{h_{-\infty}}$$

Substituting Equation (307) into Equation (295), the term of $\frac{d\theta}{ds}$ is

$$\frac{d\theta}{ds} = - \frac{n(U \cos \theta + u_\ell) \cos \theta}{qh} \quad (309)$$

In dimensionless form, Equation (309) becomes, with $Sh_{-\infty} = s$ and $Hh_{-\infty} = h$,

$$-\bar{q}U = q$$

$$\frac{d\theta}{dS} = \frac{n \left(\cos \theta + \frac{u_\ell}{U} \right) \cos \theta}{\bar{q}H} \quad (310)$$

Substituting Equation (308) into Equation (310), the term of $\frac{d\theta}{dS}$ is

$$\frac{d\theta}{dS} = \frac{n(1+n)(H-1) \cos^2 \theta}{\bar{q}H^2} \quad (311)$$

At $H = 1$ (i.e., $\theta = 0$), the local radius of curvature (i.e., $\frac{1}{d\theta/dS}$) does reduce to infinity for a finite value of $\bar{q}$.

In Equation (311), the velocity term of $\bar{q}$ is in the denominator.

Therefore, in order to remain the curvature bounded at $\bar{q} = 0$, the angle of θ must be equal to $\frac{\pi}{2}$ in the numerator in Equation (311). Consequently,

the stagnation point can occur only at $\theta = \frac{\pi}{2}$. At other angles in the range of $0 \leq \theta < \frac{\pi}{2}$, there is no stagnation point on the free boundary. The limiting condition of $\frac{d\bar{q}}{dS}$ for $\theta \rightarrow \frac{\pi}{2}$ has been proved to be zero (see Appendix B). Substituting Equations (C-4), (C-6) and (C-7) into Equation (305), Equation (305) becomes, with $p - p_a = (P - P_a)\rho U^2$,

$$\begin{aligned}
 P - P_a = \frac{1}{2} \left[\left\{ 1 + \frac{(1+n)^2}{1+2n} \frac{(1-H)^2}{H^2} + \frac{2(1-H)}{H} \right\} \cdot \left\{ 1 - \cos 2\theta \right\} \right. \\
 \left. + \left\{ 1 - H^2 \right\} \right] + \frac{C_{xc}}{2} \left[\frac{\theta}{2} + \left\{ \frac{1}{4} + \left(n - \frac{1+n}{H} \right) \right\} \sin 2\theta + \left\{ \left(n - \frac{1+n}{H} \right)^2 \theta \cos^2 \theta \right\} \right. \\
 \left. - \left\{ 1 + n \left(n - \frac{1+n}{H} \right) \right\}^2 \left\{ \ln \left(\frac{H}{\cos \theta} \right) \right\} \frac{\cos^3 \theta}{\sin \theta} \right] \quad (312)
 \end{aligned}$$

At $\theta = 0$ and $H = 1$, Equation (312) becomes

$$P - P_a = 0 \quad (313)$$

In fact, the quantity of P must be equal to P_a as $\theta = 0$ and $H = 1$. Substituting Equation (312) into Equation (294) [see Eq. (C-14)], Equation (294) in dimensionless form becomes

$$2 \frac{d\bar{q}}{dS} = N \frac{d\theta}{dS} + R_s (P - P_a) \quad (314)$$

Substituting Equation (312) into Equation (314), Equation (314) becomes

$$\begin{aligned}
2 \frac{d\bar{q}}{dS} = & N \frac{d\theta}{dS} + \frac{R_s}{2} \left[\left\{ 1 - \cos 2\theta \right\} \cdot \left\{ 1 + \frac{(1+n)^2}{1+2n} \frac{(1-H)^2}{H^2} + \frac{2(1-H)}{H} \right\} \right. \\
& + \left. \left\{ 1 - H^2 \right\} \right] + \frac{R_s C_{xc}}{2} \left[\frac{\theta}{2} + \left\{ \frac{1}{4} + \left(n - \frac{1+n}{H} \right) \right\} \sin 2\theta \right. \\
& + \left. \left\{ \left(n - \frac{1+n}{H} \right)^2 \theta \cos^2 \theta \right\} - \left\{ \ln \left(\frac{H}{\cos \theta} \right) \right\} \left\{ 1 + \left(n - \frac{1+n}{H} \right) \right\}^2 \frac{\cos^3 \theta}{\sin \theta} \right]
\end{aligned} \tag{315}$$

