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G. A. Goldin, Department of Physics and Graduate School of Education, University of Pennsylvania

J. Grodnik, Burroughs Corp., Wayne, Pa. 19087

R. T. Powers and D. H. Sharp, Department of Physics, University of Pennsylvania, Philadelphia, Pa. 19174

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G. A. Goldin
Department of Physics and Graduate School of Education
University of Pennsylvania
Philadelphia, Pa. 19174

J. Grodnik
Burroughs Corp., 200 W. Lancaster Ave.
Wayne, Pa. 19087

R. T. Powers+ and D. H. Sharp
Department of Physics
University of Pennsylvania
Philadelphia, Pa. 19174

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+ Alfred P. Sloan Foundation Fellow

Abstract

The case of a non-interacting infinite Bose gas at zero temperature is studied in the formalism of local current algebras, using the representation theory of nuclear Lie groups. The class of representations describing such a system is obtained by taking an "N/V limit" of the finite case. These representations can also be determined uniquely from the solutions of a functional differential equation, which follows in turn from a condition on the ground state vector. Finally a system of functional differential equations is formulated for a theory with interactions, using a proposed definition of indefinite functional integration.

1. Introduction.

There are two main reasons that non-relativistic models based on algebras of local currents have recently drawn the attention of theorists[1-3]. First, they provide an interesting reformulation of ordinary quantum mechanics in terms of observables such as the particle number density $\rho(\underset{\sim}{x})$ and the particle flux density $\underset{\sim}{J}(\underset{\sim}{x})$, rather than the second-quantized field operator $\psi(\underset{\sim}{x})$. In this paper we employ such a reformulation to study the properties of an infinite Bose system. For the case of non-interacting bosons at zero temperature, the local current algebra approach leads to an elegant restatement of known results[4]. When interactions are included, we develop a system of coupled functional differential equations whose solution would describe the properties of an interacting Bose gas. While these equations are not expected to yield explicit solutions to most interacting theories of interest, it is our hope that they will prove susceptible to some method of approximation.

The second reason that such non-relativistic models are studied is that they may eventually shed light on local relativistic current algebras. As emphasized by Haag and by Wightman, there are many similarities between relativistic quantum field theory and the quantum mechanics of non-relativistic systems having infinitely many degrees of freedom[5-6]. In particular, the vacuum state in quantum field theory is the analogue of the non-relativistic ground state. It is to be hoped then that the techniques of non-relativistic current algebra can be carried over and incorporated into the study of relativistic models[1,7-10].

This paper is concerned with infinite Bose systems in the "N/V limit" or thermodynamic limit, in which the total number of particles N and the volume V of the system become infinite while the average density $\bar{\rho} = N/V$ approaches a finite constant.

In Section 2 we review the case of a non-interacting infinite Bose gas at zero temperature, from the standpoint of group representation theory. The group is that obtained by exponentiating the local current commutators. Consequently the focus of attention is on the properties of the ground state expectation functional

$$L(f) = (\Omega_o, e^{i\rho(f)}\Omega_o) . \tag{1.1}$$

In Section 3 we show how a condition on the ground state vector,

$$(\underset{\sim}{\nabla}\rho + 2i\underset{\sim}{J})(\underset{\sim}{x})\Omega_o = 0 , \tag{1.2}$$

uniquely determines the class of representations obtained by other means in Section 2. Section 4 reviews the concept of functional differentiation and proposes a specific definition for a companion concept, the indefinite functional integral. The results of Section 3 are recast as the derivation and solution of a functional differential equation.

Finally, Section 5 formulates a system of such functional differential equations for a theory with interactions.

2. The Free Bose Gas at Zero Temperature.

A. Preliminaries[1-3,11]

A second-quantized non-relativistic Bose field $\psi(\underline{x})$ satisfies the canonical commutation relations:

$$[\psi(\underline{x}),\psi^*(\underline{y})] = \delta(\underline{x}-\underline{y})$$
$$[\psi(\underline{x}),\psi(\underline{y})] = [\psi^*(\underline{x}),\psi^*(\underline{y})] = 0 \ . \tag{2.1}$$

The Fock representation for such a field is defined as follows. Let $\mathcal{H}_n$ be the Hilbert space of complex square integrable functions of n vector variables which are symmetric under the exchange of particle coordinates, and let $\mathcal{H} = \bigoplus_{n=0}^{\infty} \mathcal{H}_n$ be the direct sum of the $\mathcal{H}_n$. A vector $\Psi \in \mathcal{H}_n$ has components $\Psi_n \in \mathcal{H}_n$ with $(\Psi,\Psi) = \sum_n (\Psi_n,\Psi_n) < \infty$. The action of the fields $\psi(\underline{x})$ and $\psi^*(\underline{x})$ in $\mathcal{H}$ is defined by:

$$[\psi(\underline{x})\Psi]_n(\underline{x}_1,\ldots,\underline{x}_n) = (n+1)^{\frac{1}{2}}\Psi_{n+1}(\underline{x}_1,\ldots,\underline{x}_n,\underline{x}) \ , \tag{2.2}$$

and

$$[\psi^*(\underline{x})\Psi]_n(\underline{x}_1,\ldots,\underline{x}_n)$$
$$= n^{-\frac{1}{2}} \sum_{j=1}^{n} \delta(\underline{x}-\underline{x}_j)\Psi_{n-1}(\underline{x}_1,\ldots,\hat{\underline{x}}_j,\ldots,\underline{x}_n) \ . \tag{2.3}$$

Defining the number density of particles as

$$\rho(\underset{\sim}{x}) = \psi^*(\underset{\sim}{x})\psi(\underset{\sim}{x}) \tag{2.4}$$

and the particle flux density (for particles of unit mass) as

$$\underset{\sim}{J}(\underset{\sim}{x}) = \frac{1}{2i}[\psi^*(\underset{\sim}{x})\underset{\sim}{\nabla}\psi(\underset{\sim}{x}) - (\underset{\sim}{\nabla}\psi^*(\underset{\sim}{x}))\psi(\underset{\sim}{x})] , \tag{2.5}$$

one obtains the equal time current algebra

$$[\rho(\underset{\sim}{x}),\rho(\underset{\sim}{y})] = 0 \tag{2.6}$$

$$[\rho(\underset{\sim}{x}),J_k(\underset{\sim}{y})] = -i\frac{\partial}{\partial x^k}[\delta(\underset{\sim}{x}-\underset{\sim}{y}),\rho(\underset{\sim}{x})] \tag{2.7}$$

$$[J_j(\underset{\sim}{x}),J_k(\underset{\sim}{y})] = -i\frac{\partial}{\partial x^k}[\delta(\underset{\sim}{x}-\underset{\sim}{y})J_j(\underset{\sim}{x})] + i\frac{\partial}{\partial y^j}[\delta(\underset{\sim}{x}-\underset{\sim}{y})J_k(\underset{\sim}{y})] . \tag{2.8}$$

Introducing the smeared currents

$$\rho(f) = \int \rho(\underset{\sim}{x})f(\underset{\sim}{x})\, d^3x \tag{2.9}$$

and

$$J(\underset{\sim}{g}) = \int \underset{\sim}{J}(\underset{\sim}{x})\cdot\underset{\sim}{g}(\underset{\sim}{x})\, d^3x \tag{2.10}$$

we obtain the infinite dimensional Lie algebra

$$[\rho(f),\rho(g)] = 0 \tag{2.11}$$

$$[\rho(f),J(\underset{\sim}{g})] = i\rho(\underset{\sim}{g}\cdot\underset{\sim}{\nabla}f) \tag{2.12}$$

$$[J(\underset{\sim}{f}),J(\underset{\sim}{g})] = iJ(\underset{\sim}{g}\cdot\underset{\sim}{\nabla}\underset{\sim}{f} - \underset{\sim}{f}\cdot\underset{\sim}{\nabla}\underset{\sim}{g}) \tag{2.13}$$

In Equations (2.9)-(2.13) the smearing functions (or their components) belong to Schwartz's space $\underline{\mathcal{S}}$ of C_∞ functions of rapid decrease.

The action of $\rho(f)$ and $J(\underset{\sim}{g})$ in the Fock representation (2.2)-(2.3) is given by

$$[\rho(f)\Psi]_n = \sum_{j=1}^{n} f(\underset{\sim}{x}_j)\Psi_n , \tag{2.14}$$

and

$$[J(\underset{\sim}{g})\Psi]_n = -\tfrac{1}{2}i \sum_{j=1}^{n} [\underset{\sim}{g}(\underset{\sim}{x}_j)\cdot\underset{\sim}{\nabla}_j + \underset{\sim}{\nabla}_j\cdot\underset{\sim}{g}(\underset{\sim}{x}_j)]\Psi_n . \tag{2.15}$$

The operators $\rho(f)$ and $J(\underset{\sim}{g})$ preserve $\mathcal{H}_n$ as a subspace of $\mathcal{H}$, and restricted to $\mathcal{H}_n$, define the n-particle representations of the current algebra (2.11)-(2.13).

A group is obtained by exponentiating the Lie algebra (2.11)-(2.13). Define

$$U(f) = e^{i\rho(f)} \tag{2.16}$$

and

$$V(\underset{\sim}{\varphi}_t^{\underset{\sim}{g}}) = e^{iJ(\underset{\sim}{g})} , \tag{2.17}$$

where $\underset{\sim}{\varphi}_t^{\underset{\sim}{g}}: \mathbb{R}^3 \to \mathbb{R}^3$ is the <u>flow for time t by the vector field</u> $\underset{\sim}{g}$; i.e.,

$$\frac{\partial \underset{\sim}{\varphi}_t^{\underset{\sim}{g}}(\underset{\sim}{x})}{\partial t} = \underset{\sim}{g}(\underset{\sim}{\varphi}_t^{\underset{\sim}{g}}(\underset{\sim}{x})) , \tag{2.18}$$

and $\underset{\sim}{\varphi}_{t=0}^{\underset{\sim}{g}}(\underset{\sim}{x}) = \underset{\sim}{x}$. Then U and V satisfy the group multiplication rules

$$U(f)U(g) = U(f + g) \tag{2.19}$$

$$V(\underset{\sim}{\psi})U(f) = U(f \circ \underset{\sim}{\psi})V(\underset{\sim}{\psi}) \tag{2.20}$$

$$V(\underset{\sim}{\varphi})V(\underset{\sim}{\psi}) = V(\underset{\sim}{\psi} \circ \underset{\sim}{\varphi}) , \tag{2.21}$$

where $\underset{\sim}{\psi} \circ \underset{\sim}{\varphi}$ denotes the composition of the flows.

A representation of the group satisfying (2.19)-(2.21) is in fact a representation of the <u>semidirect product</u> $\mathcal{S} \wedge \mathcal{K}$, where $\mathcal{S}$ is the group of all f's (under addition) and $\mathcal{K}$ is the group of all $\underset{\sim}{\psi}$'s (under composition). The representation theory of such a semidirect product typically focusses attention on the functional

$$L(f) = (\Omega, U(f)\Omega) , \tag{2.22}$$

where Ω is a cyclic vector for the $U(f)$'s in the representation[12].

B. The "N/V" Limit

Consider a system of N bosons in a box of volume V. The N-particle representation of the Lie algebra (2.11)-(2.13) describes such a system. Periodic boundary conditions require the smearing functions to be C_∞ functions on the torus T^3, a cube of volume V and of length $2L$ in each spatial direction with corresponding points on opposite boundaries identified. The N-particle representation of the group (2.19)-(2.21) is

$$U_{N,V}(f)\Psi(\underset{\sim}{x}_1,\ldots,\underset{\sim}{x}_N) = e^{i\sum_{j=1}^{N} f(\underset{\sim}{x}_j)}\Psi(\underset{\sim}{x}_1,\ldots,\underset{\sim}{x}_N) \tag{2.23}$$

and

$$V_{N,V}(\underset{\sim}{\psi})\Psi(\underset{\sim}{x}_1,\ldots,\underset{\sim}{x}_N) = \Psi(\underset{\sim}{\psi}(\underset{\sim}{x}_1),\ldots,\underset{\sim}{\psi}(\underset{\sim}{x}_N))\left[\det \frac{\partial\psi^k}{\partial x^\ell}(\underset{\sim}{x})\right]^{\frac{1}{2}} \tag{2.24}$$

where $\underset{\sim}{\psi}$ is a C_∞ flow on the torus. The determinant of $\partial\psi^k/\partial x^\ell$ is the Jacobian of the flow, expressed in the system of local coordinates obtained by the above-mentioned identification of the torus with the cube.

The normalized ground state wave function $\Omega_{N,V}(\underset{\sim}{x}_1,\ldots,\underset{\sim}{x}_N)$ for a system of N free bosons in a box of volume V is

$$\Omega_{N,V}(\underset{\sim}{x}_1,\ldots,\underset{\sim}{x}_N) = (1/\sqrt{V})^N . \tag{2.25}$$

The ground state $\Omega_{N,V}$ is a suitable cyclic vector with which to characterize the representation. Thus we obtain the ground state expectation functional

$$\begin{aligned} L_{N,V}(f) &= (\Omega_{N,V}, U_{N,V}(f)\Omega_{N,V}) \\ &= \int d^3x_1\cdots d^3x_N\ (1/V)^N \exp\left\{i \sum_{j=1}^{N} f(\underset{\sim}{x}_j)\right\} \\ &= \left(\frac{1}{V}\int d^3x\ e^{if(\underset{\sim}{x})}\right)^N \end{aligned} \tag{2.26}$$

The functional $L(f)$ in general determines not only the representation of $U(f)$ but also that of $V(\underset{\sim}{\psi})$, at least up to a complex phase "multiplier"[3].

Now it is not possible to take a limit of (2.25) as N and V become infinite, but we can obtain the limit of $L_{N,V}(f)$ as $N,V \to \infty$, with $N/V \to \bar{\rho}$. The constraint $N/V \to \bar{\rho}$, where $\bar{\rho}$ denotes a constant average density, suggests the name "N/V limit" for the procedure used here.

