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Final Report

PLASMA HEATING AND CONFINEMENT AT HIGH TEMPERATURES

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Introduction and Summary

Studies carried out under this contract may be divided into three parts:

- 1) Theory of Ion Cyclotron Heating
- 2) Theory of Turbulent Heating
- 3) Application of E-Beams to Plasma Confinement.

The item listed as 1) Theory of Ion Cyclotron Heating was carried out by Dr. Norman Rostoker in support of a joint experimental effort involving personnel from the University of Texas and Cornell University. The bulk of the theoretical analysis was performed at Cornell during the summer of 1972. Upon completion of this study Dr. Rostoker took up a new position at the University of California, Irvine. It was his intention to provide a final report on his investigations to Cornell University, but the press of other duties has delayed preparation of this report. Consequently, we are unable to include it in this document.

In support of C. B. Wharton's experiments on turbulent heating the following studies were performed.

2a) Modulated Beam-Plasma Interaction, which relates to showing that a density-modulated relativistic electron beam can resonantly interact with a mode of a background plasma to effect a rapid transfer of energy from the beam to the plasma. We have modeled the modulated beam with a finite train of N charged discs in uniform translation along the direction of an externally applied magnetic field. This structure is allowed to interact with a cold, uniform plasma in a magnetic field. The beam-driven fields in the plasma are computed, with particular attention given to the "synchronous" modes ($\omega/k_{\parallel} = v_0$).

In the high-density limit, $\omega_{pe}^2 \gg \Omega_e^2$, the plasma has a synchronous mode at the lower-hybrid frequency, $\omega = (\Omega_e \Omega_i)^{1/2}$. We compute the rate of resonant energy transfer from the beam to this mode, and the characteristic time required for the energy transfer. For typical parameters, this transfer time is found to be short compared with the beam pulse duration.

A full description of the calculations is attached as LPS 101 by A. A. Mondelli and P. L. Auer (September 1972). An abbreviated version of this report is under preparation and will be submitted for external publication.

In addition we have examined

2b) The Application of Quasilinear Theory to the Cornell Turbulent Heating Experiments, in which we have applied quasilinear theory to the Cornell turbulent heating experiments with the inclusion of the actual external circuitry of the experiments in our model. The resulting set of coupled, first-order, ordinary differential equations are integrated numerically by the Runge-Kutta-Gill technique.

A one-dimensional model with Maxwellian electrons and ions, assuming a current-driven ion-acoustic instability as the heating mechanism, is employed. The observed high-energy ion component is therefore excluded from our analysis. The computed current and voltage vs. time have several significant points of agreement with those in the experiments, including the "spike" structure observed in the voltage traces and the premature quenching of the instability noted in the experiments. The computed final ion temperature is in reasonable agreement with the experimentally observed bulk ion temperature (excluding the high-energy ion tail). The total electron plus ion final temperature is consistent with the diamagnetic signals observed in the experiments.

The premature turn-off of the instability appears in our calculation to be due to ions heating more efficiently than electrons, resulting in a decrease of T_e/T_i , which stabilizes the instability. As the electron drift velocity rises, the instability turns on and off several times, until T_e/T_i has decreased to the point where the drift velocity can no longer cross the threshold for instability. The observed voltage spikes are due to the successive recurrences of the instability. The duration of the computed voltage spikes is in good agreement with the experimentally observed voltage traces.

A full description of the calculations is attached as LPS 119 by Alfred Mondelli and Edward Ott (June 1973); this work has been accepted for publication in "Plasma Physics".

In support of item 3) we have studied

3a) Straight and Toroidal Plasma Equilibria with an Intense Relativistic Electron Current Component, in which we examine analytically the equilibrium of an axisymmetric, large aspect-ratio, finite temperature toroidal plasma which is enclosed by a perfectly conducting toroidal container. The new feature of the analysis is the presence of a relativistic electron beam which circulates around the torus thus producing a poloidal magnetic field in addition to the external field and the field created by currents in the plasma. This problem may be of interest in connection with recent toroidal beam injection experiments at Cornell University, and tokamaks. The equilibria are obtained from the relativistic virial theorem. Specific examples are given

wherein the plasma pressure is characterized by its dependence on the magnetic flux function and the relativistic electron beam is characterized by an electron distribution function which is a delta function in energy and canonical momentum in the toroidal direction. The results are attached as LPS 118 by Alfred Mondelli and Edward Ott (May 1973).

In addition we have studied the

3b) Synchrotron Radiation from Force-Free Equilibria, in order to examine the synchrotron radiation losses from relativistic electron beams which are close to a force-free configuration. The resulting radiation is found to be substantially less than the radiation found from a non-force-free configuration, such as the well-known Hammer-Rostoker equilibrium. We conclude that force-free beams may therefore be useful for providing magnetic confinement of plasma.

A report covering the results of these calculations is attached as a note by A. A. Mondelli and S. Yoshikawa (January 1973).

MODULATED BEAM-PLASMA INTERACTION

by

A. A. Mondelli and P. L. Auer

LPS 101

September 1972

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Abstract

A density-modulated relativistic electron beam can resonantly interact with a mode of a background plasma to effect a rapid transfer of energy from the beam to the plasma. We have modeled the modulated beam with a finite train of N charged discs in uniform translation along the direction of an externally applied magnetic field. This structure is allowed to interact with a cold, uniform plasma in a magnetic field. The beam-driven fields in the plasma are computed, with particular attention given to the "synchronous" modes ($\omega/k_{\parallel} = v_o$). In the high-density limit, $\omega_{pe}^2 \gg \Omega_e^2$, the plasma has a synchronous mode at the lower-hybrid frequency, $\omega = (\Omega_e \Omega_i)^{1/2}$. We compute the rate of resonant energy transfer from the beam to this mode, and the characteristic time required for the energy transfer. For typical parameters, this transfer time is found to be short compared with the beam pulse duration.

Introduction

Various experimental techniques for producing modulated relativistic electron beams have been proposed. Nation¹ has shown that when a relativistic electron beam is passed through a slow-wave waveguide structure, the beam becomes self-modulated. Friedman and Herndon² have obtained modulated electron beams by allowing the beam to propagate through a spatially modulated magnetic field structure, known as "Ubitron."

Recently, there has been some interest in the possibility of using a modulated electron beam to heat the ions in a background plasma. In low energy experiments, Haas and Dandl³ have demonstrated that ion heating can be accomplished when a modulated electron beam interacts with a beam-formed plasma. Haas and Eisner⁴ have studied the dependence of the ion heating on the frequency of the modulation.

A density-modulated relativistic electron beam can resonantly interact with a mode of a background plasma. The beam parametrically excites oscillations in the background plasma. Ion heating is then effected either by direct coupling between these oscillations and the plasma ions, or through the intermediary stage of plasma instabilities, driven by these oscillations, and which in turn drive energy into ion temperature. A schematic representation of these processes is shown in figure 1.

The properties of the plasma oscillations are determined from the plasma dispersion relations. The beam is most effective in driving "synchronous" plasma modes, i.e., those modes having their phase velocity along the beam equal to the beam velocity, $\omega/k_{\parallel} = V_0$. When the beam modulation frequency coincides with the frequency of one of the synchronous modes, the beam can resonantly drive energy into that mode.

We have represented the modulated relativistic electron beam by a train of N charged discs. Each disc carries charge density $-\rho$, has radius, a , and thickness, L . The thickness of the disc corresponds to the coherence length of the modulated beam, and is assumed small compared with the wavelength of the induced plasma oscillations. This model is employed in lieu of the more rigorous procedure of spatial averaging over the coherence length of the beam. The discs are equally spaced along the z -axis with spacing, D , corresponding to the periodicity of the modulation. The charge density, $-\rho$, is interpreted as the root-mean-square fluctuation in the density of the modulated beam. All the discs have velocity, $\underline{V}_0$, along the z -axis, which is also the direction of an applied uniform magnetic field, $\underline{B}_0$. The first disc crosses the origin, $z = 0$, at time $t = 0$. Figure 2 shows a pictorial representation of this model. We discuss the physical interpretation of the disc model more fully in the Appendix.

This structure is allowed to interact with a cold, uniform background plasma of infinite extent. Effects associated with finite temperature have been deferred to a later study. The background plasma is assumed to be charge neutral and quiescent until time $t = 0$ when the beam-plasma interaction is switched on.

Several Russian authors⁵⁻⁸ have considered the interaction between a modulated electron beam and a plasma. In their analyses the modulation wavelength is assumed small compared with the wavelength of the induced oscillations. The dispersion relation can then be considerably simplified by appropriate averaging over the modulation wavelength. In our analysis, however, we are seeking parametrically-driven plasma oscillations, for

which the modulation wavelength is of the same order as the wavelength of the induced oscillations. In the Appendix we will present a more detailed comparison between our model and the technique employed by the Soviet authors.

Beam-Driven Plasma Oscillations

At time $t = 0$ the discs are located on the negative z -axis, between $z = 0$ and $z = -(N - 1)D - L$. The charge density distribution of the discs is then given by

$$\rho(\underline{x}, t=0) = \begin{cases} -\rho & \text{for } -nD - L \leq z \leq -nD; r \leq a \\ 0 & \text{otherwise} \end{cases}$$

where $n = 0, 1, 2, \dots, N - 1$. Its Fourier transform is

$$\begin{aligned} \rho_{\underline{k}}(0) &= \int d\underline{x} \rho(\underline{x}, t=0) \\ &= -\rho \int_0^a r dr \int_0^{2\pi} d\phi \exp[-ik_{\perp} r \cos \phi] \sum_{n=0}^{N-1} \int_{-nD-L}^{-nD} dz \exp[-ik_{\parallel} z] \\ &= 2\pi a^2 L \rho \frac{J_1(k_{\perp} a)}{k_{\perp} a} \frac{1 - \exp[ik_{\parallel} L]}{ik_{\parallel} L} \frac{1 - \exp[iNk_{\parallel} D]}{1 - \exp[ik_{\parallel} D]} \end{aligned} \quad (4.1)$$

where $J_1(k_{\perp} a)$ is the usual Bessel function, and the coordinates have been oriented so that

$$\underline{k} = \hat{x}k_{\perp} + \hat{z}k_{\parallel}.$$

The electric fields in the plasma are found from Maxwell's equations, which we may combine to obtain

$$\nabla \times (\nabla \times \underline{E}) + \frac{1}{c^2} \frac{\partial^2 \underline{D}}{\partial t^2} = - \frac{4\pi}{c^2} \frac{\partial \underline{J}_D}{\partial t} \quad (4.2)$$

where $\underline{J}_D(\underline{x}, t)$ is the current density associated with the moving discs.

We transform this equation using a Fourier transform in the spatial variables and a Laplace transform for the time variable,

$$\underline{E}(\underline{k}, s) = \int_0^\infty dt e^{-st} \int d\underline{x} e^{-i\underline{k} \cdot \underline{x}} \underline{E}(\underline{x}, t) \quad (4.3)$$

to obtain

$$\begin{aligned} -\underline{k} \underline{k} \cdot \underline{E}(\underline{k}, s) + k^2 \underline{E}(\underline{k}, s) + \frac{s^2}{c^2} \underline{\epsilon}(\underline{k}, s) \cdot \underline{E}(\underline{k}, s) \\ = - \frac{4\pi s}{c^2} \underline{J}_D(\underline{k}, s) + \frac{4\pi}{c^2} \underline{J}_{D\underline{k}}(0) + \frac{s}{c^2} \underline{E}_{\underline{k}}(0) + \frac{1}{c^2} \dot{\underline{E}}_{\underline{k}}(0) \end{aligned} \quad (4.4)$$

The background plasma is assumed to be neutral and quiescent at $t = 0$.

Therefore, the fields $\underline{E}_{\underline{k}}(0)$ and $\dot{\underline{E}}_{\underline{k}}(0)$, at $t = 0$, are due to the moving charge density of the discs. Currents in the background plasma have been included through the plasma dielectric tensor, $\underline{\epsilon}(\underline{k}, s)$, by

$$\underline{D}(\underline{k}, s) = \underline{\epsilon}(\underline{k}, s) \cdot \underline{E}(\underline{k}, s) \quad (4.5)$$

In a more formal notation, we may re-write equation (4.4) as

$$\underline{Y}(\underline{k}, s) \cdot \underline{E}(\underline{k}, s) = \underline{S}(\underline{k}, s), \quad (4.6)$$

from which

$$\begin{aligned}\underline{E}(\underline{k}, s) &= \underline{Y}^{-1}(\underline{k}, s) \cdot \underline{S}(\underline{k}, s) \\ &= \frac{1}{|\underline{Y}(\underline{k}, s)|} \underline{R}(\underline{k}, s) \cdot \underline{S}(\underline{k}, s)\end{aligned}\quad (4.7)$$

where $|\underline{Y}(\underline{k}, s)|$ is the determinant of $\underline{Y}(\underline{k}, s)$, and $\underline{R}(\underline{k}, s)$ is the cofactor matrix of $\underline{Y}(\underline{k}, s)$.

From equations (4.4) and (4.6), the source term, $\underline{S}(\underline{k}, s)$ is given by

$$\underline{S}(\underline{k}, s) = -\frac{4\pi s}{c^2} \underline{J}_D(\underline{k}, s) + \frac{4\pi}{c^2} \underline{J}_{Dk}(0) + \frac{s}{c^2} \underline{E}_k(0) + \frac{1}{c^2} \dot{\underline{E}}_k(0) \quad (4.8)$$

For rigidly translating discs, $\underline{J}_D(\underline{k}, s) = \underline{v}_0 \rho(\underline{k}, s)$, the continuity equation implies

$$\underline{J}_D(\underline{k}, s) = \frac{\underline{v}_0 \rho_k(0)}{s + ik_v v_0} \quad (4.9)$$

Also,

$$\underline{J}_{Dk}(0) = \underline{v}_0 \rho_k(0) \quad (4.10)$$

The fields $\underline{E}_k(0)$ and $\dot{\underline{E}}_k(0)$ are due to the moving discs. In a reference frame where the discs are at rest, the electric field is given by

$$\underline{E}'(\underline{k}') = \frac{-4\pi i}{k'^2} \rho'(\underline{k}') [\hat{x} k_{\perp}' + \hat{z} k_{\parallel}']$$

The electric field $\underline{E}_k(0)$ in the laboratory frame then follows from the Lorentz transformation, evaluated at $t = 0$,

$$\underline{E}_k(0) = \frac{-4\pi i \rho_k(0)}{\gamma_o^2 v_o^2} \frac{\gamma_o \underline{v}_o \times \underline{k} \times \underline{v}_o + \underline{v}_o (\underline{k} \cdot \underline{v}_o)}{k^2 - k_{||}^2 \beta_o^2} \quad (4.11)$$

and

$$\dot{\underline{E}}_k(0) = -i(\underline{k} \cdot \underline{v}_o) \underline{E}_k(0) \quad (4.12)$$

where $\beta_o = v/c$ and $\gamma_o = (1 - \beta_o^2)^{-1/2}$.

Substitution of equation (4.9) and (4.12) into equation (4.8) yields

$$\underline{S}(k,s) = \frac{4\pi i}{c^2} \rho_k(0) \left\{ \frac{\underline{v}_o}{s + i\underline{k} \cdot \underline{v}_o} - (s - i\underline{k} \cdot \underline{v}_o) \frac{\underline{k} - \beta_o^2 k_{||} \underline{\hat{z}}}{k^2 - \beta_o^2 k_{||}^2} \right\} \quad (4.13)$$

From equation (4.4), the response tensor, $\underline{Y}(k,s)$, is

$$\underline{Y}(k,s) = k^2 \underline{I} - \underline{k} \underline{k} + \frac{s^2}{c^2} \underline{\epsilon} \quad (4.14)$$

We assume the background plasma to be a cold, uniform, infinite plasma in a uniform magnetic field $\underline{B}_o$. Both the electrons and the ions are considered as mobile species, with the ions assumed singly charged.

The dielectric tensor for such a plasma is of the form

$$\underline{\epsilon} = \begin{bmatrix} \epsilon_1 & \epsilon_2 & 0 \\ -\epsilon_2 & \epsilon_1 & 0 \\ 0 & 0 & \epsilon_3 \end{bmatrix} \quad (4.15)$$

where

$$\begin{aligned}\epsilon_1 &= 1 + \frac{\omega_{pi}^2(s + \nu)}{s[(s + \nu)^2 + \Omega_i^2]} + \frac{\omega_{pe}^2(s + \nu)}{s[(s + \nu)^2 + \Omega_e^2]} \\ \epsilon_2 &= \frac{\omega_{pi}^2 \Omega_i}{s[(s + \nu)^2 + \Omega_i^2]} - \frac{\omega_{pe}^2 \Omega_e}{s[(s + \nu)^2 + \Omega_e^2]} \\ \epsilon_3 &= 1 + \frac{\omega_{pi}^2 + \omega_{pe}^2}{s(s + \nu)}\end{aligned}\quad (4.16)$$

and $\omega_{p\alpha}^2 \equiv 4\pi n e^2 / m_\alpha$, $\Omega_\alpha \equiv |eB_0 / m_\alpha c|$, and ν is a phenomenological momentum relaxation frequency, which is included in order to properly locate poles when evaluating contour integrals. All results should be interpreted in the limit $\nu \rightarrow 0$. Therefore,

$$\underline{\underline{Y}}(\underline{k}, s) = \begin{bmatrix} k_u^2 + \frac{s^2}{c^2} \epsilon_1 & \frac{s^2}{c^2} \epsilon_2 & -k_\perp k_\parallel \\ -\frac{s^2}{c^2} \epsilon_2 & k^2 + \frac{s^2}{c^2} \epsilon_1 & 0 \\ -k_\perp k_u & 0 & k_\perp^2 + \frac{s^2}{c^2} \epsilon_3 \end{bmatrix} \quad (4.17)$$

The determinant of $\underline{\underline{Y}}(\underline{k}, s)$ is then

$$\begin{aligned}|\underline{\underline{Y}}(\underline{k}, s)| &= (k^2 + \frac{s^2}{c^2} \epsilon_1) \{ (k_u^2 + \frac{s^2}{c^2} \epsilon_1) (k_\perp^2 + \frac{s^2}{c^2} \epsilon_3) - k_\perp^2 k_\parallel^2 \} \\ &\quad + (\frac{s^2}{c^2} \epsilon_2)^2 (k_\perp^2 + \frac{s^2}{c^2} \epsilon_3)\end{aligned}\quad (4.18)$$

We note here that $|\underline{Y}(\underline{k}, s)| = 0$ is the dispersion relation of the background plasma

The cofactor matrix, $\underline{R}(\underline{k}, s)$, has the following components,

$$R_{xx} = (k^2 + \frac{s^2}{c^2} \epsilon_1) (k_{\perp}^2 + \frac{s^2}{c^2} \epsilon_3)$$

$$R_{yy} = (k_{\parallel}^2 + \frac{s^2}{c^2} \epsilon_1) (k_{\perp}^2 + \frac{s^2}{c^2} \epsilon_3) - k_{\perp}^2 k_{\parallel}^2$$

$$R_{zz} = (k_{\parallel}^2 + \frac{s^2}{c^2} \epsilon_1) (k^2 + \frac{s^2}{c^2} \epsilon_1) + (\frac{s^2}{c^2} \epsilon_2)^2$$

$$R_{xy} = -R_{yx} = -\frac{s^2}{c^2} \epsilon_2 (k_{\perp}^2 + \frac{s^2}{c^2} \epsilon_3)$$

$$R_{xz} = R_{zx} = k_{\perp} k_{\parallel} (k^2 + \frac{s^2}{c^2} \epsilon_1)$$

$$R_{yz} = -R_{zy} = \frac{s^2}{c^2} \epsilon_2 k_{\perp} k_{\parallel} \quad (4.19)$$