From Equations (299) and (300) (see Appendix C),

$$\frac{dH}{dS} = \sin \theta \tag{316}$$

$$\frac{d\xi}{dS} = \cos \theta \tag{317}$$

From Equation (311),

$$\frac{d\theta}{dS} = \frac{n(1+n)(H-1) \cos^2 \theta}{\bar{q}H^2} \tag{318}$$

Equations (315) to (318) constitute the field equations to solve for $\bar{q}$, H , ξ and θ .

The Linearized Asymptotic Solution. The linearized asymptotic solution of

Equations (315) to (318) can be obtained near the region of small surface slope where $S \rightarrow \infty$, $\theta \rightarrow 0$, $H \rightarrow 1$ and $\bar{q} \rightarrow 1$. The asymptotic solution of $\bar{q}$ and H can be assumed to be a small perturbation from unity in the region of small surface slope. Thus, at $\theta \rightarrow 0$,

$$\bar{q} = 1 - \bar{q}' \tag{319}$$

$$H = 1 + H' \quad (320)$$

Where the solutions, $\bar{q}'$ and H' , are perturbed ones compared with unity.

Substituting Equations (319) and (320) into Equations (315) to (318) and neglecting the higher order (≥ 2) terms of small perturbation quantities, Equations (315) to (320) become, respectively,

$$- 2 \frac{d\bar{q}'}{dS} = N \frac{d\theta}{dS} - R_s H' \quad (321)$$

$$\frac{dH'}{dS} = \theta \quad (322)$$

$$\frac{d\xi}{dS} = 1 \quad (323)$$

$$\frac{d\theta}{dS} = n(1+n)H' \quad (324)$$

From Equation (322), the term $\frac{d\theta}{dS}$ is

$$\frac{d\theta}{dS} = \frac{d^2 H'}{dS^2} \quad (325)$$

Substituting Equation (325) into Equation (324), one obtains

$$\frac{d^2 H'}{dS^2} - n(1+n)H' = 0 \quad (326)$$

The characteristic equation is

$$j^2 - n(1+n) = 0 \quad (327)$$

Here n is the positive real value. Hence, Equation (327) gives two distinct real roots, $\sqrt{n(1+n)}$, $-\sqrt{n(1+n)}$. Hence, the general solution of Equation (326) is

$$H' = C_1 e^{\sqrt{n(1+n)} S} + C_2 e^{-\sqrt{n(1+n)} S} \quad (328)$$

In order for H' to remain bounded as $S \rightarrow -\infty$, the constant C_2 must be zero.

Thus,

$$H' = C_1 e^{\sqrt{n(1+n)} S} \quad (329)$$

Substituting Equation (329) into (320), one obtains

$$H = 1 + C_1 e^{\sqrt{n(1+n)} S} \quad (330)$$

Substituting Equation (329) into (322), one obtains

$$\theta = C_1 \sqrt{n(1+n)} e^{\sqrt{n(1+n)} S} \quad (331)$$

The solution of Equation (323) is

$$\xi = S \quad (332)$$

Substituting Equation (324) into (321), one obtains

$$\frac{d\bar{q}'}{dS} = [R_S - Nn(1+n)] \frac{C_1}{2} e^{\sqrt{n(1+n)} S} \quad (333)$$

Integrating Equation (333) with respect to S , one obtains

$$\bar{q}' = \frac{C_1}{2} [R_S - Nn(1+n)] \frac{e^{\sqrt{n(1+n)} S}}{\sqrt{n(1+n)}} + C_3 \quad (334)$$

In order for $\bar{q}' \rightarrow 0$ as $S \rightarrow -\infty$, the constant, C_3 , must vanish. Thus,

$$\bar{q}' = \frac{C_1}{2\sqrt{n(1+n)}} [R_s - Nn(1+n)] e^{\sqrt{n(1+n)} S} \quad (335)$$