Carrying out this procedure,

$$L(f) = \lim_{\substack{N,V \to \infty \\ N/V \to \bar{\rho}}} L_{N,V}(f)$$

$$= \lim_{N \to \infty} \left(1 + \frac{\bar{\rho}}{N} \int d^3x[e^{if(\underset{\sim}{x})} - 1]\right)^N$$

$$= \exp\left\{\bar{\rho} \int (e^{if(\underset{\sim}{x})} - 1)\, d^3\underset{\sim}{x}\right\} \qquad (2.27)$$

C. Defining the Representation

The Gel'fand-Vilenkin approach to the representation theory of nuclear Lie groups discusses (continuous) representations of Schwartz's space $\mathcal{S}$ in terms of measures on $\mathcal{S}'$, the continuous dual of $\mathcal{S}$. A functional $L(f)$ is the Fourier transform of a cylindrical measure μ on $\mathcal{S}'$, and thus defines a continuous representation of $\mathcal{S}$, if and only if:

1) $L(f)$ is continuous with respect to the topology of $\mathcal{S}$,

2) $L(0) = 1$, and

3) $L(f)$ is positive definite in the sense that

$(\forall f_1,\ldots,f_m \in \mathcal{S},\ \lambda_1,\ldots,\lambda_m \in C)$

$$\sum_{j,k=1}^{m} \bar{\lambda}_k \lambda_j L(f_j - f_k) \geq 0\ . \qquad (2.28)$$

Under these conditions,

$$L(f) = \int_{F\in \mathcal{S}'} e^{i(F,f)}\, d\mu(F) \ , \tag{2.29}$$

and the representation of $\mathcal{S}$ may be realized in the Hilbert space $\mathcal{H} = L^2_\mu(\mathcal{S}')$ of μ-square integrable functions on $\mathcal{S}'$, with

$$(U(f)\Psi)(F) = e^{i(F,f)}\Psi(F) \tag{2.30}$$

for $\Psi \in L^2_\mu(\mathcal{S}')$. Furthermore, $\Omega(F) \equiv 1$ is a cyclic vector for the representation.

Therefore we need to check that Eq. (2.27) satisfies the above three conditions, and indeed defines a representation of $\mathcal{S}$.

<u>Theorem 1</u>. The functional $L(f) = \exp[\bar{\rho} \int (e^{if(\underset{\sim}{x})} - 1)\, d^3x]$ is the Fourier transform of a cylindrical measure μ on $\mathcal{S}'$, and thus defines a continuous representation $\mathcal{U}$ of $\mathcal{S}$, with a cyclic vector Ω such that $L(f) = (\Omega, U(f)\Omega)$.

<u>Proof</u>: 1) $L(f)$ is continuous with respect to the usual topology of $\mathcal{S}$; for if $f_j \to f$ in $\mathcal{S}$ as $j \to \infty$, then $(e^{if_j} - 1) \longrightarrow (e^{if} - 1)$ in $\mathcal{S}$, and

$$\int (e^{if_j(\underset{\sim}{x})} - 1)\, d^3x \longrightarrow \int (e^{if(\underset{\sim}{x})} - 1)\, d^3x \ .$$

2) $L(0) = 1$.

3) If all of the $f_1,\ldots,f_m$ in Eq. (2.28) have compact support, then $\sum_{j,k=1}^{m} \bar{\lambda}_k \lambda_j L(f_j - f_k)$ is the N/V limit of the sequence of positive functionals $\sum_{j,k=1}^{m} \bar{\lambda}_k \lambda_j L_{N,V}(f_j - f_k)$, where V contains the union of the supports of $f_1,\ldots,f_m$. Therefore Eq. (2.28) holds for functions of compact support. But any $f_1,\ldots,f_m \in \mathcal{S}$ can be approximated arbitrarily closely in $\mathcal{S}$ by C_∞ functions of compact support. Since $L(f)$ is continuous, Eq. (2.28) holds for all $f_1,\ldots,f_m \in \mathcal{S}$.

Q.E.D.

Next we shall explicitly display the representation $U(f)$ defined by the functional $L(f)$ above[13].

Let $\mathcal{H}$ be the Fock space of a second-quantized canonical non-relativistic Bose field $\psi(\underset{\sim}{x})$ satisfying Eq. (2.1). Let

$$\psi'(\underset{\sim}{x}) = \psi(\underset{\sim}{x}) + \sqrt{\bar{\rho}} \, ,$$
$$\psi'^{*}(\underset{\sim}{x}) = \psi^{*}(\underset{\sim}{x}) + \sqrt{\bar{\rho}} \, . \tag{2.31}$$

Then ψ' and ψ'^{*} also satisfy canonical commutation relations. The corresponding density is

$$\rho'(\underset{\sim}{x}) = \psi'^{*}(\underset{\sim}{x})\psi'(\underset{\sim}{x}) \, , \tag{2.32}$$

and $[\rho'(f),\rho'(g)] = 0$.

<u>Theorem 2</u>. With $\rho'(\underset{\sim}{x})$ defined in Fock space by Eq. (2.32), a representation $U_F(f) = e^{i\rho'(f)}$ is obtained for $\mathcal{S}$. The subscript F stands for Fock. The original Fock no-particle state $\Omega_F \in \mathcal{H}$ is cyclic for this representation, and defines an expectation functional $(\Omega_F, U_F(f)\Omega_F) = \exp[\bar{\rho} \int (e^{if(\underset{\sim}{x})} - 1)\, d^3x]$.

<u>Proof</u>: First let us write $\rho'(f)$ in terms of the original canonical fields;

$$\rho'(f) = \rho(f) + \bar{\rho} \int f(\underset{\sim}{x})\, d^3x + \bar{\rho}^{\frac{1}{2}}\psi^*(f) + \bar{\rho}^{\frac{1}{2}}\psi(f) \quad , \tag{2.33}$$

where $\rho(f)$ is defined in Eq. (2.4). It is clear that Ω_F is a cyclic vector for the polynomial algebra of operators generated by the identity and the $\rho'(f)$, $f \in \mathcal{S}$. In fact, for a vector Ψ which is an element of $\bigoplus_{n=0}^{N} \mathcal{H}_n$ in $\mathcal{H}$, $\rho'(f)\Psi \in \bigoplus_{n=0}^{N+1} \mathcal{H}_n$ in $\mathcal{H}$, with $(\rho'(f)\Psi)_{N+1} = \bar{\rho}^{\frac{1}{2}}\psi^*(f)\Psi_N$. Thus, by the properties of the creation operators $\psi^*(f)$ which follow from Eq. (2.3), if $\bigoplus_{n=0}^{N} \mathcal{H}_n$ is contained in the closed cyclic subspace generated by applying polynomials in the $\rho'(f)$ to Ω_F, $\bigoplus_{n=0}^{N+1} \mathcal{H}_n$ is likewise in that subspace. By induction on N, Ω_F is a cyclic vector for the representation.

Next we show that Ω_F is an analytic vector for $\rho'(f)$. In fact, from Eq. (2.33), it is certainly true that for $\Psi \in \bigoplus_{n=0}^{N} \mathcal{H}_n$ in $\mathcal{H}$,

$$\|\rho'(f)\Psi\| \leq 4(1+N)(1+\bar{\rho})(\|f\|_\infty + |\int f(\underset{\sim}{x})\, d^3x|)\|\Psi\| \;, \qquad (2.44)$$

where $\|f\|_\infty = \sup_{x \in \mathbb{R}^3} |f(\underset{\sim}{x})|$. Thus $\sum_{N=0}^{\infty} \frac{t^N}{N!}\|\rho'(f)^N \Omega_F\|$ is bounded by $\sum_{N=0}^{\infty} c^N t^N (N+1)$ where c is a constant, and converges for sufficiently small t. Similarly, all elements of $\overset{N}{\underset{n=0}{\oplus}} \mathcal{H}_n$ are analytic vectors for $\rho'(f)$, for arbitrary N. Having identified a common dense domain of analytic vectors for the $\rho'(f)$, we can now conclude the existence of a unitary representation $U_F(f) = e^{i\rho'(f)}$ in $\mathcal{H}$ with $U_F(f)U_F(g) = U_F(f+g)$.

The cyclicity of Ω_F for the $U_F(f)$ follows immediately from the fact that for Ψ in the domain of $\rho'(f)$, $\frac{1}{it}[U_F(tf)\Psi - \Psi] \longrightarrow \rho'(f)\Psi$ as $t \to 0$.

Finally it remains for us to evaluate $(\Omega_F, U_F(f)\Omega_F)$. Define the operator-valued distribution

$$A(f) = \bar{\rho} \int f(\underset{\sim}{x})\, d^3x + \bar{\rho}^{\frac{1}{2}}\psi^*(f) \; . \qquad (2.45)$$

Then

$$\rho'(f)\Omega_F = A(f)\Omega_F \;, \qquad (2.46)$$

and a simple calculation shows that

$$[\rho'(f),A(g)] = A(fg). \tag{2.47}$$

Hence

$$e^{i\rho'(f)}A(g)e^{-i\rho'(f)} = \sum_{n=0}^{\infty} \frac{i^n}{n!}(\mathrm{ad}^n \rho'(f))A(g) = A(e^{if}g) \ , \tag{2.48}$$

where

$$(\mathrm{ad}\ X)Y = [X,Y] \ . \tag{2.49}$$

Now with $L(f) = (\Omega_F, U_F(f)\Omega_F)$,

$$\begin{aligned} \frac{1}{i}\frac{d}{dt} L(tf) &= (\Omega_F, e^{it\rho'(f)}\rho'(f)\Omega_F) \\ &= (\Omega_F, e^{it\rho'(f)}A(f)\Omega_F) = (\Omega_F, A(e^{itf}f)e^{it\rho'(f)}\Omega_F) \\ &= (A^*(e^{itf}f)\Omega_F, e^{it\rho'(f)}\Omega_F) = \bar{\rho} \int e^{itf(\underset{\sim}{x})} f(\underset{\sim}{x})\ d^3x\ L(tf) \ . \end{aligned} \tag{2.50}$$

This differential equation in t, when supplemented with the boundary condition $L(0) = 1$, has the unique solution

$$L(tf) = \exp\left[\bar{\rho} \int (e^{itf(\underset{\sim}{x})} - 1)\ d^3x\right] . \tag{2.51}$$

Q.E.D.

Using the "functional derivative" to be introduced in Section 4, Eq. (2.50) may be written

$$\frac{1}{i}\ \frac{\delta L(f)}{\delta f(\underset{\sim}{x})} = \bar{\rho}\, e^{if(\underset{\sim}{x})} L(f) \ . \tag{2.52}$$

We have shown that Eq. (2.27) defines a representation of $\mathcal{S}$, the normal subgroup of the semidirect product, and have displayed the representation. Next we show that the full group $\mathcal{S} \wedge \mathcal{K}$ can be represented in the same Hilbert space.

The first step is to anticipate the form of the functional $E(f,\underset{\sim}{\psi}) = (\Omega, U(f)V(\underset{\sim}{\psi})\Omega)$ by taking another N/V limit. Again in the N-particle Fock representation in volume V,

$$E_{N,V}(f,\underset{\sim}{\psi}) = (\Omega_{N,V}, U_{N,V}(f)\, V_{N,V}(\underset{\sim}{\psi})\Omega_{N,V})$$

$$= \int d^3x_1 \cdots d^3x_N \left(\frac{1}{V}\right)^N \exp\left\{ i \sum_{j=1}^{N} f(\underset{\sim}{x}_j)\right\} \left[\prod_{n=1}^{N} \mathcal{J}_{\underset{\sim}{\psi}}(\underset{\sim}{x}_n)\right]^{\frac{1}{2}} \tag{2.53}$$

where

$$\mathcal{J}_{\underset{\sim}{\psi}}(\underset{\sim}{x}) = \det \frac{\partial \psi^k}{\partial x^\ell}(\underset{\sim}{x}) \tag{2.54}$$

is the Jacobian referred to in Eq. (2.24). Then

$$\begin{aligned} E(f,\underset{\sim}{\psi}) &= \lim_{\substack{N,V \to \infty \\ N/V \to \bar{\rho}}} E_{N,V}(f,\underset{\sim}{\psi}) \\ &= \exp\left[\bar{\rho} \int \left\{e^{if(\underset{\sim}{x})}\sqrt{\mathcal{J}_{\underset{\sim}{\psi}}(\underset{\sim}{x})} - 1\right\} d^3x\right] \ . \end{aligned} \tag{2.55}$$

A functional E on a topological group G defines a continuous representation of G, if and only if:[14]

1) E is continuous,

2) $E(1) = 1$, and

3) $$\sum_{j,k=1}^{m} \bar{\lambda}_k \lambda_j E(g_k^{-1} g_j) \geq 0 \qquad (2.56)$$

$$(\forall\, g_1, \ldots, g_m \in G,\ \lambda_1, \ldots, \lambda_m \in \mathbb{C}) .$$

Now we are ready to prove the next result.

<u>Theorem 3</u>. There exists a representation $U(f)V(\underset{\sim}{\psi})$ of $\mathcal{S} \wedge \mathcal{K}$ in a Hilbert space $\mathcal{H}$, with a cyclic vector $\Omega \in \mathcal{H}$, such that $E(f,\underset{\sim}{\psi}) = (\Omega, U(f)V(\underset{\sim}{\psi})\Omega)$ is given by Eq. (2.55).

<u>Proof</u>: We shall show that conditions 1)-3) above are satisfied by $E(f,\underset{\sim}{\psi})$.

1) It is necessary to introduce a more careful definition of $\mathcal{K}$.[15] Let $\mathcal{K}_0$ be the group of all C_∞ diffeomorphisms from $\mathbb{R}^3$ onto $\mathbb{R}^3$, having compact support. We topologize $\mathcal{K}_0$ by means of the countable family of metrics

$$\langle\langle \underset{\sim}{\varphi}, \underset{\sim}{\psi} \rangle\rangle_n = \max_{0 \leq |m| \leq n} \sup_{\underset{\sim}{x} \in \mathbb{R}^3} \left| (1 + |\underset{\sim}{x}|^2)^n (\underset{\sim}{\varphi}^{(m)}(\underset{\sim}{x}) - \underset{\sim}{\psi}^{(m)}(\underset{\sim}{x})) \right|$$

$$n = 0,1,2,\ldots \qquad (2.57)$$

for $\varphi, \psi \in \mathcal{K}_o$, where $(m) = (m_1, m_2, m_3)$, $|m| = \sum_{k=1}^{3} m_k$, and

$\varphi^{(m)}(x) = \dfrac{\partial^{|m|}\varphi(x)}{(\partial x^1)^{m_1}(\partial x^2)^{m_2}(\partial x^3)^{m_3}}$. $\mathcal{K}$ is the completion of

$\mathcal{K}_o$ with respect to this topology. The topology has a countable basis of neighborhoods of each element of $\mathcal{K}$, and is metrizable. The group operations are continuous. $\mathcal{K}$ contains diffeomorphisms which are not of compact support, but which suitably approximate the identity mapping as $|x| \to \infty$.

