ARGONNE NATIONAL LABORATORY
P. O. Box 299
Lemont, Illinois

THE KINETICS AND STABILITY OF FAST REACTORS
WITH SPECIAL CONSIDERATIONS OF NONLINEARITIES

by

Henry A. Sandmeier
Reactor Engineering Division

A Thesis
Submitted to the Department of Physics,
Swiss Federal Institute of Technology in Zurich
In Partial Fulfillment of the
Requirements for the Degree
of
Doctor of Natural Science

June, 1959

Operated by The University of Chicago
under
Contract W-31-109-eng-38

DISCLAIMER

This report was prepared as an account of work sponsored by an agency of the United States Government. Neither the United States Government nor any agency Thereof, nor any of their employees, makes any warranty, express or implied, or assumes any legal liability or responsibility for the accuracy, completeness, or usefulness of any information, apparatus, product, or process disclosed, or represents that its use would not infringe privately owned rights. Reference herein to any specific commercial product, process, or service by trade name, trademark, manufacturer, or otherwise does not necessarily constitute or imply its endorsement, recommendation, or favoring by the United States Government or any agency thereof. The views and opinions of authors expressed herein do not necessarily state or reflect those of the United States Government or any agency thereof.

DISCLAIMER

Portions of this document may be illegible in electronic image products. Images are produced from the best available original document.

Acknowledgement

I gratefully acknowledge the encouragement and guidance given to me by Professor P. Scherrer during the course of this investigation.

Thanks are especially due to the Director of the Reactor Engineering Division, Dr. B.I. Spinrad, and to the Director of the Technical Information Division, Dr. Hoylande D. Young, of Argonne National Laboratory, for giving the permission to engage in this work.

Contents

	Page
Introduction	5
Summary	7
Chapter I: LINEAR ZERO POWER KINETICS	
I.1 Zero Power Kinetics Equations	9
I.2 Linear Approximation of Zero Power Kinetics Equations and Zero Power Transfer Functions	10
I.3 Approximation to the Zero Power Transfer Function	14
I.4 Parameter Study on Zero Power Transfer Function $ZP(s)$ by Varying Delayed Neutron Parameters β_i and λ_i	15
I.5 Zero Power Transfer Function as effected by different Delayed Neutron Data and U^{235} Fission Contribution	18
Chapter II: LINEAR KINETICS AND STABILITY OF FAST POWER REACTORS	
II.1 Reactivity Coefficients	21
II.2 Doppler Effect	21
II.3 Reactivity due to Bowing of Fuel Pins	23
II.4 Reactivity Effects due to Expansion of Reactor Materials	24
II.5 Time Constant Associated with Fuel Pin	25
II.6 Structural Time Constants	31
II.7 Total Reactivity Effects due to Temperature	31
II.8 Load Power Transfer Function $LP(s)$	32
II.9 The Stability of Fast Reactor as a Function of Power Level	33
Chapter III: NONLINEAR EFFECTS IN THE KINETICS OF FAST REACTORS	
III.1 Derivation of the Exact Incremental Kinetics Equations	49
III.2 Derivation of Amplitude of m th Harmonic $X'_m/LP(jm\omega)$ and $X_m/LP(jm\omega)$ in the Expansion of $\delta n'/n_0$ and $\delta n/n_0^*$ in Implicit Form	54
III.3 Derivation of Time Independent Term ∇ in Implicit Form for X'_m and X_m	57
III.4 Solution for X_m and ∇ by Iteration	57
III.5 Fourier Components of Flux for Zero Power Reactor	60
III.6 Fourier Components of Flux for Power Reactor	63
III.7 Effect of Limited Bowing on Stability	66
III.8 Limited Bowing and Self-Sustained Oscillations	67

Appendix

A.	Hughes and Keepin Delayed Neutron Data. Effective Delayed Neutron Including U^{238} Fission for EBR-I	76
B.	Analog Computer Representation of Fast Reactors	78

Literature

82

Introduction

A fast reactor can be defined as a reactor in which the neutron energy distribution differs as little as possible from the energy distribution of the neutrons emitted in fission. The degrading of the neutrons is kept low in order to achieve a high breeding gain. If the breeding gain is more than one, we will produce more fissionable material than there is burned. Breeders will be of the utmost importance in a large scale atomic power economy. The content of this study will deal with the dynamic behavior of a fast reactor when the neutron flux is considered as a function of time.

The study of the time behavior of nuclear reactors has received much attention for two reasons. There was on the one hand the war time incentive to obtain the most violent possible explosion from a temporarily supercritical assembly of the fissionable material; and on the other hand, the following drive to devise reactors which were as safe as possible in their operation. These two lines of attack at first had little in common, not so much because of the divergence in aim, but because of the vast physical difference between the bomb type system and the large, sluggish, thermal reactor. However, in recent years there has been an increased demand for compact, highly enriched U^{235} and plutonium assemblies with a high power density, and an increased importance was attached to the understanding of the time behavior of reactors.

Questions concerning the inherent stability of chain reactors were among the first to be raised in the development of the art. If the reactivity insertions are kept well below prompt critical there is essentially no difference between the zero power kinetics of the better known thermal reactors as compared to fast reactors.) The periods in the flux which one observes are in the order of seconds and minutes. The reason for this surprisingly slow behavior of nuclear reactors is the fact that some of the neutrons released in fission do not come out of the nucleus immediately, but are delayed varying from about one second to a minute. If we assume a period of as short as 1 second in the neutron flux, the difference in excess reactivity necessary to cause this rate of rise of flux in a fast and thermal reactor is negligible. We therefore see that in this operating range the kinetics will not be dependent on the speed of the neutrons. For a reactor at prompt critical the neutron flux would indeed depend on the neutron lifetime and would rise much more rapidly in a fast reactor.

The kinetics of a fast reactor can be grouped into three distinct areas of interest. The first being normal operating conditions where all the changes are brought about in a slow manner, and the resulting flux changes being small in comparison with the steady state flux. Since the available reactivity and the power density of most large thermal reactors is so small, and the heat capacity is so large, nothing but small deviations from design conditions would occur before the control rods were inserted. Thus reactor kinetics traditionally has meant linear kinetics, which in the mathematical interpretation leads to linearized kinetics equations, where the superposition principle holds.

The second area being where there is much stronger coupling between reactivity and geometrical changes in the core. A fast reactor has a much higher power density than a thermal one and geometrical changes will therefore be more effective upon reactivity. The reason for higher power per unit volume being as follows: A fast reactor needs a greater total amount of U^{235} because the fission cross section of U^{235} is several hundred times smaller at neutron energies of the order of .1 Mev as compared to thermal energies.

For a fast reactor, therefore, we get a larger critical mass if we compare two reactors of equal volume. However, a fast reactor will always be smaller than a thermal reactor assuming the same power production. In a fast reactor we have no moderator besides that one likes to reduce the critical mass in order to keep the neutron spectrum fast. To have a fast spectrum is a requirement for a good breeding ratio. The core size will be fixed by heat transfer considerations. Practical core

designs show that a fast reactor will have a high power density, that is power per unit volume. On the other hand, since the critical mass is high, one will get a low value for the power per unit of fissionable material in a fast reactor. Because of the high power per unit volume, the possibility of resonances will be higher; also, in principle they could occur in both types of reactors. For such reactors the coupling between temperature and reactivity manifests itself very strongly.

The power production may be so fast that the stability of the reactor must depend on this coupling, rather than on relative slow acting control rods. Resonances occur because of a feedback relationship between the two dynamic variables, the neutron flux or the power, and the reactivity.

Resonances mean large flux variations which violate the linearity principle, and one has to solve therefore the exact kinetics equations where the multiplication term of the incremental neutron flux and the excess reactivity becomes important. This term can get large in two ways; first, if the reactivity insertion is high, and secondly, if we operate in a resonance region. That resonances can exist has been shown in EBR-I, Mark II, which melted partially during an experiment. Also, the reason for the meltdown was not the resonance as such, but a positive temperature coefficient due to bowing of fuel pins.

Large reactivity insertions can be visualized in accidental cases where a fuel element is dropped into the core in a sensitive region. Also, it is true that in a fast reactor there is less excess reactivity available, since we don't need to override poison as in a thermal one, where the major poison contributor is Xe^{135} . The question now arises, how does the nonlinear behavior effect the stability of the reactor, and how will the well known linear stability criteria be altered due to this effect.

An equilibrium is stable or unstable according to whether a small departure from equilibrium results or does not result in a return to equilibrium. It is useful to distinguish between linear and nonlinear stability. Linear stability implies that the disturbing accident is arbitrarily small, while nonlinear stability implies an accident which is sufficiently large for nonlinear effects to become important, although no direct damage from the large magnitude of the accident is assumed. Nonlinear stability implies linear stability and linear instability implies nonlinear instability. Linear stability can be associated with either nonlinear stability or instability, and nonlinear instability similarly permits no conclusion on linear stability.

It is important to note that both areas, the linear and nonlinear, will depend heavily on the delayed neutrons. In other words, the reactivity insertions are not so high that the delayed neutrons, or part thereof, can be neglected. For a reactivity insertion of one dollar the reactor will be critical without the delayed neutrons; that is, on the prompt neutrons alone, and the kinetic equations could be drastically simplified, provided that the reactor would keep its basic shape.

This third area of accidental and explosion type calculations which are important in estimating the maximum energy release in an accident is not discussed in this paper.

Summary

In order to get the full advantage of the corrections which have to be made due to nonlinear effects, the linear kinetics is discussed in Chapter I and Chapter II. Chapter I deals with the zero power kinetics and Chapter II with the linear power kinetics. In the linear zero power kinetics the influence of its parameters on the transfer function is discussed; i.e., neutron lifetime, delayed neutron fraction β and the delay constant λ . Besides an approximation for the zero power transfer function, we discuss the influence of different delayed neutron data by Hughes and Keepin on the kinetics; also the effect on the transfer function of U^{238} fission which takes place in a fast reactor besides the fissioning of U^{235} . In Chapter II the kinetics of fast power reactors and their stability is discussed.

The stability depends heavily on power coefficients and time constants.

A typical feedback model as one might get in EBR-I was chosen as the base of the discussion. It essentially consists of a positive and a negative contribution to reactivity with its associated time constants. Also the effect upon reactivity due to the Doppler effect, bowing of fuel pins, and expansion are shown. A time dependent temperature expression for a cylindrical pin in flowing NaK coolant is derived. It is shown that the stability depends on the power level at which one operates.

Resonances limit the power level at which the reactor can be operated. The power level of operation has to be removed far enough from the resonance power where the reactor will start oscillating on its own.

In Chapter III the influence of nonlinearities on the dynamics is discussed.

For large flux variations the product term $K_{ex} n$ in the kinetics equations becomes large and the linear assumption $K_{ex} n = K_{ex} n_0$ is no more valid. Due to the term $K_{ex} n$ harmonics are generated in the flux. There are two ways to generate harmonics. First, if we increase the amplitude of the reactivity insertion, or secondly, if the variations in the flux are large. This second case is possible even if the reactivity insertions are very small, since in a resonance region we have a very large amplification between the reactivity K_{ex} and the neutron flux n .

The analytical approach to obtain information on the harmonic content is based on an iteration method. Parallel with the analytical work a set of nonlinear kinetics equations with and without feedback was solved on analog computers, and the wave shapes were Fourier analyzed. From this Fourier analysis the fundamental component was singled out, and a new ratio was obtained between flux fundamental and reactivity insertion. The approach is an equivalent parameter technique introduced by Kryloff and Bogoliuboff in nonlinear mechanics.

As an illustration it is shown that EBR-I, Mark II may be stable for small disturbances at a certain power level, but will not be stable for sufficiently large disturbances due to the product term $K_{ex} n$ in the kinetics equations. The stability of a linear system is independent of the disturbance, but a nonlinear is not.

The second nonlinearity which is analyzed is limited bowing of fuel pins. The bowing of fuel pins gives a positive contribution to the power coefficient. This bowing is kept at a minimum, but was present in EBR-I, Mark II, and was the reason for the meltdown of the core. Future design will be more rigid, and bowing will be limited by mechanical design. It is shown that limited bowing can be visualized as a mechanism for self-sustained oscillations if the power level is high enough.

Mathematically speaking, a convergent equilibrium point is possible, or in the phase plane techniques a limit cycle exists. The reactor has to be linearly unstable first to start growing oscillations on its own. The limited bowing prevents the reactor from becoming nonlinearly unstable and furnishes the nonlinear mechanism in the feedback, so that the system can reach a convergent equilibrium point. After a decreasing

or increasing disturbance the reactor will always return on its own to oscillations of constant magnitude; that is, self-sustained oscillation. The nonlinearity was analytically represented by using similar techniques as for the term $K_{ex} n$; that is, a frequency and amplitude dependent ratio obtained where only the fundamental component of the "chopped off" positive reactivity feedback is used. That this approach prescribes the system with sufficient accuracy can be shown.

In this analysis it was assumed that the zero power kinetics is linear; that is, $K_{ex} n = K_{ex} n_0$. Actually the two nonlinearities would have to be combined into one prescribing function. To treat the two nonlinearities separately is justified, because the limited bowing which is a "chopping" action will stop high frequency, but has little effect on the fundamental component in the flux. That is, limited bowing helps stability as one would expect, physically.

The amplitude of the self-sustained oscillations depends on the power level and the bowing limit. For constant bowing limit the amplitude of the self-sustained oscillations will decrease for increasing power level. For constant power level the amplitude of oscillations will increase with increasing bowing limit; that is, if the bowing is less restrained. The frequency of oscillation will decrease with decreasing power level when the bowing limit is held constant.

For both power level and bowing limit there is a limit where no self-sustained oscillations are possible if either the bowing limit or the power level is held constant. As an illustration, a fast reactor with positive and negative terms in the power coefficient was analyzed, and the self-sustained oscillations verified on analog computers.

Chapter I

LINEAR ZERO POWER KINETICS

I.1 Zero Power Kinetics Equations

The well known zero power kinetics equations express the relationship between reactivity insertion and neutron density. The parameters involved for normal operating conditions are governed by the delayed neutrons. Equations I.1-1, 2 are two coupled differential equations involving the number of delayed neutrons precursors and the neutron density¹).

$$\frac{dn}{dt} = \frac{(1-\beta) K_{ex} - \beta}{\ell} n + \sum_i \lambda_i c_i \quad \text{I.1-1}$$

$$\frac{dc_i}{dt} = \frac{\beta_i}{\ell} (1 + K_{ex}) n - c_i \lambda_i \quad \text{I.1-2}$$

where:

n	= neutron density
c_i	= concentration of i^{th} delayed neutron precursor
β	= fraction of neutrons that are delayed
β_i	= fraction of neutrons that are delayed in i^{th} group
λ_i	= decay constant for i^{th} group
K_{ex}	= excess reactivity = $K_{\text{effective}} - 1$
ℓ	= effective prompt neutron lifetime.

Equations I.1-1, 2 are linear differential equations with variable coefficients since K_{ex} is generally a function of time. Equation I.1-1 follows directly from the time-dependent neutron diffusion equation²) which is assumed to be separable. That is, the spatial flux does not change its shape during the transient. The kinetics equations are therefore the same for each point on the steady state flux curve in the reactor. Separability is dependent on the rate of reactivity insertions³). The permissible reactivity insertion rate being proportional to the neutron lifetime. The separability is therefore probably more justified in a fast reactor than in the more frequently used assumption in thermal reactor kinetics.

Equation I.1-2 expresses the change of delayed neutron precursors being equal to the rate of generation minus the rate of decay of the precursors. The kinetics equations are valid for fast as well as for thermal reactors; that is, they are independent of the neutron speed, provided the reactor operates below prompt critical. Up to approximately 72 cents of reactivity insertion there is no appreciable difference between the kinetics behavior of a fast and thermal reactor⁴). For reactivity insertions of one dollar or more the delayed neutrons can be neglected and the kinetics behavior is determined by the neutron lifetime. For this condition the rate of rise of the neutron flux n will be several orders of magnitude faster than in a thermal reactor. Fortunately, in a fast reactor we need much less excess reactivity to override reactivity effects of temperature and fission product poisoning. A fast reactor can be designed in such a way that there will be not enough excess reactivity available to become prompt critical.