Substituting Equation (335) into Equation (319), one obtains

$$\bar{q} = 1 - \frac{C_1}{2\sqrt{n(1+n)}} [R_s - n(1+n)N] e^{\sqrt{n(1+n)} S} \quad (336)$$

Equations (330) to (332) and Equation (336) are the general linearized asymptotic solutions with arbitrary constant, C_1 . In Equations (330) to (332) and Equation (336), the constant C_1 is determined by the choice of origin for S (say $H = H_0$ at $S = 0$). Thus, from Equation (330)

$$C_1 = H_0 - 1 \quad (337)$$

Here the origin of S may be chosen to be where $H_0 = 1.0001$. Thus, Equation (319) reads

$$C_1 = 10^{-4} \quad (338)$$

Numerical Solution in the Range of $H > H_0$. Equations (315) to (318) can be numerically solved starting from $H \equiv H_0$. The starting values of $\bar{q}$, H , ξ and θ can be obtained from the linearized, asymptotic solutions in Equations (336), (330), (332) and (331), respectively. With $C_1 = 10^{-4}$ in Equation (338), these starting values are

$$\bar{q}_0 = 1 - \frac{10^{-4}}{2\sqrt{n(1+n)}} [R_s - n(1+n)N] \quad (339)$$

$$\theta_0 = \sqrt{n(1+n)} \times 10^{-4} \quad (340)$$

$$H_0 = 1.0001 \quad (341)$$

$$\xi_0 = 0 \quad (342)$$

Substituting the value of dS obtained from Equation (311) into Equations (314), (316) and (317), one obtains a set of three simultaneous first-order differential equations for three dependent variables of ξ , H and q .

$$\frac{d\xi}{d\theta} = \frac{\bar{q}H^2}{n(1+n)(H-1)\cos\theta} \quad (343)$$

$$\frac{dH}{d\theta} = \frac{\bar{q}H^2 \tan\theta}{n(1+n)(H-1)\cos\theta} \quad (344)$$

and

$$\frac{d\bar{q}}{d\theta} = \frac{N}{2} + \frac{R_s}{2} \frac{\bar{q}H^2}{n(1+n)(H-1)\cos^2\theta} (\Delta P) \quad (345)$$

where

$$\begin{aligned} \Delta P = \frac{1}{2} \left[\left\{ 1 - \cos 2\theta \right\} \left\{ 1 + \frac{(1+n)^2 (1-H)^2}{(1+2n)H^2} + \frac{2(1-H)}{H} \right\} + \left\{ 1 - H^2 \right\} \right] \\ + \frac{C_{xc}}{2} \left[\frac{\theta}{2} + \left\{ \frac{1}{4} + \left(n - \frac{1+n}{4} \right) \right\} \sin 2\theta + \left\{ \left(n - \frac{1+n}{H} \right)^2 \theta \cos^2 \theta \right\} \right. \\ \left. - \left\{ \ln \left(\frac{H}{\cos \theta} \right) \right\} \left\{ 1 + \left(n - \frac{1+n}{H} \right) \right\}^2 \frac{\cos^3 \theta}{\sin \theta} \right] \quad (346) \end{aligned}$$

Using initial conditions in Equations (339) through (342), a set of three simultaneous first-order differential equations of Equations (343) through (345) are solved for ξ , H and $\bar{q}$ by means of the Adams method. Here, the value of θ is an independent variable. The basic integrator is a self-starting variant of

this method incorporating automatic step size control. Automatic control of truncation is provided.

The properties of the three kinds of liquid metal chosen are listed in Table 33, at $t = 200^\circ\text{C}$ (above the melting point).

TABLE 33. THE PROPERTIES OF THE LIQUID METALS

	<u>μ, centipoise</u>	<u>ρ, gm/cm³</u>	<u>Γ, dynes/cm</u>
Potassium, K	0.33	0.8	119
Sodium, Na	0.457	0.9	181
Lithium, Li	0.563	0.51	395

Given the values of $U = 100 \text{ fps} = 3025 \text{ cm/sec}$, $h_{\infty} = 5 \times 10^{-4} \text{ in.} = 12.7 \times 10^{-4} \text{ cm}$ and the properties in Table 33, the values of R_s and N are calculated and listed in Table 34. The corresponding values of C_{xc} and n in this problem