Omitting the computations, it follows that if $f_j \to f$ in $\mathcal{S}$ and $\psi_k \to \psi$ in $\mathcal{K}$ as $k, j \to \infty$, then $(e^{if_j}\sqrt{\mathcal{J}_{\psi_k}} - 1) \longrightarrow (e^{if}\sqrt{\mathcal{J}_{\psi}} - 1)$ in $\mathcal{S}$, and

$$\int \{e^{if_j(x)}\sqrt{\mathcal{J}_{\psi_k}(x)} - 1\}\, d^3x \longrightarrow \int \{e^{if(x)}\sqrt{\mathcal{J}_{\psi}(x)} - 1\}\, d^3x.$$

Thus $E(f, \psi)$ is continuous.

2) Clearly $E(0,1) = 1$.

3) As in the proof of Theorem 1, choose first the elements $(f_1, \psi_1), \ldots, (f_m, \psi_m)$ to have compact support. Then with

$$\begin{aligned}(f_k, \psi_k)^{-1}(f_j, \psi_j) &= (-f_k \circ \psi_k^{-1}, \psi_k^{-1})(f_j, \psi_j) \\ &= ([f_j - f_k] \circ \psi_k^{-1}, \psi_j \circ \psi_k^{-1}), \qquad (2.58)\end{aligned}$$

the expression

$$\sum_{j,k=1}^{m} \bar{\lambda}_k \lambda_j E([f_j - f_k] \circ \psi_k^{-1}, \psi_j \circ \psi_k^{-1}) \qquad (2.59)$$

is the limit of the sequence

$$\sum_{j,k=1}^{m} \bar{\lambda}_k \lambda_j E_{N,V}([f_j - f_k] \circ \psi_k^{-1}, \psi_j \circ \psi_k^{-1}) \qquad (2.60)$$

as $N,V \to \infty$ with $N/V \to \bar{\rho}$, where the volume V contains the union of all of the supports of $f_1,\ldots,f_m$ and $\psi_1,\ldots,\psi_m$. But Eq. (2.60) is positive since $E_{N,V}$ is defined in the N-particle Fock representation in volume V by Eq. (2.53).

Therefore Eq. (2.59) is positive for elements of $\mathcal{S} \wedge \mathcal{K}$ which have compact support. But any element of $\mathcal{S} \wedge \mathcal{K}$ can be approximated arbitrarily closely by elements having compact support, due to the definition of $\mathcal{K}$ as the completion of $\mathcal{K}_o$. Since $E(f,\psi)$ is continuous, Eq. (2.59) is positive for all elements of $\mathcal{S} \wedge \mathcal{K}$.

Q.E.D.

Thus there exists a continuous representation $U(f)V(\psi)$ of $\mathcal{S} \wedge \mathcal{K}$ in a Hilbert space $\mathcal{H}$, with $\Omega \in \mathcal{H}$ cyclic for the $U(f)V(\psi)$, such that

$$E(f,\psi) = (\Omega, U(f)V(\psi)\Omega) \ . \qquad (2.61)$$

The next step is to show that Ω is a cyclic vector for the subgroup $\{U(f)\}$. We shall use the following lemma, omitting the proof which is not difficult.(16)

<u>Lemma 1</u>. Let $U(t) = e^{itA}$ be a continuous one-parameter unitary group in $\mathcal{H}$ with A self-adjoint; let $\Psi \in \mathcal{H}$ and let $f(t) = (\Psi, U(t)\Psi)$ be an entire analytic function of t. Then Ψ is an entire analytic vector for A; i.e. the series

$$\sum_{n=0}^{\infty} \frac{t^n}{n!} \|A^n \Psi\| \tag{2.62}$$

is absolutely convergent for all t; and Ψ is in the domain of $U(it)$.

<u>Theorem 4</u>. In the representation of $\mathcal{S} \wedge \mathcal{K}$ defined by Eqs. (2.55) and (2.61), Ω is cyclic for the $\{U(f)\}$.

<u>Proof</u>: Let $h \in \mathcal{S}$. Then with $U(h) = e^{i\rho(h)}$, Eq. (2.55) yields

$$(\Omega, U(th)\Omega) = e^{\bar{\rho} \int [e^{ith(\underset{\sim}{x})} - 1] d^3x} \tag{2.63}$$

which is an entire analytic function of t. Therefore by Lemma 1, Ω is in the domain of $e^{\rho(h)}$. $e^{\rho(h)}\Omega$ is of course in the closed cyclic subspace generated by the $\{U(f)\Omega\}$.

But it can be shown that $V(\psi)\Omega = e^{\rho(h)}\Omega$, where $h(x) = \ln \sqrt{\mathcal{J}_{\psi}(x)}$. In fact,

$$\|V(\psi)\Omega - e^{\rho(h)}\Omega\|^2$$

$$= 1 - (\Omega, U(-ih)V(\psi)\Omega) - (\Omega, V(\psi^{-1})U(-ih)\Omega) + (\Omega, U(-2ih)\Omega)$$

$$= 1 - \exp\left[\bar{\rho}\int (e^{h(x)}\sqrt{\mathcal{J}_{\psi}(x)} - 1)\, d^3x\right] - \exp\left[\bar{\rho}\int (e^{h\circ\psi^{-1}(x)}\sqrt{\mathcal{J}_{\psi^{-1}}(x)} - 1)\, d^3x\right] + \exp\left[\bar{\rho}\int (e^{2h(x)} - 1)\, d^3x\right] \tag{2.64}$$

$$= 0 ,$$

after some manipulation of the Jacobians. Thus Ω is cyclic for the $\{U(f)\}$.

Q.E.D.

Now we are ready to represent the full group $\mathcal{S} \wedge \mathcal{K}$ in the Fock space of Theorem 2.

Theorem 5. With $\psi'(x) = \psi(x) + \bar{\rho}^{\frac{1}{2}}$ as in Eq. (2.31), with $\rho'(x)$ given by Eq. (2.32), and with

$$J'(x) = \frac{1}{2i}[\psi'^*(x)\nabla\psi'(x) - (\nabla\psi'^*(x))\psi'(x)] \tag{2.65}$$

in the Fock space of the non-relativistic canonical Bose field $\underset{\sim}{\psi}(x)$, there exists a continuous unitary representation $U_F(f)V_F(\underset{\sim}{\psi})$ of the group $\mathcal{S} \wedge \mathcal{K}$ such that

$$U_F(f) = e^{i\rho'(f)} \tag{2.66}$$

and

$$V_F(\underset{\sim}{\varphi}_t^{\underset{\sim}{g}}) = e^{iJ'(\underset{\sim}{g})} . \tag{2.67}$$

Then with $\Omega_F \in \mathcal{H}$ the original Fock vacuum state for $\underset{\sim}{\psi}(x)$,

$$E(f,\underset{\sim}{\psi}) = (\Omega_F, U_F(f)V_F(\underset{\sim}{\psi})\Omega_F) \tag{2.68}$$

where $E(f,\underset{\sim}{\psi})$ is given by Eq. (2.55).

<u>Proof</u>: First we assert that the representation of $\mathcal{S} \wedge \mathcal{K}$ obtained in Theorem 3 can be mapped unitarily into the Fock Hilbert space.

Let $\Omega \to \Omega_F$ and $U(f)\Omega \longrightarrow U_F(f)\Omega_F$, where Ω_F and $U_F(f)$ are as in Theorem 2. Since by Theorem 4, Ω is cyclic for the $U(f)$, this mapping defines a unitary representation not only of $\mathcal{S}$ but of $\mathcal{S} \wedge \mathcal{K}$ in the Fock Hilbert space; we may write $V(\underset{\sim}{\psi}) \to V_F(\underset{\sim}{\psi})$, and

$$E(f,\underset{\sim}{\psi}) = (\Omega_F, U_F(f)V_F(\underset{\sim}{\psi})\Omega_F) .$$

It remains only to show that $J'(\underset{\sim}{g})$ as defined by Eq. (2.65) is indeed the infinitesimal generator of the one-parameter unitary group $V_F(\underset{\sim}{\varphi}_t^{g})$. By Stone's theorem, it is sufficient to show that

$$\lim_{t\to 0} \frac{V_F(\underset{\sim}{\varphi}_t^{g}) - I}{it} \Omega_F = J'(\underset{\sim}{g})\Omega_F \; ; \tag{2.69}$$

the result then follows from the fact that $\rho'(f)$ and $J'(\underset{\sim}{g})$ satisfy the correct algebra of commutation relations on the domain of polynomials in the $\rho'(f)$ applied to Ω_F.

Now by Eq. (2.65),

$$J'(\underset{\sim}{g})\Omega_F = \frac{1}{2i}\, \rho'(\underset{\sim}{\nabla}\cdot \underset{\sim}{g})\Omega_F \; , \tag{2.70}$$

and

$$\begin{aligned} &\lim_{t\to 0} \left\| \frac{V_F(\underset{\sim}{\varphi}_t^{g}) - I}{it} \Omega_F - \frac{1}{2i}\, \rho'(\underset{\sim}{\nabla}\cdot \underset{\sim}{g})\Omega_F \right\|^2 \\ &= \lim_{s,t\to 0} \left\| \frac{V_F(\underset{\sim}{\varphi}_t^{g}) - I}{it} \Omega_F + \frac{1}{2}\, \frac{U_F(s\underset{\sim}{\nabla}\cdot \underset{\sim}{g}) - I}{s} \Omega_F \right\|^2 \\ &= 0 \; , \end{aligned} \tag{2.71}$$

using Eq. (2.55) for $E(f,\psi)$. Thus Eq. (2.69) is demonstrated.

Q.E.D.

To summarize, we have obtained a representation of the group $\mathcal{S} \wedge \mathcal{K}$, the exponentiated non-relativistic current algebra, in the N/V limit. This was done by means of the expectation functional $E(f,\underset{\sim}{\psi})$. The representation thus obtained was shown to be unitarily equivalent to an explicit representation of the current commutators in a certain Fock space, with the original Fock ground state being the cyclic vector defining the functional $E(f,\underset{\sim}{\psi})$.

In the next section we show how a condition on the cyclic vector which asserts that it is the physical ground state of an infinite free Bose gas uniquely determines the class of representations obtained above; namely those defined by $E(f,\underset{\sim}{\psi})$ for an arbitrarily specified average particle density $\bar{\rho}$.

3. A Condition on the Ground State.

In this section we study representations of the current algebra, Eqs. (2.11)-(2.13), in which there exists a cyclic vector Ω_o satisfying the condition

$$[2iJ(\underset{\sim}{x}) + (\nabla\rho)(\underset{\sim}{x})]\Omega_o = 0 \ . \qquad (3.1)$$

Ω_o will usually be interpreted as the ground state of the system.

Convincing heuristic arguments that Eq. (3.1) determines the ground state of a non-interacting Bose gas have been given.(11,17-19) Here we shall explore the consequences of this constraint somewhat more systematically. We show that for a system in a box with periodic boundary conditions, Eq. (3.1) implies that the operator $\int \rho(\underset{\sim}{x})\, d^3x$ has integer eigenvalues. In Section 2 we saw that the expectation functional $(\Omega_o, e^{i\rho(f)}\Omega_o)$ is given by Eq. (2.27) in the N/V limit. In this section we show not only that Eq. (2.27) determines a representation satisfying the constraint (3.1), as has been previously indicated(19), but that it defines the unique class of representations having this property.

Let us investigate the consequences of Eq. (3.1) on the functional

$$L(f) = (\Omega_o, e^{i\rho(f)}\Omega_o) , \tag{3.2}$$

where Ω_o is a cyclic vector in a representation of the current algebra, satisfying the condition

$$2iJ(\underset{\sim}{g})\Omega_o = \rho(\underset{\sim}{\nabla}\cdot\underset{\sim}{g})\Omega_o \tag{3.3}$$

for all real vector functions $\underset{\sim}{g}$ with components in $\mathcal{S}$. Naturally we shall assume that Ω_o is in the domain of the operators $J(\underset{\sim}{g})$ and $\rho(f)$ for all $f,\underset{\sim}{g} \in \mathcal{S}$. Actually, for the sake of mathematical rigor we shall assume slightly more. We suppose in addition that the bilinear form $(\rho(f_1)\Omega_o, \rho(f_2)\Omega_o)$ is continuous in f_1 and f_2; i.e. if $f_{1n} \to f_1$ in $\mathcal{S}$, then $(\rho(f_{1n})\Omega_o, \rho(f_2)\Omega_o) \to (\rho(f_1)\Omega_o, \rho(f_2)\Omega_o)$, and similarly for f_2. It then follows that $\|\rho(f_n)\Omega_o\|^2 \to 0$ if $f_n \to 0$ as $n \to \infty$. This assumption is slightly stronger than assuming continuity of the group representation $U(f)$. It follows readily that $L(f)$ is continuous in f; i.e. if $f_n \to f$ in $\mathcal{S}$, then $L(f_n) \to L(f)$. In fact,

$$|L(f_n) - L(f)| = |(\Omega_o, (e^{i\rho(f_n)} - e^{i\rho(f)})\Omega_o)|$$

$$= |(e^{-i\rho(f)}\Omega_o, (e^{i\rho(f_n-f)} - I)\Omega_o)|$$

$$\leq \|e^{-i\rho(f)}\Omega_o\| \cdot \|(e^{i\rho(f_n-f)} - I)\Omega_o\|$$

$$\leq 1 \cdot \|\rho(f_n - f)\Omega_o\| \longrightarrow 0$$

as $f_n \to f$.

For the case of a system in a "box" with periodic boundary conditions, assume that Eq. (3.3) holds for all infinitely differentiable periodic vector functions $\underset{\sim}{g}$. The components of such functions will be said to be in $\mathcal{S}_V$, where $\mathcal{S}_V$ has the topology of a nuclear space.