I.2 Linear Approximation of Zero Power Kinetics

Equations and Zero Power Transfer Functions

For small variations of neutron level, i.e., $\delta n/n_0 \ll 1$, where n_0 is the steady state value and δn is the incremental flux, one can neglect the product term $\delta n \times K_{ex}$ in comparison with $K_{ex} n_0$ in Eqs. I.1-1, 2 if we replace n by $n_0 + \delta n$. This approximation reduces Eqs. I.1-1, 2 to a set of linear differential equations with constant variables which is readily solved by Laplace transform methods.

$$\frac{\delta n(s)}{n_0 K_{ex}(s)} = \frac{1 - s \sum_i \frac{\beta_i}{s + \lambda_i}}{s \left[\ell + \sum_i \frac{\beta_i}{s + \lambda_i} \right]} \quad \text{I.2-1}$$

where $\delta n(s)$ and $K_{ex}(s)$ are the Laplace transforms of $\delta n(t)$ and $K_{ex}(t)$. For simplicity no distinction is made in the symbolism between the function of time and the function of the transform variable. The right hand side of I.2-2 is called "Zero Power Transfer Function" of the reactor and is denoted $ZP(s)$. The term "Zero Power" indicates that the feedback due to the coupling between temperature and the reactivity is not taken into account. The significance of the transfer function lies in the fact that it gives the amplitude and phase angle of the flux variations when the reactivity varies sinusoidally. For a sinusoidally varying reactivity input the transfer function becomes

$$ZP(j\omega) = \frac{1 - j\omega \sum_i \frac{\beta_i}{j\omega + \lambda_i}}{j\omega \left[\ell + \sum_i \frac{\beta_i}{j\omega + \lambda_i} \right]} \quad \text{I.2-2}$$

$ZP(j\omega)$ is plotted in Fig. I.2, 1 in amplitude and in Fig. I.2, 2 in phase as a function of ω for different neutron lifetimes and using the delayed neutron data reported by Hughes⁵). The Hughes and Keepin⁶) delayed neutron data is reported in Appendix A, Tables I through IV. In this paper the Hughes data will be used for all numerical evaluations, also a typical neutron lifetime ℓ of 4×10^{-8} sec. In order to discuss the curves for varying ω over the whole frequency range we can make the following simplifying assumptions in I.1-1 and I.1-2.

$$\beta \ll 1 \quad \text{I.2-3}$$

$$K_{ex} \ll 1 \quad \text{I.2-4}$$

The first approximation is justified with an accuracy of less than 1%. The second approximation means that the excess reactivity in dollars, i.e., K_{ex}/β , is less than $1/\beta = 134$. If the excess reactivity is one dollar, the error is of the same order of magnitude as that of neglecting β (.00755) as compared to 1. Except in hypothetical accidental cases, the excess reactivity is always less than one dollar. Therefore, the neglect of K_{ex} as compared to 1 is well justified in normal operating conditions. By making the above assumptions Eqs. I.1-1 and I.1-2 become

Fig. I. 2, 1.

AMPLITUDE OF $ZP(s)$ AS A FUNCTION OF ANGULAR
FREQUENCY ω FOR DIFFERENT NEUTRON LIFETIMES ℓ .
ALSO APPROXIMATION TO $ZP(s)$

Fig. I. 2, 2.

PHASE ANGLE OF $ZP(s)$ AS A FUNCTION OF ANGULAR
FREQUENCY FOR DIFFERENT NEUTRON LIFETIMES ℓ .
ALSO APPROXIMATION TO $ZP(s)$

$$\frac{dn}{dt} = \frac{K_{ex} - \beta}{\ell} n + \sum_i c_i \lambda_i \quad \text{I. 2-5}$$

$$\frac{dc_i}{dt} = \frac{\beta_i}{\ell} n - c_i \lambda_i \quad \text{I. 2-6}$$

By making the same linear assumption as for I. 2-1 and solving I. 2-5 and I. 2-6 by Laplace transform, one obtains for the zero power transfer function

$$ZP(j\omega) = \frac{1}{j\omega \left[\ell + \sum_i \frac{\beta_i}{j\omega + \lambda_i} \right]} \quad \text{I. 2-7}$$

For low values of ω , i.e., for values smaller than the smallest λ_i , $ZP(j\omega)$ is approximated by

$$ZP(j\omega) = \frac{1}{j\omega \ell^*} \quad \text{I. 2-8}$$

where:

$$\ell^* = \ell + \sum_i \frac{\beta_i}{\lambda_i}$$

since $\sum_i \frac{\beta_i}{\lambda_i} \gg \ell$ one sees that in the region of low ω 's the amplitude curve I. 2, 1

does not depend on ℓ the neutron lifetime, but is proportional to $1 / \sum_i \frac{\beta_i}{\lambda_i}$, which is a quantity determined by the delayed neutrons only.

From Fig. I. 2, 2 we see that the phase angle in this region goes toward -90° , which can be readily seen by observing I. 2-8.

For moderately large values of ω , i.e., ω is larger than the largest λ_i and smaller than $1/\ell$, $ZP(j\omega)$ is approximated by

$$ZP(j\omega) = \frac{1}{\beta} \quad \text{I. 2-9}$$

which explains the plateau in the amplitude curve. The phase angle is zero in this frequency region. The lower end of the plateau is determined by the largest λ_i and the upper end by $1/\ell$. Hence, for smaller ℓ one obtains more extended plateaus, as seen in Fig. I. 2, 1. For a neutron lifetime of 4×10^{-8} sec, as for EBR-I, the plateau extends to $\omega = 10^5$ (rad/sec). One can readily see that the normal operating kinetics will not be effected by the neutron lifetime for a fast reactor. The time constants which one encounters in a reactor are in the region where the delayed neutrons are effective; that is, for $.01 < \omega < 10$ (rad/sec). For still larger values of frequency, i.e., $\omega > 1/\ell$, $ZP(j\omega)$ is given by

$$ZP(j\omega) = \frac{1}{j\omega \ell} \quad \text{I. 2-10}$$

which is obtained by replacing ℓ^* in I. 2-8 by ℓ . Hence, in both cases the decrease in amplitude is one hundred per decade of frequency. It is interesting to note that the

reponse is solely determined by ℓ in the high frequency region and the delayed neutrons have no influence in contrast to the lower frequency regions. From an experimental standpoint one can obtain in the low frequency region information by means of oscillators up to approximately 300 rad/sec. The limitations being mainly mechanical in nature. For the high frequency end of $ZP(j\omega)$ information can be obtained by means of correlation techniques⁷⁾ by analyzing the power noise in a reactor.

I.3 Approximation to the Zero Power Transfer Function

For analytical purposes one would like to have an expression for $ZP(s)$ in a less complicated form than I.2-2 and I.2-7, especially if there are no computers available. Analytical solutions for one and two-group equivalent group representations are given in the literature^{8), 9)}. It is felt that a "matching" solution of the amplitude and phase curve of $ZP(s)$ by means of "breakpoints"¹⁰⁾ will give faster and sufficiently accurate results. For a fast reactor with a neutron lifetime of 4×10^{-8} sec and the Hughes delayed neutron data, the following expression has been obtained and is plotted in Fig. I.2, 1 in amplitude and in Fig. I.2, 2 in phase for $.1 < \omega < 10$ rad/sec.

$$ZP(s) = \frac{10.8 (40 s + 1) (2.76 s + 1)}{s(9.09 s + 1)} \quad \text{I.3-1}$$

By observing I.3-1 we notice that for low ω 's, $ZP(s)$ behaves like $1/s$; that is, -100 per decade of ω in amplitude at an angle of -90° . At $1/40$ rad/sec a lead effect flattens the curve; at $1/9.09$ rad/sec an additional lag cuts in and at $1/2.76$ rad/sec another lead effect appears. For $\omega \gg 1/2.76$ rad/sec $ZP(s)$ is a constant; that is, the zero power transfer function reaches the plateau at a value of $1/\beta$ and the phase angle is 0° . The effect of the neutron lifetime which becomes effective at $\omega \sim 10^5$ rad/sec is neglected here, but could be easily added to the discussion. I.3-1 is sufficiently accurate in the region $.1 < \omega < 10$.

This approximation corresponds to a two-group equivalent representation by finding the optimum match to the correct multigroup representation in amplitude and phase.

I.4 Parameter Study on Zero Power Transfer Function ZP(s) by Varying Delayed Neutron Parameters β_i and λ_i

I.4.1 Influence of β_i on $|ZP(s)|$ and $\angle ZP(s)$

From Eqs. I.2-8 and I.2-9 we see that ZP(s) is inversely proportional to $\sum_i \frac{\beta_i}{\lambda_i}$ and to $\sum_i \beta_i$ respectively. Hence, any increase in β_i will shift the curve downward in the low and moderate frequency regions and vice versa. Equation I.2-10 shows that the amplitude does not depend on β_i in the high frequency region, as seen from Fig. I.4, 1. The phase angle is effected only in the regions where the transition from low to moderate and from moderate to high frequencies takes place, as shown in Fig. I.4, 2.

I.4.2 Influence of λ_i on $|ZP(s)|$ and $\angle ZP(s)$

From I.2-8 we see that the response is proportional to $1/\sum_i \frac{\beta_i}{\lambda_i}$ if ℓ is neglected, compared to $\sum_i \frac{\beta_i}{\lambda_i}$. Hence, any increase in λ_i shifts the amplitude curve upward in the low frequency region and vice versa, as seen in Fig. I.4, 1.

In the moderate and high frequency region λ_i has no effect on the response of $|ZP(s)|$.

The phase angle is effected only in the transition region between low and moderate frequencies, as seen in Fig. I.4, 2.

I.4.3 Influence of Neglecting Delayed Neutron Groups on $|ZP(s)|$ and $\angle ZP(s)$

It is generally seen from Eqs. I.2-8 and I.2-9 that the neglect of any delayed neutron group will shift the amplitude curve upward, since the neglect will always decrease the sums $\sum_i \frac{\beta_i}{\lambda_i}$ and $\sum_i \beta_i$ respectively. The amount of displacement in the low and moderate frequency regions depends on the values β_i/λ_i and β_i respectively for the disregarded groups. For high frequency the amplitude will not be effected, since Eq. I.2-10 does not contain $\sum_i \frac{\beta_i}{\lambda_i}$ or β_i , as seen in Fig. I.4, 1. The phase angle is effected mainly in the transition regions. See Fig. I.4, 2.

Fig. I. 4, 1.

AMPLITUDE OF $ZP(s)$ WITH DELAYED NEUTRON GROUPS
 NEGLECTED AS A FUNCTION OF ANGULAR FREQUENCY ω
 FROM THE HUGHES DATA, ALSO INFLUENCE OF
 INCREASED AND DECREASED βi AND λi

Fig. I. 4, 2.

PHASE ANGLE OF $ZP(s)$ AS A FUNCTION OF ANGULAR
FREQUENCY ω WITH DELAYED NEUTRON GROUPS NEGLECTED
FROM THE HUGHES DATA, ALSO INFLUENCE OF
INCREASED AND DECREASED β_i AND λ_i

I.5 Zero Power Transfer Function as Effected by Different Delayed Neutron Data and U^{238} Fission Contribution

Figs. I.5, 1 and I.5.2 show the transfer functions for the values of β_i and λ_i given by Hughes⁵⁾ and Keepin⁶⁾. Furthermore, a curve is shown for a modified Keepin data, including U^{238} fission for EBR-I¹¹⁾. The pertinent data is shown in Table IV, Appendix A. By observing Figs. I.5, 1 and I.5, 2 it can be concluded that the influence of the different data on the transfer function and therefore on the dynamic behavior of a reactor is negligible.

Fig. I. 5, 1.

AMPLITUDE OF $ZP(s)$ AS A FUNCTION OF ANGULAR
FREQUENCY ω FOR DIFFERENT DELAYED NEUTRON DATA

Fig. I. 5, 2.

PHASE ANGLE OF $ZP(s)$ AS A FUNCTION OF ANGULAR
FREQUENCY ω FOR DIFFERENT DELAYED NEUTRON DATA

Chapter II

LINEAR KINETICS AND STABILITY OF FAST POWER REACTORS

II.1 Reactivity Coefficients

The reactivity of a reactor is a measure of the net rate of increase of neutrons per neutrons that are present. This increase arises due to the fissioning of fissile isotopes. In the case of U^{235} , 2.5 neutrons are liberated per fission. At the same time radioactive capture processes take place in which neutrons are removed besides the neutron necessary to cause another fission event. Without capture the net increase is 1.5 neutrons per event. Clearly the magnitude of reactivity depends on the probability with which fissions take place relative to capture. A chain reaction can only be obtained if the concentration of fissionable material is such that the net increase of neutrons will be zero. In this condition the reactor is said to be critical.

Fission causes release of heat, and therefore expansion of materials present in the reactor, or other geometrical distortion effects will take place, besides the change of nuclear parameters due to temperature. Due to these changes one has to re-examine the balance of neutrons created and absorbed. This is expressed in the coefficients of reactivity $\nabla k/k$ which give us an indication of how much reactivity is gained or lost in the whole reactor, or part thereof, due to temperature.

In this paper the following effects upon reactivity will be discussed.

- a) Reactivity change due to variations in the nuclear parameters, Doppler effect.
- b) Geometrical distortion of parts of the reactor.
Bowling of fuel elements.
- c) Expansion of reactor materials. Uranium, sodium and structure.

II.2 Doppler Effect

The Doppler effect^{1), 4), 14), 15), 16)} arises from the temperature variations of the distribution of relative velocities between the neutrons and the nuclei. The well known resonances in the plot of the cross sections versus energy will be lowered in height, and are broadened out with temperature. For thermal neutron energies the resonance level parameters are well known for U^{235} and U^{238} and the Doppler effect can be theoretically predicted. For a fast reactor like EBR-I the average neutron energy¹⁷⁾ is in the .1 Mev range. There is very little information available on cross sections, particularly in resonance regions, at this energy range. Efforts have been made to estimate from the known resonances in the thermal energy range how the behavior will be in the high energy range in terms of various statistical parameters of the probability distribution.

Due to a resonance effect the neutron flux will be lowered, causing a reduction in the occurrence of fission events in comparison with the events which would occur if the cross section locus were broadened out. This decrease in effectiveness is known as the self-shielding of resonances. The self-shielding factor is by definition the ratio of effective cross section to the actual cross section. The self-shielding factor is unity for cross sections which have no resonance. For a very narrow and high resonance the self-shielding factor will be less than unity.

The thermal agitation of the uranium nuclei will cause a lowering of the resonance peak and also a flattening effect. Therefore the self-shielding will be lowered and the self-shielding factor will increase.

Since the self-shielding factor increases for increasing temperature, there will be more occurrence of fission events or the reactor will be more reactive.

For pure U^{235} the effective resonance cross section will increase with temperature, causing a positive effect. This positive coefficient is in part balanced by the negative coefficient of reactivity arising from the increased absorption in U^{235} . In fact, in intermediate reactors the temperature coefficient of reactivity because of U^{235} may be negative, because resonances with higher than usual capture to fission ratio give negative contributions to the temperature coefficient. In the study of fast reactors it is found that the U^{235} by itself must have a positive Doppler temperature coefficient of reactivity. Using the expression used by Feshbach, et al¹⁴⁾

$$DC(T) = DC(T_0) \left(\frac{T_0}{T} \right)^{3/2} \left[\frac{\nabla k}{k^{\circ}C} \right] \quad \text{II. 2-1}$$

Assume $\frac{\nabla k}{k^{\circ}C}$ at $20^{\circ}C$ is 2×10^{-6} , inlet temperature $100^{\circ}C$ and a ∇T for U^{235} of $112^{\circ}C$, which are typical values for EBR-I¹²⁾, we obtain

$$\frac{\nabla k}{k \text{ MW}} = 1.26 \times 10^{-4} \quad \text{II. 2-2}$$

For a reactor where we have nearly pure U^{235} the Doppler effect may indeed cause a small positive temperature coefficient. In EBR-II the Doppler effect has been estimated to be¹³⁾ $4 \times 10^{-7} \frac{\nabla k}{k^{\circ}C}$.

An admixture of sufficient U^{235} introduces a negative coefficient of reactivity which balances the positive coefficient. If U^{235} is diluted with at least one atom of U^{238} per atom of U^{235} a negative temperature coefficient will result. The Doppler coefficient vanishes for approximately equal numbers of U^{235} and U^{238} .

Efforts have been made to determine the Doppler effect experimentally^{18), 19)}. By varying the temperature of samples of U^{235} and U^{238} and minimizing expansion effects it was proposed to measure the resulting change in reactivity which would give a measure of the Doppler effect. Only a small sample was heated, since it would be difficult to heat up an entire assembly of the core and try to separate the Doppler effect from the expansion effects of the structure, as well as the expansion of the fuel elements.

To measure such small changes it is virtually impossible to do period measurements. Changes of ambient temperatures cause continual changes of reactivity sufficient to completely drown out the changes to be measured. In addition, in the Doppler experiment the temperature of the sample gradually heats up the surrounding assembly, causing large changes of reactivity. So it was necessary to use an oscillator technique which is a modification of that developed by Langsdorf²⁰⁾, and thereby eliminating the effects of slow drifts of reactivity on the measurements. The values found by Kato

and Butler¹⁸⁾ give values of $1.7 \times 10^{-6} \left[\frac{\nabla k}{k^{\circ}\text{C}} \right]$. Experiments by the British²¹⁾ indicate values of a factor of 2 less than the above.

