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**TRANSIENT THERMAL STRESSES IN
HEAT-GENERATING PLATES AND
HOLLOW CYLINDERS CAUSED BY SUDDEN
CHANGES IN ENVIRONMENTAL TEMPERATURE**

by

**R. A. Valentin, G. S. Rosenberg,
and D. F. Schoeberle**

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NOMENCLATURE

Symbol	Definition
t	Time
κ	Thermal diffusivity $\left(= \frac{k}{\rho c} \right)$
k	Thermal conductivity
ρ	Density
c	Specific heat
h	Boundary conductance
E	Young's modulus
ν	Poisson's ratio
α	Coefficient of thermal expansion

Section II

x	Plate thickness measured from midplane
L	Half plate thickness
$T(x, t)$	Temperature
$\tilde{T}$	Constant coolant temperature
T_0	Initial plate temperature
ΔT	$T - \tilde{T}$
ΔT_0	$T_0 - \tilde{T}$
σ	In-plane stress ($\sigma \equiv \sigma_{yy} = \sigma_{zz}$)
$\bar{\sigma}$	$\sigma(1 - \nu)/\alpha E(\Delta T_0)$
$\bar{\sigma}_1, \bar{\sigma}_2$	Dimensionless stresses of Eq. (2.20)
p	Laplace transform variable
b_n	Roots of (2.14)
m	hL/k , Biot number
t'	$\kappa t/L^2$, Fourier number
C	$QL^2/\kappa(\Delta T_0)$
Q	Constant heat-generation rate

Symbol	Definition
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Section III

$T(r, t)$	Temperature
T_1	Uniform coolant temperature
T_a	Initial uniform cylinder temperature
θ	$(T - T_1)/(T_a - T_1)$ = dimensionless temperature difference
τ	$\kappa t/a^2$ = dimensionless time (Fourier number)
Q_1	Internal heat-generation rate function
Q_a	Constant internal heat-generation rate
Q	$a^2 Q_a / \kappa (T_a - T_1)$ = dimensionless generation rate
r	Radial coordinate
a	Inner radius
b	Outer radius
y	r/a = dimensionless radial coordinate
y_0	b/a = outer dimensionless radius
h	Boundary conductance at inner cylinder surface
β	ha/k = dimensionless boundary conductance (Biot number)
$\hat{\sigma}_r, \hat{\sigma}_\theta, \hat{\sigma}_z$	Radial, tangential, and axial stress (dimensionless)
$\theta^*(y)$	Dimensionless temperature for $\tau \rightarrow \infty$, Eq. (3.24)
$\hat{\sigma}_r^*, \hat{\sigma}_\theta^*, \hat{\sigma}_z^*$	Stresses corresponding to temperature field θ^*
$H[f] \equiv \bar{f}$	Finite Hankel transform of f ; see (3.6)
ξ_i	Roots of (3.7)
J_0, J_1	Bessel functions of first kind; orders 0 and 1
Y_0, Y_1	Bessel functions of second kind; orders 0 and 1

In addition to the above, various incidental notation is introduced throughout. Definitions should be clear from the context.

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ABSTRACT

The transient thermal stresses in a free, heat-generating flat plate and a free, hollow, heat-generating cylinder, insulated at its outer surface and cooled on its inner surface through a temperature-independent boundary conductance, are analyzed within the framework of a quasi-static treatment to determine the effect of a sudden change in environmental temperature when combined with internal heat generation. The flat-plate problem is treated by Laplace transform methods, and the cylinder problem is analyzed by using a recently developed extension of the classical Hankel transform. Graphical temperature and stress results are presented for a wide range of Biot numbers and, in the case of the hollow cylinder, for a wide range of the thickness parameter. In addition, an extensive table of the characteristic roots occurring in the definition of the Hankel transform is included since these roots should find many applications beyond the particular cylinder problem considered.

I. INTRODUCTION

The problems to be treated concern the transient thermal stresses generated in various reactor-core components by the sudden introduction of coolant having an appreciably different temperature from that existing during steady-state operation. Specifically, we shall be concerned with two problems in which components of a given geometry are suddenly exposed to a change in environmental temperature through a temperature-independent boundary conductance. These problems are similar to those encountered in various common quenching operations; however, they differ significantly in that we assume the components are generating heat internally at some constant rate. The specific purpose of this study is to determine the effect of this constant internal heat-generation rate on the peak stress magnitudes

and on the time decay of the stress field as compared with what one would compute in the nongenerating case. In addition, the numerical results presented are sufficient to provide design data for a large range of practical problems.

Within the limitations of a quasi-static treatment, the transient thermal-stress field will be determined in the following two special cases:

Problem A: A thin, unconstrained, infinite, flat plate possessing a uniform internal heat-generation rate, having attained some uniform initial temperature, is suddenly exposed to a change in ambient temperature through a temperature-independent boundary conductance.

Problem B: A thick, unconstrained, infinite, hollow circular cylinder, initially at uniform temperature, is suddenly exposed to a differing, constant temperature through a temperature-independent boundary conductance at its inner surface. The cylinder has an insulated outer surface and generates heat internally at a known, uniform, temperature-independent rate.

Problem A is treated in Section II, which contains the explicit problem statement, its analytical solution by means of Laplace transforms, and the numerical evaluation of results and their discussion. Problem B is analyzed in Section III dealing with the problem statement, its finite Hankel transform solution, and the numerical analysis related to this solution.

II. THE THIN FLAT PLATE

A. Problem Statement

A flat plate, containing a time-dependent heat source, $Q(t)$, having attained a uniform temperature, T_0 , is suddenly exposed to a changed ambient temperature. The plate is thin, unconstrained, and infinite, and heat is transmitted through a uniform boundary conductance, h . In the determination of the thermal stresses as a function of time, the limiting case of an instantaneous change in plate surface temperature ($h = \infty$) provides an upper bound on the attainable stress, but this bound may be unnecessarily conservative--particularly if the plate is in contact with a coolant having a very small value of h .

The transient state to be treated is quasi-static in that the rise times of the temperatures are assumed greater than any periods associated with either plate-vibration modes or stress-wave propagation. The resulting solution will thus reduce, for $Q \equiv 0$, to that treated in Ref. 1, p. 283, for the quenching of thin flat plates.

In most physically realizable situations of the type being considered, we can assume that temperature changes in the direction of coolant flow have negligible effect on thermal stresses.^{2,3} This assumption will be made in the present treatment even though its applicability to cases involving sudden temperature changes in the coolant is not immediately obvious. We shall further assume that the plate width is small relative to the core dimensions and, hence, that temperature variations in this direction are of minor importance. The resulting problem thus contains only one space dimension and may be formulated as follows:

$$\kappa \frac{\partial^2 T}{\partial x^2} + \frac{Q}{\rho c} = \frac{\partial T}{\partial t}, \quad t > 0; \quad (2.1)$$

and

$$\left. \begin{aligned} \frac{\partial T}{\partial x} \Big|_{x=0} &= 0, \quad t > 0, \\ \left[k \frac{\partial T}{\partial x} + hT \right]_{x=L} &= h\tilde{T}, \quad t > 0, \\ T &= F(x), \quad t = 0. \end{aligned} \right\} \quad (2.2)$$

With some computational difficulty the various physical properties may be considered as functions of x ; however, only constant properties will be treated at present. Furthermore, we will assume that there is no initial forced coolant flow and that the plate conductivity is sufficiently high to justify the assumption that the temperature field for $t < 0$ is essentially constant (i.e., $F(x) \equiv T_0 = \text{constant}$). In terms of the temperature difference $\Delta T = T - \tilde{T}$, Eqs. (2.1) and (2.2) become

$$\kappa \frac{\partial^2 \Delta T}{\partial x^2} + \frac{Q}{\rho c} = \frac{\partial \Delta T}{\partial t}, \quad t > 0; \quad (2.3)$$

and

$$\left. \begin{aligned} \frac{\partial \Delta T}{\partial x} \Big|_{x=0} &= 0, \quad t > 0, \\ \left[k \frac{\partial \Delta T}{\partial x} + h\Delta T \right]_{x=L} &= 0, \quad t > 0, \\ \Delta T(x, 0) &= T_0 - \tilde{T} \equiv \Delta T_0, \quad t = 0. \end{aligned} \right\} \quad (2.4)$$