TABLE 34. THE VALUES OF R_s , $\bar{R}$, N , C_{xc} AND n

	<u>R_s</u>	<u>$\bar{R} = \frac{R_s}{4}$</u>	<u>N</u>	<u>C_{xc}</u>	<u>n</u>
K	930	233	11.92	0.016	0.5
Na	758	190	13.1	0.018	0.55
Li	348	87	23.2	0.02	0.835

are chosen approximately from Reference (10), with $n = \frac{\delta^*/(h_{\infty}/2)}{1 - \delta^*/(h_{\infty}/2)}$ and

the value of $\bar{R}$ in Table 34. Substituting the given values of R_s , N , C_{xc} and n in Table 34 into Equations (339) through (342), one obtains the initial conditions of each case as listed in Table 35.

TABLE 35. THE INITIAL CONDITIONS OF THREE CASES

	θ_o	ξ_o	H_o	$\bar{q}_o$
K	0.865×10^{-4}	0	1.0001	0.9468
Na	0.923×10^{-4}	0	1.0001	0.9595
Li	1.24×10^{-4}	0	1.0001	0.9874

Using the initial conditions in Table 35, the values of ξ , H and $\bar{q}$ are solved numerically from Equations (343) through (345), respectively. Results of ξ , H , $\bar{q}$ and $(p_a - p)/\rho U^2$ are plotted in Figures 103 through 106. As shown in Figure 103, the dimensionless distance from $h = h_\infty$, ξ increases as θ increases. The value of ξ near $\theta = 0^\circ$ becomes very sensitive to θ . In other words, the value of ξ seems to be significantly affected by θ in the neighborhood of $\theta = 0^\circ$. Figure 104 shows that the dimensionless film thickness, H , builds up as θ increases. Of course, as $\theta \rightarrow 0^\circ$, the value of H approaches its asymptotic character, $H = 1$. As $\theta = 90^\circ$, a stagnation point appears at $H = 1.48$, 1.49 and 1.58 for potassium, sodium and lithium, respectively. From Figure 106, note that the value of $(p_a - p)/\rho U^2$ builds up as the inclination angle of free surface increases. Of course, the value of $(p_a - p)/\rho U^2$ is equal to zero as θ approaches zero. Figure 105 shows the variation of $\bar{q}$ with θ . The value of $\bar{q}$ becomes significantly sensitive to the value of θ when θ becomes small. As $\theta \rightarrow 0^\circ$, the dimensionless surface velocity of the free surface of the fluid film, $\bar{q}$, is of unity.

FIGURE 103. VARIATION OF DIMENSIONLESS DISTANCE WITH INCLINATION ANGLE

FIGURE 104. VARIATION OF DIMENSIONLESS FILM THICKNESS WITH INCLINATION ANGLE

FIGURE 105. VARIATION OF DIMENSIONLESS VELOCITY AT FREE SURFACE WITH INCLINATION ANGLE

FIGURE 106. VARIATION OF $(-\Delta P)$ WITH INCLINATION ANGLE

LIST OF REFERENCES

1. D. F. Wilcock, "Turbulence in High Speed Journal Bearings," Trans. ASME, Vol 72, 1950, pp 825-834.
2. R. A. Burton, "On the Close Relationship between Turbulent Plane-Couette and Pressure Flows," J. Aeronautical Sciences, Vol 29, 1962.
3. G. I. Taylor, "Fluid Friction between Rotating Cylinders," Scientific Papers of G. I. Taylor, Vol II, Cambridge University Press, 1960.
4. John Laufer, "The Structure of Turbulence in Full Developed Pipe Flow," NACA Tech. Report 1174, 1954.
5. M. Couette, "Etudes sur le Frottement des Liquides," Ann. Chem. Phys., Vol 21, No. 6, 1890.
6. M. I. Smith and D. D. Fuller, "Journal-Bearing Operation at Superlaminar Speeds," Trans. ASME, Vol 78, 1956.
7. D. Ross, "Evaluation of the Momentum Integral Equation for Turbulent Boundary Layers," J. Aeronautical Sciences, Vol 20, July 1953.
8. H. Reichardt, "On the Velocity Distribution in a Rectilinear Turbulent Couette Flow," Zeitschrift für Angewandte Mathematik und Mechanik, Special Supplement, 1956.
9. H. J. Carper and R. A. Burton, "A Technique for Micromanometry," AIAA Journal, Vol 4, No. 3, March 1966.
10. R. A. Burton and J. Carper, "An Experimental Study of Annular Flows with Applications in Turbulent Film Lubrication," ASME Paper No. 66-Lub-14, 1966.
11. E. A. Mujdermann, "Analysis and Design of Spiral Groove Bearings," M. S. 4688, Phillips Research Laboratories, Eindhoven, Netherland.
12. H. Schlichting, Boundary Layer Theory, 4th Ed., McGraw-Hill Book Co., Inc., New York, N.Y., 1960.
13. E. B. Arwas, B. Sternlicht, and R. J. Wernick, "Analysis of Plain Cylindrical Journal Bearings in Turbulent Regime," Trans. ASME, J. Basic Engineering, Vol 86, Ser. D., No. 2, June 1964.