II.3 Reactivity Due to Bowing of Fuel Pins

The analysis of the partial meltdown of EBR-I, Mark II in November 1955²²⁾ showed that a positive prompt temperature coefficient was present. It is generally agreed that this positive coefficient is due to some features in the engineering design of this reactor and is not due to the high speed of the neutrons. The work prompt refers to the positive temperature coefficient being faster in its effect than other coefficients. This will be discussed later. EBR-I was designed to establish above all the feasibility of breeding and its associated breeding ratio, besides the study of other physical measurements.

It was not intended for engineering tests and was therefore not designed with the care, and did not have the mechanical rigidity that one should demand from a power reactor.

The fuel elements in EBR-I are rods of U^{235} which are held at their top and bottom. When the reactor operates the side of each fuel rod which is closer to the center of the reactor will become hotter than the side away from the center. This will make the inner side expand relative to the outer and the fuel elements will bow. Since its top and bottom are fixed in vertical position, the middle of the fuel elements will be pushed closer to the center of the reactor. This will lead to an increase of reactivity. EBR-I, Mark II was put purposely on a fast rising period of ~60 sec and the prompt positive reactivity coefficient due to bowing became dominant over the slow negative coefficient due to expansion which caused the reactor to "run away" before the safety scrams became effective. Because of this unfortunate incident, efforts have been made in EBR-I, Mark III and EBR-II to reduce the bowing of fuel pins. In EBR-I this was accomplished by putting on each fuel pin a spacer wire in three places, and besides this, a whole bundle of pins was put into a hex can where the individual rods could be tightened by a tightening rod in the middle of the hex can. To make the core even more rigid, one clamped the whole hex can assembly in three places along the vertical axis of the core. Recent measurements on EBR-I, Mark III indicate indeed that the purpose was accomplished, and that there is practically no bowing present. The numerical values of positive reactivity in EBR-I and EBR-II due to bowing have been quoted as follows:

$$\begin{array}{ll}
 \text{EBR-I, Mark II} & \left\{ \begin{array}{ll} .8 \times 10^{-3} \left[\frac{\nabla k}{k \text{ MW}} \right] & 1) \\ 2.0 \times 10^{-3} \left[\frac{\nabla k}{k \text{ MW}} \right] & 23) \end{array} \right. \\
 \text{EBR-I, Mark III} & \text{No measurable bowing} \\
 \text{EBR-II} & 7.4 \times 10^{-5} \left[\frac{\nabla k}{k \text{ MW}} \right] \quad 13)
 \end{array}$$

II.4 Reactivity Effects Due to Expansion of Reactor Materials

In order to determine the reactivity effects $\frac{\nabla k}{k} / \frac{\nabla \rho}{\rho}$ in fast reactors one uses multigroup diffusion codes where the uranium, sodium, stainless steel and zirconium is changed in its density $\frac{\nabla \rho}{\rho}$ in individual regions of the reactor. One starts out with a reference k for which the compositions of individual regions are known. In order to get detectable reactivity changes one has to exaggerate the density changes. Very sensitive regions have to be split up into more regions than unimportant ones with respect to reactivity. From the engineers we obtain the volume change in individual regions due to a temperature change of 1 degree, $\frac{\nabla V}{V} / ^\circ\text{C}$, which is the same as $\frac{\nabla \rho}{\rho} / ^\circ\text{C}$.

From the flux curve as a function of space we can obtain the heat generation per volume element in the active fission part of the core. Also there is some appreciable γ -heating. Once the heat generation is known a temperature distribution can be calculated which will, of course, depend on coolant flow and operating conditions. For each region in the core according to its importance one can define an average temperature. By multiplying $\frac{\nabla \rho}{\rho} / ^\circ\text{C}$ by the temperature $T_{\text{av, region}}$, one obtains the density change $\frac{\nabla \rho}{\rho}$ for this region for a particular operating condition. This $\frac{\nabla \rho}{\rho}$ is multiplied by $\frac{\nabla k}{k} / \frac{\nabla \rho}{\rho}$ obtained from the multigroup calculation and we finally get the reactivity change $\frac{\nabla k}{k}$ for this region and operating condition. By adding up individual reactivity effects $\frac{\nabla k}{k}$, the total reactivity change in the whole reactor due to temperature effects at a power level P in megawatts is obtained.

The reactivity coefficient $\frac{\nabla k}{k \text{ MW}}$ is called power coefficient.

Typical power coefficients are

EBR-I, Mark II ²³⁾	$-3 \times 10^{-3} \left[\frac{\nabla k}{k \text{ MW}} \right]$
EBR-I, Mark III ¹²⁾	$-7.9 \times 10^{-4} \left[\frac{\nabla k}{k \text{ MW}} \right]$
EBR-II ¹³⁾	$-6 \times 10^{-5} \left[\frac{\nabla k}{k \text{ MW}} \right]$

Another commonly used reactivity coefficient is the isothermal temperature coefficient.

The isothermal temperature coefficient expressed in $\frac{\nabla k}{k} / ^\circ\text{C}$ will give us an indication of how much reactivity is gained or lost in the whole reactor or part thereof due to a temperature rise of 1 degree. The isothermal temperature coefficient is calculated by assuming that the entire reactor, including the structure, heats up uniformly. Expansions and changes of density are computed and translated into reactivity effects as mentioned before.

Typical isothermal temperature coefficients are:

EBR-I, Mark II ²⁴⁾	$-3.5 \times 10^{-5} \left[\frac{\nabla k}{k} / ^\circ\text{C} \right]$
EBR-I, Mark III ¹²⁾	$-3.1 \times 10^{-5} \left[\frac{\nabla k}{k} / ^\circ\text{C} \right]$
EBR-II ¹³⁾	$-3.5 \times 10^{-5} \left[\frac{\nabla k}{k} / ^\circ\text{C} \right]$

II.5 Time Constant Associated with Fuel Pin

The reactivity effects for expansion of the fuel pins and other core materials, the reactivity coefficient for bowing, and the Doppler effect do not become effective immediately. They are closely related with the time constant of the fuel pin. In order to determine this time constant one has to obtain the temperature in the fuel pin as a function of radius and time, and also the temperature of the coolant as a function of time and space. Obviously these time constants will depend on the particular engineering design of the pins. In this paper the temperature equations are described for a cylindrical fuel pin with zirconium cladding and axial NaK cooling.

The equations for fuel and zirconium cladding are homogenized into one equation. All equations are written for one fuel pin according to Fig. II.5, 1. The power production $q(t)$ and the inlet temperature $T_{s1}(t)$ are functions of time. In the radial direction perfect mixing of the sodium is assumed. The average coolant temperature in a region is the average between inlet and outlet temperature assuming a linear rise in the axial direction.

Set of equations to be solved.

$$\nabla^2 T_1(r, t) + \frac{q(t)}{K_1} = \frac{c_1 s_1}{K_1} \frac{dT_1(r, t)}{dt} \quad \text{II. 5-1}$$

$$-\sigma K_1 \frac{dT_1(r, t)}{dr} \bigg|_{r=c} + 2Wc_2 T_{s1}(t) = 2Wc_2 T_{s2}(t) + Vg_2 c_2 \frac{dT_2(t)}{dt} \quad \text{II. 5-2}$$

Boundary Conditions

$$\frac{dT_1(r, t)}{dr} \bigg|_{r=0} = 0 \quad \text{II. 5-3}$$

$$-K_1 \frac{dT_1(r, t)}{dr} \bigg|_{r=c} = h \left[T_1(c, t) - T_2(t) \right] \quad \text{II. 5-4}$$

where:

$T_1(r, t)$	= Temperature in fuel pin in $[^{\circ}\text{C}]$
$q(t)$	= Heat generation per second per cm^3 of pin volume, $\left[\frac{\text{cal}}{\text{cm}^3 \text{sec}} \right]$
$Q(t)$	= Heat generation per second in pin $\left[\frac{\text{cal}}{\text{sec}} \right]$
c_1	= Specific heat of homogenized fuel-zirconium region, $\left[\frac{\text{cal}}{\text{gm}^{\circ}\text{C}} \right]$
ρ_1	= Density of homogenized fuel-zirconium region $\left[\frac{\text{gm}}{\text{cm}^3} \right]$
K_1	= Conductivity of homogenized fuel-zirconium region, $\left[\frac{\text{cal}}{\text{cm sec}^{\circ}\text{C}} \right]$
c	= Radius of fuel pin, $[\text{cm}]$
σ	= Surface of fuel pin, $[\text{cm}^2]$
W	= $w \times$ flow area of NaK, $\left[\frac{\text{gm}}{\text{sec}} \right]$
w	= Mass flow of NaK, $\left[\frac{\text{gm}}{\text{cm}^2 \text{sec}} \right]$
h	= Heat transfer rate, $\left[\frac{\text{cal}}{\text{cm}^2 \text{sec}^{\circ}\text{C}} \right]$
c_2	= Specific heat of NaK, $\left[\frac{\text{cal}}{\text{gm}^{\circ}\text{C}} \right]$
V	= Volume of NaK around one pin for region, $[\text{cm}^3]$
ρ_2	= Density of homogenized fuel-zirconium region, $\left[\frac{\text{gm}}{\text{cm}^3} \right]$
$T_{s1}(t)$	= Coolant inlet temperature, $[^{\circ}\text{C}]$
$T_{s2}(t)$	= Coolant outlet temperature, $[^{\circ}\text{C}]$
$T_2(t)$	= Average coolant temperature for linear temperature rise, $[^{\circ}\text{C}]$

Equations II. 5-1 through II. 5-4 represent a set of linear equations. Therefore, we can assume superposition and the set can be solved by Laplace transform. Since superposition is valid we can assume that the reactor has been operating at steady state, full power for example, when a disturbance in the power $q(t)$ or in the inlet temperature $T_{s1}(t)$ was applied and we calculate the increment above this steady level. All initial conditions are set to be zero. Since in II. 5-1 through II. 5-4 the highest order of derivatives is one, we need only the values at $t = 0^-$ and no derivatives at $t = 0^-$. Equations II. 5-1 through II. 5-4 could easily be extended for the core with an initial condition, but for the present purpose it did not seem desirable, since the main purpose of this investigation is the evaluation of the transfer function of reactivity versus temperature.

Equations II. 5-1 through II. 5-4 are solved by means of Laplace transform.

L = Laplace transform operator.

Substitutions:

$$\begin{array}{ll}
 2Wc_2 &= \alpha & L [T_1(r, t)] &= \theta_1(r, s) \\
 V\rho_2c_2 &= \beta & L [T_{s1}(r, t)] &= \theta_{s1}(r, s) \\
 c_1\rho_1 &= \delta & L [T_{s2}(r, t)] &= \theta_{s2}(r, s) \\
 x^2 &= \frac{K_1}{c_1\rho_1} & L [T_2(t)] &= \theta_2(s)
 \end{array}$$

HEAT BALANCE EQUATION IN NaK

$$\underbrace{-\sigma K_1 \left. \frac{dT_1(r, t)}{dr} \right|_{r=C}}_{\text{HEAT ACROSS BOUNDARY}} + \underbrace{WC_2 T_{S1}(t)}_{\text{HEAT CARRIED INTO SEGMENT BY INLET COOLANT}} = \underbrace{WC_2 T_{S2}(t)}_{\text{HEAT CARRIED OUT BY OUTLET COOLANT}} + \underbrace{V\rho C_2 \frac{dT_2(t)}{dt}}_{\text{CHANGE OF AVERAGE COOLANT TEMPERATURE}}$$

Fig. II. 5, 1.

Laplace transform of set of equations.

$$\theta_1(r, s) = A(s) I_0 \left(\frac{\sqrt{s}}{x} r \right) + B(s) K_0 \left(\frac{\sqrt{s}}{x} r \right) + \frac{q(s)}{\delta s} \quad \text{II. 5-5}$$

$$-\sigma K_1 \frac{d\theta_1(r, s)}{dr} \bigg|_{r=c} + \alpha \theta_{s1}(s) = \alpha \theta_2(s) + \beta s \theta_2(s) \quad \text{II. 5-6}$$

$$-\frac{d\theta_1(r, s)}{dr} \bigg|_{r=0} = 0 \quad \text{II. 5-7}$$

$$-K_1 \frac{d\theta_1(r, s)}{dr} \bigg|_{r=c} = h \left[\theta_1(c, s) - \theta_2(s) \right] \quad \text{II. 5-8}$$

$$\text{From II. 5-7 it follows } B(s) = 0 \quad \text{II. 5-9}$$

$$\begin{aligned} -\sigma K_1 A(s) \frac{\sqrt{s}}{x} I_1 \left(\frac{\sqrt{s}}{x} c \right) + \alpha \theta_{s1}(s) &= \alpha \theta_2(s) + \beta s \theta_2(s) \\ \therefore A(s) &= \frac{-\theta_2(s) (\alpha + \beta s) + \alpha \theta_{s1}(s)}{\sigma K_1 \frac{\sqrt{s}}{x} I_1 \left(\frac{\sqrt{s}}{x} c \right)} \end{aligned} \quad \text{II. 5-10}$$

Substitute II. 5-9 and II. 5-10 into II. 5-5 and solve for $\theta_2(s)$

$$\begin{aligned} \theta_2(s) &= \frac{\theta_{s1}(s) \left[K_1 \frac{\sqrt{s}}{x} \alpha I_1 \left(\frac{\sqrt{s}}{x} c \right) + h \alpha I_0 \left(\frac{\sqrt{s}}{x} c \right) \right] + \frac{q(s)}{s \delta} h K_1 \sigma \frac{\sqrt{s}}{x} I_1 \left(\frac{\sqrt{s}}{x} c \right)}{\frac{\sqrt{s}}{x} I_1 \left(\frac{\sqrt{s}}{x} c \right) \left[(\alpha + \beta s) K_1 + h \sigma K_1 \right] + I_0 \left(\frac{\sqrt{s}}{x} c \right) h (\alpha + \beta s)} \\ &\quad \text{II. 5-11} \end{aligned}$$

Using the final value theorem we obtain for the steady state

$$s \theta_2(s) \bigg|_{s=0} = T_2(\omega) = T_{s1}(\omega) + \frac{qc \sigma}{2 \alpha} \quad \text{II. 5-12}$$

$$\frac{qc \sigma}{2 \alpha} = \frac{Q}{2 W c_2} \quad \text{II. 5-13}$$

where Q = Total heat generated in a pin

$$T_2(\omega) = T_{s1}(\omega) + \frac{Q}{2 W c_2} \quad \text{II. 5-14}$$

In II. 5-14 it has been assumed that both $q(t)$ and $T_{s1}(t)$ are applied as steps at $t = 0^+$. This result can readily be obtained by solving II. 5-1 through II. 5-4 in the time domain for the steady state.

The time solution to II. 5-11 can be obtained by means of the inverse Laplace theorem.

We define:

$$D(s) = \frac{\sqrt{s}}{x} I_1 \left(\frac{\sqrt{s}}{x} c \right) \left[(\alpha + \beta s) K_1 + h \sigma K_1 \right] + I_0 \left(\frac{\sqrt{s}}{x} c \right) h(\alpha + \beta s) \quad \text{II. 5-15}$$

$$D(s) = (s-s_1)(s-s_2) \dots (s-s_n) \quad \text{II. 5-16}$$

$$Z(s) = \theta_{s1}(s) \left[K_1 \alpha \frac{\sqrt{s}}{x} I_1 \left(\frac{\sqrt{s}}{x} c \right) + h \alpha I_0 \left(\frac{\sqrt{s}}{x} c \right) \right] + \frac{q(s)}{s \delta} h K_1 \sigma \frac{\sqrt{s}}{x} I_1 \left(\frac{\sqrt{s}}{x} c \right) \quad \text{II. 5-17}$$

Using II. 5-16 and II. 5-17 we get for the time solution of II. 5-11 the coolant temperature.

$$T_2(t) = \frac{Z(s)(s-s_1)}{D(s)} \bigg|_{s=s_1} e^{-s_1 t} + \frac{Z(s)(s-s_2)}{D(s)} \bigg|_{s=s_2} e^{-s_2 t} + \dots + \frac{Z(s)(s-s_n)}{D(s)} \bigg|_{s=s_n} e^{-s_n t} \quad \text{II. 5-18}$$

In a similar manner we obtain for

$$\begin{aligned} \theta_1(s, r) = & \frac{I_0 \left(\frac{\sqrt{s}}{x} r \right) \left[\theta_{s1}(s) h \alpha - \frac{q(s)}{s \delta} h(\alpha + \beta s) \right] + I_0 \left(\frac{\sqrt{s}}{x} c \right) h(\alpha + \beta s)}{\frac{\sqrt{s}}{x} I_1 \left(\frac{\sqrt{s}}{x} c \right) \left[(\alpha + \beta s) K_1 + h \sigma K_1 \right] + I_0 \left(\frac{\sqrt{s}}{x} c \right) h(\alpha + \beta s)} \\ & + \frac{\frac{q(s)}{s \delta} \left[\frac{\sqrt{s}}{x} I_1 \left(\frac{\sqrt{s}}{x} c \right) \left[(\alpha + \beta s) K_1 + h \sigma K_1 \right] \right]}{\frac{\sqrt{s}}{x} I_1 \left(\frac{\sqrt{s}}{x} c \right) \left[(\alpha + \beta s) K_1 + h \sigma K_1 \right] + I_0 \left(\frac{\sqrt{s}}{x} c \right) h(\alpha + \beta s)} \quad \text{II. 5-19} \end{aligned}$$

In an analog manner as under II. 5-14 we can show by using the final value theorem

$$T_1(r, \infty) = \frac{q}{4K_1} \left[c^2 + \frac{2c}{h} - r^2 \right] + T_2 \quad \text{II. 5-20}$$

In II. 5-20 it has been assumed that both $q(t)$ and $T_{s1}(t)$ are applied as steps at $t = 0^+$. The time solution of II. 5-19 is found in an analog manner as II. 5-18.