ΔT having been determined from Eqs. (2.3) and (2.4), the only non-zero stress component $\sigma \equiv \sigma_{yy} = \sigma_{zz}$ is given by Ref. 1, p. 401, as

$$\sigma = \frac{\alpha E}{1-\nu} \left\{ -T + \frac{1}{2L} \int_{-L}^L T \, dx + \frac{3x}{2L^3} \int_{-L}^L Tx \, dx \right\}. \quad (2.5)$$

B. Analytical Solution

The solution of Eqs. (2.3) and (2.4) is most conveniently obtained by application of Laplace transforms in the time variable. Thus, denoting transformed quantities by a bar, we obtain from (2.3)

$$\kappa \frac{d^2 \bar{\Delta T}}{dx^2} - p \bar{\Delta T} = -\frac{\bar{Q}(p)}{\rho c} - \Delta T_0, \quad (2.6)$$

while (2.4) gives

$$\left. \begin{aligned} \frac{d\bar{\Delta T}}{dx} \Big|_{x=0} &= 0, \\ \left[k \frac{d\bar{\Delta T}}{dx} + h \bar{\Delta T} \right]_{x=L} &= 0. \end{aligned} \right\} \quad (2.7)$$

From (2.6) we have

$$\bar{\Delta T} = A \exp \left[\left(\frac{p}{\kappa} \right)^{1/2} x \right] + B \exp \left[- \left(\frac{p}{\kappa} \right)^{1/2} x \right] + p^{-1} \left(\frac{\bar{Q}(p)}{\rho c} + \Delta T_0 \right),$$

which, upon using (2.7) to evaluate the constants yields

$$\bar{\Delta T} = p^{-1} \left(\frac{\bar{Q}(p)}{\rho c} + \Delta T_0 \right) (1 - p \bar{V}), \quad (2.8)$$

where

$$\bar{V} = \frac{h \cosh \left(\frac{p}{\kappa} \right)^{1/2} x}{p \left[k \left(\frac{p}{\kappa} \right)^{1/2} \sinh \left(\frac{p}{\kappa} \right)^{1/2} L + h \cosh \left(\frac{p}{\kappa} \right)^{1/2} L \right]}. \quad (2.9)$$

Thus, upon inversion and application of the convolution theorem, one obtains

$$\Delta T(x, t) = \frac{1}{\rho c} \left[\int_0^t Q(\tau) d\tau - \int_0^t Q(\tau) V(x, t - \tau) d\tau \right] + \Delta T_0 [1 - V(x, t)]. \quad (2.10)$$

Standard methods applied to $\bar{V}$ yield

$$V(x, t) = \sum_{j=1}^{\infty} \frac{h \cosh \left[\left(\frac{p_j}{\kappa} \right)^{1/2} x \right] \exp(p_j t)}{\left(\frac{kLp_j}{2\kappa} + h \right) \cosh \left(\frac{p_j}{\kappa} \right)^{1/2} L + \frac{1}{2} \left(\frac{p_j}{\kappa} \right)^{1/2} (3k + hL) \sinh \left(\frac{p_j}{\kappa} \right)^{1/2} L}, \quad (2.11)$$

where p_j are the roots of

$$p \left[k \left(\frac{p}{\kappa} \right)^{1/2} \sinh \left(\frac{p}{\kappa} \right)^{1/2} L + h \cosh \left(\frac{p}{\kappa} \right)^{1/2} L \right] = 0. \quad (2.12)$$

By redefining various terms and simplifying, we can show that (2.11) and (2.12) are equivalent to

$$V(x, t) = 1 - 2 \sum_{n=1}^{\infty} \frac{\cos \left(\frac{b_n x}{L} \right) \exp \left(- \frac{\kappa b_n^2 t}{L^2} \right)}{\left(1 + m + \frac{b_n^2}{m} \right) \cos b_n}, \quad (2.13)$$

where b_n are the roots of

$$b \tan b = m, \quad (2.14)$$

with the Biot number, m , defined as $m = hL/k$. Thus, from (2.10) and (2.13), we obtain

$$\begin{aligned} \Delta T(x, t) = & 2m T_0 \sum_{n=1}^{\infty} \frac{\sin b_n \cos \left(\frac{b_n x}{L} \right) \exp \left(- \frac{\kappa b_n^2 t}{L^2} \right)}{b_n (\sin^2 b_n + m)} \\ & + 2m \int_0^t \frac{Q(\tau)}{\rho c} \sum_{n=1}^{\infty} \frac{\sin b_n \cos \frac{b_n x}{L} \exp \left[- \frac{\kappa b_n^2 (t - \tau)}{L^2} \right]}{b_n (\sin^2 b_n + m)} d\tau. \end{aligned} \quad (2.15)$$

For $Q \equiv 0$, (2.15) reduces to the solution of transient conduction as given in Ref. 1, p. 205.

Under the conditions of symmetry imposed by the problem formulation, Eq. (2.5) reduces to

$$\sigma = \frac{\alpha E}{1 - \nu} \left\{ -\Delta T(x, t) + \frac{1}{L} \int_0^L \Delta T(x, t) dx \right\}, \quad (2.16)$$

and, upon substitution of (2.15), we have

$$\begin{aligned} \frac{\sigma(1 - \nu)}{\alpha E(\Delta T_0)} &= \frac{2m}{L} \int_0^L \sum_{n=1}^{\infty} F_n(x, t) dx - 2m \sum_{n=1}^{\infty} F_n(x, t) \\ &+ \frac{2m}{L} \int_0^L \left[\sum_{n=1}^{\infty} F_n(x, t) \int_0^t \frac{Q(\tau)}{\Delta T_0 \rho c} \exp(-\kappa b_n^2 \tau / L^2) d\tau \right] dx \quad (2.17) \\ &- 2m \sum_{n=1}^{\infty} F_n(x, t) \int_0^t \frac{Q(\tau)}{\Delta T_0 \rho c} \exp(-\kappa b_n^2 \tau / L^2) d\tau, \end{aligned}$$

where

$$F_n(x, t) = \frac{\sin b_n \cos b_n \frac{x}{L} \exp(-\kappa b_n^2 t / L^2)}{b_n(\sin^2 b_n + m)}. \quad (2.18)$$

C. Numerical Evaluation of Results and Discussion

We shall restrict the present discussion to the case $Q(\tau) = Q = \text{constant}$ for which (2.17) and (2.18) evaluated at the plate surface reduce to

$$\bar{\sigma} = 2m \sum_{n=1}^{\infty} \frac{\sin b_n \exp(-t' b_n^2)}{b_n(\sin^2 b_n + m)} \left(\frac{\sin b_n}{b_n} - \cos b_n \right) \left[1 + C \left(\frac{1 - \exp(-t' b_n^2)}{b_n^2} \right) \right], \quad (2.19)$$

where we have introduced the notation

$$t' = \frac{\kappa t}{L^2}, \quad \bar{\sigma} = \frac{\sigma(1 - \nu)}{\alpha E(\Delta T_0)}, \quad C = \frac{QL^2}{\kappa(\Delta T_0)}.$$

Since the solution involves three independent parameters (m , t' , and C), it is inconvenient to plot the results for a complete range of parameter

combinations. We have chosen to plot the terms $\bar{\sigma}_1$ and $\bar{\sigma}_2$ as functions of t' for various values of m , where

$$\bar{\sigma} = \bar{\sigma}_1 + C\bar{\sigma}_2$$

and

$$\left. \begin{aligned} \bar{\sigma}_1 &= 2m \sum_{n=1}^{\infty} G_n(t'; m), \\ \bar{\sigma}_2 &= 2m \sum_{n=1}^{\infty} G_n(t'; m) \frac{1 - \exp(-t' b_n^2)}{b_n^2}, \\ G_n(t'; m) &= \frac{\sin b_n \exp(-t' b_n^2)}{b_n (\sin^2 b_n + m)}. \end{aligned} \right\} \quad (2.20)$$

Thus the term $\bar{\sigma}_2$, when weighted by the constant C , represents the contribution of the internal heat-generation rate to the final stress, $\bar{\sigma}$.

The evaluation of (2.20) for various values of the parameters m and t' is a relatively elementary computation. The roots b_n of (2.14) are tabulated in many texts (e.g., Ref. 4); however, since their numerical determination is not a time-consuming process, the program evaluating the stresses incorporated a simple interval-halving root-finding subprogram as an integral part of the computation routine.