14. F. W. Devirk, "Short Bearing Approximation for Full Journal Bearings," Technical Note 2808, National Advisory Committee for Aeronautics, October 1952.
15. D. F. Hays, "Plane Sliders of Finite Width," ASLE Trans., Vol 1, No. 2, 1958.
16. J. C. Coyne and H. G. Elrod, Jr., "Conditions for the Rupture of a Lubricating Film," Technical Report MTI-65 TR 58, December 1965.

APPENDIXES

Appendix A

$$\frac{d}{ds} = \frac{dx}{ds} \frac{\partial}{\partial x} + \frac{dy}{ds} \frac{\partial}{\partial y} \quad (\text{A-1})$$

$$\frac{d^2}{ds^2} = \frac{d}{ds} \left(\frac{d}{ds} \right) = \left(\frac{dx}{ds} \frac{\partial}{\partial x} + \frac{dy}{ds} \frac{\partial}{\partial y} \right) \left(\frac{dx}{ds} \frac{\partial}{\partial x} + \frac{dy}{ds} \frac{\partial}{\partial y} \right) \quad (\text{A-2})$$

where

$$\frac{dx}{ds} = \cos \Delta \theta \quad \frac{dy}{ds} = \sin \Delta \theta \quad (\text{A-3})$$

Equations (A-1) and (A-3) give

$$\frac{\partial}{\partial x} = \frac{1}{\cos \Delta \theta} \left(\frac{d}{ds} - \sin \Delta \theta \frac{\partial}{\partial y} \right) \quad (\text{A-4})$$

As $A \rightarrow B$, then $\Delta \theta \rightarrow 0$ and $\Delta s \rightarrow 0$. As $\Delta \theta \rightarrow 0$, Equation (A-4) gives

$$\left(\frac{\partial}{\partial x} \right)_{\Delta \theta = 0} = \frac{d}{ds} \quad (\text{A-5})$$

As $\Delta \theta \rightarrow 0$, Equations (A-2) and (A-3) give

$$\frac{d^2}{ds^2} = \left(\frac{\partial^2}{\partial x^2} \right)_{\Delta \theta = 0} + \frac{d\theta}{ds} \left(\frac{\partial}{\partial y} \right)_{\Delta \theta = 0} \quad (\text{A-6})$$

The velocity components u and v are related to the velocity along stream line

s , q , are

$$(u)_{y=0} = q \cos \Delta \theta \quad (\text{A-7})$$

$$(v)_{y=0} = q \sin \Delta \theta \quad (\text{A-8})$$

As $\Delta \theta \rightarrow 0$, Equation (A-5) with Equation (A-7) gives

$$\left(\frac{\partial u}{\partial x} \right)_{y=0} = \frac{dq}{ds} \quad (\text{A-9})$$

As $\Delta\theta \rightarrow 0$, Equation (A-5) with Equation (A-8) gives

$$\left(\frac{\partial v}{\partial x}\right)_{y=0} = q \frac{d\theta}{ds} \quad (\text{A-10})$$