Substitution: $\theta_1(s, r) = \frac{G(s, r)}{D(s)}$ where $G(s, r)$ Numerator of II. 5-19.

$$T_1(t, r) = \frac{G(s, r)(s-s_1)}{D(s)} \Big|_{s=s_1} e^{-s_1 t} + \frac{G(s, r)(s-s_2)}{D(s)} \Big|_{s=s_2} e^{-s_2 t} + \dots + \frac{G(s, r)(s-s_n)}{D(s)} \Big|_{s=s_n} e^{-s_n t} \quad \text{II. 5-21}$$

Note that the argument of the Bessel function $I_0\left(\frac{\sqrt{s}}{x} c\right)$ and $\frac{\sqrt{s}}{x} I_1\left(\frac{\sqrt{s}}{x} c\right)$ contains a term with a square root which would lead to a multivalued function and in order to evaluate the inverse Laplace transform we would have to integrate around the branch cut. If we observe the series expansion of $I_0\left(\frac{\sqrt{s}}{x} c\right)$ and $\frac{\sqrt{s}}{x} I_1\left(\frac{\sqrt{s}}{x} c\right)$ we see that the resulting functions are analytic everywhere. Therefore, we can evaluate the time function II. 5-18 and II. 5-21 with the residue theorem.

Using II. 5-11 and II. 5-19 we get for the total reactivity change at power level P due to expansion of the homogenized fuel pin and the coolant

$$PK(s)_{\text{expansion}} = \theta_1(s, r) \propto_{elhc} \propto_{rlhc} + \theta_2(s) \propto_{e2hc} \propto_{r2hc} \quad \text{II. 5-22}$$

where: P = power level in megawatt

$K(s)$ = power coefficient in the frequency domain $\left[\frac{\nabla k(s)}{k \text{ MW}} \right]$

$\propto_{elhc}$ = expansion coefficient of uranium in homogenized core $\left[\frac{\nabla \rho}{\rho} / ^\circ\text{C} \right]$

$\propto_{e2hc}$ = expansion coefficient of NaK in homogenized core $\left[\frac{\nabla \rho}{\rho} / ^\circ\text{C} \right]$

$\propto_{rlhc}$ = reactivity coefficient of uranium in homogenized core $\left[\frac{\nabla k}{k} / \frac{\nabla \rho}{\rho} \right]$

$\propto_{r2hc}$ = reactivity coefficient of NaK in homogenized core $\left[\frac{\nabla k}{k} / \frac{\nabla \rho}{\rho} \right]$

Note that $PK(s)_{\text{expansion}}$ is a function of the power level and also of the coolant flow velocity besides the heat transfer parameters and material constants for the homogenized fuel pin and the coolant.

Since we want to obtain information on the time constant we have to examine the denominator of II. 5-11 and II. 5-19 which is, of course, the same.

$$\frac{\sqrt{s}}{x} I_1\left(\frac{\sqrt{s}}{x} c\right) \left[(\alpha + \beta s) K_1 + h \sigma K_1 \right] + I_0\left(\frac{\sqrt{s}}{x} c\right) h(\alpha + \beta s) \quad \text{II. 5-23}$$

Equation II. 5-23 can be written in terms of their roots.

$$(s-s_1)(s-s_2) \dots (s-s_n) \quad \text{II. 5-24}$$

or

$$s_1 s_2 \dots s_n \left[\frac{s}{s_1} - 1 \right] \left[\frac{s}{s_2} - 1 \right] \dots \left[\frac{s}{s_n} - 1 \right] \quad \text{II. 5-25}$$

where:

$$1/s_1 = T_1$$

$$1/s_2 = T_2 \quad \text{II. 5-26}$$

$$1/s_n = T_n$$

$T_1, T_2 \dots T_n$ are the time constants of the homogenized fuel pin, sodium system.

In order to evaluate the roots of II. 5-23 one has to use a series expansion approximation for the Bessel functions I_0 and I_1 . In most practical cases four terms for I_0 and three terms for I_1 will be sufficient for the accuracy desired.

By observing the roots and time constants in a practical case one will see that there will be a dominant long time constant in comparison with much shorter ones.

For this paper a time constant of $T_f = .5$ seconds for the fuel pin is assumed.

II. 6 Structural Time Constants

Structural time constants are even more dependent on the engineering design of a reactor than the fuel pin time constants. Also it is very important to determine which part of the structure will have an effect on reactivity. For this investigation a structural time constant of $T_s = 5$ sec is assumed.

II. 7 Total Reactivity Effects Due to Temperature

The effects described in II. 1 through II. 6 have to be added up to obtain $PK(s)$.

We assume

$$PK(s) = P 10^{-3} \left[\frac{2}{1 + \frac{s}{2.0}} - \frac{3}{(1 + \frac{s}{2.0})(1 + \frac{s}{0.2})} \right] \quad \text{II. 7-1}$$

where: P = power level; $P = 1$, full power

2×10^{-3} = power coefficient for bowing

$1/2.0$ = time constant for fuel pin T_f

-3×10^{-3} = power coefficient for structural expansion

$1/.2$ = time constant for structural expansion T_s

II.8 Load Power Transfer Function $LP(s)$

In a power reactor the power influences the reactivity and the reactivity in turn determines the power so that these two variables form a feedback loop. The net incremental reactivity will be

$$\delta k(t) = K_{ex}(t) + FB(t) \quad \text{II.8-1}$$

where $K_{ex}(t)$ is the excess reactivity insertion into the cold reactor and $FB(t)$ is an expression for the influence of temperatures due to power on reactivity.

If the linear assumption is valid, equation II.8-1 can be written in terms of the Laplace transform.

$$\delta k(s) = K_{ex}(s) + n_0 PK(s) \frac{\delta n(s)}{n_0} \quad \text{II.8-2}$$

where:	$\delta k(s)$	Net incremental reactivity
	$\frac{\delta n(s)}{n_0}$	Relative incremental flux
	$K_{ex}(s)$	Reactivity insertion in cold reactor
	n_0	Steady state flux
	P	Power level in MW
	$K(s)$	Power coefficient $\left[\frac{\nabla k(s)}{k \text{ MW}} \right]$

Note that actually one would have to use other symbols for the variables in the time domain t and image domain s but this is not done here for simplicity.

The zero power kinetic can be combined with the expression for the incremental reactivity equation II.8-2 and shown symbolically in a feedback block diagram representation

Fig. II.8, 1

From Fig. II.8, 1 we readily obtain the load power transfer function $LP(s)$. Since we are interested in relative changes with respect to the steady state flux n_0 we are free to set $n_0 = 1$.

$$LP(s) = \frac{ZP(s)}{1 - ZP(s) PK(s)} \quad \text{II. 8-3}$$

or

$$\frac{1}{LP(s)} = \frac{1}{ZP(s)} - PK(s) \quad \text{II. 8-4}$$

II.9 The Stability of a Fast Reactor as a Function of Power Level

Oscillator tests on both EBR-I, Mark-II and Mark-III, show that a peaking occurs in the load power transfer function $LP(s)$. In EBR-I, Mark-II a distinct resonance was observed at low flow conditions²²). Recent measurements on the Mark-III core show that a small "bump" can be observed over the zero power transfer function $ZP(s)$. These phenomena will limit the power level at which the reactor can be operated. The resonance power, where the reactor will start oscillating on its own with increasing amplitude has to be sufficiently far from normal operating conditions. One of the major advantages of the oscillator test is that information can be obtained at low power for high power operation without subjecting the reactor to its associated hazards of high power operation.

Here it is assumed that $PK(s)$ in the feedback model will not change over the whole range of power, P . That is the parameter in the dynamic expression for the power coefficient $K(s)$ are constants. As oscillator tests in EBWR show, for example, this will not be the case over too large a range of power since time constants and material constants are not the same from zero to full power. Nevertheless, these variations should not be large and do not change the major conclusions of the influence of power level on the dynamics. The dynamics as affected by the power level is explained by the shifting of the roots in the characteristic equation for $LP(s)$ by means of the root-locus²⁵⁾ method known in the theory of functions.

To apply the root-locus method, equation II.8-3 has to be rewritten,

$$LP(s) = \frac{1}{PK(s)} \frac{ZP(s) PK(s)}{1 - ZP(s) PK(s)} \quad \text{II. 9-1}$$

We observe that the angle condition for the exploratory point for the locus of poles of $LP(s)$ is zero degrees. This is because the feedback in the block diagram notation is positive if we want to keep the physical true meaning of a power coefficient which is negative in steady state, at least it should be if the reactor is to be stable. The positive feedback represents itself in the minus sign in the denominator of equation II.9-1. The root-locus calls for $ZP(s)$ and $PK(s)$ given as polynomials in s and their roots have to be known.

For this reason, the approximation I.3-1 for $ZP(s)$ is used.

The root-locus postulates that $LP(s)$ is described within a constant if the poles and zeros of $ZP(s) PK(s)$ are known.

$$ZP(s) PK(s) = \frac{A(s-s_1)(s-s_3)\dots(s-s_{2n+1})}{(s-s_2)(s-s_4)\dots(s-s_{2n})} \quad \text{II. 9-2}$$

For $PK(s)$ II. 7-1 is rewritten as follows:

$$PK(s) = \frac{P \times 10^{-3} (10s - 1)}{(5s + 1)(0.5s + 1)} \quad \text{II. 9-3}$$

Note that $PK(s)$ is negative for steady state. Physically the power coefficient is only positive for fast transients but is negative if one waits long enough. The product of $ZP(s)$ then becomes:

$$ZP(s) PK(s) = \frac{P \times 10.8 \times 10^{-3} (40s + 1) (2.76s + 1)(10s - 1)}{s(9.09s + 1)(5s + 1)(0.5s + 1)} \quad \text{II. 9-4}$$

From equation II. 9-4 we observe that the zeros are at

$$s = -\frac{1}{40}; -\frac{1}{2.76}; +\frac{1}{10}$$

and the poles are at

$$s = 0; -\frac{1}{5}; -\frac{1}{0.5}; -\frac{1}{9.09}$$

Figure II. 9, 1 shows the poles and zeros of $ZP(s) PK(s)$. The poles of $LP(s)$ start at poles of $ZP(s) PK(s)$ and end at the zeros of $ZP(s) PK(s)$. The sum of phase angles to the exploratory point must be zero degrees. The root-locus is shown in Figure II. 9, 1. The advantage of the root-locus lies in the fact that the complete range of poles of $LP(s)$ as the power level, P , is increased is completely determined. From Figure II. 9, 1 we see that the root S_1 for example is bound into the limits $-2.0 \leq S_1 \leq -.326$.

Since the polynomial which describes the poles of $LP(s)$, that is the zeros of $1 - ZP(s) PK(s)$ is of 4th order we observe 4 possible roots. As the power level is increased the character of these roots will vary. At low power level all the roots are negative real, as the power level is increased two of the roots remain negative real and the other two become negative conjugate complex, at $P = 0.2$ full power. At still higher power levels the real part of the complex roots becomes zero at $P = 3.2$ full power and if the power is further increased the conjugate complex roots will have positive real parts and the reactor is unstable.

Once the root-locus of

$$\frac{ZP(s) PK(s)}{1 - ZP(s) PK(s)} \quad \text{II. 9-10}$$

is determined, one simply has to add the poles and zeros of

$$\frac{1}{PK(s)} \quad \text{II. 9-11}$$

to the pole-zero configuration of equation II. 9-10. Note that after the addition of equation II. 9-11, the pole and zero at $s = +1/10$ cancel each other. The poles and zeros at $s = -2.0$ and -0.2 cancel each other only at zero power.

$$ZP(s) = \frac{10.8(40s + 1)(2.76s + 1)}{s(9.09s + 1)}$$

$$PK(s) = \frac{P \times 10^{-3}(10s - 1)}{(5s + 1)(.5s + 1)}$$

$$LP(s) = \frac{1}{PK(s)} \frac{ZP(s)PK(s)}{1 - ZP(s)PK(s)}$$

X - POLES
 O - ZEROS
 } $ZP(s)PK(s)$
 X - POLES
 O - ZEROS
 } $\frac{1}{PK(s)}$

Fig. II. 9, 1.

ROOT LOCUS FOR FAST REACTOR

Fig. II. 9, 2.

ROOT LOCUS FOR 4 ROOTS: S_I , S_{II} , S_{III} , S_{IV}

Table II.9.I shows the numerical values²⁷⁾ of the roots S_I through S_{IV} and Figure II.9, 2 shows a detailed graph of Figure II.9, 1.

Table II.9.I

Variations of the Poles of $LP(s)$ as a Function of Power Level P ; $P = 1$, Full Power, S_I ; S_{II} ; S_{III} ; S_{IV} Roots of Polynomial $1 - ZP(s) PK(s)$

Power Level	S_I	S_{II}	S_{III}	S_{IV}
0.1	- 1.9476	- 0.0010	- 0.1262	- 0.1825
0.2	- 1.8953	- 0.0020	- 0.1538	$\pm 0.0231j$
0.5	- 1.7388	- 0.0048	- 0.1519	$\pm 0.0716j$
1.0	- 1.4805	- 0.0086	- 0.1481	$\pm 0.1246j$
1.5	- 1.2296	- 0.0113	- 0.1410	$\pm 0.1767j$
2.0	- 0.9929	- 0.0134	- 0.1271	$\pm 0.2348j$
2.5	- 0.7934	- 0.0150	- 0.0949	$\pm 0.3012j$
3.0	- 0.6529	- 0.0162	- 0.0334	$\pm 0.3652j$
3.5	- 0.5682	- 0.0172	+ 0.0559	$\pm 0.4086j$
3.8	- 0.5393	- 0.0176	+ 0.1203	$\pm 0.4181j$
4.0	- 0.5181	- 0.0179	+ 0.1624	$\pm 0.4213j$

Frequency Response $LP(s)$ at Different Power Levels

Figure II.9, 3 shows the load power frequency response as computed by BUM²⁶⁾ on the 650 where $1/ZP(s)$ is computed from the 6 group delayed neutron data by Hughes. $PK(s)$ has to be given in polynomial form. BUM only performs complex algebra. That is, it computes a frequency response even if the reactor is unstable. One has to search for a resonance peak with this code. From Figure II.9, 3 we notice that the frequency at which the peak appears is not the same for all power levels.

Figures II.9, 4 and II.9, 5 show an oscillator test on a simulated fast reactor on analog computers. From Figures II.9, 4 and II.9, 5 a frequency response as shown in Figure II.9, 3 can be obtained by repeating the same for different frequencies. Similar conclusions as under the root-locus discussion can be drawn by examining Figures II.9, 4 and II.9, 5; that is, the incremental flux with respect to the power level we operate at, $\delta n/K_{no}$ is increasing with increasing power level. Also, a positive phase shift of the resonance occurs with respect to $K_{ex} \sin \omega t$ as the power level P is increased. That is, the stability decreases with increasing power level. Instability occurs at $\omega = .46$ rad/sec and at a power level of $P = 3.02$ full power. The difference in the resonance power as compared to the root-locus, discussion $P = 3.2$ arises because of the approximation which has been used for $ZP(s)$. Here the exact 6 group $ZP(s)$ is used.

Fig. II. 9, 3.

LOAD POWER TRANSFERFUNCTION $LP(s)$ AS A
FUNCTION OF POWER LEVEL P FOR FAST REACTOR

Fig. II. 9, 4.