In Fig. 1 we have plotted $\bar{\sigma}_1$ as a function of t' for various values of the Biot number, m . This is essentially the same graph as that found in Ref. 1, p. 284. Figure 2 is a plot of $\bar{\sigma}_2$, the term due to internal heat generation, for the same values of m found in Fig. 1. Note that Fig. 2 is plotted on an expanded vertical scale; hence, unless the constant C is of the order of unity or larger, the total stress, $\bar{\sigma}_1 + C\bar{\sigma}_2$, will not differ by a very significant amount from the case of zero heat generation as given by Fig. 1. As an illustration of this, Fig. 3 shows $\bar{\sigma}_1$, $\bar{\sigma}_2$, and $\bar{\sigma}_1 + \bar{\sigma}_2$ for $m = 5.0$. Examination of the dimensionless number C reveals that, even in a very high power-density core, it would be unlikely the C would have a value as great as unity. Thus even the relatively minor effect of the term $\bar{\sigma}_2$ illustrated in Fig. 3 may be considered as a somewhat extreme case. In general, the effect of the heat-generation term is to provide a slight increase in the peak stress magnitude together with a slight broadening of the stress plot in the direction of larger times. That is, the length of time during which the stress is above some given level is increased, and the increase is due mostly to the peak value of $\bar{\sigma}_2$ occurring at a time greater than that at which the peak occurs in $\bar{\sigma}_1$.

Fig. 1. Thermal Stress at Surface of Free Plate, $\bar{\sigma}_1$, as a Function of Fourier Number, t' , for Various Biot Numbers, m

Fig. 2. Increment of Thermal Stress Due to Uniform Heat Generation, $\bar{\sigma}_2$, Evaluated at Plate Surface

Fig. 3. Superposition of $\bar{\sigma}_1$ and $\bar{\sigma}_2$ for $m = 5.0$, $c = 1.0$

III. THE INFINITE HOLLOW CYLINDER

A. Problem Statement

Transient thermal stresses in internally generating hollow cylinders have been considered for the case of known internal and external surface temperatures,⁵ and the results applied in the study of time-dependent radiation heating. The problem treated in this section is somewhat similar; however, the transients are due to a sudden change in environment temperature at the inner surface of the cylinder. Thus, other than the cylindrical geometry, the problem is analogous to that of Section II.

The problem to be stated represents an elementary model for the stress state within a particular type of fuel-element coolant channel caused by a change in coolant temperature.

We consider an infinitely long, unconstrained, hollow cylinder with uniform initial temperature. At some initial time, coolant of a temperature differing from that of the cylinder begins transfer of heat from the inner surface of the cylinder through a temperature-independent boundary conductance while the cylinder begins generating heat at a uniform rate. The external surface of the cylinder is perfectly insulated, and all occurring material properties are temperature-independent.

The Hankel-transform method to be applied in the solution of the temperature problem could be replaced by the Laplace-transform solution given by Carslaw and Jaeger.⁴ However, the former was included rather than a specialization of the already existing analysis since it may be more readily modified to include time- and space-varying heat-generation rates if a more exact analysis becomes desirable.

The treatment of a perfectly free cylinder rather than more general boundary conditions was motivated by a desire for analytical simplicity. A more realistic model of the type of coolant channels one would find in various core structures would perhaps dictate that the external surface be considered in contact with an elastic body of differing material properties, having a different internal generation rate, etc. However, the present simplifications were felt to be justified in that they provide a qualitative picture of the relative effect of the various occurring parameters for the geometry in question.

For radial symmetry in the presence of internal heat generation, the heat-conduction equation is

$$\kappa \left(\frac{\partial^2 T}{\partial r^2} + \frac{1}{r} \frac{\partial T}{\partial r} \right) + \frac{Q_1}{\rho c} = \frac{\partial T}{\partial t}, \quad (3.1)$$

while the boundary and initial conditions specifying the problem described above are

$$\left. \begin{aligned} \frac{\partial T}{\partial r} &= 0, & r &= b, & t &> 0, \\ k \frac{\partial T}{\partial r} - hT &= -hT_1, & r &= a, & t &> 0, \\ T(r, t) &= T_a, & t &= 0, & a \leq r \leq b, \\ Q_1 &= \begin{cases} 0, & t = 0, & a \leq r \leq b, \\ Q_a, & t > 0, & a \leq r \leq b. \end{cases} \end{aligned} \right\} \quad (3.2)$$

In terms of dimensionless quantities, (3.1) may be restated as

$$\frac{\partial^2 \theta}{\partial y^2} + \frac{1}{y} \frac{\partial \theta}{\partial y} + Q = \frac{\partial \theta}{\partial \tau}, \quad \tau > 0, \quad 1 \leq y \leq y_0, \quad (3.3)$$

with the boundary and initial conditions

$$\left. \begin{aligned} \frac{\partial \theta}{\partial y} &= 0, & y &= y_0, & \tau &> 0, \\ \frac{\partial \theta}{\partial y} - \beta \theta &= 0, & y &= 1, & \tau &> 0, \\ \theta &= 1, & 1 &\leq y \leq y_0, & \tau &= 0. \end{aligned} \right\} \quad (3.4)$$

Upon solution of (3.3) and (3.4), the thermal stresses in a long, hollow cylinder subjected to a radial variation in temperature are given by Ref. 6, p. 408, as

$$\left. \begin{aligned} \hat{\sigma}_r &= \frac{1}{y^2} \left[\frac{y^2 - 1}{y_0^2 - 1} \int_1^{y_0} y \theta(y) dy - \int_1^y y \theta(y) dy \right], \\ \hat{\sigma}_\theta &= \frac{1}{y^2} \left[\frac{y^2 + 1}{y_0^2 - 1} \int_1^{y_0} y \theta(y) dy + \int_1^y y \theta(y) dy - y^2 \theta(y) \right], \\ \hat{\sigma}_z &= \frac{2}{y_0^2 - 1} \int_1^{y_0} y \theta(y) dy - \theta(y). \end{aligned} \right\} \quad (3.5)$$

B. Analytical Solution

The solution of the above problem is obtained through application of the finite Hankel transform defined by[†]

$$H[f(y)] \equiv \bar{f}(\xi_i) \equiv \int_1^{y_0} y [Y_1(\xi_i y_0) J_0(\xi_i y) - J_1(\xi_i y_0) Y_0(\xi_i y)] f(y) dy, \quad (3.6)$$

where ξ_i ($i = 1, 2, \dots$) are the positive real roots of

$$Y_1(\xi y_0) [\xi J_1(\xi) + \beta J_0(\xi)] = J_1(\xi y_0) [\xi Y_1(\xi) + \beta Y_0(\xi)], \quad y_0 > 1, \quad \beta > 0. \quad (3.7)$$

The corresponding inversion theorem is

$$f(y) = \frac{\pi^2}{2} \sum_{i=1}^{\infty} \frac{\xi_i^2 [\xi_i J_1(\xi_i) + \beta J_0(\xi_i)]^2 [Y_1(\xi_i y_0) J_0(\xi_i y) - J_1(\xi_i y_0) Y_0(\xi_i y)] \bar{f}(\xi_i)}{[\xi_i J_1(\xi_i) + \beta J_0(\xi_i)]^2 - (\xi_i^2 + \beta^2) [J_1(\xi_i y_0)]^2}. \quad (3.8)$$

[†]See Appendix A for the derivation of the various properties of this transform.