As $\Delta\theta \rightarrow 0$, Equation (A-6) with Equation (A-7) gives

$$\left(\frac{\partial^2 u}{\partial x^2}\right)_{y=0} = \frac{d^2 q}{ds^2} - \frac{d\theta}{ds} \left(q \frac{d\theta}{ds} + \frac{\partial u}{\partial y} \right)_{y=0} \quad (\text{A-11})$$

Substituting Equation (A-10) into Equation (A-11), $\left(\frac{\partial^2 u}{\partial x^2}\right)_{y=0}$ gives

$$\left(\frac{\partial^2 u}{\partial x^2}\right)_{y=0} = \frac{d^2 q}{ds^2} - \frac{d\theta}{ds} \left(\frac{\partial v}{\partial x} + \frac{\partial u}{\partial y} \right)_{y=0}$$

Substitute Equation (279)

$$\left(\frac{\partial^2 u}{\partial x^2}\right)_{y=0} = \frac{d^2 q}{ds^2} \quad (\text{A-12})$$

Transformed to coordinate $(x - \eta)$ where $x = x$ and $\eta = y + \ell$ (i.e., $dx = dx$ or $d\eta = dy$), Equations (A-9), (A-10) and (A-12) read

$$\left(\frac{\partial u}{\partial x}\right)_{\eta=\ell} = \frac{dq}{ds} \quad (\text{A-13})$$

$$\left(\frac{\partial v}{\partial x}\right)_{\eta=\ell} = q \frac{d\theta}{ds} \quad (\text{A-14})$$

and

$$\left(\frac{\partial^2 u}{\partial x^2}\right)_{\eta=\ell} = \frac{d^2 q}{ds^2} \quad (\text{A-15})$$

Similarly,

$$\left(\frac{\partial p}{\partial x}\right)_{\eta=\ell} = \frac{dp}{ds} \quad (\text{A-16})$$

Appendix B

To apply L'Hospital's rule to Equation (310), at $\theta \rightarrow \pi/2$ and $\bar{q} \rightarrow 0$, reads

$$\left\{ \frac{d\theta}{ds} + \frac{2n(1+n)(H-1) \cos \theta \sin \theta \frac{d\theta}{ds}}{H^2 \frac{d\bar{q}}{ds}} \right\}_{\substack{\theta \rightarrow \pi/2 \\ \bar{q} \rightarrow 0}} = 0 \quad (\text{B-1})$$

In view of the fact that $\frac{d\theta}{ds} \neq 0$ in Equation (B-1), then

$$\begin{aligned} \left(\frac{d\bar{q}}{ds} \right)_{\theta = \pi/2} &= \lim_{\theta \rightarrow \pi/2} \left[\frac{-2n(1+n)(H-1)}{H^2} \cos \theta \right] \\ &= 0 \end{aligned} \quad (\text{B-2})$$

Appendix C

$$\begin{aligned} \frac{d \left[\int_0^{\ell} u^2 d\eta \right]}{\ell} &\equiv d \left[\int_0^1 u^2 d \left(\frac{\eta}{\ell} \right) \right] \\ &\equiv U^2 d \left[\int_0^1 \left(\frac{u}{U} \right)^2 d \left(\frac{\eta}{\ell} \right) \right] \end{aligned} \quad (\text{C-1})$$

Substituting Equation (307) into Equation (C-1) and integrating it, Equation (C-1) reduces to

$$\begin{aligned} \frac{d \left[\int_0^{\ell} u^2 d\eta \right]}{\ell} &= U^2 d \left[\cos^2 \theta - \frac{2}{(1+n)} \cos \theta \left(\cos \theta + \frac{u\ell}{U} \right) \right. \\ &\quad \left. + \frac{1}{(1+2n)} \left(\cos \theta + \frac{u\ell}{U} \right)^2 \right] \end{aligned} \quad (\text{C-2})$$

Substituting Equation (308) into Equation (C-2), Equation (C-2) reduces to

$$d \left[\int_0^1 u^2 d \left(\frac{\eta}{\ell} \right) \right] = - U^2 \left\{ 1 + \frac{(1+n)^2}{(1+2n)} \frac{(1-H)^2}{H^2} + \frac{2(1-H)}{H} \right\} \sin 2\theta d\theta \quad (C-3)$$