FREQUENCY RESPONSE AT DIFFERENT POWER LEVELS P , FOR
 CONSTANT SINUSOIDAL AMPLITUDE OF REACTIVITY INSERTION
 FOR FAST REACTOR $\ell = 4 \times 10^{-8}$; HUGHES DELAYED NEUTRON DATA,
 $K_{ex} = .05 \text{ DOLLARS}$; $\omega = .4 \text{ (rad/sec)}$

Fig. II. 9, 5.

FREQUENCY RESPONSE AT DIFFERENT POWER LEVELS P , FOR
 CONSTANT SINUSOIDAL AMPLITUDE OF REACTIVITY INSERTION
 FOR FAST REACTOR $\lambda = 4 \times 10^{-8}$; HUGHES DELAYED NEUTRON DATA,
 $K_{ex} = .05 \text{ DOLLARS}; \omega = .4 \text{ (rad/sec)}$

Stability as a Function of Power Level by Means of Nyquist Diagram

In conventional feedback theory the Nyquist diagram is extensively used. That is, we examine

$$1 + ZP(s) [-PK(s)] = 0 \quad \text{II.9-12}$$

by plotting $ZP(s) [-PK(s)]$ as a function of s in the polar plane. If the -1 point lies to the left of the locus for increasing s , that is, the -1 point is not encircled as s goes from $-j\infty$ to $+j\infty$, the system is said to be stable. All modes have therefore negative real parts as shown in the root-locus.

If the locus passes through the -1 point, resonance power is reached, and if the -1 point is encircled, the reactor is unstable. $PK(s)$ is taken negatively, that is, $-PK(s)$ in order to discuss stability with respect to the -1 point in terms of conventional negative feedbacks. The above statements with respect to the -1 point do not hold true for every complex function $ZP(s) [-PK(s)]$ and care must be taken in its interpretation. Figure II.9, 6 shows a Nyquist plot for different power levels. Figure II.9, 7 shows the entire locus for $ZP(s) [-PK(s)]$ as the s -plane is encircled in the right half plane.

As $s \rightarrow 0$, $ZP(s) [-PK(s)]$ behaves like $1/s$; that is, for $s \rightarrow j\epsilon$, $f(s)$ goes to infinity at -90 degrees. For $s \rightarrow -j\omega$ at A_4 , $f(s)$ goes to infinity at +90 degrees. For large s , that is, $s \rightarrow +j\infty$ $f(s)$ behaves like $-1/s^2$ or $f(s) \rightarrow$ zero at 0 degrees at A_2 . If $s \rightarrow -j\infty$, $f(s)$ behaves like $-1/s^2$ also, and $f(s) \rightarrow$ zero at 0 degrees at A_3 . If the power level is low enough, the -1 point is not encircled. The present model represents a multiloop feedback, that is, a positive and a negative reactivity in parallel. The number of encirclement of the -1 point in the $f(s)$ plane is therefore equal to the poles of $1 + ZP(s) [-PK(s)]$ in the right half plane minus the zeros of $1 + ZP(s) [-PK(s)]$ in the right half plane. If the -1 point in Figure II.9, 7 is encircled -2 times for power levels >3.02 the number of zeros of $1 + ZP(s) [-PK(s)]$ in the RHP is 2. The reactor is unstable since there are 2 zeros of $1 + ZP(s) [-PK(s)]$ in the RHP, or $LP(s)$ has 2 poles in the right half plane.

$LP(s)$ in Vector Form

Equation II.9-13 is a vector equation

$$\frac{\overrightarrow{1}}{LP(s)} = \frac{\overrightarrow{1}}{ZP(s)} - \overrightarrow{PK(s)} \quad \text{II.9-13}$$

A plot of $\overrightarrow{ZP(s)}$ is shown in Figure II.9, 8 and $\overrightarrow{PK(s)}$ in Figure II.9, 9, also $\overrightarrow{1/PK(s)}$ in Figure II.9, 10.

Equation II.9-13 is the reason why the reciprocal value of $\overrightarrow{ZP(s)}$ is plotted often in the literature. If the model is known both vectors $\overrightarrow{1/ZP(s)}$ and $\overrightarrow{PK(s)}$ are known and $\overrightarrow{1/LP(s)}$ can be obtained by vector addition. $\overrightarrow{1/ZP(s)}$ is independent of power and $\overrightarrow{PK(s)}$ is dependent on power. A resonance occurs if

$$\overrightarrow{ZP(s)} = \frac{\overrightarrow{1}}{PK(s)} \quad \text{II.9-14}$$

From Figures II.9, 8 and II.9, 10 we observe that at $P = 3.02$, $\vec{ZP}(s)$ and $\vec{1}/PK(s)$ have the same magnitude of 179.6 and an angle of -23.8 degrees at a resonance frequency ω_r of 0.46 [rad/sec]. In practical cases $\vec{1}/ZP(s)$ is theoretically known and $\frac{\vec{1}}{LP(s)}$ is measured from which $\vec{PK}(s)$ can be obtained, or

$$\vec{PK}(s) = \frac{\vec{1}}{ZP(s)} - \frac{\vec{1}}{LP(s)} \quad \text{II.9-15}$$

The question then arises how to synthesize $\vec{PK}(s)$ in physical terms.

Transient Response

Since we already obtained the roots of $LP(s)$ for different power levels, we can obtain the transient response, that is, the response in time to a step change of reactivity for example by obtaining a partial fraction expansion for $LP(s)$. In terms of the 4 roots, we get the following form

$$LP(t) = \frac{\delta n(t)}{n_0} = A_0 + A_1 e^{S_I t} + A_2 e^{S_{II} t} + A_3 e^{S_{III} t} + A_4 e^{S_{IV} t} \quad \text{II.9-15}$$

where the roots S_I , S_{II} , S_{III} , S_{IV} are functions of the power level as previously discussed. A_1 , A_2 , A_3 , A_4 are the residues of the poles S_I , S_{II} , S_{III} , S_{IV} ; they are directly proportional to K_{ex} but are also dependent on the roots. As an example, the coefficients A_1 , A_2 , A_3 , A_4 , have been computed for different power levels and are shown in Table II.9.II.

From equation II.9-15, it is obvious that for a stable reactor all the roots S_I , S_{II} , S_{III} , S_{IV} must have negative real parts. For higher power level, roots S_{III} and S_{IV} are conjugate complex and can be combined in an exponentially rising or decreasing sinusoidal term. Note that in Table II.9.II at $P = 4$, the third term is increasing exponentially with time. The reactor is therefore unstable.

Fig. II. 9, 8.

ZERO POWER TRANSFERFUNCTION $ZP(s)$, FAST REACTOR
 $\ell = 4 \times 10^{-8}$; HUGHES DATA; $\omega = .1$ TO $.7$ (rad/sec)

$$PK(s) = P \times 10^{-3} \left[\frac{2}{1 + \frac{s}{2.0}} - \frac{3}{\left(1 + \frac{s}{2.0}\right) \left(1 + \frac{s}{.2}\right)} \right]; P = \text{POWER LEVEL}$$

Fig. II. 9, 9.
 POWERCOEFFICIENT TRANSFERFUNCTION $PK(s)$
 FOR DIFFERENT POWER LEVELS P
 FAST REACTOR; $\omega = .1$ TO $.7$ (rad/sec)

$$PK(s) = P \times 10^{-3} \left[\frac{2}{1 + \frac{s}{2.0}} - \frac{3}{\left(1 + \frac{s}{2.0}\right) \left(1 + \frac{s}{.2}\right)} \right]; P = \text{POWER LEVEL}$$

Fig. II. 9, 10.

RECIPROCAL POWERCOEFFICIENT TRANSFERFUNCTION
 $1/PK(s)$ FOR DIFFERENT POWER LEVELS P
 FAST REACTOR; $\omega = .1$ TO $.7$ (rad/sec)

Table II.9.II

Transient Response at Various Power Levels as a Function of Time. Power level, P, in terms of full power, K_{ex} , in Dollars

Power Level	Response
0, 1	$K_{ex} [1816.83 - 1756.35e^{-.0010t} - 46.43e^{-.1263t} + 6.14e^{-.1825t} - .51e^{-1.9476t}]$
2	$K_{ex} [86 - 37.08e^{-.0134t} + 60.76e^{-.1271t} \sin(.2348t + .1922) - 14.01e^{-.9929t}]$
3	$K_{ex} [58.44 - 18.14e^{-.0162t} + 98.03e^{-.0333t} \sin(.3652t + .0499) - 11.65e^{-.6530t}]$
4	$K_{ex} [43.62 - 11.03e^{-.0179t} + 122.82e^{+.1624t} \sin(.4213t + .0430) - 4.91e^{-.5182t}]$

Chapter III

NONLINEAR EFFECTS IN THE KINETICS OF FAST REACTORS

III.1 Derivation of the Exact Incremental Kinetic Equations

Since the main interest is the stability of a fast power reactor as effected by non linearities a set of kinetic equations is derived by assuming that the reactivity insertion and the incremental flux can be expanded into a Fourier series. The first assumption is realistic because the motion of an oscillator rod might be pure sinusoidal but the reactivity effects caused by this motion will very probably not be a pure sinusoid. The worth in reactivity of the rotated half cylinder of U^{235} is a complicated function of position. Since it is very hard to get a good absorber for fast neutrons, oscillator rods in fast reactors are special fuel rods in contrast to oscillator rods of thermal reactors which are boron rods which are moved up and down in the core.

By making inhour measurements at different rod angles in a fast reactor one can obtain the reactivity insertion versus angular oscillator rod position. The reactivity variation will very likely contain harmonics besides a dominant fundamental frequency component. Since the forcing function, in our case the reactivity insertion, is written as a Fourier series it suggests itself that one would like to expand the flux into a Fourier series also.

To make the Fourier series assumption in both reactivity and incremental flux it is advantageous to rewrite the original kinetic equations.

If we make the following substitutions and assumptions in the original exact zero power kinetic equations I.1-1 and I.1-2

$$\begin{array}{ll} \beta \ll 1 & \\ K_{ex} \ll 1 & \end{array} \quad \text{III.1-1} \quad \begin{array}{ll} C_i = c_i \frac{\ell}{\beta} & \\ a_i = \frac{\beta_i}{\beta} & \end{array} \quad \text{III.1-2}$$

one obtains

$$\frac{\ell}{\beta} \frac{dn}{dt} = \left[\frac{K_{ex}}{\beta} - 1 \right] n + \sum_i \lambda_i C_i \quad \text{III.1-3}$$

$$\frac{dC_i}{dt} = a_i n - \lambda_i C_i \quad \text{III.1-4}$$

The justification for III.1-1 has been discussed before in I.2-4 and I.2-5. Equations III.1-3 and III.1-4 can be rewritten for relative increments III.1-5 and III.1-6 and if we add an additional incremental equation III.1-7 for the effect of temperature on reactivity one gets

$$\frac{\ell}{\beta} \frac{d}{dt} \left[\frac{\delta n}{n_0} \right] = \delta k \left[1 + \frac{\delta n}{n_0} \right] + \sum_i \frac{\delta C_i \lambda_i}{n_0} - \frac{\delta n}{n_0} \quad \text{III.1-5}$$

$$\frac{d}{dt} \left[\frac{\delta C_i}{n_0} \right] = a_i \frac{\delta n}{n_0} - \frac{\delta C_i}{n_0} \lambda_i \quad \text{III.1-6}$$

$$\delta k = \nabla K + FB \quad \text{III.1-7}$$

Where:

δk : Net reactivity above delayed critical in dollars.

∇K : Reactivity insertion in dollars.

FB : Expression for reactivity change due to temperature.

$$n = n_0 + \delta n$$

$$C_i = C_{oi} + \delta C_i$$

Before making the Fourier series substitution in equations III. 1-5, III.1-6 and III.1-7 let us examine physically what will happen if a reactor is oscillated.

In Figure III.1,1 it is assumed that the amplitude of reactivity insertion is very small. In addition it is assumed that there is no resonance in LP(s) at this frequency. Large flux variations can be the cause of two things, either large periodic reactivity insertions or large amplification. That is, resonance effects. In Figure III.1,1 the reactor operates in the linear range, the flux variations δn are small in comparison with the steady state flux n_0 and the linear assumption holds $\delta n/n_0 \ll 1$. Furthermore, if the reactivity insertion contains only one frequency the neutron flux will contain also just one frequency, the same frequency as the reactivity insertion. One might mention that even in a reactor with a resonance at a particular frequency one can still obtain essentially linear data by lowering the stroke of reactivity insertion which lowers the amplitude of reactivity insertion K_{ex} . This is relatively simple in thermal reactors where one moves in and out of the core with the boron rod.

In a fast reactor one has to put a U^{235} fuel-oscillator rod into the reactor. If one starts to oscillate a reactor with $K_{ex} \sin \omega t$ where the linear assumption holds Figure III.1,1 one gets a shift in neutron level which can be obtained by applying the final value theorem of the Laplace Transform to I.2-2

$$\frac{\delta n(s)}{n_0} \Big/_{s \rightarrow 0} = \frac{n(t = \infty)}{n_0} = \frac{s K_{ex} \omega}{s^2 + \omega^2} \left[\frac{1 - s \sum_i \frac{\beta_i}{s + \lambda_i}}{s \left(\ell + \sum_i \frac{\beta_i}{s + \lambda_i} \right)} \right] \Big/_{s \rightarrow 0} \quad \text{III.1-8}$$

$$\frac{n(t = \infty)}{n_0} = \frac{K_{ex}}{\omega} \frac{1}{l + \sum_i \frac{\beta_i}{\lambda_i}} \quad \text{III.1-9}$$

in a fast reactor

$$l \ll \sum_i \frac{\beta_i}{\lambda_i} \quad \text{III.1-10}$$

and the new steady power level after the oscillation is applied becomes:

$$n(t = \infty) = \frac{K_{ex}}{\omega} \frac{n_0}{\sum_i \frac{\beta_i}{\lambda_i}} \quad \text{III.1-11}$$

This shift is relatively small but has to be taken into account in the analysis. It is advantageous experimentally to start out before oscillating with a somewhat smaller value than n_0 so that after applying the reactivity variation one oscillates around n_0 the power level at which one is interested to obtain information.

In Figure III.1, 2 the reactivity insertion is increased or what would have the same effect on the flux one operates in or near a resonance. The flux oscillations will now increase exponentially. In addition, the wave shape will no longer contain just one frequency, higher harmonics are generated. The exponential increase is very inconvenient for analysis. At each period a new power level is reached and even this increase of power level is not constant over one cycle. The effect of power level on the stability has been discussed in Chapter II. Experimentally and theoretically it is therefore advantageous for stability studies to apply a bias ∇ to the reactivity insertion to hold the average power level constant. As shown in Figure III.1, 3.

If the reactivity insertion is properly biased to hold the flux variations with a negative reactivity ∇ , the following expressions can be written for the reactivity insertions.

$$\nabla K = \nabla + \frac{K_{ex}}{2} \sum_{\substack{l=-\infty \\ l \neq 0}}^{+\infty} a_l e^{j l \omega t} \quad \text{III.1-12}$$

and for the relative incremental flux variations

$$\frac{\delta n'}{n_0} = \sum_{m=-\infty}^{+\infty} \frac{X'_m}{2} e^{j m \omega t} \quad \text{III.1-13}$$

According to Figure III.1, 2 $\delta n'$ is the incremental flux around n_0 and δn is the incremental flux around the new average flux level n_0^* . This new flux value n_0^* can be experimentally determined from the flux wave shape by getting Area I equal Area II in Figure III.1, 3. In III.1-12 $K_{ex} a_l$ is the magnitude of the l th harmonic in the expansion of ∇K . X'_m in III.1-13 is the magnitude of the m th harmonic in the expansion of $\frac{\delta n'}{n_0}$. All reactivities are expressed in dollars.

$$\nabla K = K_{ex} \sin \omega t$$

SMALL SINUSOIDAL REACTIVITY
INSERTIONS

$$\frac{n(t)}{n_0} = 1 + \frac{\delta' n}{n_0}$$

OR

$$n(t) = n_0 + \delta' n$$

Fig. III. 1, 1.

$$\nabla K = K_{ex} \sin \omega t$$

LARGE SINUSOIDAL REACTIVITY
INSERTIONS

Fig. III. 1, 2.

$$\nabla K = \nabla + K_{ex} \sin \omega t$$

LARGE SINUSOIDAL REACTIVITY
INSERTIONS WITH SHIFTED ZERO LINE
(NEGATIVE BIAS ∇)

$$n(t) = n_0 \left(1 + \frac{X_0}{2} \right) + \delta n = n_0^* + \delta n$$

$$\delta n = \delta' n - \frac{X_0}{2} n_0^*$$

Fig. III. 1, 3.