The electric field, $\underline{E}(\underline{k}, s)$, given by equation (4.7), takes the following form when we substitute equations (4.13) and (4.19),

$$\begin{aligned} \underline{E}_x(\underline{k}, s) = & \frac{-4\pi i \rho_k(0)}{(s + ik_B V_0) (k^2 - \beta_0^2 k_{\parallel}^2)} \frac{1}{|\underline{Y}(\underline{k}, s)|} \\ & \left\{ k_{\perp} \left(\frac{s^2}{c^2} + k_{\parallel}^2 \beta_0^2 \right) (k^2 + \frac{s^2}{c^2} \epsilon_1) (k_{\perp}^2 + \frac{s^2}{c^2} \epsilon_3) \right. \\ & \left. + k_{\perp} k_{\parallel}^2 \beta_0^2 (k^2 + \frac{s^2}{c^2} \epsilon_1) \left(\frac{1}{\gamma_0^2 \beta_0^2} \frac{s^2}{c^2} - k_{\perp}^2 \right) \right\} \end{aligned}$$

$$E_y(\underline{k}, s) = \frac{-4\pi i \rho_{\underline{k}}(0)}{(s + ik_{\parallel} V_0)(k^2 - \beta_0^2 k_{\parallel}^2)} \frac{1}{|Y(\underline{k}, s)|}$$

$$\left\{ \frac{s^2}{c^2} \epsilon_2 k_{\perp} \left(\frac{s^2}{c^2} + k_{\parallel}^2 \beta_0^2 \right) (k_{\perp}^2 + \frac{s^2}{c^2} \epsilon_2) \right.$$

$$\left. + \frac{s^2}{c^2} \epsilon_2 k_{\perp} k_{\parallel}^2 \beta_0^2 \left(\frac{1}{Y_0^2 \beta_0^2} \frac{s^2}{c^2} - k_{\perp}^2 \right) \right\}$$

$$E_z(\underline{k}, s) = \frac{-4\pi i \rho_{\underline{k}}(0)}{(s + ik_{\parallel} V_0)(k^2 - \beta_0^2 k_{\parallel}^2)} \frac{1}{|Y(\underline{k}, s)|}$$

$$\left\{ k_{\perp}^2 k_{\parallel} (k^2 + \frac{s^2}{c^2} \epsilon_1) \left(\frac{s^2}{c^2} + k_{\parallel}^2 \beta_0^2 \right) \right.$$

$$\left. + k_{\parallel} \beta_0^2 \left(\frac{1}{Y_0^2 \beta_0^2} \frac{s^2}{c^2} - k_{\perp}^2 \right) \left[(k_{\parallel}^2 + \frac{s^2}{c^2} \epsilon_1) (k^2 + \frac{s^2}{c^2} \epsilon_1) + \left(\frac{s^2}{c^2} \epsilon_2 \right)^2 \right] \right\}$$

(4.20)

In the asymptotic limit, $t \rightarrow \infty$, we need only retain the pole at

$$s = -ik_{\parallel} V_0.$$

$$E_x(\underline{k}, t \rightarrow \infty) = \frac{4\pi i \rho_{\underline{k}}(0) e^{-ik_{\parallel} V_0 t}}{|Y(\underline{k}, -ik_{\parallel} V_0)|} k_{\perp} k_{\parallel}^2 \beta_0^2 (k^2 - k_{\parallel}^2 \beta_0^2 \epsilon_1)$$

$$E_y(\underline{k}, t \rightarrow \infty) = \frac{-4\pi i \rho_{\underline{k}}(0) e^{-ik_{\parallel} V_0 t}}{|Y(\underline{k}, -ik_{\parallel} V_0)|} \epsilon_2 k_{\perp} k_{\parallel}^4 \beta_0^4$$

$$E_z(\underline{k}, t \rightarrow \infty) = \frac{4\pi i \rho_k(0) e^{-ik_{\parallel} v_0 t}}{|Y(\underline{k}, -ik_{\parallel} v_0)|} k_{\parallel} \beta_0^2$$

$$[k_{\parallel}^2 (1 - \beta_0^2 \epsilon_1) (k^2 - k_{\parallel}^2 \beta_0^2 \epsilon_1) + k_{\parallel}^4 \beta_0^4 \epsilon_2^2] \quad (4.21)$$

where $\underline{\epsilon} = \underline{\epsilon}(s = -ik_{\parallel} v_0)$.

The natural response frequencies of the plasma are given by the solution of the dispersion relation, $|Y(\underline{k}, s)| = 0$. Using equations (4.16) and (4.18), neglecting terms of order m_e/m_i relative to order unity, we obtain

$$|Y(\underline{k}, s)| = \frac{s^6}{k^2 c^6} \left\{ \left(\eta^2 + \frac{\omega_p^2}{s(s+v)} \right) \right.$$

$$\left[\eta^2 + \frac{\eta^2 \omega_p^2 \left(\frac{s+v}{s} \right) [(s+v)^2 + \Omega_e \Omega_i] + \omega_p^4 \left(\frac{s+v}{s} \right)^2}{(s+v)^2 \Omega_e^2} \right] k_{\perp}^2$$

$$+ \left(1 + \frac{\omega_p^2}{s(s+v)} \right) \left[\eta^2 + \frac{\omega_p^2}{s} \frac{s+v}{[(s+v) + i\Omega_e][s+v - i\Omega_i]} \right]$$

$$\left. \left[\eta^2 + \frac{\omega_p^2}{s} \frac{s+v}{[s+v - i\Omega_e][s+v + i\Omega_i]} \right] k_{\parallel}^2 \right\} \quad (4.22)$$

for the regime

$$\Omega_e^2 \gg |s^2| \gg \Omega_i^2$$

$$k^2 c^2 \gg |s^2| \quad (4.23)$$

and with the notation $\eta \equiv kc/s$ and $\omega_p^2 \equiv \omega_{pe}^2$. We examine equation (4.22) in the high-density limit, $\omega_p^2 \gg \Omega_e^2$, and in the low-density limit,

$\omega_p^2 \ll \Omega_e^2$. The approximation, $k_\perp^2 \gg k_\parallel^2$, is employed in both density regimes. In the high-density regime we obtain

$$|\underline{Y}(k, -ik_\parallel v_o)| = -\frac{\omega_p^2}{\Omega_e^2} \frac{k^2}{c^2} (1 - \alpha_o^2) v_o^2 (k_\parallel - k_1)(k_\parallel - k_2)(k_\parallel - k_3)(k_\parallel - k_4) \quad (4.24)$$

where $\alpha_o \equiv \omega_p^2 / k_\perp^2 c^2$, and

$$k_{1,2} = \pm \frac{[\Omega_e^2 + \alpha_o \Omega_e \Omega_i - (1 + \alpha_o)^2 k_\perp^2 v_o^2]^{1/2}}{(1 - \alpha_o^2)^{1/2} v_o}$$

$$- \frac{i v [\Omega_e^2 + \alpha_o \Omega_e \Omega_i - \alpha_o (1 + \alpha_o)^2 k_\perp^2 v_o^2]}{(1 - \alpha_o^2) v_o [\Omega_e^2 + \alpha_o \Omega_e \Omega_i - (1 + \alpha_o)^2 k_\perp^2 v_o^2]}$$

$$k_{3,4} = \pm \frac{(\Omega_e \Omega_i)^{1/2}}{v_o} \left[\frac{\Omega_e \Omega_i - (1 + \alpha_o) k_\perp^2 v_o^2}{\Omega_e^2 + \alpha_o \Omega_e \Omega_i - (1 + \alpha_o)^2 k_\perp^2 v_o^2} \right]^{1/2}$$

$$+ \frac{i v (1 + \alpha_o) k_\perp^2 v_o}{\Omega_e^2 + \alpha_o \Omega_e \Omega_i - (1 + \alpha_o)^2 k_\perp^2 v_o^2} \quad (4.25)$$

We take the limits $\alpha_o \gg 1$ and $\alpha_o \ll 1$ in equation (4.24). For $\alpha_o \gg 1$ we obtain $\{\omega_p^2 \gg \Omega_e^2, \omega_p^2 \gg k_\perp^2 c^2\}$

$$k_{1,2} = \pm k_\perp + \frac{i v}{v_o} \frac{k_\perp^2 c^2}{\omega_p^2} \quad (4.26a)$$

$$k_{3,4} = \pm \frac{\sqrt{\Omega_e \Omega_i}}{v_o} \frac{k_\perp c}{\omega_p} - \frac{i v}{v_o} \frac{k_\perp^2 c^2}{\omega_p^2} \quad (4.26b)$$

For $\alpha_o \ll 1$ we obtain $[\Omega_e^2 \ll \omega_p^2 \ll k_\perp^2 c^2]$

$$k_{1,2} = \pm i k_\perp - \frac{i v}{v_o} \frac{\omega_p^2}{k_\perp^2 c^2} \quad (4.27a)$$

$$k_{3,4} = \pm \frac{\sqrt{\Omega_e \Omega_i}}{v_o} - \frac{i v}{v_o} \quad (4.27b)$$

In the low-density regime,

$$|\underline{Y}(k, -i k_\parallel v_o)| \approx -\frac{k_\perp^2}{c^2} (1 - \alpha_o) v_o^2 (k_\parallel - k_1)(k_\parallel - k_2)(k_\parallel - k_3)(k_\parallel - k_4) \quad (4.28)$$

where

$$k_{1,2} = \pm \frac{[\omega_p^2 - (1 + \alpha_o) k_\perp^2 v_o^2]^{1/2}}{(1 - \alpha_o)^{1/2} v_o} - \frac{i v \omega_p^2}{2 v_o [\omega_p^2 - (1 + \alpha_o) k_\perp^2 v_o^2]}$$

$$k_{3,4} = \pm i \frac{\omega_{pi} k_\perp (1 + \alpha_o)^{1/2}}{[\omega_p^2 - (1 + \alpha_o) k_\perp^2 v_o^2]^{1/2}} + \frac{i v k_\perp^2 v_o (1 + \alpha_o)}{2 [\omega_p^2 - (1 + \alpha_o) k_\perp^2 v_o^2]} \quad (4.29)$$

We examine the limits $\alpha_o \gg 1$ and $\alpha_o \ll 1$ in equation (4.28). For $\alpha_o \gg 1$ we obtain $[\Omega_e^2 \gg \omega_p^2 \gg k_\perp^2 c^2 \gg \omega_{pi}^2]$

$$k_{1,2} = \pm i \frac{k_\perp}{\gamma_o \beta_o} - \frac{i v}{2 v_o} \gamma_o^2 \quad (4.30a)$$

$$k_{3,4} = \pm i \gamma_o \frac{\omega_{pi}}{c} + \frac{i v}{2 c} \gamma_o^2 \beta_o \quad (4.30b)$$

For $\alpha_0 \ll 1$ we obtain $[\omega_p^2 \ll \Omega_e^2, \omega_p^2 \ll k_\perp^2 c^2]$

$$k_{1,2} = \pm ik_\perp + \frac{i\nu}{2V_0} \frac{\omega_p^2}{k_\perp^2 V_0^2} \quad (4.31a)$$

$$k_{3,4} = \pm \frac{\omega_{pi}}{V_0} - \frac{i\nu}{2V_0} \quad (4.31b)$$

From equation (4.21), then, the electric field is given in the high-density limit by

$$\underline{E}(k, t \rightarrow \infty) = \frac{-8\pi^2 \rho a J_1(k_\perp a) k_\perp^2 L}{(\omega_p / \Omega_e)^2 k^2 (1 - \alpha_0^2) \prod_{j=1}^4 (k_\perp - k_j)} \exp[-ik_\parallel V_0 t] \\ \frac{1 - \exp[ik_\parallel L]}{k_\parallel L} \frac{1 - \exp[iNk_\parallel D]}{1 - \exp[ik_\parallel D]} \underline{f}(k_\perp, k_\parallel)$$

where $k_1, \dots, k_4$ are given by equation (4.25) or equations (4.26) and (4.27), and

$$\underline{f}_x(k_\perp, k_\parallel) = \frac{k^2 - k_\parallel^2 \beta_0^2 (\omega_p / \Omega_e)^2 + (\omega_{pi} / c)^2}{k^2 - k_\parallel^2 \beta_0^2 (\omega_p / \Omega_e)^2 + (\omega_{pi} / c)^2}$$

$$\underline{f}_y(k_\perp, k_\parallel) = i(\omega_p / \Omega_e)^2 (\Omega_e / c) k_\parallel \beta_0$$

$$\underline{f}_z(k_\perp, k_\parallel) = (k_\parallel / k_\perp) \left\{ [1 - \beta_0^2 (\omega_p / \Omega_e)^2 + (\omega_{pi} / k_\parallel c)^2] \right.$$

$$\left. [k^2 - k_\parallel^2 \beta_0^2 (\omega_p / \Omega_e)^2 + (\omega_{pi} / c)^2] - (\omega_p / \Omega_e)^2 (\omega_p / c)^2 \beta_0^2 \right\}$$

In the low-density limit, the electric field is

$$\underline{E}(\underline{k}, t \rightarrow \infty) = \frac{-8\pi^2 \rho a J_1(k_{\perp} a) k_{\parallel}^2 L}{k^2 (1 - \alpha_0) \prod_{j=1}^4 (k_{\parallel} - k_j)} \exp[-ik_{\parallel} v_0 t]$$

$$\frac{1 - \exp[ik_{\parallel} L]}{k_{\parallel} L} \frac{1 - \exp[iNk_{\parallel} D]}{1 - \exp[ik_{\parallel} D]} \underline{f}(k_{\perp}, k_{\parallel})$$

where $k_1, \dots, k_4$ are given by equation (4.29) or equations (4.30) and (4.31), and

$$f_x(k_{\perp}, k_{\parallel}) = k^2$$

$$f_y(k_{\perp}, k_{\parallel}) = i(\omega_p / \Omega_e)^2 (\Omega_e / c) k_{\parallel} \beta_0$$

$$f_z(k_{\perp}, k_{\parallel}) = k^2 (k_{\parallel} / k_{\perp}) \left[1 - \beta_0^2 + (\omega_{pi} / k_{\parallel} c)^2 \right]$$

Energy Transfer from the Beam to Plasma Oscillations

We now compute the rate at which the beam can resonantly lose energy to a synchronous mode of the background plasma.

The time-averaged rate of energy transfer from the beam is given by

$$\langle \text{Power} \rangle = -1/2 \operatorname{Re} \int \frac{d\underline{k}}{(2\pi)^3} \underline{J}_D^*(\underline{k}, t \rightarrow \infty) \cdot \underline{E}(\underline{k}, t \rightarrow \infty)$$

$$= -1/2 \operatorname{Re} \int \frac{d\underline{k}}{(2\pi)^3} \underline{J}_D^*(\underline{k}, t \rightarrow \infty) \cdot \underline{Y}^{-1}(\underline{k}, -ik_{\parallel} v_0) \cdot \underline{S}(\underline{k}, t \rightarrow \infty)$$

$$\langle \text{Power} \rangle = 2\pi\beta_o^2 \int \frac{dk}{(2\pi)^3} k_{\parallel} v_o \text{Im} Y_{zz}^{-1} |\rho_{\underline{k}}(0)|^2 \quad (4.32)$$

Using equation (4.7),

$$Y_{zz}^{-1} = \frac{R_{zz}}{|\underline{Y}|} \quad (4.33)$$

where $|\underline{Y}| = \text{Re}|\underline{Y}| + i \text{Im}|\underline{Y}|$, and the dispersion relation is $|\underline{Y}| = 0$.

In solving for the fields, $\underline{E}(\underline{k}, t \rightarrow \infty)$, in the previous subsection, only $\text{Re} \underline{Y}^{-1}$ entered into the formalism, and the dispersion relation used there was $\text{Re}|\underline{Y}| = 0$, which we evaluated for a cold plasma. This procedure is valid for computing any quantity which is linear in the fields. To compute nonlinear quantities, such as dissipated energy, which is quadratic in the fields, we must include $\text{Im} \underline{Y}^{-1}$ in the formalism.

For parametrically-driven fields, where $\underline{k}$ is assumed real, we approximate the dispersion relation by

$$\text{Re} |\underline{Y}(\underline{k}, -ik_{\parallel} v_o)| \approx 0 \quad (4.34)$$

The solutions to equation (4.34) were obtained in various limits in the previous subsection. For example, in the regime, $\omega_p^2 \gg \Omega_e^2$, corresponding to a high-density plasma, we found (cf. equations (4.24) and (4.27))

$$\text{Re} |\underline{Y}(\underline{k}, -ik_{\parallel} v_o)| = -(\omega_p/\Omega_e)^2 k_{\parallel}^2 \beta_o^2 (k_{\parallel}^2 - k_o^2) \quad (4.35)$$

where

$$k_o^2 = \Omega_e \Omega_i / v_o^2 \quad (4.36)$$

We express $\text{Im } Y_{zz}^{-1}$, using an approach similar to that employed by Stone and Auer,⁹ by expanding equation (4.35) in the vicinity of $k_{||} = k_0$.

Let ε_0 denote $\text{Im } |Y(k_{||}=k_0)|$.

$$Y_{zz}^{-1} = \frac{R_{zz}}{-2(\omega_p/\Omega_e)^2(k_{\perp}^2 + k_0^2)^2 k_0 \beta_0^2 (k_{||} - k_0) + i\varepsilon_0} \quad (4.37)$$

where

$$R_{zz} = -k_0^2(k_{\perp}^2 + k_0^2) \quad (4.38)$$

If we now let κ denote $\varepsilon_0/[2(\omega_p/\Omega_e)^2(k_{\perp}^2 + k_0^2)^2 k_0 \beta_0^2]$, we obtain

$$Y_{zz}^{-1} = \frac{1}{2} \frac{k_0}{(k_{\perp}^2 + k_0^2) \beta_0^2} (\Omega_e/\omega_p)^2 \frac{1}{(k_{||} - k_0) - i\kappa}$$

and

$$\text{Im } Y_{zz}^{-1} = \frac{1}{2} \frac{k_0}{(k_{\perp}^2 + k_0^2) \beta_0^2} (\Omega_e/\omega_p)^2 \left[\frac{\kappa}{(k_{||} - k_0)^2 + \kappa^2} \right] \quad (4.39)$$

The expression in square brackets in equation (4.39) is a Lorentzian function, peaked at $k_{||} = k_0$, with height, $1/\kappa$, and width, 2κ . For $\kappa \ll k_0$, we may replace the square bracket with a suitably normalized delta function for purposes of integration over $k_{||}$. We then obtain

$$\text{Im } Y_{zz}^{-1} \approx \frac{\pi}{2} \frac{k_0}{(k_{\perp}^2 + k_0^2) \beta_0^2} (\Omega_e/\omega_p)^2 \delta(k_{||} - k_0) \quad (4.40)$$

Turning now to the factor, $|\rho_{\underline{k}}(0)|^2$, in equation (4.32), we use equation (4.1) to obtain

$$|\rho_{\underline{k}}(0)|^2 = 4\pi^2 a^4 \rho^2 L^2 \frac{J_1^2(k_{\perp} a)}{(k_{\perp} a)^2} \left[\frac{\sin^2(k_{\parallel} L/2)}{(k_{\parallel} L/2)^2} \frac{\sin^2(Nk_{\parallel} D/2)}{\sin^2(k_{\parallel} D/2)} \right] \quad (4.41)$$

Therefore, $|\rho_{\underline{k}}(0)|^2$ has strong peaks when $k_{\parallel} = mk_D$, where m is an integer and $k_D = 2\pi/D$ is the modulation wavenumber. There are also $N - 2$ weaker secondary peaks between each pair of strong peaks. We note as an aside that the term in square brackets in equation (4.41) is essentially the N -slit interference intensity in the Fraunhofer limit. In the language of physical optics, the first factor in the square brackets is the diffraction envelope, the single-slit diffraction pattern, while the second factor is the structure term for the interference between the N slits.