Using Eq. (3.3), we derive a functional equation for $L(f)$ as follows. We have

$$\frac{d}{dt} L(f + t\underset{\sim}{\nabla} \cdot \underset{\sim}{g})_{t=0} = i(\Omega_o, e^{i\rho(f)}\rho(\underset{\sim}{\nabla} \cdot \underset{\sim}{g})\Omega_o) ,$$

whence it follows using Eq.(3.3) that

$$\frac{d}{dt} L(f + t\underset{\sim}{\nabla} \cdot \underset{\sim}{g})_{t=0} = -2(\Omega_o, e^{i\rho(f)}\underset{\sim}{J}(\underset{\sim}{g})\Omega_o) . \qquad (3.4)$$

Similarly,

$$\frac{d}{dt} L(f + t\nabla \cdot g)_{t=0} = i(\Omega_o, \rho(\nabla \cdot g) e^{i\rho(f)} \Omega_o)$$

$$= i(\rho(\nabla \cdot g)\Omega_o, e^{i\rho(f)} \Omega_o)$$

$$= 2(J(g)\Omega_o, e^{i\rho(f)} \Omega_o) \qquad (3.5)$$

$$= 2(\Omega_o, J(g) e^{i\rho(f)} \Omega_o) \; .$$

Combining Eqs. (3.4) and (3.5), we have

$$\frac{d}{dt} L(f + t\nabla \cdot g)_{t=0} = (\Omega_o, [J(g), e^{i\rho(f)}]\Omega_o) \; . \qquad (3.6)$$

Now it follows from the current commutation relations that

$$e^{i\rho(f)} J(g) e^{-i\rho(f)} = J(g) - \rho(g \cdot \nabla f) \; . \qquad (3.7)$$

Combining Eqs. (3.6) and (3.7),

$$\frac{d}{dt} L(f + t\nabla \cdot g)_{t=0} = (\Omega_o, [J(g) - e^{i\rho(f)} J(g) e^{-i\rho(f)}] e^{i\rho(f)} \Omega_o)$$

$$= (\Omega_o, \rho(g \cdot \nabla f) e^{i\rho(f)} \Omega_o)$$

$$= -i \frac{d}{dt} L(f + t g \cdot \nabla f)_{t=0} \; . \qquad (3.8)$$

This equation can also be written in the form

$$(\Omega_o, e^{i\rho(f)} [\rho(\nabla \cdot g) + i\rho(g \cdot \nabla f)]\Omega_o) = 0 \qquad (3.9)$$

for all $f,g \in \mathcal{S}$ or for all (periodic, infinitely differentiable) $f,g \in \mathcal{S}_V$ for a system in a "box".

Thus we have obtained a functional equation for $L(f)$ from the original condition on the ground state Ω_o.

Next we shall show that Eq. (3.8) or (3.9) implies that $L(f)$ must be of the form

$$L(f) = F(K(f)) , \tag{3.10}$$

where

$$K(f) = \int (e^{if(x)} - 1) \, d^3x$$

and $F(z)$ is a holomorphic function of the complex variable z in the interior of the range of $K(f)$.

In order to prove this result we will need the following two lemmas. We say that the mapping $t \to g_t$ of the interval $[0,1]$ into $\mathcal{S}$ (respectively $\mathcal{S}_V$) is a <u>differentiable</u> mapping of $[0,1]$ into $\mathcal{S}$ (resp. $\mathcal{S}_V$) with derivative $\frac{dg_t}{dt} = k_t \in \mathcal{S}$ (resp. $\mathcal{S}_V$) if for each $t \in [0,1]$ we have that $h^{-1}(g_{t+h} - g_t) \longrightarrow k_t$ as $h \to 0$; where the convergence is in the topology of $\mathcal{S}$ (resp. $\mathcal{S}_V$).

<u>Lemma 3.1</u>. Suppose that $t \to g_t \in \mathcal{S}$ (resp. $\mathcal{S}_V$) for $0 \leq t \leq 1$ is a differentiable mapping of $[0,1]$ into $\mathcal{S}$ (resp. $\mathcal{S}_V$). Furthermore, suppose that

$$K(g_t) = \int (e^{ig_t(\underset{\sim}{x})} - 1)\, d^3x = \text{a constant} \;; \tag{3.11}$$

that $L(f) = (\Omega_o, e^{i\rho(f)}\Omega_o)$ satisfies (3.8) or equivalently (3.9); and that $L(f)$ is continuous in f with respect to the topology of $\mathcal{S}$ (resp. $\mathcal{S}_V$). Then $L(g_t)$ is a constant independent of t.

<u>Proof</u>: We shall prove the lemma by showing that $(d/dt)L(g_t) = 0$. Let $k_t = dg_t/dt$. We begin by showing that

$$\frac{d}{dt} L(g_t) = i(\Omega_o, e^{i\rho(g_t)} \rho(k_t)\Omega_o) \;. \tag{3.12}$$

Now we have

$$\begin{aligned} & h^{-1}(L(g_{t+h}) - L(g_t)) \\ & = h^{-1}(\Omega_o, (e^{i\rho(g_{t+h})} - e^{i\rho(g_t)})\Omega_o) \\ & = h^{-1}(\Omega_o, e^{i\rho(g_t)} (e^{i\rho(g_{t+h}-g_t-hk_t)} e^{ih\rho(k_t)} - I)\Omega_o) \\ & = h^{-1}(\Omega_o, e^{i\rho(g_t)} e^{i\rho(g_{t+h}-g_t-hk_t)} (e^{ih\rho(k_t)} - I)\Omega_o) \\ & \quad + h^{-1}(\Omega_o, e^{i\rho(g_t)} (e^{i\rho(g_{t+h}-g_t-hk_t)} - I)\Omega_o) \end{aligned} \tag{3.13}$$

Estimating the second term in Eq. (3.13) as $h \to 0$ we find that

$$|h^{-1}(\Omega_o, e^{i\rho(g_t)}(e^{i\rho(g_{t+h}-g_t-hk_t)} - I)\Omega_o)|$$

$$\leq \|e^{-i\rho(g_t)}\Omega_o\|\|h^{-1}(e^{i\rho(g_{t+h}-g_t-hk_t)} - I)\Omega_o\|$$

$$\leq 1\cdot\|\rho\left(\frac{g_{t+h} - g_t}{h} - k_t\right)\Omega_o\| \longrightarrow 0 \quad \text{as} \quad h \rightarrow 0 ,$$

since $h^{-1}(g_{t+h} - g_t) - k_t \longrightarrow 0$ in $\mathcal{S}$ (resp. $\mathcal{S}_V$) as $h \rightarrow 0$. Hence, we have

$$\lim_{h\to 0} h^{-1}(L(g_{t+h}) - L(g_t))$$

$$= \lim_{h\to 0}\left(e^{-i\rho(g_t)}e^{-i\rho(g_{t+h}-g_t-hk_t)}\Omega_o, \frac{(e^{ih\rho(k_t)} - I)}{h}\Omega_o\right)$$

$$= i(\Omega_o, e^{i\rho(g_t)}\rho(k_t)\Omega_o) \tag{3.14}$$

Therefore, $L(g_t)$ is differentiable and

$$\frac{dL(g_t)}{dt} = i(\Omega_o, e^{i\rho(g_t)}\rho(k_t)\Omega_o) , \tag{3.15}$$

where $k_t = dg_t/dt$.

Next we show that Eq. (3.11) implies

$$(\Omega_o, e^{i\rho(g_t)}\rho(k_t)\Omega_o) = 0 . \tag{3.16}$$

Let T be the tempered distribution defined by

$$T(f) = (\Omega_o, e^{i\rho(g_t)} \rho(f)\Omega_o) \ . \tag{3.17}$$

From Eq. (3.9) we have

$$T(\nabla \cdot g + ig \cdot \nabla f) = 0 \tag{3.18}$$

for $f, g \in \mathcal{S}$ (resp. $\mathcal{S}_V$). Let T_g be the distribution defined by

$$T_g(f) = T(e^{-ig} f) \tag{3.19}$$

for $g \in \mathcal{S}$ (resp. $\mathcal{S}_V$). Then we have

$$\begin{aligned} T_g(\nabla \cdot f) &= T(e^{-ig} \nabla \cdot f) \\ &= T(\nabla \cdot (e^{-ig} f) + ie^{-ig} f \cdot \nabla g) = 0 \end{aligned} \tag{3.20}$$

by Eq. (3.18). Hence,

$$\nabla T_g(x) = 0 \quad \text{and} \quad T_g(x) = c_g \ ,$$

where c_g is a constant depending on g. Thus from Eq. (3.19) we have

$$T(x) = c_g e^{ig(x)} \ . \tag{3.21}$$

From Eq. (3.11),

$$\frac{d}{dt}\int (e^{ig_t(\underset{\sim}{x})} - 1)\, d^3x = i\int k_t(\underset{\sim}{x})\, e^{ig_t(\underset{\sim}{x})}\, d^3x = 0 \,. \qquad (3.22)$$

Therefore by Eqs. (3.21) and (3.15),

$$\frac{d}{dt} L(g_t) = i(\Omega_o, e^{i\rho(g_t)} \rho(k_t)\Omega_o)$$
$$= iT(k_t) = 0 \,.$$

Q.E.D.

The proof of the next lemma is extremely technical; therefore we shall present a mere sketch for the infinite-volume case in the appendix.

<u>Lemma 3.2</u>. Suppose $g_1, g_2 \in \mathcal{S}$ (resp. $\mathcal{S}_V$) and

$$\int (e^{ig_1(\underset{\sim}{x})} - 1)\, d^3x = \int (e^{ig_2(\underset{\sim}{x})} - 1)\, d^3x \,. \qquad (3.23)$$

Then for any two neighborhoods N_1 of g_1 and N_2 of g_2 in $\mathcal{S}$ (resp. $\mathcal{S}_V$), there exist functions $h_1 \in N_1$ and $h_2 \in N_2$, and a continuous mapping $t \to f_t$ of $[0,1]$ into $\mathcal{S}$ (resp. $\mathcal{S}_V$), differentiable in the open interval $(0,1)$, such that $f_0 = h_1$, $f_1 = h_2$ and

$$\int (e^{if_t(\underset{\sim}{x})} - 1)\, d^3x = \text{a constant} \,.$$

This lemma is easy to prove if f_t is permitted to be complex. The requirement that f_t be real for $0 \leq t \leq 1$ complicates the proof considerably.

Proof: See Appendix.

Theorem 3.3. Suppose that $L(g) = (\Omega_o, e^{i\rho(g)}\Omega_o)$, defined for all real $g \in \mathcal{S}$ (resp. $\mathcal{S}_V$), is continuous in g with respect to the topology of $\mathcal{S}$ (resp. $\mathcal{S}_V$). Furthermore, suppose that $L(g)$ satisfies Eq. (3.8) or equivalently Eq. (3.9). Then $L(g)$ is of the form

$$L(g) = F(K(g)) ,$$

where $K(g) = \int (e^{ig(\underset{\sim}{x})} - 1)\, d^3x$ and where $F(z)$ is a holomorphic function of the complex variable z in the interior of the range of $K(g)$.

Proof: First we show that if $g_1, g_2 \in \mathcal{S}$ (resp. $\mathcal{S}_V$) and

$$\int (e^{ig_1(\underset{\sim}{x})} - 1)\, d^3x = \int (e^{ig_2(\underset{\sim}{x})} - 1)\, d^3x$$

then $L(g_1) = L(g_2)$. Suppose $\varepsilon > 0$. Since $L(g)$ is continuous in g there are neighborhoods N_1 of g_1 and N_2 of g_2 in $\mathcal{S}$ (resp. $\mathcal{S}_V$) such that

$$|L(g_1) - L(h_1)| < \varepsilon/2 \quad \text{for all} \quad h_1 \in N_1$$
$$|L(g_2) - L(h_2)| < \varepsilon/2 \quad \text{for all} \quad h_2 \in N_2 \,. \tag{3.24}$$

From Lemma 3.2 it follows that there exist functions $k_1 \in N_1$ and $k_2 \in N_2$ and a continuous mapping $t \to f_t$ of $[0,1]$ into $\mathcal{S}$ (resp. $\mathcal{S}_V$), differentiable in $(0,1)$, such that $f_0 = k_1$, $f_1 = k_2$ and

$$\int (e^{if_t(x)} - 1)\, d^3x = \text{a constant} \,.$$

Then, by Lemma (3.1), we have that $L(k_1) = L(k_2)$. By the inequality (3.24), it follows that $|L(g_1) - L(g_2)| < \varepsilon$. Since $\varepsilon > 0$ is arbitrary, $L(g_1) = L(g_2)$. Hence, $L(g)$ depends only on the number

$$K(g) = \int (e^{ig(x)} - 1)\, d^3x \,.$$

Thus we have $L(g) = F(K(g))$, where $F(z)$ is a complex function defined on the range of $K(g)$ for all $g \in \mathcal{S}$ (resp. $\mathcal{S}_V$). We note that for $\mathcal{S}$ we have that the range of $K(g)$ is $\{z;\ \text{Re}\, z < 0 \ \text{or}\ z = 0\}$, while for $\mathcal{S}_V$, the range of $K(g)$ is $\{z;\ |z + V| \le V\}$. Next we show that $F(z)$ is differentiable for z in the interior of the range of $K(g)$.

If $K(g_1)$ is a point in the interior of the range of

$K(g)$, then g_1 is not a constant function. Then there are real functions h_1 and h_2 which have the property that, as the point (t_1,t_2) runs over a two dimensional neighborhood of $(0,0)$, $K(g_1 + t_1h_1 + t_2h_2)$ runs over a complex neighborhood of $K(g)$. The mapping $(t_1,t_2) \longrightarrow K(g + t_1h_1 + t_2h_2)$ is analytic in t_1 and t_2, and since $L(g_1 + t_1h_1 + t_2h_2)$ is differentiable in t_1 and t_2, it follows from the continuity assumptions on $L(g)$ that $F(z)$ is differentiable in a neighborhood of $K(g_1)$. Hence $F(z)$ is differentiable in the interior of the range of $K(g)$. Since $L(g)$ is continuous in g, it follows that $F(z)$ is continuous on the whole range of $K(g)$.