The reason for using X'_m instead of X_m in III.1-13 will be discussed later. K_{ex} and X'_m are divided by 2 because the identity of a sinusoidal amplitude will be retained in the exponential complex notation.

Substituting III.1-12 and III.1-13 into the incremental dynamic equations with feedback III.1-5, III.1-6 and III.1-7 to obtain III.1-14, III.1-15 and III.1-16, we assume that $\delta \dot{n} = \delta n$. The term in the bracket on the right hand side of equation III.1-14 represents the net reactivity δk of III.1-7. The reactivity insertion is so adjusted that there is no shift in average power level.

The reactor kinetic equations then become

$$\frac{\delta n}{n_0} \left[\frac{1}{ZP(jm\omega)} - PK(jm\omega) n_0 \right] = \nabla K + \left[\nabla K + PK(jm\omega) n_0 \frac{\delta n}{n_0} \right] \frac{\delta n}{n_0} \quad \text{III.1-14}$$

where:

$$\left[\frac{1}{ZP(jm\omega)} - PK(jm\omega) n_0 \right] \frac{\delta n}{n_0} = \sum_{m=-\infty}^{+\infty} \frac{X'_m e^{jm\omega t}}{2 LP(jm\omega)} \quad \text{III.1-15}$$

$$\frac{1}{LP(jm\omega)} = \frac{1}{ZP(jm\omega)} - PK(jm\omega) n_0 \quad \text{III.1-16}$$

$ZP(jm\omega)$	=	Zero power transfer function for mth harmonic
$LP(jm\omega)$	=	Load power transfer function for mth harmonic
$PK(jm\omega)$	=	Temperature feedback transfer function for mth harmonic

In III.1-14, III.1-15 and III.1-16 we notice that expressions are used which are well known from the linear analysis, $ZP(jm\omega)$, $LP(jm\omega)$ and $PK(jm\omega)$.

The only difference being that they are written for each harmonic separately.

III.2 Derivation of Amplitude of mth harmonic $X'_m/LP(jm\omega)$ and $X_m/LP(jm\omega)$ in the Expression of $\delta n/n_0$ and $\delta n/n_0^*$ in Implicit Form

Substituting III.1-15 into the left hand side of III.1-14 and III.1-12 and III.1-13 into the right hand side one obtains:

$$\sum_{m=-\infty}^{+\infty} \frac{X'_m e^{jm\omega t}}{2LP(jm\omega)} = \nabla + \frac{K_{ex}}{2} \sum_{l=-\infty}^{+\infty} a_l e^{jl\omega t} + \nabla \sum_{m=-\infty}^{+\infty} \frac{X'_m}{2} e^{jm\omega t} + \sum_{l=-\infty}^{+\infty} \frac{X'_l}{2} \left[K_{ex} a_{m-l} + PK_l n_0 X'_{m-l} \right] e^{jm\omega t} \quad \text{III. 2-1}$$

Equating the coefficients of the time dependent term $e^{jm\omega t}$ in III.2-1 gives:

$$\frac{X'_m}{LP(jm\omega)} = K_{ex} a_m + \sum_{l=-\infty}^{+\infty} \frac{X'_l}{2} \left[K_{ex} a_{m-l} + PK_l n_0 X'_{m-l} \right] + \nabla X'_m \quad \text{III. 2-2}$$

From Figure III.1, 3 we observe that the average flux will change from n_0 to n_0^* if a sinusoidal reactivity variation is applied. If we rewrite III.2-2 by separating the term for $m=0$ one gets,

$$\frac{n(t)}{n_0} = 1 + \frac{\delta n}{n_0} = 1 + \frac{X_0}{2} + \sum_{\substack{m=-\infty \\ m \neq 0}}^{+\infty} \frac{X'_m}{2} e^{jm\omega t} \quad \text{III. 2-3}$$

or

$$\frac{n(t)}{n_0 \left(1 + \frac{X_0}{2}\right)} = \frac{n(t)}{n_0^*} = 1 + \sum_{\substack{m=-\infty \\ m \neq 0}}^{+\infty} \frac{X'_m}{2 \left(1 + \frac{X_0}{2}\right)} e^{jm\omega t} \quad \text{III. 2-4}$$

$$\text{where:} \quad n_0 \left(1 + \frac{X_0}{2}\right) = n_0^* \quad \text{III. 2-5}$$

Rewriting the expression for the total flux $n(t)$

$$n(t) = n_0^* + \delta n \quad \text{III. 2-6}$$

or

$$\frac{n(t)}{n_0^*} = 1 + \frac{\delta n}{n_0^*} \quad \text{III. 2-7}$$

using III. 2-7 one gets for the incremental flux component in III. 2-4

$$\frac{\delta n}{n_0^*} = \sum_{\substack{m=-\infty \\ m \neq 0}}^{+\infty} \frac{X'_m}{2 \left(1 + \frac{X_0}{2}\right)} e^{jm\omega t} \quad \text{III. 2-8}$$

if we make the following substitution in III. 2-8

$$\frac{X_m}{2} = \frac{X'_m}{2 \left(1 + \frac{X_0}{2}\right)} \quad \text{III. 2-9}$$

one obtains

$$\frac{\delta n}{n_0^*} = \sum_{\substack{m=-\infty \\ m \neq 0}}^{+\infty} \frac{X_m}{2} e^{jm\omega t} \quad \text{III. 2-10}$$

In order to obtain $\frac{X_m}{LP(jm\omega)}$ eq. III. 2-2 is rewritten by separating out the term for $\ell = 0$.

$$\begin{aligned} \frac{X'_m}{2LP(jm\omega)} &= \frac{K_{ex}}{2} a_m + \frac{\nabla X'_m}{2} + \frac{K_{ex}}{2} a_m \frac{X_0}{2} + PK_0 n_0 \frac{X_0}{2} \frac{X'_m}{2} \\ &+ \sum_{\substack{\ell=-\infty \\ \ell \neq 0}}^{+\infty} \frac{X'_\ell}{2} \left[\frac{K_{ex}}{2} a_{m-\ell} + PK_\ell n_0 \frac{X'_{m-\ell}}{2} \right] \end{aligned} \quad \text{III. 2-11}$$

dividing the whole equation by $1 + \frac{X_0}{2}$

$$\begin{aligned} \frac{X'_m}{2 \left(1 + \frac{X_0}{2}\right) LP(jm\omega)} &= \frac{K_{ex} a_m}{2} + \frac{\nabla X'_m}{2 \left(1 + \frac{X_0}{2}\right)} + PK_0 n_0 \frac{X_0}{2} \frac{X'_m}{2 \left(1 + \frac{X_0}{2}\right)} \\ &+ \sum_{\substack{\ell=-\infty \\ \ell \neq 0}}^{+\infty} \frac{X'_\ell}{2 \left(1 + \frac{X_0}{2}\right)} \left[\frac{K_{ex}}{2} a_{m-\ell} + PK_\ell n_0 \left(1 + \frac{X_0}{2}\right) \frac{X'_{m-\ell}}{\left(1 + \frac{X_0}{2}\right)^2} \right] \quad \text{III. 2-12} \end{aligned}$$

substituting III. 2-9 and III. 2-5 into III. 2-12

$$\begin{aligned} \frac{X_m}{LP(jm\omega)} &= K_{ex} a_m + \nabla X_m + \sum_{\substack{\ell=-\infty \\ \ell \neq 0}}^{+\infty} \frac{X_\ell}{2} \left[K_{ex} a_{m-\ell} + PK_\ell n_0^* X_{m-\ell} \right] \\ &+ PK_0 (n_0^* - n_0) X_m \quad \text{III. 2-13} \end{aligned}$$

III. 2-2 and III. 2-13 are implicit expressions for X'_m and X_m , that is, the amplitude of the m th harmonic in the expansion of $\frac{\delta \dot{n}}{n_0}$ and $\frac{\delta n}{n_0^*}$ respectively.

III.3 Derivation of Time Independent Term ∇ in Implicit Form for X'_m and X_m

The time independent term ∇ or the bias in reactivity to hold the flux variations constant is obtained for $m = 0$ in Eq. III.2-1.

$$\nabla = -2R_e \left\{ \sum_{\substack{m=1 \\ m \neq 0}}^{\infty} \frac{X'_m}{2 \left(1 + \frac{X_0}{2}\right)} \left[\frac{K_{ex}}{2} a_m^* + PK_m n_0 \left(1 + \frac{X_0}{2}\right) \frac{X_m^*}{2 \left(1 + \frac{X_0}{2}\right)} \right] \right\} - \frac{X_0}{2} PK_0 n_0 \quad \text{III.3-1}$$

Substituting III.2-5 and III.2-9

$$\nabla = -R_e \left[\sum_{\substack{m=1 \\ m \neq 0}}^{\infty} \frac{X_m}{2} (K_{ex} a_m^* + PK_m n_0^* X_m^*) \right] - \frac{X_0}{2} PK_0 n_0 \quad \text{III.3-2}$$

or

$$\nabla = -R_e \left[\sum_{\substack{m=1 \\ m \neq 0}}^{\infty} \frac{X_m}{2} (K_{ex} a_m^* + PK_m n_0^* X_m^*) \right] - \frac{n_0^* - n_0}{n_0} PK_0 n_0 \quad \text{III.3-3}$$

III.4 Solution for X_m and ∇ by Iteration

In the following Eq. III.2-13 and III.3-3 are going to be solved by iterative methods. Theoretically we can assume that the right bias ∇ is provided right at the beginning of the rod oscillations. In this case the average power will not rise after the transients are died out; also the incremental flux variations will therefore not increase.

$$\delta' n = \delta n \quad \text{III.4-1}$$

$$n_0^* = n_0 \quad \text{III.4-2}$$

At this point we like to point out that emphasis was placed on the change in average flux level because experimentally there will always be a change in average power level and we had to show how this information has to be processed. After recording the periodic incremental flux variation δn , one has to determine n_0^* . This can be done by a Fourier analysis of the wave shape or in most cases it will be accurate enough to draw a line where the two areas as seen in Fig. III.1-3 increasing and decreasing flux are equal. The flux level before the oscillations should be recorded too. In order to determine n_0^* analytically the transient flux as a function of time would have to be known. This is possible in a good approximation for the zero power kinetics but with feedback present the mathematical difficulties become great. Actually very little information could be obtained with respect to stability which is not already determined from the sinusoidal steady state analysis.

As a first approximation for the solution of Eq. III.2-13 and III.3-3 the well known linear relationship between neutron flux and excess reactivity is assumed.

$$X_m = LP(jm\omega) K_{ex} a_m \quad \text{III.4-3}$$

III.4-3 represents a first order solution in K_{ex} and a second order solution can be obtained by inserting the first order solution into the right hand side of III.1-14.

$\nabla(2)$ The time independent term gives a solution for ∇ in the second approximation

$$\nabla(2) = - \frac{K_{ex}^2}{2} \sum_{m=1}^{\infty} |a_m|^2 \operatorname{Re} \left\{ LP_m + n_0 PK_m \mid LP_m \right\}^2 \quad \text{III.4-4}$$

The time dependent term gives the second order solution for $X_m^{(2)}$

$$\begin{aligned} \frac{X_m^{(2)}}{LP_m} &= K_{ex} a_m + \frac{K_{ex}^2}{2} \left[\sum_{\ell=1}^{m-1} LP_{\ell} a_{\ell} a_{m-\ell} + \sum_{\ell=1}^{\infty} LP_{m+\ell} a_{m+\ell} a_{\ell}^* \right] \\ &+ \frac{K_{ex}^2}{2} n_0 \left[\sum_{\ell=1}^{m-1} LP_{m-\ell} LP_{\ell} a_{m-\ell} a_{\ell} PK_{\ell} + \sum_{\ell=1}^{\infty} LP_{m+\ell} LP_{\ell}^* PK_{\ell}^* a_{\ell}^* a_{m+\ell} \right] \quad \text{III.4-5} \end{aligned}$$

or

$$\frac{X_m^{(2)}}{LP_m} = a_m K_{ex} + \psi_m \frac{K_{ex}^2}{2} + (0) \frac{K_{ex}^3}{4} \quad \text{III.4-6}$$

where

$$\psi_m = \gamma_m + \Lambda_m \quad \text{III.4-7}$$

$$\gamma_m = \sum_{\ell=1}^{m-1} LP_{\ell} a_{\ell} a_{m-\ell} + \sum_{\ell=1}^{\infty} LP_{m+\ell} a_{m+\ell} a_{\ell}^* \quad \text{III.4-8}$$

$$\gamma_m = \sum_{\ell=1}^{m-1} n_o LP_{m-\ell} LP_{\ell} a_{m-\ell} a_{\ell} PK_{\ell} + \sum_{\ell=1}^{\infty} n_o LP_{m+\ell} LP_{\ell}^* PK_{\ell}^* a_{\ell}^* a_{m+\ell} \quad \text{III.4-9}$$

A solution of third order for $X_m^{(3)}$ in K_{ex} is obtained by inserting the second order solution III.4-4 and III.4-6 into the right hand side of III.1-14 and compare the coefficients.

For the third iteration we obtain for the time dependent term

$$\frac{X_m^{(3)}}{LP_m} = a_m K_{ex} + \psi_m \frac{K_{ex}^2}{2} + \theta_m \frac{K_{ex}^3}{4} + (0) \frac{K_{ex}^4}{8} \quad \text{III.4-10}$$

where:

$$\theta_m = \tau_m + \sigma_m + x_m - \epsilon_m \quad \text{III.4-11}$$

$$\tau_m = \sum_{\ell=1}^{m-1} LP_{\ell} \gamma_{\ell} a_{m-\ell} + \sum_{\ell=1}^{\infty} LP_{m+\ell} \gamma_{m+\ell} a_{\ell}^* \quad \text{III.4-12}$$

$$\sigma_m = \sum_{\ell=1}^{m-1} LP_{\ell} \wedge_{\ell} a_{m-\ell} + \sum_{\ell=1}^{\infty} LP_{m+\ell} \wedge_{m+\ell} a_{\ell}^* \quad \text{III.4-13}$$

$$x_m = \sum_{\ell=1}^{m-1} n_o PK_{m-\ell} LP_{m-\ell} LP_{\ell} \left\{ (\gamma_{\ell} + \wedge_{\ell}) a_{m-\ell} + a_{\ell} (\gamma_{m-\ell} + \wedge_{m-\ell}) \right\} \\ + \sum_{\ell=1}^{\infty} (n_o PK_{\ell}^* LP_{m+\ell} LP_{\ell}^*) \left\{ (\gamma_{m+\ell} + \wedge_{m+\ell}) a_{\ell}^* + a_{m+\ell} (\gamma_{\ell}^* + \wedge_{\ell}^*) \right\} \quad \text{III.4-14}$$

$$\epsilon_m = 2LP_m a_m \sum_{\ell=1}^{\infty} |a_{\ell}|^2 R_e \left[LP_{\ell} + n_o PK_{\ell} |LP_{\ell}|^2 \right] \quad \text{III.4-15}$$

III.5 Fourier Components of Flux for Zero Power Reactor

As an example of the incremental flux component analysis let us consider a fast reactor with a neutron lifetime of $\ell = 4 \times 10^{-8}$, the Hughes delayed neutron data and a sinusoidal reactivity insertion of 30 cents at an angular frequency of $\omega = 0.1$ rad/sec. If we assume a pure sinusoidal reactivity insertion $\psi_m = 0$ in III.4-7 and if we divide III.4-10 by K_{ex} we obtain

$$\frac{X_m^{(3)}}{LP_m K_{ex}} = 1 + \theta_m \frac{K_{ex}^2}{4} \quad \text{III.5-1}$$

For very small reactivity insertions III.5-1 can be written

$$\frac{X_m^{(3)}}{K_{ex}} = LP_m \quad \text{III.5-2}$$

which is the well known linear relationship between K_{ex} and the incremental flux $\frac{\delta n}{n_0}$.

From Eq. III.5-1 we can regard the term $\frac{\theta_m K_{ex}^2}{4}$ as a correction term to the linear approximation. In the following the fundamental incremental flux component is evaluated for a numerical case. For the fundamental $m = 1$, also for a pure sinusoidal reactivity insertion $a_1 = 1$; $a_2 \dots a_n = 0$. For zero power $PK(s) = 0$.

$$\psi_1 = \phi_1 = x_1 = 0 \quad \text{III.5-3}$$

and III.5-1 becomes for the fundamental

$$\frac{X_1^{(3)}}{LP(0.1j) K_{ex}} = 1 + \theta_1 \frac{K_{ex}^2}{4} \quad \text{III.5-4}$$

The terms which are non-zero in III.4-11 are ε_1 and τ_1 .

$$\tau_1 = LP(2 \times 0.1j) LP(0.1j) = 0.0716 \times 10^6 \angle -71.8^\circ \quad \text{III.5-5}$$

$$\varepsilon_1 = 2LP(0.1j) R_e [LP(0.1j)] = 0.1479 \times 10^6 \angle -39.3^\circ \quad \text{III.5-6}$$

Inserting III.5-5 and III.5-6 into III.5-4 one gets for a K_{ex} of 0.3 dollars

$$\frac{X_1^{(3)}}{K_{ex}} = LP(0.1j) \left[1 + 0.091 \angle -71.8^\circ - .19 \angle -39.3^\circ \right] \quad \text{III. 5-7}$$

In the case with no feedback

$$LP(0.1j) = ZP(0.1j) \quad \text{III. 5-8}$$

which one knows from linear theory to be $308.8 \angle -39.3^\circ$ for $\omega = 0.1$ rad/sec.