Combining equations (4.32), (4.39) and (4.41), and integrating over the delta function in $k_{\parallel}$, we obtain

$$\langle \text{Power} \rangle = \pi^2 a^4 \rho^2 L^2 (\Omega_e / \omega_p)^2 k_o^2 V_o \frac{\sin^2(k_o L/2)}{(k_o L/2)^2} \frac{\sin^2(Nk_o D/2)}{\sin^2(k_o D/2)} \Gamma \quad (4.42)$$

where

$$\Gamma \equiv \int_0^{\infty} \frac{dx J_1^2(x)}{x(x^2 + k_o^2 a^2)} \quad (4.43)$$

We may re-write equation (4.42) in terms of $\bar{I}$, the average current carried by the charged discs,

$$\bar{I} = \pi a^2 \rho V_o (L/D), \quad (4.44)$$

as

$$\langle \text{Power} \rangle = \bar{I}^2 (k_o^2 v_o) D^2 (\Omega_e / \omega_p)^2 \frac{\sin^2(k_o L/2)}{(k_o L/2)^2} \frac{\sin^2(N k_o D/2)}{\sin^2(k_o D/2)} \Gamma \quad (4.45)$$

The total kinetic energy carried by N charged discs is given by

$$\begin{aligned} E_D &= (\gamma_o - 1) \frac{mc^2}{e} \bar{I} \frac{(N-1)D}{v_o} \\ &= I_A \bar{I} \frac{\gamma_o - 1}{\gamma_o} \frac{(N-1)D}{v_o^2} \end{aligned} \quad (4.46)$$

for $L \ll (N-1)D$. Here I_A is the Alfvén current,

$$I_A = \frac{mc^3}{e} \beta_o \gamma_o \approx 17,000 \beta_o \gamma_o \text{ Amps} \quad (4.47)$$

From equations (4.45) and (4.46), we may estimate the time, τ , for the beam to lose half of its initial kinetic energy

$$\frac{E_D}{2\tau} = \langle \text{Power} \rangle \quad (4.48)$$

or

$$\begin{aligned} \tau &= \frac{1}{2} (I_A / \bar{I}) (\omega_p / \Omega_e)^2 \left(\frac{\gamma_o - 1}{\gamma_o} \right) \frac{N-1}{(k_o D) (k_o v_o) \Gamma} \\ &\quad \frac{(k_o L/2)^2 \sin^2(k_o D/2)}{\sin^2(k_o L/2) \sin^2(N k_o D/2)} \end{aligned} \quad (4.49)$$

We now consider resonant energy transfer from the beam to the plasma oscillations, i.e., $k_o = mk_D$, $m = 1, 2, \dots$. We assume m to be sufficiently small that $k_o L/2 = mk_D L/2 \ll 1$. Then,

$$\tau = \frac{1}{4\pi m^2} (I_A/\bar{I}) (\omega_p/\Omega_e)^2 \left(\frac{\gamma_o - 1}{\gamma_o} \right) \frac{N-1}{N^2} \frac{1}{k_D v_o} \frac{1}{\Gamma} \quad (4.50)$$

For a beam with $\gamma_o = 2$, beam radius $a = 5$ cm, carrying 1 MAmp of current with ten percent modulation and $L/D = 0.1$ (i.e., an average disc current, $\bar{I} = 10$ kAmps), interacting with a background plasma with a density, $n = 10^{13} \text{ cm}^{-3}$, in a magnetic field, $B_o = 1$ kG, the beam would transfer half of the kinetic energy carried by the modulation in a time

$$\tau = (33.5 \text{ nsec}) \frac{N-1}{mN^2} \quad (4.51)$$

If the beam is four modulation wavelengths in length ($N = 5$), the resonant energy transfer at $m = 1$ ($k_o = k_D$) has $\tau = 5.35$ nsec, which is considerably shorter than typical beam lengths of 50 to 100 nsec. The factor, Γ , was calculated by a numerical integration, and is approximately, $\Gamma \approx 0.895$, for these parameters.

With the above parameters, the amount of energy transferred to the plasma in the time $\tau = 5.35$ nsec is $W = \frac{1}{2} E_D = 160$ J.

In summary, we conclude that the resonant energy transfer from modulated relativistic electron beams to plasma oscillations can occur on a time scale which is shorter than the beam pulse duration time. We have deferred the question of finite temperature effects, which would arise in the interaction between a beam and a warm background plasma. The problem of energy transfer from the plasma oscillations to ion temperature has also been left to a later study.

Plasma heating by modulated relativistic electron beams presents an attractive alternative to previously proposed¹¹ plasma heating by return currents, so-called "return current heating," in that modulated beam heating allows a more rapid transfer of beam energy through resonant coupling of the beam and the background plasma.

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Figure 1 : Schematic Representation of Energy Flow
from a Modulated Electron Beam to Ion Temperature

Figure 2: Model of Modulated Relativistic Electron Beam

Appendix

In this appendix we explore the implicit assumptions associated with the disc model of a modulated beam, which we have employed in the text. We will also examine a different approach to modulated beam-plasma interactions, which has been developed by several investigators in the Soviet Union. 5-8

We consider the modulated beam-plasma interaction as a dressed test-particle calculation. We imagine a single electron from the beam to be on a rectilinear trajectory, starting from an initial position, $\underline{x}_0$, at time $t = 0$. We form the test current,

$$\underline{J}_T(\underline{x}, t; \underline{x}_0) = -eV_0 \hat{z} \delta(x-x_0) \delta(y-y_0) \delta(z-z_0 - V_0 t) \quad (C.1)$$

which can drive fields in the plasma.

The fields due to the test particle are found from Maxwell's equations,

$$\nabla \times (\nabla \times \underline{E}) = -\frac{4\pi}{c^2} \frac{\partial}{\partial t} [\underline{J}_p + \underline{J}_B + \underline{J}_T(\underline{x}, t; \underline{x}_0)] - \frac{1}{c^2} \frac{\partial^2 \underline{E}}{\partial t^2} \quad (C.2)$$

where $\underline{J}_p$ and $\underline{J}_B$ are the plasma and beam current densities respectively.

By Fourier analysis of equation (C.2) in space and time, we obtain

$$[k^2 \underline{I} - \underline{k} \underline{k} - \frac{\omega^2}{c^2 \epsilon_p} - \frac{4\pi i \omega}{c^2 \epsilon_B}] \cdot \underline{E}(\underline{k}, \omega; \underline{x}_0) = \frac{4\pi i \omega}{c^2} \underline{J}_T(\underline{k}, \omega; \underline{x}_0) \quad (C.3)$$

where we have used

$$\underline{J}_{p,B}(\underline{k}, \omega) = \underline{\sigma}_{p,B}(\underline{k}, \omega) \cdot \underline{E}$$

for the plasma and beam current densities, and

$$\underline{\epsilon}_p(\underline{k}, \omega) = \underline{I} + \frac{4\pi i}{\omega} \underline{\sigma}_p(\underline{k}, \omega) \quad (C.4)$$

for the plasma dielectric tensor.

In the text we tacitly assumed that the average beam density was small compared with the plasma density, which allowed us to neglect σ_B with respect to σ_p in equation (C.3).

The Fourier transform of the test current density is given by

$$\underline{J}_T(\underline{k}, \omega; \underline{x}_0) = -2\pi e V_0 \hat{z} \exp[-i\underline{k} \cdot \underline{x}_0] \delta(\omega - k_u V_0) \quad (C.5)$$

We may re-write equation (C.3) in formal notation, after substituting equation (C.5), as

$$\underline{Y}(\underline{k}, \omega) \cdot \underline{E}(\underline{k}, \omega; \underline{x}_0) = \frac{8\pi^2 i e}{c^2} \omega V_0 \exp[-i\underline{k} \cdot \underline{x}_0] \delta(\omega - k_u V_0) \hat{z}$$

and solving for the j-th component of $\underline{E}$,

$$E_j(\underline{k}, \omega; \underline{x}_0) = \frac{8\pi^2 i e}{c^2} \omega V_0 \exp[-i\underline{k} \cdot \underline{x}_0] Y_{jz}^{-1}(\underline{k}, \omega) \delta(\omega - k_u V_0) \quad (C.6)$$

The total electric field, $\underline{E}(\underline{k}, \omega)$, is then obtained by integrating equation (C.6) over a distribution of test electrons

$$\begin{aligned} E_j(\underline{k}, \omega) &= \int d\underline{x}_0 n(\underline{x}_0) E_j(\underline{k}, \omega; \underline{x}_0) \\ &= \frac{8\pi^2 i e}{c^2} \omega V_0 Y_{jz}^{-1}(\underline{k}, \omega) \delta(\omega - k_u V_0) n(\underline{k}) \end{aligned} \quad (C.7)$$

where $n(\underline{x}_0)$ is the initial density distribution for the modulated beam, and $n(\underline{k})$ is its Fourier transform.

Similarly, the total beam current is found from a superposition of test currents,

$$\underline{J}_B(\underline{k}, \omega) = -2\pi e V_0 \hat{z} \delta(\omega - k_u V_0) n(\underline{k}) \quad (C.8)$$

The time-averaged rate of energy transfer from the beam to the plasma fields is then given by

$$\begin{aligned} & -1/2 \operatorname{Re} \int \frac{d\underline{k}}{(2\pi)^3} \underline{J}_B^*(\underline{k}, t) \cdot \underline{E}(\underline{k}, t) \\ & = 2\pi e^2 \beta_o^2 \int \frac{d\underline{k}}{(2\pi)^3} k_{\parallel} V_o \operatorname{Im} Y_{zz}^{-1} |n(\underline{k})|^2 \end{aligned} \quad (C.9)$$

which depends on the modulated beam parameters through the factor, $|n(\underline{k})|^2$.

We assume that the beam density may be written as the sum of an unmodulated component, $n_o(r)$, and a modulated component, $\delta n(r, z)$,

$$n(\underline{x}_o) = n_o(r) + \delta n(r, z) \quad (C.10)$$

The Fourier transform of this structure is then

$$n(\underline{k}) = 2\pi n_o(k_{\perp}) \delta(k_{\parallel}) + \delta n(\underline{k}) \quad (C.11)$$

Since the unmodulated component contributes only to the $k_{\parallel} = 0$ term of the spectrum, it provides only a dc level, and we have neglected it in the text.

The remaining term, $\delta n(\underline{k})$, due to the modulation of the beam, drives oscillating fields in the plasma. From equation (C.9), neglecting the dc term in equation (C.11), the energy transferred to the plasma oscillations will depend on the quantity, $|\delta n(\underline{k})|^2$.

For a beam which is uniform in cross-section, with radius, a , and modulated along the z -axis, with line density, $\delta N(z)$,

$$\delta n(r, z) = \begin{cases} \delta N(z) / \pi a^2 & r \leq a \\ 0 & r > a \end{cases} \quad (C.12)$$

Its Fourier transform is

$$\delta n(k) = \frac{2}{k_{\perp} a} J_1(k_{\perp} a) \delta N(k_{\parallel}) \quad (C.13)$$

and taking the absolute square,

$$|\delta n(k)|^2 = \frac{4}{k_{\perp}^2 a^2} J_1^2(k_{\perp} a) |\delta N(k_{\parallel})|^2 \quad (C.14)$$

We can express $|\delta N(k_{\parallel})|^2$ as

$$\begin{aligned} |\delta N(k_{\parallel})|^2 &= \int d\lambda \exp[-ik_{\parallel}\lambda] \int dz \delta N(z) \delta N(z + \lambda) \\ &= \int d\lambda \exp[-ik_{\parallel}\lambda] C(\lambda) \end{aligned} \quad (C.15)$$

where $C(\lambda)$ is the correlation function,

$$C(\lambda) = \int dz \delta N(z) \delta N(z + \lambda) \quad (C.16)$$

Equation (C.15), together with equations (C.14) and (C.9), forms a statement of the Wiener-Khintchin theorem.¹⁰

The coherence length, denoted by L in the text, is just the width of $C(\lambda)$, or equivalently the inverse width of $|\delta N(k_{\parallel})|^2$. If the wavelength of the modulation is long compared to L , $L \ll D$, we may approximate $\delta N(z)$ by a periodic step function, having period D and step width L . This approximation is the disc model employed in the text.

Since the power transferred from the beam is proportional to $|\delta N|^2$, we may interpret the height of the steps, i.e., the amount of charge carried by each disc, as the square root of the average of δN^2 over the length L . Therefore, the charge density, $-\rho$, on each disc is the root-mean-square of the fluctuation in density of the modulated beam.

For parametrically-driven oscillations and resonant energy transfer from the beam to the plasma oscillations, we require that the modulation wavelength be of the same order as the wavelength of the induced plasma oscillations. Therefore, L is assumed to be small compared with the induced plasma wavelength.

In a different regime, namely when the modulation wavelength is short compared with the induced plasma wavelength, it is possible to use the method of averages developed by several Russian authors.⁵⁻⁸ In this regime, the induced fields consist of a slowly-varying component, E_0 , and a component, E_1 , which oscillates with wavelengths on the order of the modulation wavelength. Expanding the rapidly oscillating quantities in a Fourier series in $k_D = 2\pi/D$, the oscillating quantities are found to be small, $E_1 \ll E_0$. By averaging the dielectric tensor for the beam and plasma over the modulation wavelength, a simple dispersion relation for beam-plasma interactions can be obtained.

The main differences, then, between these two approaches lies in the scale length used for spatial averaging. The Russian authors average over the modulation wavelength. In their work the modulated beam is assumed to be electrostatically neutralized by the plasma in zero-order, the fields then being due solely to the first-order perturbations in the beam and plasma densities. In our work, the beam electrons are assumed to travel on rectilinear trajectories, and the entire modulated beam is not considered as electrostatically neutral, i.e., we include the electric field due to the charged discs as a zero-order field. The averaging implied by the disc model is then

over the coherence length, which is short compared with both the modulation wavelength and with the wavelength of the induced plasma oscillations, which allows us to work in the regime where resonant energy transfer from the beam to the plasma can occur.

**The Application of Quasilinear Theory to the
Cornell Turbulent Heating Experiments**

by

Alfred Mondelli and Edward Ott

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Abstract

We have applied quasilinear theory to the Cornell turbulent heating experiments with the inclusion of the actual external circuitry of the experiments in our model. The resulting set of coupled, first-order, ordinary differential equations are integrated numerically by the Runge-Kutta-Gill technique.

A one-dimensional model with Maxwellian electrons and ions, assuming a current-driven ion-acoustic instability as the heating mechanism, is employed. The observed high-energy ion component is therefore excluded from our analysis. The computed current and voltage vs. time have several significant points of agreement with those in the experiments, including the "spikey" structure observed in the voltage traces and the premature quenching of the instability noted in the experiments. The computed final ion temperature is in reasonable agreement with the experimentally observed bulk ion temperature (excluding the high-energy ion tail). The total electron plus ion final temperature is consistent with the diamagnetic signals observed in the experiments.

The premature turn-off of the instability appears in our calculation to be due to ions heating more efficiently than electrons, resulting in a decrease of T_e/T_i , which stabilizes the instability. As the electron drift velocity rises, the instability turns on and off several times, until T_e/T_i has decreased to the point where the drift velocity can no longer cross the threshold for instability. The observed voltage spikes are due to the successive recurrences of the instability. The duration of the computed voltage spikes is in good agreement with the experimentally observed voltage traces.

Introduction

We have attempted to apply quasilinear theory to describe the Cornell turbulent heating experiments (Wharton, et al. 1971; Robertson, et al. 1972). The theory may be represented by an infinite set of coupled, first-order, ordinary differential equations describing both the plasma and the external circuitry. By picking a sample, discrete wave spectrum out of the actual continuum spectrum, we can reduce the theory to a finite set of differential equations, which we then integrate numerically by the well-known Runge-Kutta-Gill technique (cf. Appendix I). Numerical solution of the quasilinear equations was first applied to the problem of anomalous resistivity by Field and Fried (1964). Subsequent work along these lines has been done by Fried and Ossakow (1966), and by Davidson and Caponi (to be published). The contribution of the present work is an attempt to make detailed application and comparison to a particular experiment with the realistic inclusion in the model of the actual external circuitry of the experiment. In view of the many competing theories of anomalous resistivity, it is only by such a comparison that reasonable credibility of a model can be established.

The Cornell turbulent heating facility (ETH-III) and the experimental results have been described in detail (Wharton, et al. 1971; Robertson, et al. 1972). The experiment may be represented by an equivalent circuit, shown in Fig. 1. At $t=0$ the switch is closed, and the capacitor bank, C, is allowed to discharge through the plasma. In Fig. 2 we show some typical voltage and current traces obtained in the experiments. The instability sets in at a predictable threshold (Stringer, 1964), accompanied by a sharp, sudden rise in the voltage across the plasma column. The voltage traces in Fig. 2 (particularly 2(c) and 2(d)) show a "spikey" structure, corresponding to the instability repeatedly turning on and off. The instability is finally

quenched, usually before the peak current is reached. This premature quenching of the instability is a significant feature of the turbulent heating experiments since it can severely degrade the amount of plasma heating. The purpose of our theoretical study has been two-fold: (i) to determine whether the quasilinear theory forms a valid basis for discussion of the turbulent heating experiments, and (ii) to explain the premature quenching of the instability, mentioned above.

The qualitative theoretical picture which we have assumed, and which appears to be consistent with both our computer results and the experimental observations is that the heating is effected by a current-driven ion-acoustic instability. The initial plasma in the experiments is a column 150 cm in length with a line density, $nA = 2 \times 10^{14} \text{ cm}^{-1}$ (A is the cross-sectional area of the column), and density, $n = 5-8 \times 10^{13} \text{ cm}^{-3}$. Initially $T_e/T_i \sim 1$, which precludes the possibility of an ion-acoustic instability at very early times. Some other mechanism (possibly the two-stream instability) must be present at these early times to heat the electrons and raise T_e/T_i to values large compared with unity. As T_e increases, the ion-acoustic instability becomes possible for drift velocities in excess of a critical velocity of order $c_s = (T_e/n_i)^{1/2}$.

Our computer analysis follows the evolution of a one-dimensional wave spectrum from the onset of the ion-acoustic instability. The "initial" temperature ratio, T_{e0}/T_{i0} , at the time of onset of the ion-acoustic instability is an input parameter in the computer program. This parameter is not known experimentally, but reasonable choices of T_{e0}/T_{i0} all lead to approximately the same behavior. The frequencies and growth rates for the various modes are computed with the approximation, $T_e/T_i \gg 1$. Comparison with Stringer's work (Stringer, 1964) shows that this approximation is satisfactory for $T_{e0}/T_{i0} \geq 15$.