The pertinent transformation property is

$$H \left[f'' + \frac{1}{y} f' \right] = - \frac{2}{\pi \xi_i} [f'(y_0)] - [Y_1(\xi_i y_0) J_0(\xi_i) - J_1(\xi_i y_0) Y_0(\xi_i)] [f'(1) - \beta f(1)] - \xi_i^2 \bar{f}(\xi_i). \quad (3.9)$$

Applying the above transform to the y variable in (3.3) and to the boundary conditions (3.4), we have the elementary ordinary differential equation,

$$\frac{d\bar{\theta}}{d\tau} + \xi_i^2 \bar{\theta} - QF(\xi_i) = 0, \quad (3.10)$$

with the boundary condition

$$\bar{\theta}(0) = F(\xi_i), \quad (3.11)$$

where the $F(\xi_i)$, the transform of unity, is given by

$$H[1] = F(\xi_i) = \frac{1}{\xi_i} [J_1(\xi_i y_0) Y_1(\xi_i) - Y_1(\xi_i y_0) J_1(\xi_i)]. \quad (3.12)$$

Equations (3.11) and (3.12) yield

$$\bar{\theta}(\xi_i) = F(\xi_i) \left\{ e^{-\xi_i^2 \tau} + Q \left[\frac{1}{\xi_i^2} \left(1 - e^{-\xi_i^2 \tau} \right) \right] \right\}; \quad (3.13)$$

hence, upon application of the inversion formula (3.8) and use of the relation[†]

$$F(\xi_i) [\xi_i J_1(\xi_i) + \beta J_0(\xi_i)] = \frac{-2\beta J_1(\xi_i y_0)}{\pi \xi_i^2}, \quad (3.14)$$

one obtains as the temperature field

$$\theta(y, \tau) = \sum_{i=1}^{\infty} B_i C_i(y) D_i(\tau), \quad (3.15)$$

[†]See Appendix A for the derivation of (3.14).

where

$$B_i = - \frac{\pi \beta J_1(\xi_i y_0) [\xi_i J_1(\xi_i) + \beta J_0(\xi_i)]}{[\xi_i J_1(\xi_i) + \beta J_0(\xi_i)]^2 - (\xi_i^2 + \beta^2) [J_1(\xi_i y_0)]^2}, \quad (3.16)$$

$$C_i(y) = Y_1(\xi_i y_0) J_0(\xi_i y) - J_1(\xi_i y_0) Y_0(\xi_i y), \quad (3.17)$$

$$D_i(\tau) = e^{-\xi_i^2 \tau} + Q \left[\frac{1}{\xi_i^2} \left(1 - e^{-\xi_i^2 \tau} \right) \right]. \quad (3.18)$$

Upon substitution of (3.15) into (3.5), we have

$$\hat{\sigma}_r = \sum_{i=1}^{\infty} B_i D_i(\tau) \left[\frac{1}{\xi_i y^2} \left\{ \left(\frac{y^2 - y_0^2}{y_0^2 - 1} \right) [J_1(\xi_i y_0) Y_1(\xi_i) - Y_1(\xi_i y_0) J_1(\xi_i)] \right. \right. \\ \left. \left. - y [J_1(\xi_i y) Y_1(\xi_i y_0) - Y_1(\xi_i y) J_1(\xi_i y_0)] \right\} \right] \quad (3.19)$$

$$\equiv \sum_{i=1}^{\infty} B_i D_i(\tau) P_i(y),$$

$$\hat{\sigma}_\theta = \sum_{i=1}^{\infty} B_i D_i(\tau) \left[\frac{1}{\xi_i y^2} \left\{ \left(\frac{y^2 + y_0^2}{y_0^2 - 1} \right) [J_1(\xi_i y_0) Y_1(\xi_i) - Y_1(\xi_i y_0) J_1(\xi_i)] \right. \right. \\ \left. \left. + y [J_1(\xi_i y) Y_1(\xi_i y_0) - Y_1(\xi_i y) J_1(\xi_i y_0)] \right. \right. \\ \left. \left. - y^2 \xi_i [Y_1(\xi_i y_0) J_0(\xi_i y) - J_1(\xi_i y_0) Y_0(\xi_i y)] \right\} \right] \quad (3.20)$$

$$\equiv \sum_{i=1}^n B_i D_i(\tau) Q_i(y),$$

$$\hat{\sigma}_z = \sum_{i=1}^{\infty} B_i D_i(\tau) \left\{ \frac{2}{\xi_i (y_0^2 - 1)} [J_1(\xi_i y_0) Y_1(\xi_i) - Y_1(\xi_i y_0) J_1(\xi_i)] \right. \\ \left. - [Y_1(\xi_i y_0) J_0(\xi_i y) - J_1(\xi_i y_0) Y_0(\xi_i y)] \right\} \quad (3.21)$$

$$\equiv \sum_{i=1}^{\infty} B_i D_i(\tau) R_i(y).$$

The problem outlined in Section I is solved by the temperature field (3.15) and the resultant stresses as given in (3.19), (3.20), and (3.21). The numerical evaluation of these results in each case of interest centers primarily on the problem of determining the roots of Eq. (3.7) for given values of the parameters y_0 and β and performing the required summations to the accuracy desired.

As a check on the numerical results, it is useful to obtain the limiting values of the stress field for large τ . For $Q \equiv 0$, the temperature and stresses all decay to zero as $\tau \rightarrow \infty$; for $Q \neq 0$, we would expect the limiting values of (3.15), (3.19), (3.20), and (3.21) to approach the solution obtained from

$$\frac{d^2\theta^*}{dy^2} + \frac{1}{y} \frac{d\theta^*}{dy} + Q = 0 \quad (3.22)$$

with the boundary conditions

$$\left. \begin{aligned} \frac{d\theta^*}{dy} &= 0, & y &= y_0; \\ \frac{d\theta^*}{dy} - \beta\theta^* &= 0, & y &= 1. \end{aligned} \right\} \quad (3.23)$$

Equations (3.22), and (3.23) yield

$$\theta^*(y) = \frac{Q}{2} \left[\frac{1}{\beta}(y_0^2 - 1) - \frac{1}{2}(y^2 - 1) + y_0^2 \ln y \right], \quad (3.24)$$

which, when substituted into Eqs. (3.5), gives

$$\hat{\sigma}_r^*(y) = \frac{Q}{4} \frac{1}{y^2} \left\{ \frac{1}{4}(y^2 - 1)(y^2 - y_0^2) + \frac{y_0^2}{y_0^2 - 1} [y_0^2(y^2 - 1) \ln y_0 - y^2(y_0^2 - 1) \ln y] \right\}, \quad (3.25)$$

$$\begin{aligned} \hat{\sigma}_\theta^*(y) &= \frac{Q}{4} \frac{1}{y^2} \left\{ \frac{y^2}{2} \left(\frac{3}{2} y^2 - \frac{5}{2} y_0^2 - 1 \right) - \frac{1}{8}(y_0^2 - y^2) \right. \\ &\quad \left. + \frac{y_0^2}{y_0^2 - 1} \left[(y^2 + 1) y_0^2 \ln y_0 - (y_0^2 - 1) y^2 \ln y \right] \right\}, \end{aligned} \quad (3.26)$$

$$\hat{\sigma}_z^*(y) = \frac{Q}{2} \left\{ \frac{1}{2} \left(y^2 - \frac{1}{2} - \frac{3}{2} y_0^2 \right) + \frac{y_0^2}{y_0^2 - 1} \left[y_0^2 \ln y_0 - (y_0^2 - 1) \ln y \right] \right\}. \quad (3.27)$$

Note that the temperature field (3.24) is a function of the Biot number, β , while the stresses (3.25), (3.26), and (3.27) are independent of β . Thus, for fixed $Q \neq 0$, and given values of y_0 and y , as $\tau \rightarrow \infty$ one would expect each time-dependent stress component to approach the corresponding stress value given by the above equations independent of the value of β .

C. Numerical Evaluation of Results and Discussion

The temperature and stress expressions, (3.15) through (3.21), were evaluated for five values of the dimensionless thickness parameter y_0 ranging from $y_0 = 1.20$ to $y_0 = 5.0$, and for each value of y_0 , results were computed for 11 values of the Biot number, β , ranging from $\beta = 0.01$ to $\beta = 100.0$. Before these evaluations, a separate computation for the required roots of (3.7) was performed by applying Newton's method in conjunction with Bessel-function subroutines using the polynomial approximations given in (Ref. 10, Section 9). The initial root values used in the iterations were based on qualitative properties of (3.7) derived from selected hand computations and information contained in Ref. 11, where the roots of (3.7) are tabulated for certain cases having $\beta < 0$ and $y_0 < 1$. The resulting table of roots is given in Appendix B to six decimal places. A complete error analysis of these roots was not performed; however, representative hand checks were made and in all cases resulted in at least five-decimal-place accuracy. This table is, to the author's knowledge at the time of writing, the only extensive tabulation of the roots of (3.7) for $\beta > 0$ and $y_0 > 1$ and, hence, has intrinsic interest in that it is applicable to any problem that may be treated in terms of the Hankel transform of Appendix A.