Next we show that $F(z)$ is holomorphic for z in the interior of the range of $K(g)$. To prove this it is sufficient to show that

$$\frac{\partial}{\partial u} F(u + iv) = -i \frac{\partial}{\partial v} F(u + iv)$$

for $z = u+iv$ in the interior of the range of $K(g)$. Since $F(z)$ is differentiable we have

$$\frac{d}{dt} L(g + th)\Big|_{t=0} = \frac{\partial F(K(g))}{\partial u} \frac{\partial}{\partial t}[\mathrm{Re}(K(g + th))]\Big|_{t=0}$$

$$+ \frac{\partial F(K(g))}{\partial v} \frac{\partial}{\partial t}[\mathrm{Im}(K(g + th)]\Big|_{t=0}$$

$$= - \frac{\partial F(K(g))}{\partial u} \int h(\underset{\sim}{x}) \sin g(\underset{\sim}{x})\, d^3x$$

$$+ \frac{\partial F(K(g))}{\partial v} \int h(\underset{\sim}{x}) \cos g(\underset{\sim}{x})\, d^3x \; . \qquad (3.25)$$

Then from Eq. (3.8) we obtain

$$- \frac{\partial F(K(g))}{\partial u} \int (\underset{\sim}{\nabla} \cdot \underset{\sim}{f})(\underset{\sim}{x}) \sin g(\underset{\sim}{x})\, d^3x$$

$$+ \frac{\partial F(K(g))}{\partial v} \int (\underset{\sim}{\nabla} \cdot \underset{\sim}{f})(\underset{\sim}{x}) \cos g(\underset{\sim}{x})\, d^3x$$

$$+ i \frac{\partial F(K(g))}{\partial u} \int (\underset{\sim}{f} \cdot \underset{\sim}{\nabla} g)(\underset{\sim}{x}) \sin g(\underset{\sim}{x})\, d^3x \qquad (3.26)$$

$$+ i \frac{\partial F(K(g))}{\partial v} \int (\underset{\sim}{f} \cdot \underset{\sim}{\nabla} g)(\underset{\sim}{x}) \cos g(\underset{\sim}{x})\, d^3x = 0 \; .$$

From the divergence theorem, we obtain the relationships

$$\int (\underset{\sim}{\nabla} \cdot \underset{\sim}{f})(\underset{\sim}{x}) \sin g(\underset{\sim}{x})\, d^3x = -\int (\underset{\sim}{f} \cdot \underset{\sim}{\nabla} g)(\underset{\sim}{x}) \cos g(\underset{\sim}{x})\, d^3x$$

$$\int (\underset{\sim}{\nabla} \cdot \underset{\sim}{f})(\underset{\sim}{x}) \cos g(\underset{\sim}{x})\, d^3x = \int (\underset{\sim}{f} \cdot \underset{\sim}{\nabla} g)(\underset{\sim}{x}) \sin g(\underset{\sim}{x})\, d^3x \qquad (3.27)$$

Combining (3.26) and (3.27) we find

$$\left(\frac{\partial F(K(g))}{\partial u} + i \frac{\partial F(K(g))}{\partial v}\right) \int (\underset{\sim}{\nabla} \cdot \underset{\sim}{f})(\underset{\sim}{x}) e^{-ig(\underset{\sim}{x})}\, d^3x = 0 \; .$$

If g is not a constant, one can find an $\underset{\sim}{f}$ with components in $\mathcal{S}$ (resp. $\mathcal{S}_V$) such that

$$\int (\nabla \cdot \underset{\sim}{f})(\underset{\sim}{x}) e^{-ig(\underset{\sim}{x})} d^3x \neq 0 .$$

Hence, $\frac{\partial F}{\partial u} = -i \frac{\partial F}{\partial v}$, and $F(z)$ is holomorphic for z in the interior of the range of $K(g)$. This completes the proof of the theorem.

Q.E.D.

Next we shall determine the explicit form of $L(f) = (\Omega_0, e^{i\rho(f)}\Omega_0)$ under the further assumption that $U(f)V(\underset{\sim}{\psi})$ determines a factor representation of the current algebra. The importance of factor representations lies in the fact that every representation of a C*-algebra (in particular, the C*-algebra associated with currents) can be uniquely decomposed into a direct integral of factor representations.[21] Roughly speaking, if one knows all of the factor representations of a C*-algebra, one can construct all representations by taking direct integrals.

Suppose we have a continuous unitary representation of $\mathcal{S} \wedge \mathcal{K}$. We denote by $\mathfrak{U}$ the *-algebra of polynomials in $U(f)$ and $V(\underset{\sim}{\psi})$, with $f \in \mathcal{S}$ and $\underset{\sim}{\psi} \in \mathcal{K}$, and by $\mathfrak{U}'$ the commutant of $\mathfrak{U}$, i.e. the set of all bounded operators which commute with the elements of $\mathfrak{U}$. Finally we denote by $\mathfrak{U}''$ the bicommutant of $\mathfrak{U}$, i.e. the commutant of $\mathfrak{U}'$. It follows from a theorem of von Neumann[20] that $\mathfrak{A}''$ is the strong closure of $\mathfrak{A}$, i.e. for $A \in \mathfrak{A}''$ and any finite set of vectors $\{\Psi_i; i = 1,\ldots,n\}$ in the Hilbert space of the representation, and for $\varepsilon > 0$, there exists a $B \in \mathfrak{A}$ such that $\|(A - B)\Psi_i\| < \varepsilon$. The representation $U(f)V(\underset{\sim}{\psi})$ is said to be a factor representation if $\mathfrak{A}' \cap \mathfrak{A}'' = \{\lambda I\}$; i.e. if the only operators common to both $\mathfrak{A}'$ and $\mathfrak{A}''$ are multiples of the identity. Every irreducible representation of the current algebra is a factor representation, since for irreducible representations $\mathfrak{A}' = \{\lambda I\}$.

Let us turn to the question of determining $L(g) = (\Omega_o, e^{i\rho(g)}\Omega_o)$ for a factor representation with a vector Ω_o satisfying Eq. (3.1) or equivalently Eq. (3.3).

We begin with the case of a system in a box. Since the function $e_o(\underset{\sim}{x}) \equiv 1$ is in $\mathcal{S}_V$, we can consider the operator $U(\lambda e_o) = \exp[i\lambda\rho(e_o)]$. Since $U(\lambda e_o)$ commutes with $U(f)V(\underset{\sim}{\psi})$ for all $(f,\underset{\sim}{\psi}) \in \mathcal{S} \wedge \mathcal{K}$, $U(\lambda e_o)$ is in the center of the current algebra. Then for a factor representation we must have $U(\lambda e_o) = \omega(\lambda)I$, where $|\omega(\lambda)| = 1$. By the group property and by Stone's theorem, we then have $\exp[i\lambda\rho(e_o)] = \exp[i\lambda Q]I$, and $\rho(e_o) = QI$, where Q is to be interpreted as the total number of particles in the system. But we have already seen that $L(g) = F(K(g))$ where $F(z)$ is analytic for $|z+V|<V$ and continuous for $|z+V|\leq V$. Now we have

$$F(K(\lambda e_o)) = F(Ve^{i\lambda} - V) = e^{iQ\lambda} . \tag{3.28}$$

Since F is single-valued, $F(Ve^{i\lambda} - V) = F(Ve^{i(\lambda+2\pi)} - V)$. Hence $e^{i2\pi Q} = 1$ and $Q = 0, \pm 1, \pm 2, \ldots$. If $F(z)$ is to be holomorphic for $|z+V|<V$ we must have $Q = 0,1,2,\ldots$; hence it follows that $F(z)$ is of the form

$$F(z) = \left(\frac{z+V}{V}\right)^Q , \qquad Q = 0,1,2,\ldots . \tag{3.29}$$

Therefore

$$L(f) = \left(\int_V e^{if(x)}\, d^3x\right)^Q . \tag{3.30}$$

Every representation of the current algebra can be expressed as a direct integral of factor representations; thus for an arbitrary representation of the current algebra in a box with the ground state satisfying Eq. (3.1) or (3.3), $L(f)$ is of the form

$$L(f) = \sum_{Q=0}^{\infty} \mu_Q \left(\int_V e^{if(x)}\, d^3x\right)^Q , \tag{3.31}$$

with $Q = 0,1,2,\ldots$, $\mu_Q \geq 0$, and $\sum_{Q=0}^{\infty} \mu_Q = 1$.

Next we determine the form of $L(f)$ for the case of infinite volume. Again we assume that we have a factor representation, and obtain the general case by taking a direct integral of factor representations.

Consider the expression

$$L(f_1 + f_2^n) = (\Omega_o, e^{i\rho(f_1)} e^{i\rho(f_2^n)} \Omega_o) \,, \tag{3.32}$$

where $f_1 \in \mathcal{S}$; $f_2^n(\underset{\sim}{x}) = f_2(\underset{\sim}{x} - n\underset{\sim}{a})$ for $n = 0,1,2,\ldots$ and $f_2 \in \mathcal{S}$; and where $\underset{\sim}{a}$ is a vector of unit length.

Now we have

$$L(f_1 + f_2^n) = F(K(f_1 + f_2^n))$$

and

$$K(f_1 + f_2^n) = \int (e^{if_1(\underset{\sim}{x})} e^{if_2(\underset{\sim}{x}-n\underset{\sim}{a})} - 1)\, d^3x$$

$$\longrightarrow K(f_1) + K(f_2) \quad \text{as} \quad n \to \infty. \tag{3.33}$$

Since F is continuous we have

$$L(f_1 + f_2^n) \longrightarrow F(K(f_1) + K(f_2)) \quad \text{as} \quad n \to \infty \,.$$

Now the set of all operators of norm not greater than one in a Hilbert space (i.e. the unit ball) is compact in the weak operator topology. Therefore the sequence $e^{i\rho(f_2^n)}$ has at least one cluster point in the weak operator topology which we shall call G . Since $[e^{iJ(g)}, e^{i\rho(f_2^n)}] \longrightarrow 0$ strongly as $n \to 0$ (i.e. the $e^{i\rho(f_2^n)}$ tend to commute with elements of the current algebra as $n \to \infty$) , it follows that G is in the commutant of the current algebra. Since G is a cluster point of a sequence of elements of the current algebra, G is also in the weak closure of the current algebra. Hence, by the assumption of a factor representation G is a multiple of the identity, i.e. $G = \lambda I$.

Since we have the existence of the limit

$$L(f_1 + f_2^n) \longrightarrow F(K(f_1) + K(f_2)) \quad \text{as} \quad n \to \infty$$

and since

$$L(f_2^n) = F(K(f_2)) \quad \text{for all} \quad n ,$$

it follows from Eq. (3.32) that

$$L(f_1 + f_2^n) \longrightarrow \lambda L(f_1) \quad \text{as} \quad n \to \infty$$

and

$$L(f_2^n) = \lambda \ .$$

Combining these equations, we find

$$F(K(f_1) + K(f_2)) = F(K(f_1))F(K(f_2)) \tag{3.34}$$

for all $f_1, f_2 \in \mathcal{S}$. Hence,

$$F(z_1 + z_2) = F(z_1)F(z_2) \ , \tag{3.35}$$

and it follows that $F(z)$ is of the form $F(z) = A \exp\{\bar{\rho} z\}$. Since $F(0) = 1$ we have $A = 1$ and since $|L(f)| = |F(K(f))| \leq 1$ for all $f \in \mathcal{S}$, we have $|F(z)| \leq 1$ for all z with $\text{Re}\{z\} \leq 0$. Hence $\bar{\rho} \geq 0$ and $F(z) = \exp\{\bar{\rho} z\}$. Thus

$$L(f) = \exp \bar{\rho} \int (e^{if(\underset{\sim}{x})} - 1) \, d^3x \ . \tag{3.36}$$

For the general case of a representation with ground state satisfying Eq. (3.1) or (3.3), $L(f)$ is a direct integral of functionals of the above form, i.e.

$$L(f) = \int_0^\infty \exp\left\{\bar{\rho}\int (e^{if(x)} - 1)\, d^3x\right\} d\mu(\bar{\rho}) \ , \qquad (3.37)$$

where μ is a positive measure on $[0,\infty)$ normalized so that $\int_0^\infty d\mu(\bar{\rho}) = 1$.

We summarize these results as follows:

Theorem 3.3. Suppose $g \to J(g)$ and $f \to \rho(f)$ is a *-representation of the non-relativistic current algebra of Eqs. (2.11)-(2.13) with a cyclic vector Ω_o. Suppose Ω_o is in the domain of $\rho(f)$ and $J(g)$ for all $f, g \in \mathcal{S}$, and that $(\rho(f_1)\Omega_o, \rho(f_2)\Omega_o)$ is a continuous bilinear form on $\mathcal{S} \times \mathcal{S}$ (resp. $\mathcal{S}_V \times \mathcal{S}_V$). Finally, suppose that

$$2iJ(g)\Omega_o = \rho(\nabla \cdot g)\Omega_o \qquad (3.3)$$

for all g with components in $\mathcal{S}$ (resp. $\mathcal{S}_V$). Then if $L(f) = (\Omega_o, e^{i\rho(f)}\Omega_o)$, $L(f)$ is of the form

$$L(f) = \sum_{Q=0}^{\infty} \mu_Q\left(\int_V e^{if(x)}\, d^3x\right)^Q \qquad (3.31)$$

in a box of volume V, and of the form

$$L(f) = \int_0^\infty \exp\left\{\bar{\rho}\int (e^{if(x)} - 1)\, d^3x\right\} d\mu(\bar{\rho}) \qquad (3.37)$$

in the infinite volume case; where $\mu_Q \geq 0$ for $Q = 0,1,2,\ldots$, $\sum_{Q=0}^{\infty} \mu_Q = 1$ and where μ is a positive measure on $[0,\infty)$ normalized to unity.

We remark that the form of $L(f,\underset{\sim}{g}) = (\Omega_o, e^{i\rho(f)} e^{iJ(\underset{\sim}{g})}\Omega_o)$ is completely determined by the form of $L(f)$ together with Eq. (3.3).

Theorem 3.4. Representations corresponding to Eq. (3.30) and Eq. (3.36) respectively are irreducible.

Proof: Supposing the contrary, there exists a closed invariant subspace $\mathfrak{m}$ of $\mathcal{H}$ with $U(f)\mathfrak{m} \subseteq \mathfrak{m}$, $V(\psi)\mathfrak{m} \subseteq \mathfrak{m}$; and we can decompose Ω_o into $\lambda^{\frac{1}{2}}\Omega_1 + (1-\lambda)^{\frac{1}{2}}\Omega_2$ with $\Omega_1 \in \mathfrak{m}$, $\Omega_2 \in \mathfrak{m}^{\perp}$, $0 < \lambda < 1$. Then $U(f)\Omega_2$ and $V(\underset{\sim}{\psi})\Omega_2$ are likewise in $\mathfrak{m}^{\perp}$. Since Ω_o is a cyclic vector for the $U(f)$, it follows that $\{U(f)\Omega_1\}$ generates a dense subspace of $\mathfrak{m}$, and $\{U(f)\Omega_2\}$ a dense subspace of $\mathfrak{m}^{\perp}$. Furthermore Ω_1 and Ω_2 are in the domains of $\rho(f)$ and $J(\underset{\sim}{g})$ by Stone's theorem, with $\rho(f)\Omega_1 \in \mathfrak{m}$, $\rho(f)\Omega_2 \in \mathfrak{m}^{\perp}$, etc. Since $\|\rho(f_n)\Omega_o\|^2 \longrightarrow 0$ if $f_n \to 0$ as $n \to \infty$, $\|\rho(f_n)\Omega_1\|^2 \longrightarrow 0$ and $\|\rho(f_n)\Omega_2\|^2 \longrightarrow 0$, whence $(\rho(f_1)\Omega_1, \rho(f_2)\Omega_1)$ and $(\rho(f_1)\Omega_2, \rho(f_2)\Omega_2)$ are continuous in f_1 and f_2. Evidently, $2iJ(\underset{\sim}{g})\Omega_1 = \rho(\nabla\cdot\underset{\sim}{g})\Omega_1$ and similarly for Ω_2.