The expression in the bracket in Eq. III. 5-7 can be regarded as the change in this linear vector due to non-linear effects. Eq. III. 5-7 is plotted in Fig. III. 5, 1.

Fig. III. 5, 1

In Fig. III. 5, 2 we record the incremental flux on an analog computer. A description of the block diagram for the simulation of a fast power reactor is given in Appendix B. The exact kinetic equations are simulated and the incremental flux is recorded. The flux curve was Fourier analyzed and the fundamental component and the first harmonic is shown.

By observing the fundamental component and the total incremental flux we see that there is little phase shift between them; also the amplitude difference is not very large. This is an inherent characteristic of this type of non-linearity that the fundamental component differs little from the flux in amplitude and phase. The wave shape, of course, being quite different. In the positive half wave a "peaking" occurs and in the negative half wave a "rounding out" effect. Experimentally if the non-linear effects are not too dominant one might take the amplitude and phase of the incremental flux as a good approximation for the fundamental flux component. This phenomena is due to the first harmonic which is approximately in phase with the positive peak, but 180° out of phase in the negative half wave.

From Fig. III. 5, 2 we note that the fundamental flux component describes essentially the flux. This is important because if this is true one can use a linear parameter technique introduced by Kryloff and Bogulioboff³⁰⁾ in non-linear mechanics. The same approach was introduced in the U.S. by R.J. Kochenburger³¹⁾ as the prescribing function method.

If the fundamental component prescribes the system with enough accuracy one can define a new ratio.

Fig. III. 5, 2.

$$\frac{\text{Fundamental Flux Component}}{\text{Excess Reactivity}} = \frac{\delta n}{n_0 K_{ex}} \quad \text{III. 5-9}$$

where δn is the amplitude of the incremental fundamental flux component. Once the fundamental is extracted one can use the ratio in III. 5-9 as it were, a linear transfer function. The only difference is that this ratio will now be a function of the excess reactivity K_{ex} . A linear system is independent of the disturbance but a non-linear is not.

III. 6 Fourier Components of Flux for Power Reactor

For a power reactor at $P = 2$ full power the vectors for the fundamental in III. 4-11, III. 4-12, III. 4-13, III. 4-14 and III. 4-15 become

$$\theta_1 = \tau_1 + \phi_1 + x_1 - \epsilon_1 \quad \text{III. 6-1}$$

where:

$$\tau_1 = LP_{2FP}(2j\omega) LP_{2FP}(j\omega) \quad \text{III. 6-2}$$

$$\phi_1 = LP_{2FP}(2j\omega) 2n_0 PK(j\omega) [LP_{2FP}(j\omega)]^2 \quad \text{III. 6-3}$$

$$x_1 = 2n_0 PK^*(j\omega) LP_{2FP}^*(j\omega) LP_{2FP}(2j\omega) \left[LP_{2FP}(j\omega) + 2n_0 PK(j\omega) [LP_{2FP}(j\omega)]^2 \right] \quad \text{III. 6-4}$$

$$\epsilon_1 = 2LP_{2FP}(j\omega) R_e \left[LP_{2FP}(j\omega) + 2n_0 PK(j\omega) |LP(j\omega)|^2 \right] \quad \text{III. 6-5}$$

where:

$PK(j\omega)$	Expression for power coefficient as a function of ω
$PK^*(j\omega)$	Conjugate complex expression for power coefficient as a function of ω
$LP_{2FP}^*(j\omega)$	Conjugate complex linear load power transfer function as a function of ω

Fig. III. 6, 1 and III. 6, 2 show the amplitude and phase of the ratio of the fundamental amplitude to the amplitude of K_{ex} , the excess reactivity as a function of ω and K_{ex} .

From Fig. III. 6, 1 we observe that for $K_{ex} = 0.3$ dollars sinusoidal amplitude a 55% and for $K_{ex} = 0.5$ dollars a 286% increase in amplification between reactivity and flux occurs at the resonance frequency.

Since the peak in Fig. III. 6, 1 increases with increasing K_{ex} we conclude that the reactor is more unstable for increasing reactivity insertion or what is equivalent for large flux variations.

Fig. III. 6, 1.

RATIO OF AMPLITUDE; FUNDAMENTAL COMPONENT
 OF NEUTRON FLUX $x_1^{(3)}$ AND SINUSOIDAL EXCESS
 REACTIVITY K_{ex} FOR $K_{ex} = .3$ AND $.5$ DOLLARS
 EBR-I; 2 X FULL POWER

Fig. III. 6, 2.

INCREMENTAL FLUX AND PRODUCT TERM $\delta k n/n_0$ FOR .3
AND .1 DOLLARS STEP CHANGE IN REACTIVITY. FAST
REACTOR, $\ell = 4 \times 10^{-8}$, HUGHES DELAYED NEUTRON DATA

FULL POWER, $PK(j\omega) = 10^{-3}$

$$\left[\frac{2}{1 + \frac{j\omega}{2.0}} - \frac{3}{\left(1 + \frac{j\omega}{2.0}\right)\left(1 + \frac{j\omega}{.2}\right)} \right]$$

Fig. III.6, 2 shows a step response for a power reactor at full power. The linear and non-linear response is recorded on analog computer. Also the important term for linear and non-linear operation $\delta k n/n_0$ is shown.

The figures are self-explanatory and show clearly that for large reactivity insertion one has to take the quadratic non-linearity $K_{ex} n$ into account. For $K_{ex} = 0.3$ dollars step input we note that the first swing in $\delta' n/n_0$ has a period T approaching 13.5 sec which we know is the resonance period.

Transient responses or rod drop experiments are frequently performed on fast reactors especially because this experiment is much simpler to perform than a frequency response. Mathematically speaking no more information can be obtained from a rod drop experiment than from a frequency response. In the frequency response method one obtains information on individual frequencies separately. On the other hand in a rod drop experiment one generates several harmonics in the reactivity insertion. If, for example, one were able to apply a perfect step change of reactivity, an infinite set of frequencies is generated according to the Fourier expansion of a step. Logically the incremental flux will also contain an infinite number of frequencies. The problem is now to extract information on individual frequencies from this superposition of frequencies. To process this data will be much more laborious and also more subject to experimental errors.

III.7 Effect of Limited Bowing on Stability

Since the partial meltdown of EBR-I Mark II was due to bowing¹⁾ of fuel pins, efforts have been made to restrict the movement of the pins in the radial direction. In EBR-I Mark III this is accomplished by bundling the pins into hexagonal cans. Furthermore, to each pin there are spacers attached and in the center of each hex can there is a central tightening rod. This construction reduces the movement of pins drastically. Due to tolerances in the fabrication and thermal cycling it is possible that a limited amount of bowing will be present. The question therefore arises what limited bowing will do to the stability. Bowing is detected experimentally in reactors by stopping the coolant flow and observing the flux variations. If there are no geometrical distortions the reactivity should drop, since the temperature increases and the flux should therefore decrease. If there is bowing present the temperature gradient across the pins will increase and the pins will move into a more reactive region and the flux will increase. In terms of a frequency response the positive contribution in the power coefficient

$$\frac{2}{1 + \frac{j\omega}{2.0}}$$

as shown in Chapter II is limited, which can be written as

$$\left| \frac{2}{1 + \frac{j\omega}{2.0}} \right| \leq L \quad \text{III. 7-1}$$

In Fig. III.7, 1 it is assumed that the forcing function before being limited or the positive reactivity will have a sinusoidal shape. Two of the reactivity amplitudes, 1 and 2, are not sufficient in amplitude that the limit L is reached and the system would

be linear for this amplitude and frequency. The amplitude of 3 is such that the limit L is reached and the positive half wave is chopped off as shown in Fig. III.7, 1c. This chopped off contribution due to bowing can be Fourier analyzed and the fundamental component can be regarded as the "essential" behavior.

The amplitude and phase of this fundamental component as a function of frequency and amplitude will give us information how the stability is effected. In Fig. III.7, 1c the fundamental component $3F$ is reduced in amplitude and has a negative phase shift. What one is interested in is the ratio between curve 1 in a and curve 1 in c. For 1 and 2 this amplitude ratio remains constant, but for 3 it is not the same as for 1 and 2. $3F$ is effected by the limit of the amplitude. In Fig. III.7, 1 it is assumed that the wave shape is pure sinusoidal. We know that for large reactivity insertion the flux will contain harmonics and that the chopping will be on a wave shape as shown in Fig. III.5, 2. For Fig. III.7, 2 the same case was used as in Fig. III.5, 2 for $\omega = 0.1$ rad/sec and a sinusoidal reactivity insertion of 0.3 dollars. The positive peak was limited at different flux levels and from the resultant chopped off flux we extracted the fundamental component. In this qualitative representation it was assumed for simplicity that there is no frequency dependent term between flux and positive reactivity due to bowing. From III.7, 2 we conclude that the amplitude of the fundamental component decreases as the bowing limit is set at a lower value. The phase is effected very little. The stability of the reactor will be better since the loop gain is lowered. In terms of a Nyquist plot the reactor behaves the same way as one for a lower power level since in the linear representation the power level can be visualized as directly proportional to the loop gain.

III.8 Limited Bowing and Self-Sustained Oscillations

Self-sustained oscillations can be visualized in a fast reactor operating on a high power level when the bowing of the fuel pins is limited. The reactor has to be linearly unstable first to start growing oscillations on its own. The limited bowing prevents the reactor from becoming non-linearly unstable and furnishes the non-linear mechanism in the feedback so that the system can reach a convergent equilibrium point. After a decreasing or increasing disturbance the reactor will always return on its own to oscillations of constant magnitude, that is self-sustained oscillations.

In order to explain the mechanisms of self-sustained oscillations let us for the time being assume that the zero power kinetics equations are linear, or $K_{ex} n \cong K_{ex} n_0$. Actually we have a superposition of 2 non-linearities, the effect of $K_{ex} n$ and the limiting in the power coefficient. The linear assumption for the zero power kinetic will not change the essential conditions. For limiting in the bowing the load power transfer function $LP(s)$ can be written in the following way.

$$LP(\alpha, j\omega) = \frac{ZP(j\omega)}{1 - ZP(j\omega) PK(\alpha, j\omega)} \quad \text{III.8-1}$$

where:

- $LP(\alpha, j\omega)$ = Load power transfer function with variable α for limited bowing
- $ZP(j\omega)$ = Zero power transfer function
- $PK(\alpha, j\omega)$ = Power coefficient with variable α for limited bowing

Fig. III. 7, 1.

LIMITED AND UNLIMITED POSITIVE REACTIVITY DUE TO BOWING IN FAST REACTOR. ALSO FUNDAMENTAL FREQUENCY COMPONENT OF LIMITED REACTIVITY DUE TO BOWING

Fig. III. 7, 2.

EFFECT OF ASSYMETRIC LIMITING OF FLUX ON
 AMPLITUDE AND PHASE OF FUNDAMENTAL COMPONENT
 OF FLUX; $ZP(s)$; $\omega = .1(\text{rad/sec})$; $K_{ex} = .3$ (DOLLARS)

$PK(\alpha, j\omega)$ is the power coefficient which depends not only on frequency but also on amplitude.

From equation III.8-1 we observe that self-sustained oscillations may exist when

$$1 - ZP(j\omega) PK(\alpha, j\omega) = 0 \quad \text{III.8-2}$$

or when

$$ZP(j\omega) = \frac{1}{PK(\alpha, j\omega)} \quad \text{III.8-3}$$

The curves as shown in Fig. III.8, 1 are obtained from a simulated fast reactor, $\ell = 4 \times 10^{-8}$, Hughes delayed neutron data and a feedback with a positive and negative contribution as outlined in Chapt. II. The power level was assumed to be 3.1 full power. From Chapt. II we know that the reactor will be linearly unstable for this power level. $ZP(s)$ and $PK(s)$ are plotted in Fig. III.8, 1 for the frequency region of interest. The two curves intersect at a frequency somewhat higher than for $P = 3.02$ which is discussed before.

Fig. III.8, 2 shows the section in the resonance region in Fig. III.8, 1 enlarged with a possible locus for $1/PK(\alpha, j\omega)$ which will cause self-sustained oscillations. For an analytical evaluation both curves for $ZP(j\omega)$ and $1/PK(\alpha, j\omega)$ would have to be plotted as a family of loci. Also $ZP(j\omega)$ would have to be written as $ZP(\alpha, j\omega)$ since there is a dependence on amplitude in this term too. Another approach would be to write the two non-linearities in one describing function. In Fig. III.8, 2 the locus for $1/PK(\alpha, j\omega)$ by virtue of Eq. III.8-3 can be considered to be the locus of the critical point, which in conventional theory is the -1 or -180 degree point. When the critical point lies to the left of $ZP(j\omega)$ the poles of $LP(\alpha, j\omega)$ have negative real parts and the system is said to be stable at this operating condition in the sense that any disturbances which appear tend to die out. On the other hand, when the critical point lies to the right of $ZP(j\omega)$, the poles have positive real parts and operation is unstable. If the $ZP(j\omega)$ locus passes through the critical point, as at the intersection in Fig. III.8, 2 the real parts are zero, indicating a periodic solution.

Such a periodic solution may be stable or unstable with respect to a slight disturbance in the amplitude of oscillation. If the solution is stable, the amplitude tends to return to its original value following a disturbance; if unstable, the oscillation either increases or decreases indefinitely, depending on the direction of the disturbance.

Let us now assume that the reactor is operating at the point of intersection in Fig. III.8, 2 and the amplitude is increased slightly. The critical point is therefore shifted to the left and stable side of $ZP(j\omega)$. The amplitude consequently tends to decrease and operation will go back to the point of intersection. On the other hand, if the amplitude tends to decrease slightly, the critical point shifts to the right side of $ZP(j\omega)$ and the system gets more unstable, and the amplitude of oscillations will increase, and the operating point will move back to the original operating point at the intersection.

The point of intersection is therefore a stable one and is called a convergent equilibrium point. The amplitude of the periodic solution is analogous to the position of a ball lying in the bottom of a hollow as shown in Fig. III.8, 2. The ball always rolls back to its original position after a disturbance.

The same discussions could, in an analogous way, be done by using phase plane techniques. The convergent equilibrium point would correspond to a limit cycle where the phase trajectories converge towards the limit cycle from both sides. Also, in the center of the limit cycle we have to have an unstable equilibrium point which will start the oscillations.

Fig. III. 8, 1.

$ZP(j\omega)$; $1/PK(j\omega)$ IN GAIN PHASE PLANE FOR FAST REACTOR
 $\ell = 4 \times 10^{-8}$; HUGHES DELAYED NEUTRON DATA

$$PK(j\omega) = 3.1 \times 10^{-3} \left[\frac{2}{1 + \frac{j\omega}{2.0}} - \frac{3}{\left(1 + \frac{j\omega}{2.0}\right)\left(1 + \frac{j\omega}{.2}\right)} \right]; P = 3.1 \text{ FULL POWER}$$

Fig. III. 8, 2.

$ZP(j\omega)$; $1/PK(j\omega)$ AND $1/PK(\alpha_1 j\omega)$
 IN GAIN PHASE PLANE

In order to verify self-sustained oscillations a fast reactor was simulated on the analog computer and the results are shown in Figs. III.8, 3 to III.8, 8.

We observe that the amplitude of the oscillations is a function of the bowing limit and also of the power level.

For a given bowing limit the oscillations decrease with increasing average power level. If the power level is too high, no self-sustained oscillations are possible.

If we put the bowing limit higher, the amplitude of oscillations will increase for constant average power level. Also, the frequency of oscillations decreases with decreasing average power level, which is in line with Fig. III.8, 1 where the locus $1/PK(j\omega)$ is shifted to the right side for decreasing power level and the intersection being therefore at a lower frequency. Fig. III.8, 7 and III.8, 8 show that if the bowing limit is set at $0.6/\beta$ [cents], no self-sustained oscillations are possible even if the oscillations are forced upon at high power level and then removed.

Similarly there will be no self-sustained oscillations possible if the power level is below 3 x full power, that is the linear stability limit.