Graphical results are presented for the above-mentioned values of y_0 and β only for the case $y = 1.0$ (the inner surface of the cylinder) and only the stress component $\hat{\sigma}_\theta$ is given.[†] We have factored the expression for D_1 in (3.18) to give

$$\hat{\theta} = \hat{\theta}_1 + Q \hat{\theta}_2,$$

$$\hat{\sigma}_\theta = (\hat{\sigma}_\theta)_1 + Q(\hat{\sigma}_\theta)_2,$$

and have plotted the terms $\hat{\theta}_1$, $\hat{\theta}_2$, $(\hat{\sigma}_\theta)_1$, and $(\hat{\sigma}_\theta)_2$ as functions of the dimensionless time, τ . Figures 4-23 consist of five sets of four graphs each, one set for each of the five values of y_0 . The terms $\hat{\theta}_1$ and $(\hat{\sigma}_\theta)_1$ refer to the case of no internal heat generation, while $\hat{\theta}_2$ and $(\hat{\sigma}_\theta)_2$, when weighted by the appropriate value of Q , give the increment of temperature and stress due to constant internal heat generation.

[†]Note that $\hat{\sigma}_\theta(y = 1.0) = \hat{\sigma}_z(y = 1.0)$ and $\hat{\sigma}_r(y = 1.0) \equiv 0$. Also, $\hat{\sigma}_\theta(y = 1.0)$ has the largest magnitude of all stress components over the range of parameters for which computations were performed.

In general, the graphical results are self-explanatory, and qualitative trends due to changes in various parameter combinations can be easily deduced; however, two points deserve special mention. First we note that peak values in the terms due to internal heat generation occur at larger times than the peak values in the nongenerating terms, and also that the increment due to internal generation is of relatively minor importance unless Q is quite large[†] or unless the cylinder is quite thick (say, $y_0 \geq 2$).

Fig. 4. Temperature Function, $\hat{\theta}_1$ at Inner Surface of Hollow Cylinder as a Function of Fourier Number, τ , for Various Biot Numbers, β ; Thickness Parameter $y_0 = 1.20$

Fig. 5

Increment of Temperature Caused by Uniform Heat Generation, $\hat{\theta}_2$, $y = 1.0$, as a Function of τ and β for $y_0 = 1.20$

[†]A value of $Q \geq 1$ may be seen to be "large" by evaluating this constant in some typical high power-density case.

Fig. 6. Stress $(\hat{\sigma}_\theta)_1$ at Inner Surface of Hollow Cylinder as a Function of τ and β for $y_0 = 1.20$

Fig. 7. Increment of Stress Caused by Uniform Heat Generation, $(\hat{\sigma}_\theta)_2$, $y = 1.0$, as a Function of τ and β for $y_0 = 1.20$

Fig. 8. Temperature Function $\hat{\theta}_1$ at Inner Surface of Hollow Cylinder as a Function of Fourier Number τ for Various Biot Numbers β ; Thickness Parameter $y_0 = 1.50$

Fig. 9
Increment of Temperature Caused by Uniform Heat Generation, $\hat{\theta}_2(y = 1.0)$, as a Function of τ and β for $y_0 = 1.50$

Fig. 10. Stress $(\hat{\sigma}_{\theta})_1$ at Inner Surface of Hollow Cylinder as a Function of τ and β for $y_0 = 1.50$

Fig. 11. Increment of Stress Caused by Uniform Heat Generation, $(\hat{\sigma}_{\theta})_2$, $y = 1.0$, as a Function of τ and β for $y_0 = 1.50$

Fig. 12. Temperature Function $\hat{\theta}_1$ at Inner Surface of Hollow Cylinder as a Function of Fourier Number τ for Various Biot Numbers β ; Thickness Parameter $y_0 = 2.0$

Fig. 13
Increment of Temperature Caused by Uniform Heat Generation, $\hat{\theta}_2(y = 1.0)$, as a Function of τ and β for $y_0 = 2.0$

Fig. 14. Stress $(\hat{\sigma}_{\theta})_1$ at Inner Surface of Hollow Cylinder as a Function of τ and β for $y_0 = 2.0$

Fig. 15. Increment of Stress Caused by Uniform Heat Generation, $(\hat{\sigma}_{\theta})_2$, $y = 1.0$, as a Function of τ and β for $y_0 = 2.0$

Fig. 16. Temperature Function $\hat{\theta}_1$ at Inner Surface of Hollow Cylinder as a Function of Fourier Number τ for Various Biot Numbers β ; Thickness Parameter $y_0 = 3.0$

Fig. 17

Increment of Temperature Caused by Uniform Heat Generation, $\hat{\theta}_2(y = 1.0)$, as a Function of τ and β for $y_0 = 3.0$

Fig. 18. Stress $(\hat{\sigma}_\theta)_1$ at Inner Surface of Hollow Cylinder as a Function of τ and β for $y_0 = 3.0$

Fig. 19. Increment of Stress Caused by Uniform Heat Generation, $(\hat{\sigma}_\theta)_2$, $y = 1.0$, as a Function of τ and β for $y_0 = 3.0$

Fig. 20. Temperature Function $\hat{\theta}_1$ at Inner Surface of Hollow Cylinder as a Function of Fourier Number τ for Various Biot Numbers β ; Thickness Parameter $y_0 = 5.0$

Fig. 21

Increment of Temperature Caused by Uniform Heat Generation, $\hat{\theta}_2(y = 1.0)$, as a Function of τ and β for $y_0 = 5.0$

Fig. 22. Stress $(\hat{\sigma}_{\theta})_1$ at Inner Surface of Hollow Cylinder as a Function of τ and β for $y_0 = 5.0$

Fig. 23. Increment of Stress Caused by Uniform Heat Generation, $(\hat{\sigma}_{\theta})_2$, $y = 1.0$, as a Function of τ and β for $y_0 = 5.0$

The second point is that the values of $\hat{\theta}_2$ and $(\hat{\sigma}_\theta)_2$ converge to a nonzero value for large time ($\tau \rightarrow \infty$). This latter effect provides a convenient check on the numerical evaluations by enabling one to compare the computed results for large τ with the analytical expressions for θ^* and σ_θ^* ($\tau = \infty$) derived in Section III-B. For example, consider a problem in which the thickness and heat-transfer parameters are $y_0 = 1.20$ and $\beta = 2.0$. For $\tau = 10$ (the largest time computed in this case), evaluation of the series solutions (3.15) and (3.20) gave

$$\begin{aligned}\hat{\theta}_2(y=1.0) &= 0.109999, \\ (\hat{\sigma}_\theta)_2(y=1.0) &= 0.014613,\end{aligned}$$

while (3.24) and (3.26) for $\tau = \infty$ gave

$$\begin{aligned}\theta^*(y=1.0) &= 0.110000, \\ \sigma_\theta^*(y=1.0) &= 0.014616.\end{aligned}$$

This close correlation between the numerical results and analytical results, while providing no definitive proof of the accuracy of the computations, does lend confidence of sufficient accuracy for all practical engineering purposes.

APPENDIX A

The Finite Hankel Transform

The required finite Hankel transform is a direct result of the Fourier-Bessel expansion theorem and certain orthogonality properties of Bessel functions. General details of the Hankel transform method are contained in Sneddon,⁷ while the specific transform used in Section III may be developed from specialization of results given in Marchi and Zgrablich,⁸ or, more directly, from the paper by Cinelli.⁹ For completeness and because its recent development precludes wide familiarity with the various expressions defining this transform, we include a short derivation of the properties used in the problem considered.

The transform is based on the following orthogonality property of Bessel functions (Ref. 10, p. 485):

Let

$$C_\nu(z) = AJ_\nu(z) + BY_\nu(z), \quad (\text{A.1})$$

where A and B are real constants. Then

$$\int_a^b t C_\nu(\xi_m t) C_\nu(\xi_n t) dt = \begin{cases} 0, & m \neq n, \\ \left[\frac{1}{2} t^2 \left\{ \left(1 - \frac{\nu^2}{\xi_n^2 t^2} \right) C_\nu^2(\xi_n t) + C_\nu'^2(\xi_n t) \right\} \right]_a^b, & m = n, 0 < a < b, \end{cases} \quad (\text{A.2})$$

provided ξ_n is a real zero of

$$h_1 \xi C_{\nu+1}(\xi a) - h_2 C_\nu(\xi a) = 0 \quad (\text{A.3})$$

and there exist numbers k_1 and k_2 (both not zero) such that, for all n ,

$$k_1 \xi_n C_{\nu+1}(\xi_n b) - k_2 C_\nu(\xi_n b) = 0. \quad (\text{A.4})$$

We shall be concerned with solutions of the eigenvalue problem for the first-order Bessel equation

$$f'' + \frac{1}{y} f' + \xi^2 f = 0, \quad 1 \leq y \leq y_0, \quad (\text{A.5})$$

subject to the homogeneous boundary conditions

$$f' - \beta f = 0, \quad y = 1, \quad (\text{A.6})$$

$$f' = 0, \quad y = y_0. \quad (\text{A.7})$$

Using the notation of (A.1), we have as the solution to (A.5)

$$f(y) = AJ_0(\xi y) + BY_0(\xi y) = C_0(\xi y), \quad (\text{A.8})$$

while the boundary conditions (A.6) and (A.7) become, respectively,

$$\xi C_1(\xi) + \beta C_0(\xi) = 0, \quad (\text{A.9})$$

$$C_1(\xi y_0) = 0. \quad (\text{A.10})$$

From (A.10) and (A.8), we thus have

$$B = -A \frac{J_1(\xi y_0)}{Y_1(\xi y_0)}, \quad (\text{A.11})$$

which, upon substitution into (A.9), yields the characteristic equation

$$Y_1(\xi y_0)[\xi J_1(\xi) + \beta J_0(\xi)] = J_1(\xi y_0)[\xi Y_1(\xi) + \beta Y_0(\xi)], \quad (\text{A.12})$$

whose real roots determine the eigenvalues ξ_i .