Thus the functionals $L_1(f) = (\Omega_1, U(f)\Omega_1)$ and $L_2(f) = (\Omega_2, U(f)\Omega_2)$ satisfy all of the assumptions made earlier in this section, with $L(f) = \lambda L_1(f) + (1 - \lambda)L_2(f)$. Consequently, L_1 and L_2 must each be of the form of Eq. (3.31) or (3.37), which is impossible unless $L_1 = L_2 = L$. Therefore the representations are irreducible.

Q.E.D.

4. Functional Calculus.

We have shown that in order to describe a free Bose gas at zero temperature, one takes a representation of the current algebra, Eqs. (2.11)-(2.13), with a cyclic vector Ω_o satisfying Eq. (3.1). This corresponds to making the assumption that the expectation functional $L(f) = (\Omega_o, e^{i\rho(f)}\Omega_o)$ satisfies a certain functional differential equation.[19] In fact, with

$$\underset{\sim}{K}(\underset{\sim}{x}) = \underset{\sim}{\nabla}\rho(\underset{\sim}{x}) + 2i\underset{\sim}{J}(\underset{\sim}{x}) \tag{4.1}$$

and

$$(\Omega_o, e^{i\rho(f)}\underset{\sim}{K}(\underset{\sim}{x})\Omega_o) = 0 , \tag{4.2}$$

together with the commutation relation

$$[e^{i\rho(f)}, \underset{\sim}{K}(\underset{\sim}{x})] = -2i\underset{\sim}{\nabla}f(\underset{\sim}{x})\rho(\underset{\sim}{x})e^{i\rho(f)} , \tag{4.3}$$

one easily obtains

$$-i\underset{\sim}{\nabla}f(\underset{\sim}{x})(\Omega_o, e^{i\rho(f)}\rho(\underset{\sim}{x})\Omega_o) + (\Omega_o, \underset{\sim}{\nabla}\rho(\underset{\sim}{x})e^{i\rho(f)}\Omega_o) = 0, \tag{4.4}$$

which is the unsmeared form of Eq. (3.9).

Equation (4.4) may be rewritten as a functional differential equation as follows. We use the standard notation for functional derivatives. If $L(f)$ is a continuous functional on Schwartz's space $\mathcal{S}$ we say L has a functional derivative at f if there is a tempered distribution $T_f(\underset{\sim}{x})$ such that

$$\lim_{t\to 0} \frac{L(f + tg) - L(f)}{t} = T_f(g) \ . \tag{4.5}$$

We denote the functional derivative by

$$\frac{\delta L(f)}{\delta f(\underset{\sim}{x})} = T_f(\underset{\sim}{x}) \ . \tag{4.6}$$

It is a consequence of the assumptions we made on $L(f)$ in the beginning of Section 3 that L has a functional derivative at all $f \in \mathcal{S}$, and

$$\frac{\delta L(f)}{\delta f(\underset{\sim}{x})} = i(\Omega, e^{i\rho(f)} \rho(\underset{\sim}{x})\Omega) \ . \tag{4.7}$$

Higher functional derivatives are defined in exactly the same fashion.

In this notation, Eq. (4.4) reads

$$-i[\underset{\sim}{\nabla} f(\underset{\sim}{x})]\frac{\delta L(f)}{\delta f(\underset{\sim}{x})} + \underset{\sim}{\nabla}_{\underset{\sim}{x}} \frac{\delta L(f)}{\delta f(\underset{\sim}{x})} = 0 \ . \tag{4.8}$$

A unique solution to Eq. (4.8) is determined when the following boundary conditions are imposed on $L(f)$:

(i) $L(f)$ is a positive functional in the Bochner sense, Eq. (2.28). This condition is a consequence of the interpretation of $L(f)$ as an inner product in a Hilbert space of positive norm. It establishes that the measures μ_Q and $\mu(\bar{\rho})$ appearing in Eqs. (3.31) and (3.37) are positive.

(ii) $L(0) = 1$. This condition normalizes the inner product to one.

(iii) $|L(f)| \leq 1$. This is a consequence of the unitarity of $U(f)$, Eq. (2.16). This condition guarantees that the average density $\bar{\rho}$ appearing in Eqs. (3.36) and (3.37) is a positive number.

(iv) $L(f)$ is an extremal solution in the sense that it cannot be written as a convex linear combination of two other solutions.

(v) $\left.\dfrac{\delta L}{\delta f(x)}\right|_{f=0} = \bar{\rho}$ = a specified number.

Conditions (i)-(iii) were employed to prove Theorem 3.3, which implies that Eq. (4.8) has the unique class of solutions specified by Eqs. (3.31) or (3.37). Condition (iv), as demonstrated in Theorem 3.4, is used to restrict the general solution to the forms (3.30) and (3.36) defining irreducible representations of the algebra (2.11)-(2.13). Finally, condition (v) selects the particular irreducible representation corresponding to a physical system having a specific value for the average density.

Having written a functional differential equation for $L(f)$ whose solutions describe the infinite free Bose gas at a specified average density, it is natural to seek an equation or system of coupled equations whose solution would describe an infinite Bose gas with an interaction. Such a system is proposed in Section 5.

In this section we derive a mathematical relation between two of the quantities which appear in Section 5. This relation proves helpful in completing the system of coupled equations, and introduces the concept of indefinite functional integration.

Define

$$R_{ij}(f,\underset{\sim}{x},\underset{\sim}{y}) = (\Omega_o, K_i{}^*(\underset{\sim}{x})e^{i\rho(f)}K_j(\underset{\sim}{y})\Omega_o) \;, \qquad (4.9)$$

where $\underset{\sim}{K}(\underset{\sim}{x}) = \underset{\sim}{\nabla}\rho(\underset{\sim}{x}) + 2i\underset{\sim}{J}(\underset{\sim}{x})$ is an operator-valued distribution, and Ω_o is a cyclic vector for the $\rho(f)$.

Consider the expression

$$N'_{ij}(f,\underset{\sim}{x}) = (\Omega_o,\ K_i{}^*(x)e^{i\rho(f)}\frac{1}{\rho(\underset{\sim}{x})}K_j(\underset{\sim}{x})\Omega_o)$$

$$= (\Omega_o,\ e^{i\rho(f)}K_i{}^*(\underset{\sim}{x})\frac{1}{\rho(\underset{\sim}{x})}K_j(\underset{\sim}{x})\Omega_o)$$

$$- 2i(\partial_i f)(\underset{\sim}{x})(\Omega_o, e^{i\rho(f)}K_j(\underset{\sim}{x})\Omega_o) \ . \qquad (4.10)$$

One way to define such an expression has previously been proposed.[11] Here we shall, roughly speaking, functionally integrate R_{ij} to obtain N'_{ij}.

In representations of non-relativistic systems of physical interest one usually has that $\rho(f) \geq 0$ if $f(\underset{\sim}{x}) \geq 0$ for all $\underset{\sim}{x}$. This corresponds to the fact that ρ usually describes the number density for a single species of particle. Let us for the moment pretend that $\rho(\underset{\sim}{x})$ is a well defined self-adjoint operator at each point $\underset{\sim}{x}$, with positive spectrum. Then we could write

$$\frac{1}{\rho(\underset{\sim}{x})} e^{i\rho(f)} = \int_0^\infty dt\ e^{-t\rho(\underset{\sim}{x})} e^{i\rho(f)} = \int_0^\infty dt\ e^{i\rho(f+it\delta_{\underset{\sim}{x}})} , \qquad (4.11)$$

where $\delta_{\underset{\sim}{x}}(\underset{\sim}{y}) = \delta(\underset{\sim}{x} - \underset{\sim}{y})$, the Dirac delta function. The relation between $R_{ij}(f,\underset{\sim}{x},\underset{\sim}{y})$ and $N'_{ij}(f,\underset{\sim}{x})$ would be given by

$$N'_{ij}(f,\underset{\sim}{x}) = \int_0^\infty dt\ R_{ij}(f + it\delta_{\underset{\sim}{x}}, \underset{\sim}{x}, \underset{\sim}{x}) \ . \qquad (4.12)$$

Now in general $\rho(\underset{\sim}{x})$ is not well-defined as an operator at a point, and an expression such as $R_{ij}(f + it\delta_{\underset{\sim}{x}}, \underset{\sim}{x}, \underset{\sim}{x})$ is not well defined. In fact, let us compute R_{ij} for an infinite free Bose gas at zero temperature of average density $\bar{\rho}$, with Ω_o the ground state. We have $L(f) = (\Omega_o, e^{i\rho(f)}\Omega_o)$ given by Eq. (3.36). Suppose that $h \in \mathcal{S}$ and $\Omega_h = e^{i\rho(h)}\Omega_o$. We shall compute $R_{ij}^{(h)}(f,\underset{\sim}{x},\underset{\sim}{y})$ with respect to the cyclic vector Ω_h:

$$R_{ij}^{(h)}(f,\underset{\sim}{x},\underset{\sim}{y}) = (\Omega_h, K_i^*(\underset{\sim}{x})e^{i\rho(f)}K_j(\underset{\sim}{y})\Omega_h)$$

$$= (K_i(\underset{\sim}{x})e^{i\rho(h)}\Omega_o, e^{i\rho(f)}K_j(\underset{\sim}{y})e^{i\rho(h)}\Omega_o) \qquad (4.13)$$

From Eq. (4.3) together with $K_i(\underset{\sim}{x})\Omega_o = 0$, we have

$$K_i(\underset{\sim}{x})e^{i\rho(h)}\Omega_o = 2ie^{i\rho(h)}(\partial_i h)(\underset{\sim}{x})\rho(\underset{\sim}{x})\Omega_o \qquad (4.14)$$

or

$$K_i(\underset{\sim}{x})\Omega_h = 2i(\partial_i h)(\underset{\sim}{x})\rho(\underset{\sim}{x})\Omega_h . \qquad (4.15)$$

Hence,

$$R_{ij}^{(h)}(f,\underset{\sim}{x},\underset{\sim}{y}) = 4(\partial_i h)(\underset{\sim}{x})(\partial_j h)(\underset{\sim}{y})(\Omega_o, e^{i\rho(f)}\rho(\underset{\sim}{x})\rho(\underset{\sim}{y})\Omega_o)$$

$$= -4(\partial_i h)(\underset{\sim}{x})(\partial_i h)(\underset{\sim}{y}) \frac{\delta^2 L(f)}{\delta f(\underset{\sim}{x})\delta f(\underset{\sim}{y})} . \qquad (4.16)$$

A straightforward computation of the functional derivatives yields

$$R_{ij}^{(h)}(f,\underset{\sim}{x},\underset{\sim}{y}) =$$
$$4(\partial_i h)(\underset{\sim}{x})(\partial_j h)(\underset{\sim}{y})(\bar{\rho}^2 e^{if(\underset{\sim}{x})} e^{if(\underset{\sim}{y})} + \bar{\rho}\delta(\underset{\sim}{x}-\underset{\sim}{y})e^{if(\underset{\sim}{x})})L(f) \tag{4.17}$$

It is clear that $R_{ij}^{(h)}(f + it\delta_{\underset{\sim}{x}}, \underset{\sim}{x}, \underset{\sim}{x})$ is ill-defined, since it contains exponentials of delta functions, as well as a delta function evaluated at zero.

Instead we propose to interpret Eq. (4.11) as follows. Let $\delta_{\underset{\sim}{x}}^n$ be a sequence of functions in $\mathcal{S}$ which converges to $\delta_{\underset{\sim}{x}}$ in the sense of a distribution, i.e. for all $f \in \mathcal{S}$,

$$\lim_{n\to\infty} \int \delta_{\underset{\sim}{x}}^n(\underset{\sim}{y}) f(\underset{\sim}{y})\, d^3y = f(\underset{\sim}{x}) .$$

We now interpret Eq. (4.12) by means of the limiting procedure

$$N'_{ij}(f,\underset{\sim}{x}) = \lim_{n\to\infty} \int_0^\infty dt\; R_{ij}(f + it\delta_{\underset{\sim}{x}}^n, \delta_{\underset{\sim}{x}}^n, \delta_{\underset{\sim}{x}}^n) . \tag{4.18}$$

We shall show that this definition works for the case of $R_{ij}^{(h)}$ in Eq. (4.17). Notice that $R_{ij}^{(h)}(f,g,k)$ can be extended from functions $f,g,k \in \mathcal{S}$ to bounded Borel functions which decrease faster than any polynomial in $\underset{\sim}{x}$ at infinity. Then we can take our approximating sequence for a delta function to be the more convenient set of functions

$$\delta^n_{\underset{\sim}{x}}(\underset{\sim}{y}) = \begin{cases} n^3 & \text{if } |x_i - y_i| \leq 1/2n \text{ for } i = 1,2,3 \\ 0 & \text{otherwise} \end{cases} \tag{4.19}$$