Integrating Equation (C-3) with respect to θ from 0 to θ , the term of

$$\begin{aligned} & \left[\int_0^1 u^2 d \left(\frac{\eta}{\ell} \right) \right]_{\theta=0} - \left[\int_0^1 u^2 d \left(\frac{\eta}{\ell} \right) \right]_{\theta=0} \text{ is} \\ & \left[\int_0^1 u^2 d \left(\frac{\eta}{\ell} \right) \right]_{\theta=\theta} - \left[\int_0^1 u^2 d \left(\frac{\eta}{\ell} \right) \right]_{\theta=0} \\ & = \frac{U^2}{2} \left\{ 1 + \frac{(1+n)^2}{(1+2n)} \frac{(1-H)^2}{H^2} + \frac{2(1-H)}{H} \right\} \left\{ \cos 2\theta - 1 \right\} \quad (C-4) \end{aligned}$$

$$\begin{aligned} & \int_{h-\infty}^{h/\cos \theta} \frac{(U \cos \theta + u_\ell)^2}{\ell \tan \theta} d\ell - \int_0^\theta (U \cos \theta + u_\ell)^2 d\theta \\ & = U^2 \left[- \left\{ \left(2 \frac{u_\ell}{U} + \frac{\cos \theta}{2} \right) \sin \theta + \left(\frac{u_\ell^2}{U^2} + \frac{1}{2} \right) \theta \right\} + \left\{ \ln \frac{H}{\cos \theta} \right\} \times \right. \\ & \quad \left. \times \left\{ \frac{\left(\frac{u_\ell^2}{U} \right) + \frac{2u_\ell}{U} \cos \theta + \cos^2 \theta}{\tan \theta} \right\} \right] \quad (C-5) \end{aligned}$$

Substituting Equation (308) into Equation (C-5), Equation (C-5) reduces to

$$\begin{aligned} & \int_{h-\infty}^{h/\cos \theta} \frac{(U \cos \theta + u_\ell)^2}{\ell \tan \theta} d\ell - \int_0^\theta (U \cos \theta + u_\ell)^2 d\theta = U^2 \left[- \left\{ \frac{1}{2} \theta \right\} \right. \\ & \quad - \left\{ \frac{1}{4} + \left(n - \frac{1+n}{H} \right) \right\} \sin 2\theta - \left\{ \left(n - \frac{1+n}{H} \right)^2 \theta \cos^2 \theta \right\} \\ & \quad \left. + \left\{ 1 + \left(n - \frac{1+n}{H} \right) \right\}^2 \left\{ \ln \left(\frac{H}{\cos \theta} \right) \right\} \cdot \frac{\cos^3 \theta}{\sin \theta} \right] \quad (C-6) \end{aligned}$$

$$- h^2 \int_{h_{-\infty}}^{h/\cos \theta} \frac{d\ell}{\ell^3} + \frac{1}{2} \int_0^\theta \sin 2\theta \, d\theta = \frac{1 - H^2}{2} \quad (C-7)$$

From Equation (294), it becomes

$$\Gamma \frac{d\theta}{ds} + (p - p_a) = - 2\mu \frac{dq}{ds} \quad (C-8)$$

Introduce the dimensionless quantities

$$N = \frac{\Gamma}{\mu U} \quad (C-9)$$

$$R_s = \frac{U h_{-\infty}}{\nu} = 4 \frac{U(h_{-\infty}/2)}{2\nu} = 4\bar{R} \quad (C-10)$$

$$P - P_a = (p - p_a)/\rho U^2 \quad (C-11)$$

$$\bar{q} = \frac{q}{-U} \quad (C-12)$$

$$S = \frac{s}{h_{-\infty}} \quad (C-13)$$

Using Equations (C-9) to (C-13), Equation (C-8) reduces to the dimensionless form as

$$N \frac{d\theta}{dS} + R_s (P - P_a) = 2 \frac{d\bar{q}}{dS} \quad (C-14)$$

Similarly,

$$\frac{dH}{dS} = \sin \theta \quad (C-15)$$

$$\frac{d\xi}{dS} = \cos \theta \quad (C-16)$$

where

$$\xi = x'/h_{-\infty} \quad (C-17)$$