The solution of (A.5) subject to (A.6) and (A.7) may thus be written

$$f(y) = \sum_{i=1}^{\infty} \frac{A_i}{Y_1(\xi_i y_0)} [Y_1(\xi_i y_0) J_0(\xi_i y) - J_1(\xi_i y_0) Y_0(\xi_i y)]. \quad (\text{A.13})$$

Multiplying (A.13) by

$$\frac{A_j}{Y_1(\xi_j y_0)} [Y_1(\xi_j y_0) J_0(\xi_j y) - J_1(\xi_j y_0) Y_0(\xi_j y)],$$

and integrating over y yields

$$\frac{A_j \bar{f}(\xi_j)}{Y_1(\xi_j y_0)} = \int_1^{y_0} y C_0^2(\xi_j y) dy, \quad (\text{A.14})$$

where we have made use of the orthogonality property (A.2) and have defined the transform

$$H[f] \equiv \bar{f}(\xi_j) \equiv \int_1^{y_0} y [Y_1(\xi_j y_0) J_0(\xi_j y) - J_1(\xi_j y_0) Y_0(\xi_j y)] f(y) dy. \quad (\text{A.15})$$

Using the integral expression (A.2) and the boundary conditions (A.9) and (A.10) then yields

$$\int_1^{y_0} y C_0^2(\xi_j y) dy = \frac{1}{2} \left[y_0^2 C_0^2(\xi_j y_0) - \left(1 + \frac{\beta^2}{\xi_j^2}\right) C_0^2(\xi_j) \right],$$

which, upon consideration of (A.8), (A.11), and (A.12), and the Wronskian relationship,

$$Y_1(\xi_j y_0) J_0(\xi_j y_0) - J_1(\xi_j y_0) Y_0(\xi_j y_0) = -\frac{2}{\pi \xi_j y_0}, \quad (\text{A.16})$$

reduces to

$$\int_1^{y_0} y C_0^2(\xi_j y) dy = \frac{2}{\pi^2} \frac{A_j^2}{[Y_1(\xi_j y_0)]^2} \frac{1}{\xi_j^2} \left\{ 1 - (\xi_j^2 + \beta^2) \frac{[J_1(\xi_j y_0)]^2}{[\xi_j J_1(\xi_j) + \beta J_0(\xi_j)]^2} \right\}. \quad (\text{A.17})$$

Hence, upon substitution of (A.17) into (A.14), the solution (A.13) becomes

$$f(y) = \frac{\pi^2}{2} \sum_{j=1}^{\infty} \frac{\xi_j^2 [\xi_j J_1(\xi_j) + \beta J_0(\xi_j)]^2 [Y_1(\xi_j y_0) J_0(\xi_j y) - J_1(\xi_j y_0) Y_0(\xi_j y)] \bar{f}(\xi_j)}{[\xi_j J_1(\xi_j) + \beta J_0(\xi_j)]^2 - (\xi_j^2 + \beta^2) [J_1(\xi_j y_0)]^2}. \quad (\text{A.18})$$

Equation (A.18) thus provides the inversion theorem for the transform defined by (A.15).

The transformation property,

$$\begin{aligned} H\left[f'' + \frac{1}{y} f'\right] &= -\frac{2}{\pi \xi_j} [f'(y_0)] - [Y_1(\xi_j y_0) J_0(\xi_j) - J_1(\xi_j y_0) Y_0(\xi_j)] [f'(1) - \beta f(1)] \\ &\quad - \xi_j^2 H[f], \end{aligned} \quad (\text{A.19})$$

may be easily verified by repeated integration by parts of the expression $H[f'' + (1/y) f']$, use of (A.16) and (A.12), and application of the identities,

$$\frac{d}{dt} [J_1(\xi t)] = \xi J_0(\xi t) - \frac{1}{t} J_1(\xi t),$$

$$\frac{d}{dt} [Y_1(\xi t)] = \xi Y_0(\xi t) - \frac{1}{t} Y_1(\xi t).$$

The transform of unity needed in solution of the given problem is obtained from the elementary integrals,

$$\int_1^{y_0} t J_0(\xi_i t) dt = \frac{1}{\xi_i} [y_0 J_1(\xi_i y_0) - J_1(\xi_i)],$$

$$\int_1^{y_0} t Y_0(\xi_i t) dt = \frac{1}{\xi_i} [y_0 Y_1(\xi_i y_0) - Y_1(\xi_i)],$$

which yield

$$\begin{aligned} H[1] &= \int_1^{y_0} t [Y_1(\xi_i y_0) J_0(\xi_i t) - J_1(\xi_i y_0) Y_0(\xi_i t)] dt \\ &= \frac{1}{\xi_i} [J_1(\xi_i y_0) Y_1(\xi_i) - Y_1(\xi_i y_0) J_1(\xi_i)] \equiv F(\xi_i) \end{aligned}$$

as in Eq. (3.12).

Similarly, Eq. (3.14) may be derived as follows:

$$\begin{aligned} F(\xi_i) [\xi_i J_1(\xi_i) + \beta J_0(\xi_i)] &= \frac{J_1(\xi_i)}{\xi_i} [\xi_i J_1(\xi_i y_0) Y_1(\xi_i) + \beta J_1(\xi_i y_0) Y_0(\xi_i) \\ &\quad - \xi_i J_1(\xi_i) Y_1(\xi_i y_0) - \beta J_0(\xi_i) Y_1(\xi_i y_0)] \\ &\quad - \frac{2 \beta J_1(\xi_i y_0)}{\pi \xi_i^2}, \end{aligned}$$

where (A.16) has been used. The bracketed term on the right above is identically zero by (A.12), and thus one obtains Eq. (3.14).

APPENDIX B

Roots of the Equation

$$Y_1(\xi y_0)[\xi J_1(\xi) + \beta J_0(\xi)] = J_1(\xi y_0)[\xi Y_1(\xi) + \beta Y_0(\xi)]$$

$y_0 = 1.20$						
i	β					
	0.01	0.05	0.10	0.20	0.50	1.0
1	.213129	.475940	.671968	.947175	1.482984	2.063792
2	15.730917	15.743582	15.759379	15.790874	15.884500	16.037681
3	31.427452	31.433808	31.441754	31.457624	31.505131	31.583928
4	47.131578	47.135819	47.141120	47.151722	47.183480	47.236299
5	62.837626	62.840807	62.844788	62.852737	62.876580	62.916269
6	78.544443	78.546984	78.550167	78.556533	78.575617	78.607399
7	94.251633	94.253754	94.256410	94.261712	94.277620	94.304117

i	β					
	2.0	5.0	10.0	50.0	100.0	
1	2.829146	4.105975	5.146183	6.860620	7.193832	
2	16.333152	17.132861	18.203063	21.451155	22.368930	
3	31.739980	32.193919	32.894177	36.105625	37.428777	
4	47.341468	47.652508	48.151904	50.984017	52.528062	
5	62.995453	63.231073	63.615345	66.062055	67.689772	
6	78.670856	78.860233	79.171457	81.292241	82.918219	
7	94.357051	94.515276	94.776386	96.632936	98.210091	