The electrons and ions are both assumed to be described by

Maxwellian distributions. The observed high-energy ion tail, or "two-temperature" distribution, is therefore excluded from our analysis. The final ion temperature which we obtain is comparable to the "bulk" ion temperature in the "two-temperature" model of the experiments. Also, the final computed $T_e + T_i$ is consistent with the diamagnetic loop signals in the experiments.

During the ion-acoustic instability, computed ion and electron heating rates are such that T_e/T_i decreases, which quenches the instability. As the drift velocity continues to rise, the instability reappears, and T_e/T_i is reduced still further, again quenching the instability. This process continues until T_e/T_i has been sufficiently reduced that the drift velocity can no longer cross the instability threshold. The voltage across the plasma column therefore shows a spikey behavior, each spike lasting approximately 50 nsec, in agreement with the experimentally observed voltage traces. The final spike, beyond which T_e/T_i is too small for the instability to recur, typically occurs substantially earlier than the maximum drift velocity, which is the "premature turn-off phenomenon" observed in the experiments. Thus the anomalous resistivity observed in the Cornell turbulent heating experiments cannot be explained solely on the basis of a steady-state anomalous resistivity. Indeed, the time dependence appears to contain the dominant physical processes as previously noted by Field and Fried (1964).

Formalism

The electrons have drift velocity, v_d , while the ions are assumed to be at rest. By conservation of energy and momentum for the electrons and waves we have

$$\frac{d}{dt} \left[n T_e + \frac{1}{2} m_e V_d^2 \right] = -2 \sum_k \gamma_k^e N_k \omega_k - n e E_0 V_d \quad (1)$$

$$\frac{d}{dt} [n m_e V_d] = -2 \sum_k \gamma_k^e N_k k - n e E_0 \quad (2)$$

where n is the ambient electron number density, γ_k^e is the electron contribution to the linear growth rate and $N_k \omega_k$ is the wave energy density for the mode with wavenumber k . The spectrum is one-dimensional with wave propagation only along v_d . E_0 is the applied electric field which drives the electron current. Equ. (2) may be used to define an anomalous collision frequency, ν , by

$$\begin{aligned} \frac{dV_d}{dt} &= - \left[\frac{2}{n m_e V_d} \sum_k \gamma_k^e N_k k \right] V_d - \frac{e}{m_e} E_0 \\ &\equiv - \nu V_d - \frac{e}{m_e} E_0 \end{aligned} \quad (3)$$

where

$$\begin{aligned} \nu &\equiv \sum_k \nu_k \\ \nu_k &\equiv \frac{2}{n m_e V_d} \gamma_k^e N_k k \end{aligned} \quad (4)$$

Combining Equ.'s (1) and (2), we obtain an expression for the evolution of the electron temperature

$$\begin{aligned} \frac{d}{dt} n T_e &= -2 \sum_k \gamma_k^e N_k (\omega_k - k V_d) \\ &= n m_e V_d^2 \sum_k \nu_k \left(1 - \frac{\omega_k}{k V_d} \right) \end{aligned} \quad (5)$$

Similarly, from the energy and momentum equations for the ions, we obtain an equation for the ion temperature

$$\begin{aligned} \frac{d}{dt} n T_i &= -2 \sum_k \gamma_k^i N_k \omega_k \\ &= n m_e v_d^2 \sum_k (-\gamma_k^i / \gamma_k^e) \nu_k \frac{\omega_k}{k v_d} \end{aligned} \quad (6)$$

Returning to Equ. (4), we evaluate our expression for ν_k as

$$\begin{aligned} \nu_k &= \frac{2}{n m_e v_d} \gamma_k^e N_k k \\ &= \frac{2}{n m_e v_d} k \frac{-\text{Im } \epsilon_e}{\partial \epsilon / \partial \omega} \left[\frac{1}{2} \epsilon_0 \frac{\partial \epsilon}{\partial \omega} |E_k|^2 \right] \\ &= 2 \left(\frac{\pi}{2} \right)^{1/2} \frac{\epsilon_k}{n T_e} \frac{\omega_{pe}}{k \lambda_D} \left[1 - \frac{\omega_k}{k v_d} \right] \end{aligned} \quad (7)$$

where we have defined $\epsilon_k \equiv \frac{1}{2} \epsilon_0 |E_k|^2$, and we have assumed a Maxwellian distribution for the electrons with $v_0 = (T_e/n_e)^{1/2} \gg \omega_k/k$, and $T_e \gg T_i$. λ_D is the electron Debye length; ϵ is the plasma dielectric constant, and ϵ_e is the electron contribution to $\epsilon = 1 + \epsilon_e + \epsilon_i$.

We represent the turbulent heating experiment with an equivalent circuit shown in Fig. 1. This circuit is described by two first-order ordinary differential equations

$$\frac{dI}{dt} = \frac{1}{L} (V_c - V) - \frac{R}{L} I \quad (8)$$

$$\frac{dV_c}{dt} = -\frac{1}{C} I \quad (9)$$

where V_c is the voltage on the capacitor, $V = -E_0 \ell$ is the voltage drop across the plasma column, of length ℓ , and $I = n e v_0 A$ is the current flowing in the circuit, where A is the cross-sectional area of the plasma column.

We combine Equ.'s (8) and (3) to find an expression for V ,

$$V = \frac{L}{L+L_p} \frac{m_e \ell}{e} [\nu - R/L] V_d + \frac{L_p}{L+L_p} V_c \quad (10)$$

where $L_p \equiv m_e \ell / n e^2 A$ is the inertial inductance of the plasma column. In practice $L_p \ll L$. Using Equ. (10) to eliminate V , we obtain

$$\frac{dV_d}{dt} = - \left[\frac{L_p}{L+L_p} \nu + \frac{R}{L+L_p} \right] V_d + \frac{e}{m_e \ell} \frac{L_p}{L+L_p} V_c \quad (11)$$

$$\frac{dV_c}{dt} = - \frac{n e A}{C} V_d \quad (12)$$

The remaining equations consist of the evolution equation for $\mathcal{E}_k$ for each mode,

$$\frac{d\mathcal{E}_k}{dt} = 2 \gamma_k \mathcal{E}_k ; \quad \gamma_k \equiv \gamma_k^e + \gamma_k^i \quad (13)$$

and the solution to the dispersion relation for the ion-acoustic instability,

$$\omega_k = \frac{k C_s}{\sqrt{1 + k^2 \lambda_D^2}} \quad (14)$$

$$\gamma_k^e = \left(\frac{\pi}{8} \right)^{1/2} \frac{m_i}{m_e} \frac{\omega_k^3}{k^2 v_e^2} \frac{V_d}{V_e} \left[1 - \frac{\omega_k}{k V_d} \right] \quad (15)$$

$$\gamma_k^i = - \left(\frac{\pi}{8} \right)^{1/2} \frac{\omega_k^4}{k^3 v_i^3} \exp \left[- \frac{1}{2} \frac{\omega_k^2}{k^2 v_i^2} \right] \quad (16)$$

which are good approximations for $T_e \gg T_i$, $v_e \gg \omega_k/k \gg v_i$, and Maxwellian distributions for the electrons and ions.

The following two tables present the basic equations which are solved numerically. The equations are written in the normalized form given in Table 1. In these tables the subscript zero is used to indicate the initial value of a quantity (e.g., T_{e0} is the initial electron temperature).

Table 1

Normalized Variables

$$\bar{t} = \omega_{pi} t \quad \bar{\nu}_k = \nu_k / \omega_{pi} \quad \bar{\omega}_k = \omega_k / \omega_{pi} \quad \bar{\gamma}_k = \gamma_k / \omega_{pi}$$

$$\bar{T}_e = T_e / T_{eo} \quad \bar{T}_i = T_i / T_{io} \quad \bar{\epsilon}_k = \epsilon_k / \eta T_{eo}$$

$$\bar{V}_d = V_d / c_{so} \quad \bar{k} = k \lambda_{D0}$$

$$\bar{V}_c = (C / ne A \lambda_{D0}) V_c$$

where $c_{so} = (T_{eo}/m_i)^{1/2}$, $v_{eo} = (T_{eo}/m_e)^{1/2}$, $\lambda_{D0} = v_{eo}/\omega_{pe}$.

Table 2

Summary of Basic Equations

$$\bar{\nu} = \sum_k \bar{\nu}_k$$

$$\bar{\nu}_k = 2 \left(\frac{\pi}{2} \right)^{1/2} \frac{\bar{\epsilon}_k}{\bar{k} \bar{T}_e^{3/2}} \left(\frac{m_i}{m_e} \right)^{1/2} \left[1 - \frac{\bar{\omega}_k}{\bar{k} \bar{V}_d} \right]$$

$$\bar{\omega}_k = \left[1 + \frac{1}{\bar{k}^2 \bar{T}_e} \right]^{-1/2}$$

$$\bar{\gamma}_k^e = \left(\frac{\pi}{8} \right)^{1/2} \left(\frac{m_e}{m_i} \right)^{1/2} \frac{\bar{\omega}_k^3}{\bar{k}^2 \bar{T}_e^{3/2}} \bar{V}_d \left[1 - \frac{\bar{\omega}_k}{\bar{k} \bar{V}_d} \right]$$

$$\bar{\gamma}_k^i = - \left(\frac{\pi}{8} \right)^{1/2} \left(\frac{\bar{T}_e}{\bar{T}_i} \cdot \frac{T_{eo}}{T_{io}} \right)^{3/2} \frac{\bar{\omega}_k^4}{\bar{k}^3 \bar{T}_e^{3/2}} \exp \left[-\frac{1}{2} \frac{\bar{\omega}_k^2}{\bar{k}^2 \bar{T}_e} \frac{\bar{T}_e}{\bar{T}_i} \frac{T_{eo}}{T_{io}} \right]$$

$$\bar{\gamma}_k = \bar{\gamma}_k^e + \bar{\gamma}_k^i$$

$$\frac{d\bar{\epsilon}_k}{d\bar{t}} = 2 \bar{\gamma}_k \bar{\epsilon}_k$$

Table 2 (continued)

$$\frac{d\bar{T}_e}{d\bar{t}} = \frac{m_e}{m_i} \bar{V}_d^2 \sum_k \bar{V}_k \left[1 - \frac{\bar{\omega}_k}{k \bar{V}_d} \right]$$

$$\frac{d\bar{T}_i}{d\bar{t}} = \frac{\bar{T}_{eo}}{\bar{T}_{io}} \frac{m_e}{m_i} \bar{V}_d^2 \sum_k \left(-\bar{V}_k^i / \bar{V}_k^e \right) \bar{V}_k \frac{\bar{\omega}_k}{k \bar{V}_d}$$

$$\frac{d\bar{V}_c}{d\bar{t}} = -\bar{V}_d$$

$$\frac{d\bar{V}_d}{d\bar{t}} = - \left[\frac{L_p}{L+L_p} \bar{V} + \bar{C}_{R/L}^{-1} \right] \bar{V}_d + \bar{C}_{LC}^{-2} \bar{V}_c$$

where

$$\bar{C}_{R/L}^{-1} \equiv \frac{1}{\omega_{pi}} \frac{R}{L+L_p}, \quad \bar{C}_{LC}^{-2} \equiv \frac{1}{\omega_{pi}^2} \frac{1}{(L+L_p) \bar{C}}$$

The basic input parameters for this model are therefore $\bar{T}_{eo}/\bar{T}_{io}$, $L_p/(L+L_p)$, $\bar{C}_{LC}$, $\bar{C}_{RL}$, and the initial value of $\bar{V}_c$. The initial values of $\bar{T}_e$ and $\bar{T}_i$ are unity, and the initial value of $\bar{V}_d$ is zero. In addition, the program requires the initial values of $\bar{V}_k$ for each k .

Computer Result

A computer program (cf. Appendix II) using the fourth-order Runge-Kutta-Gill technique (cf. Appendix I) is employed to integrate the coupled set of first-order ordinary differential equations summarized in Table 2. The continuum wave spectrum is approximated by a discrete spectrum, thereby reducing the infinite set of

differential equations in Table 2 to a finite set, which can be treated on the computer. The discrete k -spectrum, consisting of twenty modes evenly spaced, is chosen broad enough that the entire unstable spectrum is represented at all times during the run, and narrow enough that at least half the modes are strongly unstable during each voltage spike. Since T_e/T_i decreases from one voltage spike to the next, the wavenumbers for the most unstable modes also decrease. The instability turns off in our computer analysis when the mode having the lowest value of k can no longer become unstable. This "lowest- k " mode has $k\lambda_{De} = 0.03$, and differs only slightly from the $k = 0$ mode in its stability.

The results of the computer analysis are shown graphically in Fig.'s 3 and 4, for the case $T_{e0}/T_{i0} = 20$ and 30 respectively. The other parameters are as given in the figure captions. The results are found to be insensitive to the initial values of $\bar{\epsilon}_k$, which we varied by three orders of magnitude with no change in the final temperatures. The two figures show the same qualitative behavior. The instability turns on and off repeatedly, each repetition accompanied by a spike in the voltage across the plasma column, which in normalized units is given by $\bar{v}/\bar{v}_0$. Simultaneously, the current (drift velocity) is depressed, due to the sudden increase of the anomalous resistance during the turbulence. The electron and ion temperatures both increase sharply during the voltage spikes, in such a way that T_e/T_i decreases during each spike. The turbulent collision frequency reaches values as high as $0.2 \omega_{pi}$ during the instability, in rough agreement with the experimental value deduced from the measured voltage and current traces. Each voltage spike lasts for approximately 50 nsec, in good agreement with the experiments.

In the experiments the plasma is created with approximately 40 eV per electron-ion pair, and has sufficient time to completely thermalize before the main capacitor banks are fired. The initial electron heating phase is assumed to raise T_e so that T_{e0}/T_{i0} is 20 to 30 by the time the ion-acoustic instability sets in. The

final electron and ion temperatures are then computed to be $T_e \approx 1500$ eV and $T_i \approx 200$ eV, in rough agreement with the experimental results for the "bulk" ion temperature and the diamagnetic signal level. In addition, the experiments find a high-energy ion component ($T_i \approx 1500$ eV) which is excluded by our model. To properly include this additional feature would require a direct integration of the kinetic equations themselves, rather than using moments of the kinetic equations as we have done. Techniques for evolving the kinetic equations have been described by Field and Fried (1964).

Our model excludes the possibility of particle trapping by the waves. At the peaks of the instability we find that the ion bounce frequency never exceeds $\omega_s \sim Y$, where ω_s is the ion bounce frequency, $\omega_s = (ekE_L/m_i)^{1/2}$, which implies that ion trapping is only marginally significant even if the wave spectrum is assumed coherent. In fact, the wave spectrum is far from coherent (particularly in two or three dimensions) and this further decreases the effect of particle trapping. The incoherence of the fluctuating fields is particularly important for the electrons and justifies our neglect of electron trapping (Manheimer, to be published). We expect, however, that ion trapping may be significant in parameter regimes other than those considered here. Indeed ion trapping does appear to play an important role in recent two-dimensional plasma-particle simulations of anomalous resistivity (Orens, to be published; Biskamp and Chodura, 1971).

The treatment given here is one-dimensional. In two or three dimensions waves can be convected to the edge of the plasma column. However, we estimate that the waves may not have sufficient time to escape from the column in these experiments. Even in the absence of such wave convection effects, allowance for the three-dimensional character of the wave spectrum (Field and Fried, 1964) would be an important improvement of the model used here.

Conclusion

We have shown that quasilinear theory with the inclusion of the external circuitry forms a basis for discussing many interesting features of the Cornell turbulent heating experiments. In particular, the voltage spikes and the premature quenching of the instability before the current maximum is reached appear to be described by our model. An important generalization of this work would be the inclusion of non-Maxwellian distributions and a three-dimensional wave spectrum in the manner described by Field and Fried (1964). Other possibly important effects that could be included in a future anomalous resistivity computer program are thermal conductivity to the end plates, skin effect, ion trapping, and a self-consistent treatment of the expansion of the plasma column as it is heated. We emphasize, however, that significant agreement with the experiments has been obtained with the present model which includes only one-dimensional quasilinear effects. Inclusion of the additional effects, mentioned above, would not be expected to alter our results qualitatively, but might be necessary if, for example, one wished to reproduce the voltage vs. time traces in detail. It is intriguing to speculate on the possible use of anomalous resistivity computer programs, such as the one described in this paper, or a more sophisticated successor, for the purpose of aiding in the design of turbulent heating schemes in scaled-up fusion devices (such as Tokamak), or for predicting their behavior.

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Figure Captions

Fig. 1: Equivalent Circuit for Turbulent Heating Experiments

Fig. 2: Experimental Voltage and Current Oscilloscope Traces

Fig. 3: Results of Computer Analysis

$T_{eo}/T_{io}=20$, $\bar{V}_c=3 \times 10^5$ initially, $\bar{\epsilon}_k=10^4$, $\bar{\epsilon}_{kL}=4.5 \times 10^5$,
 $L_p/(L+L_p)=0.05$, and $\bar{\epsilon}_k=5 \times 10^{-9}$ initially for each k

Fig. 4: Results of Computer Analysis

$T_{eo}/T_{io}=30$, with the other parameters the same as in
Fig. 3.

Figure 1

(a)

$$C = .09 \mu f$$

$$V_{bank} = \pm 40 kV$$

(b)

$$C = .09 \mu f$$

$$V_{bank} = \pm 80 kV$$

(c)

$$C = .13 \mu f$$

$$V_{bank} = \pm 80 kV$$

(d)

$$C = .35 \mu f$$

$$V_{bank} = \pm 80 kV$$

Time $\rightarrow$ (0.5 μ sec/div)

Vertical Scales
Voltage: V $\rightarrow$ (25 kV/div)
Current: I $\rightarrow$ (4 μ A/div)

Figure 2

Figure 3

Appendix I

The Runge-Kutta-Gill Formalism

The Runge-Kutta-Gill formalism is a fourth-order technique for integrating a finite set of n first-order, ordinary differential equations,

$$y_1' = dy_1/dx = f_1(x, y_1, y_2, \dots, y_n)$$

$$y_2' = dy_2/dx = f_2(x, y_1, y_2, \dots, y_n)$$

⋮

$$y_n' = dy_n/dx = f_n(x, y_1, y_2, \dots, y_n)$$

starting with the initial values, $y_1(x_0) = y_{10}$, $y_2(x_0) = y_{20}$,
 $\dots$, $y_n(x_0) = y_{no}$.

We rewrite these equations in a vector notation,

$$Y'(x) = F(x, Y) \text{ with } Y(x_0) = Y_0$$

where

$$Y = \begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{bmatrix}; \quad F(x, Y) = \begin{bmatrix} f_1 \\ f_2 \\ \vdots \\ f_n \end{bmatrix}; \quad Y_0 = \begin{bmatrix} y_{10} \\ y_{20} \\ \vdots \\ y_{no} \end{bmatrix}$$

Starting at x_0 with $Y(x_0)=Y_0$, the object of a single Runge-Kutta step is to generate $Y(x_0+h)$ for a given increment, h . This is done in four stages. The quantity Q_0 , appearing below, is initially zero and is a correction for numerical round-off errors.