Computing $R^{(h)}_{ij}(f + it\delta^n_{\underset{\sim}{x}}, \delta^n_{\underset{\sim}{x}}, \delta^n_{\underset{\sim}{x}})$, we find

$$R^{(h)}_{ij}(f + it\delta^n_{\underset{\sim}{x}}, \delta^n_{\underset{\sim}{x}}, \delta^n_{\underset{\sim}{x}}) =$$

$$A_1(t,n)L(f + it\delta^n_{\underset{\sim}{x}}) + A_2(t,n)L(f + it\delta^n_{\underset{\sim}{x}}) ,$$

where

$$A_1(t,n) =$$

$$4\bar{\rho}^2 \int_{V_n} d^3y \int_{V_n} d^3z \; n^6 (\partial_i h)(\underset{\sim}{y})(\partial_j h)(\underset{\sim}{z}) e^{if(\underset{\sim}{y})} e^{if(\underset{\sim}{z})} e^{-2tn^3}$$

and

$$A_2(t,n) = 4\bar{\rho} \int_{V_n} d^3y \; n^6 (\partial_i h)(\underset{\sim}{y})(\partial_j h)(\underset{\sim}{y}) e^{if(\underset{\sim}{y})} e^{-tn^3} ,$$

and where $V_n = \{\underset{\sim}{y};\ |y_i - x_i| \leq 1/2n \text{ for } i = 1,2,3\}$. A straightforward computation shows that

$$|L(f + it\delta^n_{\underset{\sim}{x}}) - L(f)| \leq 1 - e^{-\bar{\rho}/n^3}$$

for all $\underset{\sim}{x} \in R^3$, $t \geq 0$ and $f \in \mathcal{S}$. Hence, $L(f + it\delta^n_{\underset{\sim}{x}})$ converges uniformly to $L(f)$ as $n \to \infty$. Since $A_1(t,n)$ and $A_2(t,n)$ converge in the L^1 topology to absolutely integrable functions, it follows that

$$\lim_{n\to\infty} \int_0^\infty dt\, R_{ij}(f + it\delta^n_{\underset{\sim}{x}}, \delta^n_{\underset{\sim}{x}}, \delta^n_{\underset{\sim}{x}})$$

$$= \left(\lim_{n\to\infty} \int_0^\infty (A_1(t,n) + A_2(t,n))\, dt\right) L(f) \ . \tag{4.20}$$

Now performing the integration over t, we have

$$\int_0^\infty A_1(t,n)\, dt \tag{4.21}$$

$$= 2\bar{\rho}^2 n^3 \int_{V_n} d^3y \int_{V_n} d^3z\, (\partial_i h)(\underset{\sim}{y})(\partial_j h)(\underset{\sim}{z})\, e^{if(\underset{\sim}{y})} e^{if(\underset{\sim}{z})} \longrightarrow 0$$

as $n \to \infty$, since the square of the volume of V_n goes as n^{-6}. Furthermore,

$$\int_0^\infty A_2(t,n)\, dt = 4\bar{\rho} n^3 \int_{V_n} (\partial_i h)(\underset{\sim}{y})(\partial_j h)(\underset{\sim}{y})\, e^{if(\underset{\sim}{y})}\, d^3y$$

$$\longrightarrow 4\bar{\rho}(\partial_i h)(\underset{\sim}{x})(\partial_j h)(\underset{\sim}{x})\, e^{if(\underset{\sim}{x})} \ . \tag{4.22}$$

Hence we have

$$\lim_{n\to\infty} \int_0^\infty dt\, R_{ij}(f + it\delta^n_{\underset{\sim}{x}}, \delta^n_{\underset{\sim}{x}}, \delta^n_{\underset{\sim}{x}})$$

$$= 4\bar{\rho}(\partial_i h)(\underset{\sim}{x})(\partial_j h)(\underset{\sim}{y})e^{if(\underset{\sim}{x})}L(f) \quad . \tag{4.23}$$

Next we compute $N'^{(h)}_{ij}(f,\underset{\sim}{x})$ directly, using the interpretation of $1/\rho(\underset{\sim}{x})$ proposed earlier by Goldin and Sharp.[11] They interpret $1/\rho(\underset{\sim}{x})$ as the map $\frac{1}{\rho(\underset{\sim}{x})}: \mathcal{V} \times \mathcal{V} \longrightarrow \mathcal{S}'$, where $\mathcal{S}'$ is the continuous dual of $\mathcal{S}$ and $\mathcal{V}$ is the linear span of the vector-valued distributions $\{g(\underset{\sim}{x})\rho(\underset{\sim}{x})\Phi \mid \Phi \in D,\ g \in \mathcal{O}_M\}$, with D a dense invariant domain for $\rho(f)$, $f \in \mathcal{S}$, and $\mathcal{O}_M$ the real-valued C_∞ functions which together with all derivatives are polynomially bounded at ∞.

Now, using Eq. (4.15)

$$\left(\Omega_h, e^{i\rho(f)}K_i^*(\underset{\sim}{x}) \frac{1}{\rho(\underset{\sim}{x})} K_j(\underset{\sim}{x})\Omega_h\right) = \left(K_i(\underset{\sim}{x})\Omega_{h-f}, \frac{1}{\rho(\underset{\sim}{x})} K_j(\underset{\sim}{x})\Omega_h\right)$$

$$= 4\left((\partial_i h - \partial_i f)(\underset{\sim}{x})\rho(\underset{\sim}{x})\Omega_{h-f}, \frac{1}{\rho(\underset{\sim}{x})}(\partial_j h)(\underset{\sim}{x})\rho(\underset{\sim}{x})\Omega_h\right)$$

$$= 4(\partial_i h - \partial_i f)(\underset{\sim}{x})(\partial_j h)(\underset{\sim}{x})(\Omega_o, e^{i\rho(f)}\rho(\underset{\sim}{x})\Omega_o) \quad , \tag{4.24}$$

and

$$N_{ij}^{(h)\prime}(f,\underset{\sim}{x}) = (\Omega_h,\ e^{i\rho(f)} K_i{}^*(\underset{\sim}{x}) \frac{1}{\rho(\underset{\sim}{x})} K_j(\underset{\sim}{x})\Omega_h)$$

$$- 2i(\partial_i f)(\underset{\sim}{x})(\Omega_h, e^{i\rho(f)} K_j(\underset{\sim}{x})\Omega_h)$$

$$= 4(\partial_i h)(\underset{\sim}{x})(\partial_j h)(\underset{\sim}{x})(\Omega_o, e^{i\rho(f)} \rho(\underset{\sim}{x})\Omega_o)$$

$$= -4i(\partial_i h)(\underset{\sim}{x})(\partial_j h)(\underset{\sim}{x}) \frac{\delta L(f)}{\delta f(\underset{\sim}{x})}$$

$$= 4\bar{\rho}(\partial_i h)(\underset{\sim}{x})(\partial_j h)(\underset{\sim}{x}) e^{if(\underset{\sim}{x})} L(f) , \qquad (4.25)$$

where $L(f)$ is given by Eq. (3.36) in evaluating the functional derivative. Hence we see that Eq. (4.18) gives the correct relation between R_{ij} and N'_{ij} for the case at hand. We leave unanswered at this time the important problem of determining a general set of sufficient conditions to be imposed on $R_{ij}(f,\underset{\sim}{x},\underset{\sim}{y})$, in order to ensure that the limiting procedure of Eq. (4.18) leads to a well-defined expression.

5. Determining L(f) when the Particles Interact.

The preceding work has shown how the functional L(f) of Eq. (3.36), which determines an irreducible representation of the local current algebra (2.11)-(2.13), can be defined uniquely as the solution to a functional differential equation satisfying the appropriate boundary conditions (conditions (i)-(v) following Eq. (4.8)).

These results apply only to non-interacting bosons. Next we ask whether the same pattern of results persists when interactions are included. Can one find a set of functional differential equations which, when supplemented with suitable boundary conditions, determine a ground state expectation functional L(f)? In this section we suggest the possibility of an affirmative answer to this question.

The functional equation which defined L(f) for non-interacting bosons was Eq. (4.8), obtained from the condition (3.1),

$$\underset{\sim}{K}(\underset{\sim}{x})\Omega_o = (\underset{\sim}{\nabla}\rho + 2i\underset{\sim}{J})(\underset{\sim}{x})\Omega_o = 0 .$$

The first step towards deriving a corresponding set of functional equations in the interacting case is to find conditions replacing Eq. (3.1), since the latter correctly expresses the action of the Hamiltonian in a representation only for non-interacting bosons. These conditions take the form of equations which relate, and ultimately determine, the following quantities:

$$L(f) \equiv (\Omega_0, e^{i\rho(f)}\Omega_0) \tag{5.1}$$

$$M(f,\underset{\sim}{x}) \equiv (\Omega_0, e^{i\rho(f)}\underset{\sim}{K}(\underset{\sim}{x})\Omega_0) \tag{5.2}$$

$$R_{ij}(f,\underset{\sim}{x},\underset{\sim}{y}) \equiv (\Omega_0, K_i{}^*(\underset{\sim}{x})e^{i\rho(f)}K_j(\underset{\sim}{y})\Omega_0) \tag{5.3}$$

$$N_{ij}(f,\underset{\sim}{x}) \equiv$$

$$\tfrac{1}{2}\left(\Omega_0,\ e^{i\rho(f)}\left[K_i{}^*(\underset{\sim}{x})\frac{1}{\rho(\underset{\sim}{x})}K_j(\underset{\sim}{x}) + K_j{}^*(\underset{\sim}{x})\frac{1}{\rho(\underset{\sim}{x})}K_i(\underset{\sim}{x})\right]\Omega_0\right) . \tag{5.4}$$

We assume that the particles interact through a central two-body potential $V(|\underset{\sim}{x} - \underset{\sim}{y}|)$ and write the Hamiltonian (for particles of unit mass) as[1]

$$H = \frac{1}{8}\int d^3x\, K_i{}^*(\underset{\sim}{x})\,\frac{1}{\rho(\underset{\sim}{x})}\,K_i(\underset{\sim}{x})$$

$$+ \frac{1}{2}\iint d^3x\, d^3y\, \rho(\underset{\sim}{x})V(|\underset{\sim}{x} - \underset{\sim}{y}|)\rho(\underset{\sim}{y}) , \tag{5.5}$$

where the repeated index i is summed over $i = 1,2,3$. For the Hamiltonian to be well-defined it may be necessary to subtract from Eq. (5.5) an infinite constant corresponding to its ground state expectation value, thus establishing a zero of the energy, $(H - E_o)\Omega_o = 0$.

It should also be noted that we have no guarantee that Eq. (5.4) defining $N_{ij}(f,\underset{\sim}{x})$ makes sense as it stands. Nevertheless there is reason to hope that the ensuing system of equations ultimately lends itself to a meaningful interpretation and we shall proceed as though the quantities under discussion are all well-defined.

1. The first condition replacing (3.1) follows from the requirement that the cyclic vector be an eigenvector of the energy operator,

$$(H - E_o)\Omega = 0 \ , \tag{5.6}$$

which we write in the form

$$(\Omega_0, e^{i\rho(f)} H\Omega_o) - E_o L(f) = 0 \ . \tag{5.7}$$

To write Eq. (5.7) as a relationship between functionals, we introduce $N_{ij}(f,\underset{\sim}{x})$, Eq. (5.4), and note that

$$\frac{1}{i}\frac{\delta}{\delta f(\underset{\sim}{x})}\,\frac{1}{i}\frac{\delta}{\delta f(\underset{\sim}{y})}\,L(f) = (\Omega_o, \rho(\underset{\sim}{x})\rho(\underset{\sim}{y})e^{i\rho(f)}\Omega_o) \tag{5.8}$$

to obtain

$$\frac{1}{8} \sum_{i=1}^{3} \int d^3x \, N_{ii}(f,\underset{\sim}{x})$$
$$- \frac{1}{2} \iint d^3x \, d^3y \, V(|\underset{\sim}{x} - \underset{\sim}{y}|) \frac{\delta^2 L(f)}{\delta f(\underset{\sim}{x}) \delta f(\underset{\sim}{y})} - E_o L(f) = 0 \, . \tag{5.9}$$

2. A second equation follows from the requirement that Ω_o be invariant under time reversal,

$$T\Omega_o = \Omega_o \; ; \tag{5.10}$$

where T is the anti-unitary time reversal operator satisfying $(T\Psi, T\Phi) = (\Phi, \Psi)$. To derive the desired equation from (5.10), consider

$$(\Omega_o, e^{i\rho(f)} \underset{\sim}{J}(\underset{\sim}{x}) \Omega_o) = (Te^{i\rho(f)} \underset{\sim}{J}(\underset{\sim}{x}) \Omega_o, T\Omega_o) \; . \tag{5.11}$$

Since $T\rho(f)T^{-1} = \rho(f)$, $T\underset{\sim}{J}(\underset{\sim}{x})T^{-1} = -\underset{\sim}{J}(\underset{\sim}{x})$, $TiT^{-1} = -i$, and $T\Omega_o = \Omega_o$, we find

$$(\Omega_o, e^{i\rho(f)} \underset{\sim}{J}(\underset{\sim}{x}) \Omega_o) = -(\Omega_o, \underset{\sim}{J}(\underset{\sim}{x}) e^{i\rho(f)} \Omega_o) \; . \tag{5.12}$$

Recalling that $e^{i\rho(f)} \underset{\sim}{J}(\underset{\sim}{x}) e^{-i\rho(f)} = \underset{\sim}{J}(\underset{\sim}{x}) - \underset{\sim}{\nabla} f(\underset{\sim}{x}) \rho(\underset{\sim}{x})$, Eq. (3.7), we may write Eq. (5.12) in the form

$$(\Omega_o, e^{i\rho(f)} \underset{\sim}{K}(\underset{\sim}{x}) \Omega_o) =$$
$$(\Omega_o, e^{i\rho(f)} \underset{\sim}{\nabla} \rho(\underset{\sim}{x}) \Omega_o) - i(\Omega_o, e^{i\rho(f)} \rho(\underset{\sim}{x}) \underset{\sim}{\nabla} f(\underset{\sim}{x}) \Omega_o) \; . \tag{5.13}$$

Finally we may introduce $\underset{\sim}{M}(f,\underset{\sim}{x})$, Eq. (5.2), and the appropriate functional derivatives, to find

$$\underset{\sim}{M}(f,\underset{\sim}{x}) = \underset{\sim}{\nabla}_{\underset{\sim}{x}}\left(\frac{1}{i}\frac{\delta L(f)}{\delta f(\underset{\sim}{x})}\right) - [\underset{\sim}{\nabla} f(\underset{\sim}{x})]\frac{\delta L(f)}{\delta f(\underset{\sim}{x})} . \tag{5.14}$$

All of the dynamical information about a system of interacting bosons is expressed in Eqs. (5.9) and (5.14), when these equations are supplemented with suitable boundary conditions. However, the two equations relate three unknown functionals. The additional relationships among the functionals (5.1)-(5.4) needed to complete the system of equations are obtained entirely from consideration of the mathematical properties of the functionals.