$\bar{y}_0 = 1.50$						
i	β					
	0.01	0.05	0.10	0.20	0.50	1.0
1	.126387	.281690	.396757	.556601	.859525	1.170663
2	6.324986	6.337407	6.352849	6.383447	6.472965	6.614592
3	12.587703	12.594029	12.601924	12.617676	12.664614	12.741692
4	18.863834	18.868066	18.873352	18.883914	18.915501	18.967793
5	25.143464	25.146643	25.150614	25.158551	25.182320	25.221786
6	31.424511	31.427054	31.430233	31.436589	31.455635	31.487300
7	37.706268	37.708388	37.711039	37.716337	37.732223	37.758652
8	43.988432	43.990252	43.992522	43.997066	44.010688	44.033364
9	50.270853	50.272443	50.274432	50.278408	50.290332	50.310185
10	56.553442	56.554857	56.556626	56.560159	56.570761	56.588415
11	62.836152	62.837424	62.839015	62.842197	62.851741	62.867635
12	69.118947	69.120105	69.121551	69.124443	69.133121	69.147574
13	75.401807	75.402868	75.404195	75.406846	75.414801	75.428051
14	81.684715	81.685694	81.686919	81.689370	81.696709	81.708943
15	87.967666	87.968578	87.969712	87.971987	87.978806	87.990165
16	94.250646	94.251496	94.252557	94.254680	94.261045	94.271644

i	β					
	2.0	5.0	10.0	50.0	100.0	
1	1.544788	2.061870	2.384942	2.768360	2.827620	
2	6.870380	7.452210	8.024624	8.988809	9.161123	
3	12.891018	13.294021	13.811508	15.077696	15.355783	
4	19.070832	19.364135	19.787122	21.161894	21.534077	
5	25.300046	25.527700	25.874624	27.256825	27.709136	
6	31.550281	31.735458	32.026067	33.367173	33.884938	
7	37.811309	37.967056	38.215722	39.494222	40.063269	
8	44.078587	44.212844	44.429531	45.637669	46.245104	
9	50.349806	50.467715	50.659405	51.796410	52.431020	
10	56.623666	56.728743	56.900436	57.968992	58.621342	
11	62.899378	62.994121	63.149498	64.153874	64.816234	
12	69.176446	69.262692	69.404526	70.349562	71.015735	
13	75.454528	75.533668	75.664091	76.554682	77.219797	
14	81.733389	81.806503	81.927189	82.768015	83.428307	
15	88.012868	88.080805	88.193089	88.988499	89.641106	
16	94.292841	94.356281	94.461246	95.215202	95.858013	

i	$y_0 = 2.0$					
	β					
	0.01	0.05	0.10	0.20	0.50	1.0
1	.081521	.181147	.254202	.354058	.535887	.709038
2	3.199554	3.211378	3.225981	3.254603	3.335875	3.456861
3	6.313911	6.320149	6.327919	6.343369	6.388958	6.462164
4	9.445517	9.449722	9.454970	9.465438	9.496602	9.547634
5	12.581995	12.585161	12.589116	12.597013	12.620600	12.659516
6	15.720489	15.723027	15.726197	15.732532	15.751484	15.782862
7	18.860006	18.862123	18.864768	18.870055	18.885885	18.912147
8	22.000112	22.001928	22.004197	22.008733	22.022321	22.044890
9	25.140587	25.142177	25.144163	25.148135	25.160035	25.179817
10	28.281310	28.282724	28.284490	28.288021	28.298606	28.316211
11	31.422207	31.423479	31.425069	31.428248	31.437779	31.453637
12	34.563230	34.564387	34.565833	34.568723	34.577391	34.591816
13	37.704348	37.705408	37.706734	37.709384	37.717331	37.730561
14	40.845538	40.846516	40.847740	40.850188	40.857525	40.869741
15	43.986786	43.987695	43.988831	43.991103	43.997917	44.009265
16	47.128080	47.128928	47.129988	47.132110	47.138470	47.149064
17	50.269411	50.270207	50.271201	50.273189	50.279152	50.289086
18	53.410773	53.411522	53.412459	53.414329	53.419943	53.429293
19	56.552161	56.552868	56.553752	56.555520	56.560823	56.569654
20	59.693570	59.694240	59.695077	59.696752	59.701775	59.710143
21	62.834997	62.835634	62.836430	62.838020	62.842792	62.850743
22	65.976442	65.977049	65.977807	65.979321	65.983868	65.991439
23	69.117898	69.118476	69.119199	69.120646	69.124985	69.132215
24	72.259367	72.259920	72.260612	72.261995	72.266145	72.273061
25	75.400845	75.401377	75.402039	75.403366	75.407344	75.413970
26	78.542333	78.542843	78.543479	78.544753	78.548570	78.554933
27	81.683828	81.684317	81.684929	81.686154	81.689825	81.695944
28	84.825333	84.825804	84.826392	84.827572	84.831108	84.836998
29	87.966841	87.967295	87.967863	87.968999	87.972410	87.978091
30	91.108357	91.108796	91.109344	91.110442	91.113734	91.119220
31	94.249877	94.250302	94.250832	94.251891	94.255074	94.260378
32	97.391402	97.391811	97.392325	97.393352	97.396433	97.401563

$y_0 = 2.0$					
i	β				
	2.0	5.0	10.0	50.0	100.0
1	.894723	1.107914	1.217505	1.328809	1.344565
2	3.652378	4.002408	4.252642	4.554137	4.599252
3	6.597059	6.909078	7.208058	7.662484	7.736779
4	9.645630	9.899778	10.193749	10.756504	10.859116
5	12.735509	12.943752	13.213833	13.846845	13.976568
6	15.844647	16.019031	16.261866	16.936604	17.091965
7	18.964099	19.113299	19.330837	20.027099	20.206425
8	22.089665	22.219674	22.415187	23.119022	23.320492
9	25.219136	25.334144	25.510879	26.212763	26.434473
10	28.351248	28.454258	28.615048	29.308548	29.548561
11	31.485226	31.578448	31.725653	32.406500	32.662888
12	34.620571	34.705667	34.841232	35.506664	35.777546
13	37.756945	37.835196	37.960712	38.609036	38.892606
14	40.894115	40.966524	41.083302	41.713574	42.008118
15	44.031912	44.099281	44.208404	44.820209	45.124128
16	47.170213	47.233189	47.335563	47.928858	48.240663
17	50.308921	50.368039	50.464421	51.039423	51.357749
18	53.447968	53.503668	53.594701	54.151802	54.475404
19	56.587295	56.639950	56.726182	57.265890	57.593640
20	59.726861	59.776782	59.858682	60.381584	60.712464
21	62.866628	62.914084	62.992058	63.498781	63.831881
22	66.006570	66.051791	66.126191	66.617384	66.951894
23	69.146661	69.189848	69.260980	69.737297	70.072498
24	72.286881	72.328207	72.396345	72.858430	73.193684
25	75.427215	75.466836	75.532213	75.980704	76.315454
26	78.567649	78.605698	78.668531	79.104037	79.437797
27	81.708173	81.744768	81.805243	82.228356	82.560700
28	84.848775	84.884023	84.942311	85.353597	85.684156
29	87.989449	88.023448	88.079695	88.479692	88.808152
30	91.130185	91.163017	91.217365	91.606587	91.932673
31	94.270980	94.302724	94.355294	94.734228	95.057706
32	97.411824	97.442550	97.493452	97.862564	98.183242