A single Runge-Kutta step then proceeds as follows:

$$K_1 = h \cdot F(x_0, Y_0); \quad Y_1 = Y_0 + \frac{1}{2}(K_1 - 2Q_0) \\ Q_1 = Q_0 + 3 \left[\frac{1}{2}(K_1 - 2Q_0) \right] - \frac{1}{2}K_1$$

$$K_2 = h \cdot F(x_0 + \frac{1}{2}h, Y_1); \quad Y_2 = Y_1 + (1 - 1/\sqrt{2})(K_2 - Q_1) \\ Q_2 = Q_1 + 3 \left[(1 - 1/\sqrt{2})(K_2 - Q_1) \right] - (1 - 1/\sqrt{2})K_2$$

$$K_3 = h \cdot F(x_0 + \frac{1}{2}h, Y_2); \quad Y_3 = Y_2 + (1 + 1/\sqrt{2})(K_3 - Q_2) \\ Q_3 = Q_2 + 3 \left[(1 + 1/\sqrt{2})(K_3 - Q_2) \right] - (1 + 1/\sqrt{2})K_3$$

$$K_4 = h \cdot F(x_0 + h, Y_3); \quad Y_4 = Y_3 + (K_4 - 2Q_3)/6 \\ Q_4 = Q_3 + 3 \left[(K_4 - 2Q_3)/6 \right] - \frac{1}{2}K_4$$

On the next Runge-Kutta step, $x_0 + h$ and Y_4 are the initial values, and Q_4 is used as Q_0 to compensate for numerical errors. Note that $Q_4 = 0$ when the calculation is performed with arbitrarily good numerical precision.

Appendix II

The Computer Program

The equations in Table 2 constitute the problem to be solved numerically. They consist of a set of coupled, first-order, ordinary differential equations which are solved by the fourth-order Runge-Kutta-Gill integration procedure described in Appendix I.

Since the instability evolves on a much faster time scale than the external circuit, the program uses two separate time scales. A large time step is used whenever the instability is completely off, and a small time step is used during the instability. During the large time step phase, only the external circuit parameters evolve. The fields are held at a specified "thermal" level, and the temperatures remain constant. Whenever any node becomes unstable, the program moves to the small time step. In calling the small time step, the program backspaces one large time step in order to accurately follow the turning on process using the finer time scale. All quantities then evolve on the small time scale until all the nodes which became unstable have stabilized, and all the fields have decayed back to the "thermal" level. The program then returns to the large time step until the instability reappears, causing the program to again use the small time step. This procedure continues until the specified "end-time" for the run has been reached, and the program terminates.

The program has four separate parts: a MAIN program and three subroutines, REGS, OUPF and ECT.

1) MAIN program

This program reads the input parameters and picks the appropriate time step for each portion of the run. It coordinates the three subroutines.

2) RKGS ... the Runge-Kutta-Gill subroutine

This subroutine is called by the MAIN program. The calling procedure specifies the time step to be used. To change the time step, MAIN must recall RKGS. RKGS uses the equations in Appendix I to integrate the differential equations. The derivatives and growth rates required by RKGS are calculated by subroutine FCT. The result of each Runge-Kutta step in RKGS is reported to subroutine OUTF.

3) OUTF ... the output subroutine

This subroutine is called by RKGS after the completion of each Runge-Kutta step. OUTF then prints the results. OUTF monitors messages from FCT which indicate whether the instability has either turned on or terminated, in which case OUTF causes RKGS to return to MAIN to allow the time step to be altered.

4) FCT ... the function subroutine

This subroutine is called by RKGS. FCT computes the frequency and growth rates for each mode as well as all the derivatives required by RKGS. In addition, FCT monitors the onset and termination of the instability and sets various parameters used by OUTF to allow the time step to be altered.

Straight and Toroidal Plasma Equilibria with
an Intense Relativistic Electron Current Component

by

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**Straight and Toroidal Plasma Equilibria with
an Intense Relativistic Electron Current Component***

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We study analytically the equilibrium of an axisymmetric, large aspect-ratio, finite temperature toroidal plasma which is enclosed by a perfectly conducting toroidal container. The new feature of the analysis is the presence of a relativistic electron beam which circulates around the torus thus producing a poloidal magnetic field in addition to the external field and the field created by currents in the plasma. This problem may be of interest in connection with recent toroidal beam injection experiments at Cornell University, and tokamaks. The equilibria are obtained from the relativistic virial theorem. Specific examples are given wherein the plasma pressure is characterized by its dependence on the magnetic flux function and the relativistic electron beam is characterized by an electron distribution function which is a delta function in energy and canonical momentum in the toroidal direction.

I. Introduction

In this paper we shall discuss the equilibrium of an axisymmetric, large aspect-ratio, toroidal plasma in the presence of a relativistic electron beam which circulates around the torus thus contributing a large toroidal component of current. The toroidal plasma plus beam system is assumed to be enclosed by a perfectly conducting toroidal shell. This problem is of interest for at least two classes of experiments:

(1) Experimental programs are presently underway in several laboratories to inject an intense relativistic electron beam into a toroidal conducting shell.¹ The poloidal field produced by the beam would support confined plasma equilibria which may be of interest for controlled fusion applications.

(2) In tokamak devices, toroidal currents are produced by a toroidal, inductively produced electric field. This field can in turn lead to large runaway electron current components which may be relativistic. Indeed the ORMAK facility at Oak Ridge National Laboratory has produced runaway electrons with energies in the MeV range and currents of the order of 10^5 Amperes.²

Our formulation of the problem will be closely related to that employed by Anderson et al.,³ who have employed numerical methods to obtain astron equilibria including both E-layer and plasma. Their problem differs from that considered here in that for their case, the bounding conductor is a large cylinder enclosing the E-layer, whereas in our case the bounding conductor is toroidal. While our results have the benefit of being analytical, they apply only for large aspect-ratio. To obtain results for arbitrary aspect ratios, numerical methods such as those used in Ref. 3 can be employed.⁴ Another related work is that of Ott and Sudan,⁵ who have calculated the large aspect-ratio toroidal equilibrium of a relativistic electron beam with the background plasma pressure neglected.

In Section II, we use the relativistic virial theorem to obtain some results for large aspect-ratio, axisymmetric toroidal equilibria of a plasma with an intense, relativistic, electron current component. Application of the virial theorem to non-relativistic isotropic, toroidal plasma equilibria has previously been made by Yoshikawa⁶ and by Shafranov.⁷ Our Section II reviews these previous works but includes relativistic electrons.

In Section III we obtain and study some specific toroidal equilibria for a plasma with an embedded relativistic electron beam. Both plasma currents and beam currents are assumed to flow only along the column, and the beam distribution function is assumed to be a delta function in particle energy and canonical momentum along the beam. The plasma pressure depends only on the magnetic flux function. Since both the beam and plasma currents of these equilibria are only in the toroidal direction, a vacuum toroidal magnetic field of arbitrary strength, $B_0(R/r)\hat{\phi}$, may be included without affecting the equilibria at all because no $\underline{J} \times \underline{B}$ force is produced. Such a field will, however, play a very important role in any considerations of stability.

II. Toroidal Equilibria for Relativistic Systems

The basic equation for equilibrium is

$$\nabla \cdot (\underline{P} + \underline{T}) = 0 \quad (1)$$

where

$$\begin{aligned} \underline{P} &= \sum_{\sigma} n^{\sigma} m^{\sigma} \langle \underline{v} \underline{v} \rangle^{\sigma} \\ \underline{T} &= \frac{1}{8\pi} B^2 \underline{I} - \frac{1}{4\pi} \underline{B} \underline{B} \end{aligned} \quad (2)$$

Here γ is the relativistic factor, $\gamma = (1 - v^2/c^2)^{-1/2}$, the summation in equation (2) is over particle species, and the brackets, $\langle \rangle^\sigma$, denote an average of the given quantity over the distribution function, f^σ , for the σ -th species. We assume electrostatic neutrality so that $\underline{E} = 0$. Equation (1) follows easily from the stationary, relativistic Vlasov equation. Note that the streaming motion of the relativistic electrons is included in $\underline{P}$ and makes $\underline{P}$ anisotropic even when the background plasma (through which the relativistic electrons stream) is isotropic.

The virial theorem for an equilibrium configuration is written as

$$\nabla \cdot [\underline{r} \cdot (\underline{P} + \underline{T})] = (\nabla \underline{r}) : (\underline{P} + \underline{T}) = \text{Trace} [\underline{P} + \underline{T}] \quad (3)$$

which follows from (1).

We may re-write equation (3) in integral form as

$$\int_V (\text{Trace} [\underline{P}] + \frac{B^2}{8\pi}) dV = \oint_S \underline{r} \cdot (\underline{P} + \frac{B^2}{8\pi} \underline{I}) \cdot d\underline{S} \quad (4)$$

where the closed surface S is chosen to be a magnetic surface, $\underline{B} \cdot d\underline{S} = 0$, and V is the volume enclosed by the closed surface S . We may also express equation (1) in integral form as

$$\oint_S \underline{P} \cdot d\underline{S} + \oint_S \left[\frac{B^2}{8\pi} d\underline{S} - \frac{\underline{B}\underline{B}}{4\pi} \cdot d\underline{S} \right] = 0, \quad (5)$$

which we apply to the volume enclosed by the given magnetic surface between the infinitesimally close surfaces, $\phi = \text{const.}$ and $\phi + d\phi = \text{const.}$

Taking the $\hat{e}_r$ component of equation (5), we have

$$d\phi \int \left(P_{\phi\phi} + \frac{B^2}{8\pi} - \frac{B_\phi^2}{4\pi} \right) dS_\phi = \int \left(P_{\theta\theta} + \frac{B^2}{8\pi} \right) \cos \Theta dS_\theta \quad (6)$$

where $dS_\phi = \rho d\rho d\Theta$, and $dS_\theta = (R + \rho \cos \Theta) \rho d\Theta d\phi$ in toroidal coordinates, (ρ, Θ, ϕ) , depicted in figure (1).

Equations (4) and (6) form a basic set of equations. Let the magnetic surface have radius, r_m . To first order in an expansion in the inverse aspect ratio, r_m/R , we write the longitudinal and azimuthal components of the magnetic field on the magnetic surface as

$$\begin{aligned} B_\phi &= B_e (1 - r_m/R \cos \Theta) \\ B_\theta &= B_a (1 + \frac{r_m}{R} \Lambda \cos \Theta) \end{aligned} \quad (7)$$

Equation (7) defines the coefficient of asymmetry, Λ , which we need only obtain in zero order to evaluate B_θ to first order in the inverse aspect ratio. Equations (4) and (6) then become

$$\begin{aligned} \bar{P}_{\theta\theta} + \bar{P}_{\phi\phi} - 3P_{\theta\theta}(r_m) &= \frac{3\bar{B}_a^2}{8\pi} + \frac{\bar{B}_e^2 - \bar{B}_i^2}{8\pi} - \frac{\bar{B}_\theta^2}{8\pi} + 2\Lambda \frac{\bar{B}_a^2}{8\pi} \\ \bar{P}_{\phi\phi} - P_{\theta\theta}(r_m) &= \frac{\bar{B}_a^2}{8\pi} - \frac{\bar{B}_e^2 - \bar{B}_i^2}{8\pi} - \frac{\bar{B}_\theta^2}{8\pi} + 2\Lambda \frac{\bar{B}_a^2}{8\pi} \end{aligned} \quad (8)$$

(9)

where bar over a quantity denotes the average of that quantity over the cross-section within the magnetic surface, and $\bar{B}_i^2$ is the average longitudinal field within the magnetic surface. If the plasma plus beam current density is $\underline{J} = J_\phi \hat{\phi}$, as we shall subsequently assume, then $\bar{B}_e^2 - \bar{B}_i^2 = 0$.

Equations (8) and (9) may be combined to yield the pressure balance equation

$$\frac{1}{2}(\bar{P}_{\theta\theta} + \bar{P}_{\phi\phi}) - P_{pp}(r_w) - \frac{B_a^2}{8\pi} = \frac{B_e^2 - \bar{B}_i^2}{8\pi} \quad (10)$$

which may then be employed with either (8) or (9) to yield

$$\Lambda = \frac{\frac{1}{2}\bar{P}_{\phi\phi} + \frac{1}{4}(\bar{P}_{\theta\theta} + \bar{P}_{\phi\phi}) - P_{pp}(r_w)}{B_a^2/8\pi} + \frac{L_i}{2} - 1 \quad (11)$$

where $L_i = \bar{B}_i^2/B_a^2$ = the internal inductance per unit length.

Equations (7) and (11) apply to any magnetic surface of the configuration. We now specialize our discussion to the "plasma surface," the magnetic surface with radius, r_p , beyond which the current density vanishes. The magnetic field outside the plasma surface then takes the form $\underline{B} = \nabla\psi$, where $\nabla^2\psi = 0$. The solution to this equation which satisfied the continuity condition at the plasma surface is given by⁷

$$\begin{aligned} B_\theta &= B_a \frac{r_p}{r} + B_a \frac{r_p}{2R} \left[\left(1 + \frac{r_p^2}{R^2}\right) \left(\Lambda + \frac{1}{2}\right) + \ln \frac{r}{r_p} - 1 \right] \cos \theta \\ B_r &= \frac{r_p}{2R} B_a \left[\left(1 - \frac{r_p^2}{R^2}\right) \left(\Lambda + \frac{1}{2}\right) + \ln \frac{r}{r_p} \right] \sin \theta \end{aligned} \quad (12)$$

to first-order in r/R .

Let r_c denote the radius of a perfect conducting wall. Then, the shift, δ , between the magnetic surface of radius r_p and the wall is determined from the condition that the normal component of the magnetic field vanish at the wall,

$$B_p^{(1)} - B_0^{(0)} \frac{\delta}{r_c} \sin \Theta = 0 \quad (13)$$

which yields

$$\delta = \frac{r_c^2}{2R} \left[\ln \frac{r_c}{r_p} + \left(\Lambda + \frac{1}{2} \right) \left(1 - r_p^2/r_c^2 \right) \right] \quad (14)$$

Addition of a vertical magnetic field, $B_z \hat{z}$, causes an additional shift, δ_v , which can be used to control the position of the plasma. The field B_z results in another term in (13), $B_z \sin \Theta$. Thus the additional shift, $\delta_v = r_c B_z / B_0^{(0)}$, results.

III. Application to a Beam-Plasma System

A. Model

Here we apply the results of the previous section to obtain toroidal beam-plasma equilibria. The beam is specified by its distribution function, $f(H, P_\phi)$, which is assumed to depend only on the energy, $H = \gamma mc^2$, and on the canonical toroidal momentum, $P_\phi = \gamma mrv_\phi - \frac{e}{c} rA_\phi$, where $r = R + \rho \cos \Theta$. The plasma is specified by its pressure, $p(\psi)$, where $\psi = rA_\phi$ is the magnetic flux function.

For definiteness we will choose the beam distribution function, $f(H, P_\phi)$, to have the Hammer-Rostoker form,⁸

$$f(H, P_\phi) = \frac{n_B r}{2\pi \gamma_0 m} \delta(H - \gamma_0 mc^2) \delta(P_\phi - \gamma_0 m R V_\phi) \quad (15)$$

All the beam electrons have the same energy, $\gamma_0 mc^2$. For $A_\phi(\varphi=0) = 0$,

V_ϕ is then the toroidal velocity for a particle on the toroidal axis ($\phi=0$). Ott and Sudan⁵ have considered the above beam equilibrium in the presence of a cold background plasma. In contrast, in the present work the background plasma has a finite temperature and pressure.

The beam density for this equilibrium is given by

$$n_b = n_b \Theta(\alpha - 1) \quad (16)$$

where $\Theta(x)$ is the unit step function, which is unity for $x \geq 0$ and zero otherwise, and α is given by

$$\alpha \equiv \gamma_0 \left\{ 1 + (mc)^{-2} \left[\frac{\gamma_0 m R V_\phi}{r} + \frac{e}{c} A_\phi \right]^2 \right\}^{-1/2} \quad (17)$$

We should note at this point that the beam surface is not a magnetic surface, since magnetic surfaces are given by $\psi = r A_\phi = \text{constant}$, while the beam surface is $\frac{e}{c} A_\phi + \gamma_0 m R V_\phi / r = \text{constant}$ from equation (17).

The beam current density corresponding to equation (15) is given by

$$\vec{J}_B = - \frac{\omega_b^2}{4\pi c} \left[A_\phi + \frac{\gamma_0 m c R V_\phi}{e r} \right] \Theta(\alpha - 1) \quad (18)$$

where $\omega_b^2 = 4\pi n_b e^2 / \gamma_0 m$. The beam carries current only in the $\hat{\phi}$ direction.

The plasma pressure, $p(\psi)$, is chosen to be of the form,

$$p(\psi) = p_0 \left[1 - \frac{\psi}{\psi_0} \right]^n \Theta(\psi_0 - \psi) \quad (19)$$

where n is an integer, and ψ_0 is a constant, chosen so that $\psi = \psi_0$.

at the plasma surface. The plasma surface is therefore a magnetic surface.

p_0 is the plasma pressure at $\psi = 0$.

The plasma is assumed to carry current in the $\hat{\phi}$ direction only. Then, the relationship $\nabla p = \frac{1}{c} \mathbf{J}_p \times \mathbf{B}$ may be written as $\frac{dp}{d\psi} \nabla \psi = \frac{1}{rc} \mathbf{J}_p \nabla \psi$ or $J_p = rc \, dp/d\psi$. Thus

$$J_p = \begin{cases} -\frac{nrcp_0}{\psi_0} \left[1 - \frac{\psi}{\psi_0}\right]^{n-1} \Theta(\psi_0 - \psi) & \text{for } n > 0 \\ -rcp_0 \delta(\psi_0 - \psi) & \text{for } n = 0. \end{cases} \quad (20)$$

Anderson, Killeen, and Rensink³ have considered a beam-plasma model similar to that employed here.

From $\nabla \times \mathbf{B} = \frac{4\pi}{c} (\mathbf{J}_B + \mathbf{J}_p)$, the equilibrium solution satisfies

$$\begin{aligned} \nabla^2 A_\phi - \frac{1}{r^2} A_\phi &= \frac{\omega_0^2}{c^2} \left[A_\phi + \frac{\gamma_0 mc R V_\phi}{er} \right] \Theta(\alpha - 1) \\ &+ \frac{4\pi p_0 n r}{\psi_0} \left[1 - \frac{r A_\phi}{\psi_0} \right] \Theta(\psi_0 - r A_\phi) \end{aligned} \quad (21)$$

For definiteness, we will assume that the beam lies within the plasma.

We expand equation (21) in powers of β/R . To zero order, we obtain the straight cylindrical equilibrium, in which the toroidal axis, $\hat{\phi}$, becomes the z-axis. The zero-order vector potential, $A_z(\rho)$, then satisfies

$$\begin{aligned} \frac{1}{r} \frac{d}{d\rho} \rho \frac{dA_z}{d\rho} &= \frac{\omega_0^2}{c^2} \left[A_z + \frac{mc^2}{e} \gamma_0 \beta_z \right] \Theta(A_1 - A_z) \\ &+ \frac{4\pi p_0 n}{A_0} \left[1 - \frac{A_z}{A_0} \right]^{n-1} \Theta(A_0 - A_z) \end{aligned} \quad (22)$$

where $A_1 = \frac{mc^2}{e} \gamma_0 (\beta_0 - \beta_z)$, $\beta_0 = \gamma_0^{-1} \sqrt{\gamma_0^2 - 1}$, $\beta_z = V_z/c$, $\beta_0^2 = \beta_z^2 + \langle \beta_\perp^2 \rangle_{p=0}$, $A_0 = \psi_0/R$, and $A_z(\rho=0) = 0$.