In summary we can say that the amplitude of oscillations depends on the power level and bowing limit. For constant bowing limit the amplitude of the self-sustained oscillations will decrease for increasing power level. For constant power level the amplitudes will increase with increasing bowing limit. The frequency of oscillations will decrease with decreasing power level when the bowing limit is held constant. For both power level and bowing limit there is a limit where no self-sustained oscillations are possible if either the bowing limit or the power level is held constant.

Two assumptions have been made in this paper. First it is assumed that the input into the non-linearity is sinusoidal. The wave shape of the flux has been recorded on the analog and the effect of limited bowing on the wave shape was very minor. Limited bowing acts mostly as a gain reduction in the loop.

Due to the limiting action the power coefficients, or better the reactivity feedback, contains higher harmonics. These harmonics are returned to the flux so that the assumption of sinusoidal input into the non-linearity is strictly not true. Also there is a production of higher harmonics due to the K_{ex} term in the zero power kinetics.

Therefore, due to the neglect of the non-linearity in the zero power kinetics, an error is introduced. From Figs. III.8, 3 through III.8, 6 we observe that all frequencies of sustained oscillations are between 0.4 and 0.5 rad/sec, the same value one might expect by examining Fig. III.8, 1. It is a question of a slight shift of $ZP(j\omega)$ and $1/PK(\infty, j\omega)$ to the left or to the right.

Fig. III. 8, 3.

Fig. III. 8, 4.

SELF-SUSTAINED OSCILLATIONS IN FAST REACTOR:

$\ell = 4 \times 10^{-8}$; HUGHES DELAYED NEUTRON DATA;

$$PK(j\omega) = P \times 10^{-3} \left[\frac{2}{1 + \frac{j\omega}{2.0}} - \frac{3}{\left(1 + \frac{j\omega}{2.0}\right) \left(1 + \frac{j\omega}{.2}\right)} \right] \text{ AT DIFFERENT}$$

POWER LEVELS P; $n_0 = 1$; BOWING LIMIT $\frac{.695}{\beta}$ [cts]

Fig. III. 8, 5.

Fig. III. 8, 6.

SELF-SUSTAINED OSCILLATIONS IN FAST REACTOR:

$\ell = 4 \times 10^{-8}$; HUGHES DELAYED NEUTRON DATA;

$$PK(j\omega) = P \times 10^{-3} \left[\frac{2}{1 + \frac{j\omega}{2.0}} - \frac{3}{\left(1 + \frac{j\omega}{2.0}\right) \left(1 + \frac{j\omega}{.2}\right)} \right] \text{ AT DIFFERENT}$$

POWER LEVELS P ; $n_0 = 1$; BOWING LIMIT $\frac{.745}{\beta}$ cts

Fig. III. 8, 7.

Fig. III. 8, 8.

SELF-SUSTAINED OSCILLATIONS IN FAST REACTOR;
 $\ell = 4 \times 10^{-8}$; HUGHES DELAYED NEUTRON DATA;

$$PK(j\omega) = P \times 10^{-3} \left[\frac{2}{1 + \frac{j\omega}{2.0}} - \frac{3}{\left(1 + \frac{j\omega}{2.0}\right) \left(1 + \frac{j\omega}{.2}\right)} \right] \text{ AT DIFFERENT}$$

POWER LEVELS P ; $n_0 = 1$; BOWING LIMIT $\frac{.6}{\beta} [\text{cts}]$

Appendix

Appendix A

Table I
Hughes Delayed Neutron Data⁵⁾

i	λ_i	β_i	a_i/a	$T^{1/2}, i$
1	.01246	.00025	.034	55.6
2	.0315	.00166	.22	22.0
3	.1537	.00213	.282	4.51
4	.456	.00241	.319	1.52
5	1.612	.00085	.112	.43
6	13.86	<u>.00025</u>	.033	.05
		$\beta = .00755$		

Table II
Keepin Fast Fission Delayed Neutron Data⁶⁾
A value of $\nu = 2.56$ was used for β_i

i	λ_i	β_i	a_i/a	$T^{1/2}, i$
1	.01272	.000247	.038	54.51
2	.031738	.0013845	.213	21.84
3	.115525	.0012222	.188	6.0
4	.31083	.0026455	.407	2.23
5	1.39748	.000832	.128	.496
6	3.87235	<u>.000169</u>	.026	.179
		$\beta = .0065$		

Table III

Keepin Thermal Fission Delayed Neutron Data

A value of $\nu = 2.47$ was used for β_i

i	λ_i	β_i	a_i/a	$T^{1/2}, i$
1	.01244	.0002112	.033	55.72
2	.03051	.0014016	.219	22.72
2	.11144	.0012544	.196	6.22
4	.30137	.002528	.395	2.3
5	1.13631	.000736	.115	.61
6	3.01370	<u>.000268</u>	.042	.23

$$\beta = .0064$$

Table IV

Effective Delayed Neutrons Including $U^{238^{11}}$ Fissions for EBR-I

i	a_i	β_i	λ_i	$T^{1/2}, i$
1	.03425	.000234	.0127	54.51
2	.2016	.001377	.0318	21.84
3	.1841	.001257	.1153	6.0
4	.40415	.002760	.311	2.23
5	.14255	.000974	1.4	.496
6	.03335	<u>.000228</u>	3.87	.179

$$\beta = .006830$$

Appendix B

Analog Computer Representation of a Fast Reactor

To make the computer diagram simpler, the following substitutions are made in I.1-1 and I.1-2:

$$C_{1i} = \frac{c_i}{C_{oi}} \quad K_{ex} \rightarrow \delta k \quad C_{oi} = \frac{\beta_i}{\lambda_i \ell} n_o \quad B-1$$

C_{oi} is evaluated from equilibrium conditions applied to I.1-2; that is, $\frac{dc_i}{dt} = 0$; $n = n_o$; $K_{ex} = 0$; $c_i(\infty) = C_{oi}$. All quantities of reactivity are expressed in dollars. For a power reactor the net reactivity δk is the sum of the "cold" reactivity insertion K_{ex} plus an expression due to the feedback reactivity FB, as shown in Eq. B-4. We can

neglect the term $\frac{\ell}{\beta} \frac{d\left(\frac{n}{n_o}\right)}{dt}$ as compared to $\frac{n}{n_o}$. This is justified because $\frac{\ell}{\beta}$ is extremely small; for a fast reactor about 10^{-5} sec, and for a thermal reactor about 10^{-2} sec. Thus, we assume that the change of neutron level is slow compared with 10^{-5} sec, an assumption which is well justified in all circumstances except the most severe accidents. However, after this assumption, n/n_o is no longer necessarily a continuous function of time. If K_{ex} changes as a step function, the reactor flux will change discontinuously. The intensities of the delayed neutron c_i will still be a continuous function of time.

After the proper manipulations, we obtain

$$\frac{n}{n_o} = (1-\beta) \delta k \frac{n}{n_o} + \sum_i \frac{\beta_i}{\beta} C_{1i} \quad B-2$$

$$\frac{dC_{1i}}{dt} = \lambda_i \beta \delta k \frac{n}{n_o} + \lambda_i \frac{n}{n_o} - \lambda_i C_{1i} \quad B-3$$

$$\delta k = K_{ex} + FB \quad B-4$$

For the feedback the previously used model representing a positive and negative contribution to the power coefficient is simulated. In the frequency domain we get

$$PK(s) = P \times 10^{-3} \left[\frac{2}{1 + \frac{s}{\omega_f}} - \frac{3}{\left(1 + \frac{s}{\omega_f}\right) \left(1 + \frac{s}{\omega_s}\right)} \right] \quad B-5$$

B-2 through B-5 are expressed in Figure B.1. Figure B.1 is self-explanatory, since each term appearing in the equation is marked on the graph.

Amplifier I sums up the different reactivity effects in dollars. All quantities going into Amplifier I are multiplied by a factor of 100, so that the reactivities are expressed in cents. The top lead corresponds to the driving in of control rods as the power level goes up. One has to add reactivity to hold the same power level when the temperature goes up. Potentiometer 3 is also used to compensate for inaccuracies in the computer. After the switch is put on "operate", one might not get the exact desired power level, even if the initial condition Pot. 24 through 29 are set correctly. The second lead provides the negative bias ∇ to hold the flux level constant for the non-linear investigation. The third lead provides $K_{ex}(t)$ which can be an arbitrary function of time. For rod drop experiments we apply a step change of reactivity and for frequency response measurements a sinusoidal reactivity insertion is generated.

This can be done either by using a commercial oscillator or by making an oscillator circuit on the computer itself. The fourth lead represents the feedback on the reactivity due to power. The multiplier performs the multiplication of $\delta k(t) \frac{n(t)}{n_0}$, the net excess reactivity $\delta k(t)$ with the relative neutron flux $\frac{n(t)}{n_0}$, both terms being functions of time. If one lowers the gain in the multiplier, the neutron flux will be a constant, since the shaft will not move anymore and we get the linear approximation $\delta k(t)n_0$.

Here only $\delta k(t)$ is a function of time and n_0 is the steady state flux level. Before making any measurements, it is always advisable to check the computer with this well known linear approximation. The integrators (1) through (6) generate the intensities of the delayed neutrons C₁₁ through C₁₆. Note that at the input of all integrators a factor of $1/a$ appears. The factor a is a constant relating actual time to computer time.

where:

$$\tau = at$$

$$\tau = \text{computer time}$$

$$t = \text{actual time}$$

$$a = \text{constant}$$

B-6

At the output of Amplifier III we generate n/n_0 , the relative neutron flux. Amplifier IV which is connected in a special manner has a gain of one because of the feedback around the amplifier going into a gain of 10 at the input. This reduces the through gain by a factor of 10, and if the inputs into the amplifier go into a gain of 10, we will have an over-all gain in Amplifier IV of unity. At the output of Amplifier IV we generate the incremental flux δn at Terminal C.

If the term $\frac{\ell}{\beta} \frac{d\left(\frac{n}{n_0}\right)}{dt}$ were not neglected, Amplifier III would have to be an

integrator. For a zero power reactor switch S is left open and for a power reactor, where feedback is present, switch S is closed. In order to change the power level one has to change the initial condition Pots. 24 through 29. C₀₁ through C₀₆ are functions of the steady state flux level n_0 . Note that in the exact non-linear representation the gain increase is fed back to the multiplier and also through the temperature feedback. In the linear case the lead going to the multiplier shaft can be considered disconnected, since the time variations of $n(t)$ will not move the shaft.

For increasing power level we first have to increase the term $\delta k(t) n_0$ by a constant factor -- whatever power level we operate at. The other place where we have to increase the gain would be in the feedback loop. It is obvious that it makes no difference where we put this additional gain into the loop.

We conclude, therefore, that in the exact non-linear simulation the power level must be changed with the initial conditions at Integrators (1) through (6); whereas in the linear case we can introduce the gain wherever it is advantageous in the feedback loop beside the constant gain change at the "quasi" multiplier where the shaft is not moving anymore.

The feedback shows a representation of Eq. B5. Here $FB(t)$ is the time function of reactivity which is fed back due to the relative flux $\frac{n(t)}{n_0}$ as a function of time. We obtain $PK(s)$ if we apply a sinusoidal flux change at Terminal B and record at Terminal F the response and form the ratio of response to input. If a sinusoidal excitation is applied to Terminal B and the response is measured at Terminal D, we get for the ratio of response to excitation:

$$= \frac{3Pn_0}{\left(1 + \frac{s}{\omega_f}\right) \left(1 + \frac{s}{\omega_s}\right)} \frac{1}{10} \quad B-7$$

where $1/10$ is a scale factor.

In a similar way we obtain for the ratio of response at Terminal E and sinusoidal excitation at Terminal B:

$$\frac{2Pn_0}{1 + \frac{s}{\omega_f}} \quad B-8$$

The two terminals D and E represent the individual contributions due to expansion and bowing. Since there is at time $t = 0$ no voltage at the output of the integrators, we have to apply initial conditions at Integrators (7) and (8).

The two diode limiters at Integrator (8) and Amplifier VI limit the quantity at the output, according to the Pot. settings 44 and 45. Here the expansion term cannot get more negative than a certain amount and the bowing cannot get more positive than a set value.

In the expansion the positive side is not limited and in the bowing there is no limit on the negative side. Care has to be taken if one changes the scaling of the relative flux. If we change, for example, the output of Amplifier III from $25 n/n_0$ to $10 n/n_0$, we should not change the loop gain. We, therefore, have to change the factor $1/2.5$ in the setting of Pot. 46 to 1.0 for the above case.

Literature

- 1) H. A. Bethe: "Reactor Safety and Oscillator Test", APDA-117, October 17, 1956.
- 2) S. Glasstone and M. C. Edlund: "The Elements of Nuclear Reactor Theory", D. Van Nostrand.
- 3) A. F. Henry: "The Application of Reactor Kinetics to the Analysis of Experiments", Nuclear Science and Engineering, 3, 52-70 (1958).
- 4) H. A. Bethe: "Testimony in the Matter of Power Reactor Development Company", Docket No. F-16, U. S. Atomic Energy Commission, 1957.
- 5) Hughes, Dobbs, Kahn and Hall: Physical Review 73, III, 1958.
- 6) G. R. Keepin: "Delayed Neutrons", A review as of October, 1955, Los Alamos Scientific Laboratory.
- 7) M. N. Moore: "On the Determination of Reactor Transfer Functions from Measurement of Steady Operation, June, 1957, American Nuclear Society Meeting.
- 8) H. Soodak, NDA: "One Delay Group Parameters, October, 1957, Nuclear Society Meeting.
- 9) R. E. Skinner and E. R. Cohen, NDA: "Reduced Delayed Neutron Group Representation", October, 1957, Nuclear Society Meeting.
- 10) G. S. Brown and D. P. Campbell: "Principles of Servomechanisms", John Wiley and Sons, New York, 1948.
- 11) D. Okrent: "Effective Delayed Neutron and Zero Power Transfer Function for EBR-I, Mark-III. Geneva Conference Paper, 1958.
- 12) A. Fenstermacher, D. Okrent and H. Sandmeier: "Power Coefficient Calculation for EBR-I, Mark-III, Internal ANL Memorandum, November 11, 1957.
- 13) "Summary Report of the Hazards of the EBR-II", ANL-5719, May, 1957.
- 14) H. Feshbach, G. Goertzel and H. Yamauchi: "Estimation of Doppler Effect in Fast Reactors", Nuclear Science and Engineering 1, 1 p.4, March, 1956.
- 15) A. M. Lane, J. E. Lynn and J. S. Story: "An Estimation of the Doppler Effect in Fast Neutron Reactors", AERE T/M 137, July, 1956.
- 16) H. A. Bethe: "On the Doppler Effect in Fast Reactors", APDA-119.
- 17) D. Okrent, R. Avery and H. H. Hummel: "Survey of the Theoretical and Experimental Aspects of Fast Reactor Physics", Geneva Conference Paper, 1955, P/609.

- 18) W. Y. Kato and D. K. Butler: "Doppler Effect Measurement of Uranium-235 Resonances in the Fast Spectrum of ZPR-III, Internal Memorandum, ANL, April 26, 1956.
- 19) G. Goertzel and C. Klahr: "Interpretation of an Experiment to Measure Doppler Effect", NDA 14-127, June 29, 1956.
- 20) A. Langsdorf: "Thermal Neutron Reactor as an Instrument for Measuring Neutron Cross Sections", ANL-4342.
- 21) R. D. Smith: AERE Harwell, Private Communication, 1958.
- 22) R. O. Brittan: "Some Problems in the Safety of Fast Reactors", ANL-5577.
- 23) G. H. Kinchin: "The Instability of Fast Reactors", RP/M 83, Sub. Ref. FRDC/P. 174, AERE Harwell, June, 1956.
- 24) H. V. Lichtenberger et al: Geneva Conference Paper P/813, 1955.
- 25) T. A. Truxal: "Automatic Feedback Control System Synthesis", Chapter 4, McGraw Hill, 1955.
- 26) Argonne National Laboratory Code BUM 650.
- 27) Argonne National Laboratory Code 445-Re-118.
- 28) R. Siegel and H. Hurwitz, Jr.: "The Effect of Temperature Coefficients on Reactor Stability and Reactor Transfer Function", March 1, 1955, Appendix by M. Nelkin.
- 29) T. J. Higgins: "A Resume of the Development and Literature of Non-Linear Control-System Theory", ASME Transactions, 1957.
- 30) N. Kryloff and N. Bogoliuboff: "Introduction to Non-Linear Mechanics", Translation by Solomon Lefschetz, Princeton University Press, 1947.
- 31) Ralph J. Kochenburger: "A Frequency Response Method for Analysis and Synthesizing Contactor Servomechanisms", AIEE Transactions, Vol. 69, Part I, 1950.