$y_0 = 3.0$						
i	β					
	0.01	0.05	0.10	0.20	0.50	1.0
1	.049856	.110217	.153710	.211590	.310605	.395271
2	1.638270	1.648739	1.661493	1.685925	1.751284	1.838113
3	3.180343	3.186328	3.193747	3.208368	3.250418	3.314142
4	4.739123	4.743245	4.748377	4.758570	4.788518	4.836047
5	6.303502	6.306632	6.310536	6.318311	6.341355	6.378657
6	7.870339	7.872858	7.876002	7.882273	7.900938	7.931454
7	9.438457	9.440563	9.443193	9.448443	9.464105	9.489856
8	11.007325	11.009134	11.011393	11.015905	11.029386	11.051629
9	12.576667	12.578252	12.580231	12.584187	12.596015	12.615576
10	14.146328	14.147738	14.149500	14.153020	14.163554	14.181003
11	15.716214	15.717484	15.719071	15.722242	15.731735	15.747478
12	17.286265	17.287420	17.288862	17.291747	17.300386	17.314725
13	18.856438	18.857497	18.858820	18.861466	18.869391	18.882554
14	20.426707	20.427684	20.428907	20.431350	20.438670	20.450834
15	21.997051	21.997959	21.999094	22.001363	22.008164	22.019469
16	23.567454	23.568302	23.569362	23.571480	23.577830	23.588389
17	25.137908	25.138703	25.139696	25.141683	25.147637	25.157542
18	26.708401	26.709149	26.710084	26.711954	26.717560	26.726887
19	28.278928	28.279634	28.280518	28.282284	28.287579	28.296392
20	29.849483	29.850152	29.850989	29.852662	29.857680	29.866031
21	31.420062	31.420698	31.421493	31.423083	31.427851	31.435787
22	32.990662	32.991268	32.992025	32.993539	32.998081	33.005641
23	34.561279	34.561858	34.562581	34.564027	34.568362	34.575580
24	36.131913	36.132466	36.133158	36.134541	36.138688	36.145593
25	37.702560	37.703090	37.703753	37.705076	37.709053	37.715672
26	39.273218	39.273727	39.274363	39.275636	39.279452	39.285807
27	40.843887	40.844377	40.844989	40.846212	40.849882	40.855993
28	42.414565	42.415037	42.415626	42.416804	42.420339	42.426224
29	43.985253	43.985708	43.986277	43.987412	43.990820	43.996496
30	45.555948	45.556386	45.556935	45.558032	45.561323	45.566804
31	47.126649	47.127073	47.127604	47.128665	47.131846	47.137143
32	48.697356	48.697767	48.698280	48.699307	48.702385	48.707512

$y_0 = 3.0$					
i	β				
	2.0	5.0	10.0	50.0	100.0
1	.474428	.550983	.585156	.616933	.621227
2	1.955326	2.111960	2.196267	2.280522	2.292163
3	3.418377	3.600751	3.722246	3.856703	3.875844
4	4.920930	5.098419	5.241715	5.421244	5.447857
5	6.448179	6.610569	6.763431	6.981915	7.015930
6	7.989651	8.135413	8.289735	8.540888	8.582203
7	9.539634	9.670123	9.821135	10.099025	10.147516
8	11.094997	11.212245	11.357426	11.656743	11.712261
9	12.653934	12.759915	12.898127	13.214271	13.276650
10	14.215356	14.311774	14.442682	14.771750	14.840807
11	15.778564	15.866841	15.990546	16.329275	16.404811
12	17.343100	17.424399	17.541234	17.886909	17.968712
13	18.908644	18.983921	19.094319	19.444701	19.532548
14	20.474976	20.545014	20.649446	21.002685	21.096346
15	22.041929	22.107380	22.206311	22.560889	22.660125
16	23.609385	23.670789	23.764664	24.119333	24.223900
17	25.177251	25.235064	25.324293	25.678032	25.787685
18	26.745456	26.800062	26.885025	27.236999	27.351487
19	28.313945	28.365672	28.446713	28.796242	28.915317
20	29.882673	29.931803	30.009232	30.355766	30.479181
21	31.451607	31.498382	31.572479	31.915576	32.043084
22	33.020716	33.065347	33.136366	33.475673	33.607032
23	34.589977	34.632649	34.700817	35.036057	35.171030
24	36.159370	36.200245	36.265768	36.596726	36.735080
25	37.728879	37.768101	37.831164	38.157678	38.299186
26	39.298491	39.336185	39.396958	39.718910	39.863352
27	40.868192	40.904473	40.963108	41.280417	41.427578
28	42.437975	42.472943	42.529578	42.842196	42.991869
29	44.007830	44.041575	44.096337	44.404238	44.556227
30	45.577750	45.610354	45.663358	45.966540	46.120652
31	47.147727	47.179264	47.230616	47.529095	47.685146
32	48.717755	48.748295	48.798091	49.091897	49.249711

$y_0 = 5.0$						
i	β					
	0.01	0.05	0.10	0.20	0.50	1.0
1	.028727	.063026	.087111	.117982	.166718	.203621
2	.849254	.857438	.867162	.885047	.928391	.977096
3	1.612429	1.617799	1.624372	1.637047	1.671345	1.717184
4	2.386294	2.390185	2.394994	2.404430	2.431163	2.470207
5	3.164966	3.167987	3.171739	3.179155	3.200629	3.233495
6	3.946027	3.948487	3.951549	3.957624	3.975422	4.003395
7	4.728417	4.730488	4.733069	4.738201	4.753338	4.777517
8	5.511615	5.513401	5.515629	5.520066	5.533207	5.554417
9	6.295337	6.296907	6.298865	6.302769	6.314365	6.333215
10	7.079418	7.080817	7.082563	7.086047	7.096416	7.113355
11	7.863754	7.865016	7.866591	7.869735	7.879108	7.894474
12	8.648278	8.649427	8.650862	8.653726	8.662274	8.676327
13	9.432945	9.433999	9.435316	9.437946	9.445801	9.458741
14	10.217722	10.218697	10.219913	10.222344	10.229608	10.241596
15	11.002587	11.003493	11.004623	11.006883	11.013638	11.024801
16	11.787522	11.788367	11.789423	11.791534	11.797847	11.808290
17	12.572514	12.573307	12.574298	12.576277	12.582202	12.592011
18	13.357553	13.358300	13.359233	13.361097	13.366677	13.375924
19	14.142632	14.143337	14.144219	14.145980	14.151254	14.159999
20	14.927744	14.928413	14.929248	14.930917	14.935917	14.944211
21	15.712885	15.713520	15.714314	15.715900	15.720652	15.728539
22	16.498050	16.498655	16.499411	16.500922	16.505450	16.512967
23	17.283236	17.283814	17.284535	17.285979	17.290302	17.297483
24	18.068440	18.068993	18.069683	18.071064	18.075201	18.082074
25	18.853660	18.854190	18.854852	18.856176	18.860141	18.866731
26	19.638895	19.639404	19.640039	19.641310	19.645118	19.651448
27	20.424142	20.424631	20.425242	20.426464	20.430127	20.436216
28	21.209401	21.209871	21.210460	21.211637	21.215164	21.221030
29	21.994668	21.995123	21.995690	21.996825	22.000228	22.005885
30	22.779945	22.780384	22.780932	22.782029	22.785314	22.790779
31	23.565231	23.565654	23.566185	23.567244	23.570420	23.575704
32	24.350523	24.350933	24.351446	24.352472	24.355546	24.360661

$y_0 = 5.0$					
i	β				
	2.0	5.0	10.0	50.0	100.0
1	.234011	.259911	.270498	.279873	.281108
2	1.030190	1.085397	1.110266	1.133081	1.136126
3	1.777965	1.854431	1.892773	1.929267	1.934187
4	2.529306	2.617680	2.667635	2.717483	2.724289
5	3.287610	3.380790	3.440174	3.502862	3.511558
6	4.051995	4.145415	4.212023	4.286903	4.297486
7	4.821034	4.912075	4.983881	5.070208	5.082674
8	5.593522	5.680832	5.756099	5.853071	5.867410
9	6.368562	6.451559	6.528862	6.635649	6.651852
10	7.145512	7.224054	7.302267	7.418037	7.436092
11	7.923914	7.998098	8.076357	8.200295	8.220187
12	8.703438	8.773483	8.851143	8.982460	9.004175
13	9.483843	9.550019	9.626613	9.764562	9.788083
14	10.264950	10.327546	10.402741	10.546621	10.571928
15	11.046625	11.105923	11.179495	11.328649	11.355725
16	11.828764	11.885032	11.956837	12.110661	12.139484
17	12.611287	12.664772	12.734728	12.892663	12.923213
18	13.394130	13.445061	13.513128	13.674664	13.706916
19	14.177245	14.225827	14.292000	14.456669	14.490600
20	14.960590	15.007009	15.071307	15.238683	15.274268
21	15.744133	15.788557	15.851014	16.020710	16.057924
22	16.527847	16.570427	16.631091	16.802753	16.841569
23	17.311709	17.352581	17.411506	17.584816	17.625206
24	18.095701	18.134989	18.192234	18.366900	18.408836
25	18.879807	18.917621	18.973248	19.149009	19.192462
26	19.664014	19.700456	19.754526	19.931143	19.976085
27	20.448311	20.483472	20.536048	20.713304	20.759706
28	21.232687	21.266651	21.317794	21.495494	21.543325
29	22.017135	22.049976	22.099747	22.277715	22.326945
30	22.801648	22.833435	22.881892	23.059968	23.110565
31	23.586218	23.617016	23.664213	23.842252	23.894186
32	24.370841	24.400707	24.446699	24.624570	24.677809

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