3. One of the remaining equations we need has been derived in Section 4. It relates the indefinite functional integral of $R_{ij}(f,\underset{\sim}{x},\underset{\sim}{y})$, Eq. (5.3), to the quantity $N'_{ij}(f,\underset{\sim}{x})$, Eq. (4.10). We can write $N_{ij}(f,\underset{\sim}{x})$, Eq. (5.4), in terms of $R_{ij}(f,\underset{\sim}{x},\underset{\sim}{y})$ and $\underset{\sim}{M}(f,\underset{\sim}{x})$ as follows:

$$N_{ij}(f,\underset{\sim}{x}) = \frac{1}{2}(N'_{ij}(f,\underset{\sim}{x}) + N'_{ji}(f,\underset{\sim}{x})) + i(\partial_i f)(\underset{\sim}{x})M_j(f,\underset{\sim}{x}) + i(\partial_j f)(\underset{\sim}{x})M_i(f,\underset{\sim}{x}) , \tag{5.15}$$

where as in Eq. (4.18)

$$N'_{ij}(f,\underset{\sim}{x}) = \lim_{n\to\infty} \int_0^\infty R_{ij}(f + it\delta^n_{\underset{\sim}{x}}, \delta^n_{\underset{\sim}{x}}, \delta^n_{\underset{\sim}{x}})\ dt\ . \qquad (5.16)$$

4. The final equation relates $R_{ij}(f,\underset{\sim}{x},\underset{\sim}{y})$ to $\underset{\sim}{M}(f,\underset{\sim}{x})$. Referring to Section 2.C, we may write $L(f)$, $\underset{\sim}{M}(f,\underset{\sim}{x})$ and $R_{ij}(f,\underset{\sim}{x},\underset{\sim}{y})$ as

$$L(f) = \int_{\mathcal{S}'} e^{i(F,f)}\ d\mu(F)\ , \qquad (5.17)$$

$$\underset{\sim}{M}(f,\underset{\sim}{x}) = \int_{\mathcal{S}'} \overline{\Omega_0(F)}\, e^{i(F,f)}\, (\underset{\sim}{K}(\underset{\sim}{x})\Omega_0)(F)\ d\mu(F)\ , \qquad (5.18)$$

and

$$R_{ij}(f,\underset{\sim}{x},\underset{\sim}{y}) = \int_{\mathcal{S}'} \overline{(K_i(\underset{\sim}{x})\Omega_0)(F)}\, e^{i(F,f)}\, (K_j(\underset{\sim}{y})\Omega_0)(F)\ d\mu(F)\ , \qquad (5.19)$$

where $\mathcal{S}'$ is the continuous dual of Schwartz's space and μ is a cylindrical measure on $\mathcal{S}'$ uniquely determined by $L(f)$.

We now define $\underset{\sim}{\tilde{M}}(F,\underset{\sim}{x})$ to be the inverse Fourier transform of $\underset{\sim}{M}(f,\underset{\sim}{x})$; i.e.

$$\underset{\sim}{M}(f,\underset{\sim}{x}) = \int_{\mathcal{S}'} e^{i(F,f)}\, \underset{\sim}{\tilde{M}}(F,\underset{\sim}{x})\ d\mu(F)\ . \qquad (5.20)$$

It is not difficult to establish the existence of $\underset{\sim}{\tilde{M}}(F,\underset{\sim}{x})$ using standard methods in the Gel'fand-Vilenkin approach.[3,12] One may prove first that $\underset{\sim}{M}(f,\underset{\sim}{x})$ in

Eq. (5.2) is the Fourier transform of a (not necessarily positive) measure μ_1 on $\mathcal{S}'$; then that every set of measure zero in μ is of measure zero in μ_1; and conclude that the Radon-Nikodym derivative $d\mu_1(F)/d\mu(F)$ exists and defines $\tilde{M}(F,x)$. The assumptions needed to carry through these arguments amount to the statement that the ground state vector Ω_0 is in the domain of $K(x)$.

Similarly, define the inverse Fourier transform $\tilde{R}_{ij}(F,x,y)$ of $R_{ij}(f,x,y)$ by

$$R_{ij}(f,x,y) = \int_{\mathcal{S}'} e^{i(F,f)}\tilde{R}_{ij}(F,x,y)\, d\mu(F) \; . \qquad (5.21)$$

Then, since $\Omega_0(F) \equiv 1$ almost everywhere,

$$\tilde{R}_{ij}(F,x,y) = \overline{\tilde{M}_i(F,x)}\tilde{M}_j(F,y) \qquad (5.22)$$

almost everywhere, or

$$R_{ij}(f,x,y) = \int_{\mathcal{S}'} e^{i(F,f)}\overline{\tilde{M}_i(F,x)}\tilde{M}_j(F,y)\, d\mu(F) \; . \qquad (5.23)$$

To summarize, we have the following system of coupled functional equations.

1. "Schroedinger Equation" $((H - E_o)\Omega_o = 0)$:

$$\frac{1}{8} \sum_{i=1}^{3} \int d^3x \, N_{ii}(f,\underset{\sim}{x}) \tag{5.9}$$

$$- \frac{1}{2} \iint d^3x \, d^3y \, V(|\underset{\sim}{x} - \underset{\sim}{y}|) \frac{\delta^2 L(f)}{\delta f(\underset{\sim}{x}) \delta f(\underset{\sim}{y})} - E_o L(f) = 0 \, .$$

2. Time-reversal invariance $(T\Omega_o = \Omega_o)$:

$$\underset{\sim}{M}(f,\underset{\sim}{x}) = \underset{\sim}{\nabla}_{\underset{\sim}{x}} \left(\frac{1}{i} \frac{\delta L(f)}{\delta f(\underset{\sim}{x})} \right) - (\underset{\sim}{\nabla} f(\underset{\sim}{x})) \frac{\delta L(f)}{\delta f(\underset{\sim}{x})} \, . \tag{5.14}$$

3. Indefinite functional integration relationship:

$$N_{ij}(f,\underset{\sim}{x}) = \frac{1}{2}(N'_{ij}(f,\underset{\sim}{x}) + N'_{ji}(f,\underset{\sim}{x})) \tag{5.15}$$

$$+ i(\partial_i f)(\underset{\sim}{x}) M_j(f,\underset{\sim}{x}) + i(\partial_j f)(\underset{\sim}{x}) M_i(f,\underset{\sim}{x}) \, ,$$

where $N'_{ij}(f,\underset{\sim}{x}) = \lim_{n \to \infty} \int_0^\infty R_{ij}(f + it\delta^n_{\underset{\sim}{x}}, \delta^n_{\underset{\sim}{x}}, \delta^n_{\underset{\sim}{x}}) \, dt$.

4. Fourier transform relationship:

$$R_{ij}(f,\underset{\sim}{x},\underset{\sim}{y}) = \int_{g'} e^{i(F,f)} \overline{\tilde{M}_i(F,\underset{\sim}{x})} \tilde{M}_j(F,\underset{\sim}{y}) \, d\mu(F) \, , \tag{5.23}$$

where $L(f) = \int_{g'} e^{i(F,f)} \, d\mu(F)$ and $\underset{\sim}{M}(f,\underset{\sim}{x}) = \int_{g'} e^{i(F,f)} \underset{\sim}{\tilde{M}}(F,\underset{\sim}{x}) du(F)$

Equations (5.15) and (5.23) together express $N_{ij}(f,\underset{\sim}{x})$ in terms of $\underset{\sim}{M}(f,\underset{\sim}{x})$ and the measure μ of which $L(f)$ is the Fourier transform. Then, substituting for $N_{ii}(f,\underset{\sim}{x})$, Eqs. (5.9) and (5.14) relate the two functionals $L(f)$ and $\underset{\sim}{M}(f,\underset{\sim}{x})$.

The above system of equations can be expected to determine $L(f)$ uniquely only if it is supplemented by appropriate boundary conditions, just as in the case of Eq. (4.8) which defined the free system. The boundary conditions which applied to $L(f)$ in the free case clearly apply in the interacting case as well. We do not know at this point whether these five boundary conditions suffice to determine a unique solution to Eqs. (5.9), (5.14), (5.15) and (5.23) or whether additional boundary conditions are necessary.

In contrast to the non-interacting case discussed in Sections 3 and 4, we have no means of obtaining a solution to the above system of equations, nor do we have techniques to demonstrate that a solution exists or, if it exists, that it is unique.

There are other ways to supply some of the additional information needed to complete the system of equations begun with (5.9) and (5.14). For instance one can use the equation of motion for $\underset{\sim}{J}(\underset{\sim}{x})$ in the form

$$(\Omega_o, [e^{i\rho(f)} \underset{\sim}{J}(\underset{\sim}{x}), H]\Omega_o) = 0 \tag{5.24}$$

and the condition that the ground state be rotationally invariant

$$\mathfrak{L}\Omega_o = \Omega_o \quad , \quad \mathfrak{L} = \int \underset{\sim}{x} \times \underset{\sim}{J}(\underset{\sim}{x})\, d^3x \, , \tag{5.25}$$

in the form

$$(\Omega_o, [e^{i\rho(f)} N_{ij}(f, \underset{\sim}{x}), \mathfrak{L}]\Omega_o) = 0 \, . \tag{5.26}$$

Thus one obtains equations(19) which can be solved so as to express $N_{ij}(f,\underset{\sim}{x})$ in terms of $R_{ij}(f,\underset{\sim}{x},\underset{\sim}{y})$, $\underset{\sim}{M}(f,\underset{\sim}{x})$ and $L(f)$; and Eqs. (5.24) and (5.26) can replace Eq. (5.15).

In whatever fashion one chooses to complete Eqs. (5.9) and (5.14), one can be sure that the resulting set of equations will not be amenable to exact solution for $L(f)$ in most situations of practical interest. Therefore one would like to have techniques for its approximate determination. The approach via functional differential equations is most inviting because it is suggestive of such techniques. An approximate functional $L(f)$ would be one which was an approximate solution in some well-defined way to a system of equations whose exact solution defined an irreducible representation of a local current algebra. This is one sense in which it might have meaning to talk about an "approximate representation" of a Lie algebra of local currents.

Finally, we would like to mention that it is possible to develop systems of functional equations whose solutions determine representations of the canonical commutation relations, as has been done in references (19) and (22).

Appendix

In this appendix we sketch a proof of Lemma 3.2, for $g_1, g_2 \in \mathcal{S}$. We believe that Lemma 3.2 is also valid for $\mathcal{S}_V$, but it appears the proof would be still more involved. Throughout this section we let $K(g)$ denote the functional K evaluated at $g \in \mathcal{S}$ where

$$K(g) = \int (e^{ig(\underset{\sim}{x})} - 1)\, d^3x \, .$$

<u>Lemma 3.2</u>. Suppose $g_1, g_2 \in \mathcal{S}$ and $K(g_1) = K(g_2)$. Then for any two neighborhoods N_1 of g_1 and N_2 of g_2 in $\mathcal{S}$, there exist functions $h_1 \in N_1$, $h_2 \in N_2$ and a continuous mapping $t \to f_t$ of $[0,1]$ into $\mathcal{S}$, differentiable in $(0,1)$, such that $f_0 = h_1$, $f_1 = h_2$ and $K(f_t) =$ a constant.

Our sketch of a proof consists of a sequence of lemmas stated without proof.

<u>Lemma A.1</u>. Suppose $g_1, g_2 \in \mathcal{S}(\mathbb{R}^3)$ with $K(g_1) = K(g_2)$, and N_1 and N_2 are neighborhoods of g_1 and g_2 respectively, in the Schwartz space topology of $\mathcal{S}$. Then there exist functions $h_1 \in N_1$ and $h_2 \in N_2$ such that $K(h_1) = K(h_2)$ and h_1 and h_2 have compact support.

Lemma A.2. Suppose $t \to z(t)$ is a differentiable mapping of the closed interval $[0,1]$ into the left half complex z-plane. Suppose that $z(0) = z(1) = 0$ and that there is a $\delta > 0$ such that $|\mathrm{Re}\{z(t)\}| \geq \delta |\mathrm{Im}\{z(t)\}|$ for all $t \in [0,1]$. Then there is a continuous mapping $t \to k_t$ of $[0,1]$ into $\mathcal{S}$, differentiable in $(0,1)$, such that $K(k_t) = z(t)$ for all $t \in [0,1]$, and $k_0 = k_1 = 0$. Furthermore, the functions k_t may all be chosen to have support in a single compact region of $\mathbb{R}^3$.

Lemma A.3. Suppose $h_1, h_2 \in \mathcal{S}(\mathbb{R}^3)$ and $K(h_1) = K(h_2)$. Then there is a real-valued differentiable function $s(t)$ for $t \in [0,1]$ with $s(0) = s(1) = 1$ and $s(t) > 0$ for all $t \in [0,1]$, such that if $g_t(\underset{\sim}{x}) = (1-t)h_1(s(t)\underset{\sim}{x}) + th_2(s(t)\underset{\sim}{x})$ then $z(t) = K(h_1) - K(g_t)$ satisfies the hypotheses of Lemma A.2.

Proof of Lemma 3.2: Suppose $g_1, g_2 \in \mathcal{S}$ and $K(g_1) = K(g_2)$. Let N_1 and N_2 be neighborhoods of g_1 and g_2 respectively. By Lemma A.1, we can choose $h_1 \in N_1$ and $h_2 \in N_2$ with $K(h_1) = K(h_2)$ and with h_1 and h_2 of compact support. By Lemma A.3 there exists a differentiable function $s(t)$ on the interval $[0,1]$, allowing us to construct $g_t(\underset{\sim}{x}) = (1 - t)h_1(s(t)\underset{\sim}{x}) + th_2(s(t)\underset{\sim}{x})$, with $z(t) = K(h_1) - K(g_t)$

satisfying the hypotheses of Lemma A.2. Since h_1 and h_2 have compact support, the g_t all have support in some compact region S. By Lemma A.2, there is a continuous mapping $t \to k_t$ of $[0,1]$ into $\mathcal{S}(\mathbb{R}^3)$, differentiable in the open interval $(0,1)$, such that $K(k_t) = z(t) = K(h_1) - K(g_t)$.

By translating the functions k_t we can ensure that the functions k_t and g_t have disjoint supports, without changing the values of $K(k_t)$. Then let $f_t = g_t + k_t$; we have $K(f_t) = K(g_t + k_t) = K(g_t) + K(k_t) = K(h_1)$. Since $f_0 = h_1$ and $f_1 = h_2$, the lemma is proved.

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