The beam radius, $\rho = a$, is given by $A_z(a) = A_1$, and the plasma radius, $\rho = b$, by $A_z(b) = A_0$. The assumption that the beam lies within the plasma is $A_1 \leq A_0$.

Once we have obtained the solution to equation (22), we may apply the results of the previous section to obtain the toroidal equilibrium to first-order in ρ/R .

B. Beam with Cold Plasma

To illustrate the technique developed in Section II, we will first examine the problem of a beam in a cold plasma ($p_0 = 0$). The plasma here serves only to provide electrostatic neutralization of the beam. This problem has been solved by Ott and Sudan using a different technique. We will compare our result with theirs. The zero-order solution for this case is the well-known Hammer-Rostoker⁸ equilibrium,

$$\frac{1}{\rho} \frac{d}{d\rho} \rho \frac{dA_z}{d\rho} = \frac{\omega_b^2}{c^2} \left[A_z + \frac{mc^2}{e} \gamma_0 \beta_z \right] \oplus (A_1 - A_z) \quad (23)$$

The solution to equation (23) with $A_z(\rho=0) = 0$ is given by

$$A_z(\rho) = \begin{cases} \frac{mc^2}{e} \gamma_0 \beta_z \left[I_0\left(\frac{\omega_b \rho}{c}\right) - 1 \right] & \rho \leq a \\ \frac{mc^2}{e} \gamma_0 \beta_z \left[I_0\left(\frac{\omega_b a}{c}\right) - 1 + \frac{\omega_b a}{c} I_1\left(\frac{\omega_b a}{c}\right) \ln \frac{\rho}{a} \right] & \rho > a \end{cases} \quad (24)$$

where I_0 , I_1 are modified Bessel functions. The magnetic field is then

$$B_{\theta}(r) = - \frac{mc^2}{e} \gamma_0 \beta_z \frac{\omega_b}{c} I_1\left(\frac{\omega_b r}{c}\right) \quad r \leq a \quad (25)$$

Using equations (17) and (24), we have

$$A_z(a) = A_1 = \frac{mc^2}{e} \gamma_0 (\beta_0 - \beta_z) = \frac{mc^2}{e} \gamma_0 \beta_z \left[I_0\left(\frac{\omega_b a}{c}\right) - 1 \right]$$

or

$$\beta_0 = \beta_z I_0\left(\frac{\omega_b a}{c}\right) \quad (26)$$

The total beam current may be expressed in terms of the Alfvén current,

$I_A = (mc^3/e) \gamma_0 \beta_0$, with equations (25) and (26)

$$I = \frac{ca}{2} B_{\theta}(a) = -\frac{1}{2} I_A \frac{\omega_b a}{c} I_1\left(\frac{\omega_b a}{c}\right) / I_0\left(\frac{\omega_b a}{c}\right) \quad (27)$$

We now apply the results of Section II to obtain the first-order toroidal correction. We begin by evaluating the quantity, Δ , given by equation (11). For a cold plasma, $P_{ff}(a) = 0$ at the beam surface. The remaining pressure terms are given by

$$P_{\phi\phi}(r) = \gamma_0 n_B m v_{\phi}^2 = \frac{\omega_b^2}{4\pi c^2} \left(\frac{I_A}{c}\right)^2 I_0^2\left(\frac{\omega_b r}{c}\right) / I_0^2\left(\frac{\omega_b a}{c}\right) \quad (28)$$

where we used $v_{\phi} = (e/\gamma_0 mc)(A_z + \frac{mc^2}{e} \gamma_0 \beta_z)$, and the perpendicular pressure components are given by

$$\frac{1}{2} (P_{ff} + P_{\theta\theta}) = \gamma_0 \beta_0^2 n_B mc^2 - P_{\phi\phi}(r) \quad (29)$$

Hence,

$$P_{\phi\phi} + \frac{1}{2}(P_{\rho\rho} + P_{\theta\theta}) = \frac{1}{2} \frac{\omega_b^2}{4\pi c^2} \left(\frac{I_A}{c}\right)^2 \left[1 + I_0^2\left(\frac{\omega_b \rho}{c}\right) / I_0^2\left(\frac{\omega_b a}{c}\right) \right] \quad (30)$$

where we have noted that $\gamma_0 \beta_0^2 n_B m c^2 = (\omega_b^2 / 4\pi c^2) (I_A / c)^2$. From equation (25), we have

$$B_\theta^2(\rho) = \frac{\omega_b^2}{c^2} \left(\frac{I_A}{c}\right)^2 I_1^2\left(\frac{\omega_b \rho}{c}\right) / I_0^2\left(\frac{\omega_b a}{c}\right) \quad (31)$$

Using equations (30) and (31) in equation (11), we obtain

$$\begin{aligned} \Lambda &= \frac{1}{2 I_1^2\left(\frac{\omega_b a}{c}\right)} \left\{ I_0^2\left(\frac{\omega_b a}{c}\right) + \frac{1}{\pi a^2} \int_0^a 2\pi \rho d\rho \left[I_0^2\left(\frac{\omega_b \rho}{c}\right) + I_1^2\left(\frac{\omega_b \rho}{c}\right) \right] \right\} - 1 \\ &= \frac{1}{2} \frac{I_0\left(\frac{\omega_b a}{c}\right) I_2\left(\frac{\omega_b a}{c}\right)}{I_1^2\left(\frac{\omega_b a}{c}\right)} - \frac{I_A}{I} - 1 \end{aligned} \quad (32)$$

where the last line employs equation (27) for I_A/I , and we have used the identity, $I_2(x) = I_0(x) - (2/x)I_1(x)$. In performing the integral in equation (32), we used

$$\frac{d}{dx} [x I_0(x) I_1(x)] = x I_0^2(x) + x I_1^2(x)$$

Having computed Λ , we now turn to equation (14) to compute the shift, δ , of the magnetic surface with radius, a . Here, r_c denotes the radius of a perfectly conducting wall. From (32) and (14)

$$\delta(a) = \frac{r_c^2}{2R} \left[\ln \frac{r_c}{a} - \frac{I_A}{I} \left(1 - \frac{a^2}{r_c^2}\right) + \frac{1}{2} \left(1 - \frac{a^2}{r_c^2}\right) \left(\frac{I_0 I_2}{I_1^2} - 1\right) \right] \quad (33)$$

The argument of the Bessel function is $(\omega_b a/c)$.

The shift, $\delta(a)$, given by equation (33) applies to the magnetic surface with radius, a . As we noted earlier, the beam surface is not a magnetic surface. To find the additional shift of the beam surface with respect to the magnetic surface of the same radius, we must re-examine the definitions of these two surfaces. The beam surface is defined as $\alpha = 1$, or $\frac{e}{c} A_\phi + \gamma_0 m r v_\phi / r = \text{constant}$, while the magnetic surfaces are given by $\psi = r A_\phi = \text{constant}$. We expand $r A_\phi = \text{constant}$ in the neighborhood of $r = a$ to obtain the equation for the magnetic surface of radius, a , to first order in a/R ,

$$r = a - \delta_b \cos \theta + O(a/R)^2 \quad (34)$$

where

$$\delta_b = \frac{a}{R} \frac{A_z(a) + (mc^2/e) \gamma_0 \beta_z}{A_z'(a)} = -\frac{1}{2} \frac{a^2}{R} \frac{I_A}{I} \quad (35)$$

The total shift of the beam surface relative to the conducting wall of the torus is then given by

$$\delta_{\text{total}}(a) = \delta(a) + \delta_b = \frac{r_c^2}{2R} \left[\ln \frac{r_c}{a} - \frac{I_A}{I} + \frac{1}{2} \left(1 - \frac{a^2}{r_c^2}\right) \left(\frac{I_0 I_2}{I_1^2} - 1\right) \right] \quad (36)$$

which agrees with the calculation by Ott and Sudan⁵ (see equation (24) of their paper). The shift given by (36) is always positive.⁵

C. Beam-Plasma System ($n = 0$)

We now turn to a beam-plasma system with finite plasma pressure, $p_0 \neq 0$. The simplest example in the model we are using is the $n = 0$ case, where the plasma pressure is constant, and the plasma carries only a toroidal surface current density, J_p . In the straight, cylindrical, zero-order equilibrium, the beam surface is $\rho = a$ and the plasma surface is $\rho = b$. For $\rho < b$, the solutions for $A_z(\rho)$ and $B_\theta(\rho)$ are the same as those obtained above for a cold plasma. At $\rho = b$ the plasma pressure falls suddenly to zero, and the B_θ field changes discontinuously from $B_\theta = 2I_B/cb$ at $\rho = b - \epsilon$ to $B_\theta = 2I_B/cb + 2I_p/cb$ at $\rho = b + \epsilon$, where $I_p = 2\pi b J_p$ is the plasma current, I_B is the beam current, and ϵ is an arbitrarily small, positive quantity. The average value of B_θ across $\rho = b$ is therefore

$$B_\theta(b) \equiv \frac{1}{2} [B_\theta(b^-) + B_\theta(b^+)] = \frac{2I_B}{cb} + \frac{I_p}{cb} \quad (37)$$

The condition for equilibrium is $\frac{1}{c} J_p B_\theta(b) = p_0$ or

$$I_p \left(I_B + \frac{1}{2} I_p \right) = \pi b^2 c^2 p_0 \quad (38)$$

In terms of total current, $I = I_p + I_B$, equation (38) becomes

$$I^2 - I_B^2 = 2\pi b^2 c^2 p_0 \quad (39)$$

which defines the equilibrium value of p_0 .

From Equation (27), the beam current may be written as

$$\frac{I_B}{I_A} = -\frac{1}{2} \frac{\omega_b a}{c} I_1\left(\frac{\omega_b a}{c}\right) / I_0\left(\frac{\omega_b a}{c}\right) \quad (40)$$

where $I_A = (mc^3/e) \gamma_0 \beta_0$ is the Alfvén current.

We now apply the results of Section II to find the shifts in the plasma surface when the above cylindrical equilibrium is bent into a large aspect ratio torus. Equation (14), which gives the shift of the plasma surface with respect to a perfectly conducting wall at radius r_c is derived from matching the fields on the plasma surface to the vacuum field, the normal component of which must vanish at the conducting wall, equation (13). The appropriate Λ for the plasma surface must therefore be computed at

$\rho = b + \epsilon$ in order to properly match with the vacuum fields for $\rho \gg b$. In equation (11), defining Λ , we have $P_{\rho\rho}(b) = 0$ and $B_a^2 = B_0^2(\rho + \epsilon)$, or

$$B_a^2 = B_0^2(\rho + \epsilon) = B_0^2(\rho - \epsilon) + 8\pi p_0 \quad (41)$$

which follows from equation (39), and also follows from the general pressure balance equation (10), when we notice that the longitudinal field satisfies $B_e^2 = \overline{B_1^2}$ in the cylindrical equilibrium.

From the cold plasma example,

$$B_0^2(\rho) = \begin{cases} \frac{\omega_b^2}{c^2} \left(\frac{I_A}{c}\right)^2 I_1^2\left(\frac{\omega_b \rho}{c}\right) / I_0^2\left(\frac{\omega_b a}{c}\right) & \rho \leq a \\ \frac{\omega_b^2}{c^2} \left(\frac{I_A}{c}\right)^2 \left[I_1^2\left(\frac{\omega_b a}{c}\right) / I_0^2\left(\frac{\omega_b a}{c}\right) \right] \frac{a^2}{\rho^2} & a \leq \rho \leq b - \epsilon \end{cases}$$

and hence,

$$B_a^2 = \frac{\omega_b^2}{c^2} \left(\frac{I_A}{c} \right)^2 \left[I_1^2 \left(\frac{\omega_b a}{c} \right) / I_0^2 \left(\frac{\omega_b a}{c} \right) \right] \frac{a^2}{b^2} + 8\pi p_0 \quad (43)$$

from equation (41).

Using equation (30) for the beam pressure ($\rho \leq a$) and p_0 for the plasma pressure ($\rho \leq b$), together with equations (42) and (43), we evaluate equation (11) to obtain

$$\Lambda(b) = \frac{\frac{1}{b^2} \left(\frac{I_A}{c} \right)^2 \left[\eta^2 + 2\eta I_1/I_0 + 2\eta^2 (I_1/I_0)^2 \ln \frac{b}{a} \right] + 16\pi p_0}{\frac{2}{b^2} \left(\frac{I_A}{c} \right)^2 \eta^2 (I_1/I_0)^2 + 16\pi p_0} - 1 \quad (44)$$

where $\eta = \omega_b a/c$, which is also the argument of the Bessel functions.

With equations (39) and (40), we obtain

$$\begin{aligned} \Lambda(b) &= \frac{1}{8} \left(\frac{I_A}{I} \right)^2 \left[\eta^2 + 2\eta I_1/I_0 + 2\eta^2 (I_1/I_0)^2 \left(\ln \frac{b}{a} - 1 \right) \right] \\ &= \frac{\eta^2}{8} \left(\frac{I_A}{I} \right)^2 - \frac{1}{2} \left(\frac{I_B}{I} \right) \left(\frac{I_A}{I} \right) + \left(\frac{I_B}{I} \right)^2 \left(\ln \frac{b}{a} - 1 \right) \end{aligned} \quad (45)$$

where

$$\frac{I_B}{I} = -\frac{\eta}{2} \frac{I_1}{I_0} \frac{I_A}{I} \quad (46)$$

follows from equation (40).

The behavior of Λ as the fractional beam current, I_B/I , is allowed to vary for fixed total current I/I_A , is of interest. Since I_A/I is negative,

Λ will be monotonically increasing with I_B/I provided $\ln \frac{b}{a} \geq 1$. From equation (46) we see that for fixed I/I_A the equilibrium exists for $\eta \leq \eta_{\max}$ given by the requirement $I_B/I \leq 1$,

$$-\frac{1}{2} \eta_{\max} I_1(\eta_{\max}) / I_0(\eta_{\max}) = I / I_A \quad (47)$$

Therefore, for $(I/I_A)^2 \ll 1$, we have $\eta_{\max} \ll 1$, and Λ is monotonically increasing with I_B/I even for $\ln \frac{b}{a} < 1$. If $(I/I_A)^2 \gg 1$, however, then $\eta_{\max} \gg 1$ and for $\ln \frac{b}{a} < 1$, Λ will increase over small values of I_B/I , and subsequently decrease as I_B/I approaches unity. For $\ln \frac{b}{a} = 0$ and $(I/I_A)^2 \rightarrow \infty$, the worst case, $\Lambda \rightarrow -1/2$ as $I_B/I \rightarrow 1$.

The shift, δ , is given by equation (14),

$$\delta = \frac{V_c^2}{2R} \left[\ln \frac{V_c}{b} + \left(\Lambda + \frac{1}{2} \right) \left(1 - \frac{b^2}{V_c^2} \right) \right] \quad (48)$$

which depends on I_B/I only through Λ , and is therefore a monotonically increasing function of I_B/I when $(I/I_A)^2 \ll 1$ or when $\ln \frac{b}{a} \geq 1$, but is not monotonically increasing with I_B/I if $\ln \frac{b}{a} < 1$ and $(I/I_A)^2$ is sufficiently large.

From equation (45) we have

$$\Lambda(b) > -\frac{1}{4} \left(\frac{I_A}{I} \right)^2 \eta^2 \left(\frac{I_1}{I_0} \right)^2 = -\left(\frac{I_B}{I} \right)^2 > -1$$

or

$$\Lambda(b) + 1 > 0$$

(49)

Using equation (48), we note that $\delta(r_c = b) = 0$, and

$$\frac{d\delta}{dr_c} = \frac{r_c}{R} \left[\ln \frac{r_c}{b} + (\Lambda + 1) \right] \geq \frac{r_c}{R} (\Lambda + 1) > 0.$$

Since $r_c \geq b$ and $d\delta/dr_c > 0$, $\delta(r_c = b) = 0$ implies that $\delta > 0$. Thus the toroidal shift of the plasma surface is always outward, as expected.

Equation (48) gives the shift of the plasma surface relative to a conducting wall at radius r_c . The relative shift between the beam surface and the plasma surface is given by equation (36) with r_c replaced by b . The total shift of the beam surface with respect to the conducting wall is then the sum of equations (48) and (36), with r_c set equal to b in equation (36). The shifts of the various surfaces described here are shown in Figure (2).

D. Beam-Plasma System ($n = 1$)

The next simplest example of the model we are using is the $n = 1$ case, where the plasma pressure is a linear function of the flux function, and the plasma current is constant. For this case, equation (22) for the straight, zero-order equilibrium becomes

$$\frac{1}{\eta^2 x} \frac{d}{dx} x \frac{dA_2}{dx} = \left[A_2 + \frac{mc^2}{e} \gamma_0 \beta_z \right] \Theta(A_1 - A_2) + \frac{4\sigma a^2}{\eta^2} \Theta(A_0 - A_2),$$

(50)

where we have defined the variables,

$$x \equiv \rho/a, \quad \eta \equiv \omega_0 a/c, \quad \sigma \equiv \pi \beta_0/A_0. \quad (51)$$

We solve equation (50) in three regions, $\rho \leq a$, $a \leq \rho \leq b$, and $\rho \geq b$, where $\rho = a$ is the beam radius, $A_z(a) = A_1$, and $\rho = b$, is the plasma radius, $A_z(b) = A_0$. At the boundaries, $\rho = a$ and $\rho = b$, we require $A_z(x)$ and $A'_z(x)$ to be continuous.

The solution for A_z which satisfies $A_z(\rho=0) = 0$ is then

$$A_z(x) = \begin{cases} \left[\frac{mc^2}{e} \chi_0 \beta_z + \frac{4\sigma a^2}{\eta^2} \right] [I_0(\eta x) - 1] & x \leq 1 \\ A_1 + \left[\eta \left(\frac{mc^2}{e} \chi_0 \beta_z + \frac{4\sigma a^2}{\eta^2} \right) I_1(\eta) - 2\sigma a^2 \right] \ln x + \sigma a^2 (x^2 - 1) & 1 \leq x \leq b/a \\ A_0 + \left[\eta \left(\frac{mc^2}{e} \chi_0 \beta_z + \frac{4\sigma a^2}{\eta^2} \right) I_1(\eta) + 2\sigma (b^2 - a^2) \right] \ln \frac{ax}{b} & x \geq b/a \end{cases} \quad (52)$$

Using equations (17) and (52), we find

$$A_1 \equiv A_z(a) = \frac{mc^2}{e} \chi_0 (\beta_0 - \beta_z) = \left[\frac{mc^2}{e} \chi_0 \beta_z + \frac{4\sigma a^2}{\eta^2} \right] [I_0(\eta) - 1]$$

or

$$\beta_0 = \beta_z I_0(\eta) + \frac{4\sigma a^2}{\eta^2} \cdot \frac{e}{\chi_0 mc^2} [I_0(\eta) - 1] \quad (53)$$

and

$$I_A \equiv \frac{mc^3}{e} \chi_0 \beta_0 = \frac{mc^3}{e} \chi_0 \beta_z I_0(\eta) + \frac{4\sigma a^2 c}{\eta^2} [I_0(\eta) - 1] \quad (54)$$

The magnetic field corresponding to (52) is given by $B_0(x) = -\frac{1}{a} \frac{dA_2}{dx}$,

$$B_0(x) = \begin{cases} -\frac{\eta}{a} \left[\frac{mc^2}{e} \gamma_0 \beta_z + \frac{4\sigma a^2}{\eta^2} \right] I_1(\eta x) & x \leq 1 \\ -\frac{\eta}{ax} \left[\left(\frac{mc^2}{e} \gamma_0 \beta_z + \frac{4\sigma a^2}{\eta^2} \right) I_1(\eta) - \frac{2\sigma a^2}{\eta} \right] - 2\sigma ax & 1 \leq x \leq b/a \\ -\frac{\eta}{ax} \left[\left(\frac{mc^2}{e} \gamma_0 \beta_z + \frac{4\sigma a^2}{\eta^2} \right) I_1(\eta) + \frac{2\sigma}{\eta} (b^2 - a^2) \right] & x \geq b/a \end{cases} \quad (55)$$

The currents carried by this equilibrium then follow from

$$B_0(a) = \frac{2I_B}{ca} + \frac{2I_p}{cb^2} a = -\frac{\eta}{a} \left[\frac{mc^2}{e} \gamma_0 \beta_z + \frac{4\sigma a^2}{\eta^2} \right] I_1(\eta)$$

$$B_0(b) = \frac{2}{cb} [I_B + I_p] = -\frac{\eta}{b} \left[\frac{mc^2}{e} \gamma_0 \beta_z + \frac{4\sigma a^2}{\eta^2} \right] I_1(\eta) - 2\sigma a \left(\frac{b}{a} - \frac{a}{b} \right) \quad (56)$$

yielding the following expressions for the total current, I , the beam current, I_B , and the plasma current, I_p ,

$$I = I_B + I_p = -\frac{\eta c}{2} \left[\frac{mc^2}{e} \gamma_0 \beta_z + \frac{4\sigma a^2}{\eta^2} \right] I_1(\eta) - \sigma c (b^2 - a^2) \quad (57)$$

$$I_B = -\frac{\eta c}{2} \left[\frac{mc^2}{e} \gamma_0 \beta_z + \frac{4\sigma a^2}{\eta^2} \right] I_1(\eta) + \sigma c a^2 \quad (58)$$

$$I_p = -\sigma c b^2 \quad (59)$$

The various quantities associated with this equilibrium may be conveniently expressed in terms of the currents and η . For example, using equation (54) together with equations (57) through (59), we can

re-write equation (52) for the vector potential in the form

$$\frac{A_2(x)}{I_A/c} = \begin{cases} \left[1 - \frac{4}{\eta^2} \frac{a^2}{b^2} \frac{I_p}{I_A} \right] \frac{I_0(\eta x) - 1}{I_0(\eta)} & x \leq 1 \\ \frac{A_1}{I_A/c} - \frac{2I_B}{I_A} \ln x - \frac{a^2}{b^2} \frac{I_p}{I_A} (x^2 - 1) & 1 \leq x \leq b/a \\ \frac{A_0}{I_A/c} - \frac{2I}{I_A} \ln \frac{xa}{b} & x \geq b/a \end{cases} \quad (60)$$

We should point out that all the quantities in equation (60) are not independent. For example, if a/b and γ_0 (and hence I_A) are taken as fixed quantities, then knowledge of η and I_p/I_A would suffice to determine I_B/I_A . In general, specifying any two of $(\eta, I_p/I_A, I_B/I_A, I/I_A)$ will determine the remaining two quantities in the set. The currents and η are related by

$$\frac{I_B}{I_A} = -\frac{\eta}{2} \frac{I_1(\eta)}{I_0(\eta)} - \frac{a^2}{b^2} \frac{I_p}{I_A} \frac{I_2(\eta)}{I_0(\eta)} \quad (61)$$

$$\frac{I}{I_A} = -\frac{\eta}{2} \frac{I_1(\eta)}{I_0(\eta)} + \frac{I_p}{I_A} \left[1 - \frac{a^2}{b^2} \frac{I_2(\eta)}{I_0(\eta)} \right] \quad (62)$$

which are equivalent to equations (57) and (58).

The beam current density, equation (18), may be expressed with the help of equation (60), as

$$\frac{\pi a^2 J_B(x)}{I_A} = -\frac{\eta^2}{4} \frac{I_0(\eta x)}{I_0(\eta)} + \frac{a^2}{b^2} \frac{I_p}{I_A} \left[\frac{I_0(\eta x)}{I_0(\eta)} - 1 \right] \quad (63)$$

for $x \leq 1$. The plasma pressure may be written as

$$p(x) = \begin{cases} p_0 + \frac{1}{\pi b^2} \left(\frac{I_A}{c} \right)^2 \frac{I_p}{I_A} \left[1 - \frac{4}{\eta^2} \frac{a^2}{b^2} \frac{I_p}{I_A} \right] \frac{I_0(\eta x) - 1}{I_0(\eta)} & x \leq 1 \\ p_0 + \frac{1}{\pi b^2} \left(\frac{I_A}{c} \right)^2 \frac{I_p}{I_A} \left[\frac{A_1}{I_A/c} - \frac{2I_B}{I_A} \ln x - \frac{a^2}{b^2} \frac{I_p}{I_A} (x^2 - 1) \right] & 1 \leq x \leq b/a \end{cases} \quad (64)$$

From the requirement $p(x = \frac{b}{a}) = 0$, we have

$$p_0 = - \frac{1}{\pi b^2} \left(\frac{I_A}{c} \right)^2 \frac{I_p}{I_A} \left[\frac{A_1}{I_A/c} - \frac{2I_B}{I_A} \ln \frac{b}{a} - \frac{I_p}{I_A} \left(1 - \frac{a^2}{b^2} \right) \right] \quad (65)$$

Graphs of equations (60) through (64) are presented in figures (3) through (6) to illustrate the behavior of the cylindrical equilibrium which we are using. The case $I_p/I_A = 0$ corresponds to the Hammer-Rostoker equilibrium.⁸ From the figures we see that increasing I_p/I_A has the effect of depressing I_B/I_A and $J_B(x)$. In particular, for large values of η , where the Hammer-Rostoker equilibrium carries current only in an annular shell, our equilibrium with plasma can carry a small reverse beam current near the axis, in addition to the usual annular current as shown in figure (3). Since the single-particle angular momentum is a conserved quantity, only those particles with very small values of angular momentum can have orbits which approach the beam axis. The analytical expression for the beam particle trajectories are generally obtainable only as quadratures. For the special case, $I_p \gg I_B$, however, the B_θ field due to the uniform plasma current dominates the self-consistent fields due to the beam particles, and the beam particle trajectories become approximately the well-known Alfvén

orbits.⁹ These trajectories are drawn in figure (4) for beam particles which do not carry angular momentum, i.e., for those particles which may cross the beam axis. For $I_p \gg I_B$ the beam particle trajectory depends on the relative magnitudes of I_p and I_A . Type a and b beam particle orbits (cf. figure (4)) occur for $|I_p| < I_A$. Type c orbits, where the beam particles cross the axis with zero axial velocity, occur for $|I_p| = I_A$. For larger values of I_p , $|I_p| > I_A$, type d, e, and f orbits are obtained. The curves in figure (3) for the beam current density with $I_p/I_A = -2$ correspond to d-type orbits, in which the particles cross the beam axis with negative axial velocity, but yet have a net forward motion over their entire trajectories. This explains the reverse current near $x = 0$ in figure (3).

The vector potential, $A_z(x)$, is plotted in figure (5). As the total current is increased, $A_z(x)$ rises to higher values as expected. For $\eta = 5$, the vector potential is depressed for small values of x due to the annular beam current density profile (shown in figure (3)). The plasma pressure, displayed in figure (6), likewise shows a flattening for $x < 1$ at $\eta = 5$. If η were allowed to become arbitrarily large, the plasma pressure would approach a step function, having the value p_0 for $x < 1$ and zero for $x > 1$.

We now apply the techniques of Section II to generalize the equilibrium to toroidal geometry. The first-order expansion coefficient, Λ , is given by equation (11), which we evaluate for the plasma surface, i.e., the magnetic surface with radius, b . We note that both the beam and plasma pressures vanish at $\rho = b$, therefore $P_{ff}(b) = 0$. The components of the beam pressure tensor which are required are

$$\begin{aligned} P^b_{\phi\phi} &\equiv \frac{1}{2} \left[P^b_{\phi\phi}(\rho) + \frac{1}{2} (P^b_{\rho\rho}(\rho) + P^b_{\theta\theta}(\rho)) \right] \\ &= \frac{\eta^2}{16\pi a^2} \left[\left(\frac{I_A}{c} \right)^2 + \left(A_z + \frac{mc^2}{e} \chi_0 \beta_z \right)^2 \right] \end{aligned} \quad (66)$$

for $\rho \leq a$, and $P^b(\rho) = 0$ for $\rho > a$. Using equation (60), we obtain

$$P^b(x) = \begin{cases} \frac{\eta^2}{16\pi a^2} \left(\frac{I_A}{c}\right)^2 \left\{ 1 + \left[\frac{I_0(\eta x)}{I_0(\eta)} - \frac{4}{\eta^2} \frac{a^2}{b^2} \frac{I_p}{I_A} \left(\frac{I_0(\eta x)}{I_0(\eta)} - 1 \right) \right]^2 \right\} & x \leq 1 \\ 0 & x > 1 \end{cases} \quad (67)$$

The plasma pressure, $p(x)$, is given by equation (64).

From equation (55) we have

$$B_\theta^2(x) = \begin{cases} \frac{\eta^2}{a^2} \left(\frac{I_A}{c}\right)^2 \left[1 - \frac{4}{\eta^2} \frac{a^2}{b^2} \frac{I_p}{I_A} \right]^2 \frac{I_1^2(\eta x)}{I_0^2(\eta)} & x \leq 1 \\ \frac{1}{a^2} \left(\frac{I_A}{c}\right)^2 \left[\frac{2I_B}{I_A} \frac{1}{x} + \frac{2a^2}{b^2} \frac{I_p}{I_A} x \right]^2 & 1 \leq x \leq b/a \end{cases} \quad (68)$$

and

$$B_a^2 \equiv B_\theta^2(x = b/a) = \frac{4I^2}{c^2 b^2} \quad (69)$$

The expression for Λ is then

$$\Lambda = \frac{16\pi \bar{P} + \bar{B}_\theta^2}{2B_a^2} - 1 \quad (70)$$

when $\bar{P} = \overline{P^b(x)} + \overline{p(x)}$, and

$$\begin{aligned} 16\pi \bar{P} + \bar{B}_\theta^2 &= \frac{32\pi a^2}{b^2} \int_0^1 x dx P^b(x) + \frac{32\pi a^2}{b^2} \int_0^{b/a} x dx p(x) \\ &\quad + \frac{2a^2}{b^2} \int_0^{b/a} x dx B_\theta^2(x) \end{aligned} \quad (71)$$

Upon substitution and integration in equation (71), the result may be combined with equation (70) to yield

$$\Lambda(b) = \left(\frac{I_A}{I}\right)^2 \left\{ \frac{\eta^2}{8} [1 + (1-\zeta)^2] + \frac{3}{2} \eta \zeta (1 - \frac{5}{6} \zeta) \frac{I_1(\eta)}{I_0(\eta)} \right. \\ \left. + \left(\frac{I_b}{I_A}\right)^2 \ln \frac{b}{a} + \frac{5}{4} \left(\frac{I_p}{I_A}\right)^2 \left(1 - \frac{a^4}{b^4}\right) + \frac{3 I_B I_p}{I_A^2} (1 - a^2/b^2) \right. \\ \left. - 2 \frac{a^2}{b^2} \zeta \frac{I_p}{I_A} \right\} - 1 \quad (72)$$

where

$$\zeta \equiv 1 - \frac{4}{\eta^2} \frac{a^2}{b^2} \frac{I_p}{I_A}$$

A plot of $\Lambda(b)$ vs. the fractional current carried by the beam, I_B/I , is given in figure (7). One might expect $\Lambda(b)$ to be a monotonically increasing function of I_B/I since the outward centrifugal force of the circulating electrons increases with I_B . We note from figure (7), however, that Λ need not be a monotonically increasing function of I_B/I . This non-increasing behavior is due to the non-uniform, annular nature of the beam current density for large η . For $I/I_A = -0.8$, the beam and plasma current densities are both practically uniform, and the monotonically increasing $\bar{P}_{\phi\phi}$ term in equation (11) dominates the behavior of Λ as a function of I_B/I . For $|I/I_A| \gtrsim 2$, however, the beam current density becomes strongly non-uniform as I_B/I increases. This non-uniform character of $J_B(x)$ causes the B_e^{-2} term in equation (11) to decrease as I_B/I increases. For $|I/I_A|$ large, the decreasing B_e^{-2} term dominates the increasing $\bar{P}_{\phi\phi}$ term in equation (11), leading to the non-increasing behavior of $\Lambda(b)$ seen in figure (7) for $I/I_A = -3.0$. For I/I_A in the vicinity of -2 , we see that

the shape of $\Lambda(b)$ vs. I_B/I is rather sensitive to I/I_A . The three curves shown in figure 7 are for three values of I/I_A which differ by roughly a factor of two between adjacent curves. In the ORMAK experiments, relativistic runaways with 10^5 A of current and MeV energy were reported. For these parameters, $\gamma \approx 3$, and $I_B/I_A \approx 2$, which would be in the range spanned by curves in figure (7).

The shift in the plasma surface is then given by equation (14),

$$\delta(b) = \frac{r_c^2}{2R} \left[\ln \frac{r_c}{b} + \left(\Lambda + \frac{1}{2} \right) \left(1 - \frac{b^2}{r_c^2} \right) \right] \quad (73)$$

As seen from figure (7) and equation (73), $\delta(b)$ is always positive, i.e., the shift is always toward the outer wall of the torus. This is as expected since both the centrifugal force of the relativistic circulating electrons and the toroidal magnetic forces are outward. The non-increasing character of $\Lambda(b)$ as a function of I_B/I will occur also as a non-increasing behavior of $\delta(b)$.

Finally, we note that analytical solutions of equation (22) with $n = 2$ for straight cylindrical equilibria with background plasma are also straightforward to obtain but are omitted here.

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Figure Captions

- Fig. 1: Toroidal Coordinates
- Fig. 2: Solution of $n = 0$ Equilibrium
- Fig. 3: Current Density in straight equilibrium with $n = 1$; $a^2/b^2 = 0.5$.
- Fig. 4: Alfvén Orbits (after Hammer and Rostoker)
- Fig. 5: Vector Potential in straight equilibrium with $n = 1$; $a^2/b^2 = 0.5$.
- Fig. 6: Plasma Pressure " " " " " "
- Fig. 7: Solution for $\Lambda(b)$ in $n = 1$ Model with $a^2/b^2 = 0.5$.

Toroidal Coordinates

Figure 1

$$n = 0$$

Figure 2

Figure 3

Trajectories of Particles Starting in the z Direction
at Various Distances from the Axis of a Uniform, Neutralized Particle Beam
(After Alfvén)

Solid (dashed) curves represent particle trajectories with net motion forward (backward).

Figure 4

Figure 5

Figure 6

Figure 7

Synchrotron Radiation from
Force-Free Equilibria

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Abstract

We have examined the synchrotron radiation losses from relativistic electron beams which are close to a force-free configuration. The resulting radiation is found to be substantially less than the radiation found from a non-force-free configuration, such as the well-known Hamner-Rostoker equilibrium. We conclude that force-free beams may therefore be useful for providing magnetic confinement of plasma.

Introduction

Yoshikawa¹ has observed that high-current equilibria for relativistic electron beams can be constructed, with axial current ~~for~~^{far} in excess of the Alfvén critical current,²⁻³ $I_A \equiv \frac{mc}{4\pi} \beta \approx 17 \beta \gamma \text{ kA}$, without resorting to "hollowed-out" current distributions as proposed by Hammer and Rostoker.³ Recently, Kan and Lai⁴ have constructed a family of fully-relativistic self-consistent equilibria, incorporating Yoshikawa's suggestions, and have noted that such equilibria can carry current in excess of those in Hammer and Rostoker's equilibrium at the same radius.

The Alfvén limit² arises from the pinching of a uniform, electrostatically neutralized relativistic electron beam in its self-consistent magnetic field. When the beam current exceeds I_A , the magnetic force, $\frac{1}{2} J_z B_\theta$, is sufficient to turn the beam electrons around, and the beam then no longer propagates. Hammer and Rostoker³ showed that a beam carrying $J_z(r)$ can propagate with $I > I_A$ by becoming non-uniform. At high currents, their equilibrium has J_z flowing in an annular shell of thickness, c/ω_p , i.e., a "hollowed-out" current distribution. Here, $\omega_p^2 = 4\pi n_0 e^2/m$, and c/ω_p is the collisionless skin-depth of the beam. In this model, the beam can propagate because the electrons leave the high field sheath region before the magnetic pinching force can reverse their direction of travel.

Yoshikawa¹ suggested that the beam carry macroscopic currents, J_θ and J_z . This model allows the possibility that $\underline{J} \times \underline{B} = 0$, the force-free condition, in which the magnetic pinching force vanishes, and the Alfvén limit no longer applies. It is clear^{1,4} that a real beam cannot be force-free

over its entire cross section. The J_θ implies the existence of a radial centrifugal force, $\frac{1}{2}mv_s^2/r$. Also, a beam of finite radius would have a pressure gradient in the radial direction. These two forces must be balanced by the magnetic force for the beam to have an equilibrium. Kan and Lai⁴ have shown, however, that it is possible to construct equilibria which are close to force-free over a significant fraction of the beam cross section.

The synchrotron radiation losses from force-free equilibria in an applied axial magnetic field, B_0 , become very small in the limit that B_0 becomes large. This observation has important ramifications for any experiment in which the relativistic beam is used to provide magnetic confinement of a plasma, as in applications to controlled thermonuclear fusion research. One such device is the "Astron-Spherator."⁵

In this paper we present a calculation of the synchrotron radiation losses from an intense, relativistic electron beam which is almost force-free. The results are displayed as a function of applied magnetic field, and compared with the radiation losses from Harner and Rostoker's equilibrium. The radiated power from beams which are close to force-free is found to be substantially less^{than} that from the Harner-Rostoker equilibrium.

Force-Free Equilibria

Yoshikawa¹ has studied the force-free equilibria of a cold, charge neutralized beam, for which the equilibrium equation is

$$\frac{1}{c} [\bar{J}_\theta \bar{B}_z - \bar{J}_z \bar{B}_\theta] = - \frac{\gamma_m \bar{J}_\theta^2}{n_0^2 r} = - \pi a^2 \left(\frac{\gamma}{\nu} \right) \frac{\bar{J}_\theta^2}{r c^2} \quad (1)$$

Here, a = the beam radius, and $\nu \equiv \pi a^2 n_0^2 / m c^2$. For intense beams, $\nu/\gamma \gg 1$, we see that the right-hand side of equation (1) becomes small, i.e. the beam becomes almost force-free when $\nu/\gamma \gg 1$.

To include the perpendicular pressure of the beam, we generalize equation (1) to

$$\frac{\gamma_m \bar{J}_\theta^2}{n_0^2 r} + \frac{1}{c} [\bar{J}_\theta \bar{B}_z - \bar{J}_z \bar{B}_\theta] = \frac{d}{dr} \bar{p}_\perp \quad (2)$$

All quantities are assumed to depend only on r ($\frac{\partial}{\partial z} = \frac{\partial}{\partial t} = 0$). From Maxwell's equations, we have

$$\begin{aligned} \frac{d\bar{J}_\theta}{c} &= - \frac{d\bar{B}_z}{dr} \\ \frac{d\bar{J}_z}{c} &= \frac{1}{r} \frac{d}{dr} (r \bar{B}_\theta) \end{aligned} \quad (3)$$

and equation (2) may be written as

$$\frac{\gamma c^2}{\omega_p^2 r} \left(\frac{d\bar{B}_z}{dr} \right)^2 - \frac{1}{2} \frac{d\bar{B}_z^2}{dr} - \frac{1}{2r^2} \frac{d}{dr} (r \bar{B}_\theta)^2 = 4\pi \frac{d\bar{p}_\perp}{dr} \quad (4)$$

We now specify the following functions, for $r \leq a$,

$$\begin{aligned} \bar{B}_z(r) &= B_0 + B_1 (1 - r^2/a^2) \\ \bar{p}_\perp(r) &= \bar{p}_\perp^0 (1 - r^2/a^2) \\ n(r) &= n_0 \end{aligned} \quad (5)$$

and for $r > a$

$$\begin{aligned} B_z(r) &= B_0 \\ p_z(r) &= 0 \\ n(r) &= 0 \end{aligned} \quad (b)$$

Where B_0, B_1, p_z^0 and n_0 are constants.

From equation (4) we have

$$\frac{d}{dr} (r B_\theta)^2 = \frac{16\pi r^3}{a^2} p_z^0 + \frac{4r^3}{a^2} B_1 [B_0 + B_1(1 - r^2/a^2)] + \frac{8r^3}{a^2} \frac{v_z^2}{\omega_p^2 a^2} B_1^2 \quad (7)$$

for $r \leq a$, which yields

$$B_\theta^2(r) = \frac{r^2}{a^2} \left\{ 4\pi p_z^0 + B_1(B_0 + B_1) - \frac{2}{3} \frac{r^2}{a^2} B_1^2 + \frac{2V_z^2}{\omega_p^2 a^2} B_1^2 \right\} \quad (8)$$

The solution for $r > a$ is

$$B_\theta^2(r) = B_a^2 \frac{a^2}{r^2} \quad (9)$$

where $B_a = B_\theta(r=a)$

To find the current density,

$$\begin{aligned} J_z(r) &= \frac{c}{4\pi} \frac{1}{r} \frac{d}{dr} (r B_\theta) \\ &= \frac{c}{2\pi B_\theta(r)} \left[\frac{B_\theta^2(r)}{r} - \frac{1}{3} B_1^2 \frac{r^3}{a^4} \right] \\ &= n_0 e V_z(r) \end{aligned}$$

from which

$$V_z(r) = \frac{c}{2\pi n_0 e} \left[\frac{B_\theta^2(r)}{r} - \frac{1}{3} \frac{B_1^2}{B_\theta(r)} \frac{r^3}{a^4} \right] \quad (10)$$

Also,

$$\begin{aligned} \bar{J}_\theta(r) &= \frac{-c}{4\pi} \frac{dB_z}{dr} \\ &= \frac{c}{2\pi} B_1 \frac{r}{a^2} \end{aligned}$$

and

$$V_\theta(r) = \frac{c}{2\pi n_0 e} B_1 \frac{r}{a^2} \quad (11)$$

The functions $V_z(r)$ and $V_\theta(r)$, given by equations (10) and (11), are the first moments of the equilibrium distribution function. The second moment of the distribution function is given by the perpendicular pressure, $\dot{p}_\perp(r)$, as

$$\dot{p}_\perp = \frac{1}{2} \frac{q}{4\pi n_0} V_\perp^2 \quad (12)$$

If $V_\perp^2/c^2 \ll 1$, we can approximate $V_\perp^2$ by

$$V_\perp^2 = \frac{2\dot{p}_\perp^0}{q/4\pi n_0} \left(1 - r^2/a^2\right) \quad (13)$$

Radiation

The single-particle equations of motion for the beam electrons in a charge-neutralized beam are

$$\begin{aligned}\frac{dv_x}{dt} &= \frac{e}{\gamma m c} v_z B_\theta \frac{x}{r} - \frac{e}{\gamma m c} v_y B_z \\ \frac{dv_y}{dt} &= \frac{e}{\gamma m c} v_z B_\theta \frac{y}{r} + \frac{e}{\gamma m c} v_x B_z \\ \frac{dv_z}{dt} &= -\frac{e}{\gamma m c} \left[\frac{x v_x}{r} + \frac{y v_y}{r} \right] B_\theta\end{aligned}\quad (14)$$

This yields

$$\begin{aligned}\dot{v}^2 &= \left(\frac{dv_x}{dt} \right)^2 + \left(\frac{dv_y}{dt} \right)^2 + \left(\frac{dv_z}{dt} \right)^2 \\ &= \frac{e^2 B^2}{\gamma^2 m^2 c^2} v_r^2 + \left[\frac{e B_\theta}{\gamma m c} v_z \frac{\partial}{\partial} - \frac{e B_z}{\gamma m c} v_\theta \right]^2\end{aligned}\quad (15)$$

where $B^2 = B_\theta^2 + B_z^2$

The single-particle radiation is given by

$$\begin{aligned}P_p &= \frac{2}{3} \frac{e^2}{c^3} \gamma^4 \dot{v}^2 \\ &= \frac{2}{3} \frac{e^2}{c^3} \gamma^4 \left[\left(\frac{e B_\theta}{\gamma m c} \right)^2 v_r^2 + \left(\frac{e B_\theta}{\gamma m c} v_z - \frac{e B_z}{\gamma m c} v_\theta \right)^2 \right]\end{aligned}\quad (16)$$

and the total radiated power per unit length of beam is given by

$$P = 2\pi \int_0^a r dr \int d\mathbf{p} F(r, \mathbf{p}) P_p(r, \mathbf{p}) \quad (17)$$

where $F(r, \mathbf{p})$ is the equilibrium distribution function for the beam.

In terms of the moments of the distribution function, $F(r, \mathbf{p})$, we have

$$P = \frac{4\pi}{3} \frac{e^2}{c^3} \int_0^a r dr n(r) \left\{ \Omega^2 \langle \gamma^2 v_r^2 \rangle + \langle (\Omega_\theta \gamma v_z - \Omega_z \gamma v_\theta)^2 \rangle \right\} \quad (18)$$

where

$$n(r) \langle Q \rangle = \int d\vec{r} Q(r, \vec{r}) F(r, \vec{r})$$

and

$$\Omega = \frac{e^2}{mc} , \quad \Omega_\theta = \frac{e^2 \rho_\theta}{mc} , \quad \Omega_z = \frac{e^2 \rho_z}{mc}$$

We assume that the beam is monoenergetic, and that the first velocity moments are non-zero in the z and θ directions only,

$$\langle v_z \rangle = V_z(r) , \quad \langle v_\theta \rangle = V_\theta(r) , \quad \langle v_r \rangle = 0$$

The second moments are assumed given by

$$\begin{aligned} \langle v_z^2 \rangle &= V_z^2(r) \\ \langle v_\theta^2 \rangle &= V_\theta^2(r) + \langle \delta V_\theta^2 \rangle \\ \langle v_r^2 \rangle &= \langle \delta V_r^2 \rangle \\ \langle v_\theta v_z \rangle &= V_\theta(r) V_z(r) \end{aligned} \tag{19}$$

Furthermore,

$$\langle \delta V_\theta^2 \rangle \simeq \langle \delta V_r^2 \rangle = \frac{1}{2} V_z^2(r)$$

Hence,

$$\begin{aligned} P \simeq \frac{4\pi}{3} \frac{e^2}{c^3} \frac{n_0^2}{\epsilon_0} \int_0^a r dr n(r) \left\{ \Omega_z^2 V_z^2(r) + \frac{1}{2} \Omega_\theta^2 V_z^2(r) \right. \\ \left. + \left(\Omega_\theta V_z(r) - \Omega_z V_\theta(r) \right)^2 \right\} \end{aligned} \tag{20}$$

The assumption that $n(r) = n_0 = \text{constant}$ is not strictly compatible with our earlier assumption that $n(r) = n_0 = \text{constant}$. However, it can be shown^{1,5} that for constant V_z , the density $n(r)$ in our equilibrium would vary by < 7% over the beam cross-section. We will therefore assume that $n(r)$ in equation (20) is constant over the beam cross-section.

Radiation from the Force-Free Equilibrium

We can use equation (20) to compute the radiated power from the equilibrium presented in section 2. Using equation (2), we have

$$\Omega_0 V_0(r) - \Omega_2 V_2(r) = \frac{2\dot{p}_1^2 r}{n_0 m a^2} + \frac{\gamma_0 V_0^2(r)}{r} \quad (21)$$

We then obtain

$$P = \frac{4\pi}{3} \frac{e^2}{c^3} n_0 \gamma_0^2 \int_0^a r dr \left\{ \left(\frac{e}{m c} \right)^2 \frac{2\dot{p}_1^2}{\gamma_0 m n_0} \left[B_z^2(r) + \frac{1}{2} B_\theta^2(r) \right] (1 - r^2/a^2) \right. \\ \left. + \left[\frac{2\dot{p}_1^2 r}{n_0 m a^2} + \frac{\gamma_0 V_0^2(r)}{r} \right]^2 \right\} \quad (22)$$

After substitution of equations (5), (6), and (11), we perform the integration in equation (22) to obtain

$$P = \frac{2}{3} N_0 \frac{e^2}{c^3} \Omega_0^2 a^2 \left\{ 2\pi^2 \overline{B^2} \left(\frac{\gamma}{v} \right)^2 + \frac{1}{32} b_1^4 \left(\frac{\gamma}{v} \right)^4 \right. \\ \left. + \frac{\pi}{2} \left(\frac{\gamma}{v} \right)^2 b_1^2 \overline{B^2} + 2\pi^2 \frac{\gamma}{v} \left[\frac{1}{2} b_2^2 + \frac{11}{32} b_1^2 + \frac{3}{8} b_0 b_1 \right] \right. \\ \left. + \frac{1}{24} b_1^2 \frac{\gamma}{v} + \frac{\pi}{3} \overline{B^2} \right\} \quad (23)$$

Here we have written

$$N_0 \equiv n_0 \pi a^2, \quad B_z \equiv B_0(a), \quad \Omega_0 \equiv \frac{e \dot{\phi}_1}{m a}, \quad b_0 \equiv \frac{B_0}{B_z}, \\ b_1 \equiv \overline{B^2}/B_z, \quad \overline{B^2} \equiv \overline{B^2}/B_z^2, \quad \text{and} \quad \frac{v}{\gamma} = \frac{\pi a^2 n_0 e^2}{2 B_0 c^2} \quad (24)$$

The quantities $\overline{B^2}$ and B_1 are related for a given current, I , by

$$\overline{B^2}(a) = \frac{I^2}{2\pi^2 a^2} = 4\pi^2 B_z^2 + 2 B_0^2 + \frac{1}{3} B_1^2 + \frac{1}{2} \frac{\gamma}{v} B_z^2 \quad (25)$$

We now solve equation (25) for B_1/B_0

$$b_1 = \frac{-b_0}{\frac{2}{3} + \frac{\gamma}{\nu}} \left\{ 1 - \left[1 + \frac{2 \left(\frac{2}{3} + \frac{\gamma}{\nu} \right) (1 - 4\pi \bar{p}_\perp^0)}{b_0^2} \right]^{1/2} \right\} \quad (25)$$

Equations (23) and (26) determine the radiation loss from the beam.

As we noted earlier, this type of equilibrium is close to force-free when $\gamma/\nu \ll 1$ and $\bar{p}_\perp^0 \ll 1$. We first consider the cold beam limit ($\bar{p}_\perp^0 = 0$), in which the beam particles have only their guiding-center motion. The single-particle orbits are then helices, and in the limit of large applied field, $b_0 \gg 1$, the helix is straightened out and the radiation from the beam should sharply decline. To investigate this assertion analytically, we write equations (23) and (26) for the case $\bar{p}_\perp^0 = 0$, $\gamma/\nu \ll 1$, $b_0 \gg 1$. Then,

$$b_1 \approx 1/b_0$$

and hence

$$P \approx \frac{2}{3} N_0 \frac{e^2}{c} \Omega_A^4 \omega^2 \left\{ \frac{1}{32} \frac{1}{b_0^4} \left(\frac{\gamma}{\nu} \right)^4 \right\} \quad (27)$$

In this limit the radiated power falls off as $1/b_0^4$ for large b_0 .

If we now include a small pressure term, $\bar{p}_\perp^0 \ll 1$, we find

$$P \approx \frac{2}{3} N_0 \frac{e^2}{c} \Omega_A^4 \omega^2 \left\{ \frac{1}{32} \frac{1}{b_0^4} \left(\frac{\gamma}{\nu} \right)^4 + \pi \bar{p}_\perp^0 \frac{1}{b_0^2} \frac{\gamma}{\nu} \right\} \quad (28)$$

The pressure term, a departure from the force-free condition, leads to an enhancement of the radiation loss.

For large applied fields, $b_0 \gg 1$, we would therefore expect that the radiation losses would tend to drive the beam into the force-free state. That is, the force-free component of the beam radiates only slightly, ^{while} ~~which~~ even small departures from the force-free configuration lead to enhanced radiation, causing the beam to evolve toward a force-free state.

In figure 1 we present a graphical representation of equations (23) and (26) as a function of applied field, b_0 , for various values of $\frac{\sqrt{a_0}}{b_0}$. For comparison with a non-force-free configuration, we have also plotted the radiation loss expected from the Hammer and Rostoker equilibrium with the same radius, including an applied magnetic field. The calculation of the radiation from the Hammer-Rostoker equilibrium is presented in the appendix.

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Figure Caption

Plot of radiated power per unit length in units of $\frac{2}{3} N_b \frac{e^2}{c^3} N_a^4 a^2$ versus applied longitudinal magnetic field, B_0/B_a . The upper curve is the radiated power from Hammer and Rostoker's equilibrium. The lower curves are for various ratios of perpendicular to parallel single-particle energy. The lowest curve corresponds to a completely cold, essentially force-free beam. All the curves were obtained for $I/I_A=10$.

Appendix

In this appendix we present a calculation of the radiated power from Hammer and Rostoker's equilibrium, generalized to include an applied longitudinal magnetic field. The applied field introduces $A_0(r)$, but since the equilibrium has no macroscopic J_θ current, the applied field will not affect the self-consistent equilibrium fields. It will, of course, affect the equilibrium orbits and the radiated power.

For an electrostatically neutralized beam, the equilibrium distribution function is

$$F(r, \frac{1}{r}) = \frac{n_0 c^2}{2\pi H_0} \delta(H - H_0) \delta(P_z - P_0) \quad (A-1)$$

where

$$\begin{aligned} H &= \gamma m c^2 \\ \gamma &= [1 + \beta^2/c^2]^{1/2} \\ P_z &= \frac{1}{r} - \frac{e}{2} A_0(r) \\ \beta &= \gamma u/r \end{aligned} \quad (A-2)$$

and n_0 , H_0 , and P_0 are constants. The distribution function is normalized to n_0 ,

$$\int d\frac{1}{r} F(r, \frac{1}{r}) = \begin{cases} n_0 & r \leq a \\ 0 & r > a \end{cases} \quad (A-3)$$

The equilibrium magnetic field is then found from Maxwell's equations,

$$\begin{aligned} B_z &= B_0 \\ B_\theta(r) &= \begin{cases} -\frac{\gamma_0 m c V_z}{e L} I_1(r/L) & r \leq a \\ -\frac{\gamma_0 m c V_z}{e L} I_1(a/L) \frac{a}{r} & r > a \end{cases} \end{aligned} \quad (A-4)$$

where

$$\begin{aligned} a &= \text{the beam radius} \\ H_0 &= \gamma_0 m c^2 \\ P_0 &= \gamma_0 m V_z \\ \frac{1}{r} &= \frac{\gamma_0 m c^2}{\gamma_0 m c^2} = \frac{\gamma_0^2}{c^2} \end{aligned} \quad (A-5)$$

and I_1 is the usual modified Bessel function.

The single-particle radiation is then given by

$$\begin{aligned} P_{sp} &= \frac{2}{3} \frac{e^2}{c^3} \dot{\gamma}^2 \dot{\gamma}^2 \\ &= \frac{2}{3} \frac{e^2}{c^3} \dot{\gamma}^2 \left\{ (\dot{\gamma}_r^2 + \dot{\gamma}_\theta^2) \Omega_0^2 + (\dot{\gamma}_r^2 + \dot{\gamma}_\theta^2) \frac{\gamma_0^2 \gamma_1^2}{L^2} I_1^2(\gamma/L) \right. \\ &\quad \left. + 2 \dot{\gamma}_r \dot{\gamma}_\theta \Omega_0 \frac{\gamma_0 \gamma_1}{L} I_1(\gamma/L) \right\} \end{aligned} \quad (A-6)$$

where $\Omega_0 \equiv \frac{e B_0}{m c}$.

The total radiated power per unit length is

$$\begin{aligned} P &= \int_0^L 2\pi r dr \int d\gamma \dot{\gamma}^2 P_{sp}(r, \dot{\gamma}) F(\gamma, \dot{\gamma}) \\ &= \frac{2}{3} \frac{e^2}{c^3} n_0 \int_0^L 2\pi r dr \left\{ (\gamma_0^2 - 1) \Omega_0^2 + \frac{\gamma_0^2 \gamma_1^2}{2L^2} (\gamma_0^2 - 1) I_1^2(\gamma/L) \right. \\ &\quad \left. - \frac{\gamma_0^2 \gamma_1^2}{c^2} \Omega_0^2 I_0^2(\gamma/L) + \frac{\gamma_0^4 \gamma_1^4}{2L^2 c^2} I_0^2(\gamma/L) I_1^2(\gamma/L) \right\} \end{aligned} \quad (A-7)$$

We now define

$$\begin{aligned} \gamma &\equiv \gamma/L \\ N_0 &\equiv n_0 \pi a^2 \\ \beta_\lambda &\equiv \beta_0(a) = - \frac{\gamma_0 c}{2} \frac{\gamma_0 \gamma_1}{L} I_1(\gamma/L) \\ \Omega_\lambda^2 &\equiv \left(\frac{e B_0}{m c} \right)^2 = \frac{\gamma_0^2 \gamma_1^2}{L^2} I_1^2(\gamma/L) \\ b_0 &\equiv B_0/B_L \end{aligned} \quad (A-8)$$

The total current carried by the equilibrium is given by

$$\frac{I}{I_A} = - \frac{1}{2} \frac{d}{L} \frac{I_1(\gamma/L)}{I_0(\gamma/L)} \quad (A-9)$$

where

$$I_A \equiv \frac{\gamma_0 c^2}{a} \sqrt{\gamma_0^2 - 1}$$

Equation (A-7) then becomes

$$\begin{aligned} \gamma = \frac{2}{3} N_0 \frac{e^2}{c^2} \hbar \omega_a^4 \omega^2 \left\{ \left(\frac{I_A}{I} \right)^2 b_0^2 + \frac{1}{4} \left(\frac{I_A}{I} \right)^4 \frac{\int_0^{a/L} x dx I_1^2(x)}{I_0^2(a/L)} \right. \\ \left. - \frac{1}{2} \left(\frac{I_A}{I} \right)^4 b_0^2 \frac{I_1^2(a/L)}{I_0^2(a/L)} \int_0^{a/L} x dx I_0^2(x) + \frac{1}{16} \left(\frac{I_A}{I} \right)^4 \frac{\int_0^{a/L} x dx I_0^2(x) I_1^2(x)}{I_0^2(a/L)} \right\} \quad (A-12) \end{aligned}$$