2

BROOKHAVEN NATIONAL LABORATORY
Associated Universities, Inc.
Upton, New York

CRISP 74-15

ACCELERATOR DEPARTMENT
Informal Report

RESONANCE EXCITATION BY DISTORTION OF THE β -FUNCTION
COUPLED WITH A LONG RANGE SPACE CHARGE FORCE

M. Month

August 12, 1974

NOTICE

MASTER

This report was prepared as an account of work sponsored by the United States Government. Neither the United States nor the United States Atomic Energy Commission, nor any of their employees, nor any of their contractors, subcontractors, or their employees, makes any warranty, express or implied, or assumes any legal liability or responsibility for the accuracy, completeness or usefulness of any information, apparatus, product or process disclosed, or represents that its use would not infringe privately owned rights.

27

ABSTRACT

It is shown how a long range space charge force, such as that produced by image fields of a single beam, can combine with a lattice design having large β variations, such as occurs in machines with low β insertions, to induce characteristic resonance behavior. The significant aspect of the image force, distinguishing it from beam self forces or beam-beam forces, is its long range nature. This means that the force field does not diminish outside the beam, but extends, and increases, for amplitudes approaching the image boundary. Although the image force is highly nonlinear, it will, by itself, excite no resonances, since its azimuthal Fourier decomposition is essentially composed of 0th harmonic. It is pointed out that low periodicity, large β variations are harmonically rich and can provide the necessary azimuthal harmonics for resonance excitation. The resonance characteristics of this type of system are developed, emphasizing the influence of the two main parameters, the maximum β value and the image tune shift. The impact on storage ring design is discussed.

1. Introduction

Particle behavior in accelerators or storage rings can be characterized by betatron oscillations about a fixed equilibrium orbit.¹ Under certain circumstances, particles will exhibit resonance behavior or a growth in these betatron oscillations, which are, in general, induced by azimuthal harmonics of specific perturbing field components² (i.e. derivatives of these perturbing fields with respect to a transverse dimension, horizontal or vertical). Thus, if the perturbation has a p th derivative on the equilibrium orbit and this component has an n th azimuthal harmonic, then for the betatron tune near $\nu = n/(p+1)$, a resonance is excited with strength proportional to the n th harmonic of the perturbing field. This is an incomplete description in that particle motion in the vicinity of the resonance tune is for the case of nonlinear resonances ($p \geq 2$) significantly affected by nonlinear detuning arising from 0th azimuthal harmonics of all even ordered field components (i.e. odd p), the lowest and in many cases the most substantial component being the octupole term, corresponding to $p = 3$. Although this nonlinear detuning may dampen a

potentially explosive resonance, it cannot altogether suppress the growth characteristics. Particles whose tune is amplitude dependent can still "lock-into" a resonance and be drawn to large betatron amplitudes, although the time scale and amplitude growth are rather different from the case of explosive growth.³

There is a second aspect, in which the excitation of a resonance deviates from the simple picture of excitation through an n th azimuthal harmonic of a field component. A strict analysis demonstrates that, rather than the n th harmonic of a field component, it is the n th harmonic of a field component weighted with some power of the betatron amplitude function [i.e. $\beta^{p/2}(\theta)$].² In physical terms, it is clear that for a given perturbing nonlinear field, if particles are constrained to move at larger amplitudes, which results if the β -function is larger, then the resonance characteristics will be altered. However, in accelerators with a high periodicity, the β -function is composed primarily of a 0th azimuthal harmonic, the next contributing harmonic being related to the number of β -function oscillations per revolution. Thus, if a machine has 60 β -function periods, as in the Brookhaven AGS, then a Fourier decomposition produces the azimuthal components 0, 60, 120, With such a large separation between harmonics, it is not difficult to design a machine so that the higher harmonics of the β -function play no role in resonance excitation. The simple picture therefore holds. However, in machines designed with low symmetry⁴, as in storage rings or high energy accelerators, where a small number of specialized insertions are included in the lattice structure, this situation does not necessarily prevail and β -function variations must be dealt with.

There are 3 general features of the β -function influence on resonance excitation: (1) the β -function periodicity or lattice symmetry; (2) the β -function variation in magnitude, determining both the size of harmonic contribution and the richness in the harmonic content (i.e. the number of contributing harmonics); and (3) the extent to which the β -function harmonics and perturbing field harmonics are orthogonal, for it is the harmonic content of the appropriate product of β -function and perturbing field which actually induces the resonance.

All three of these features enter in a rather strong way in the specific case of low periodicity, high current storage rings with low β insertions.⁴ In particular, the high current provides large nonlinearities of many orders

arising from space charge fields; the large β -function required to obtain low β crossings provides the richness in azimuthal harmonic content; while the low periodicity makes the resonance tunes difficult to avoid.

Now, it is important to emphasize that we are considering here the space charge fields that arise from images resulting from the surrounding environment, in particular the electric image field from an infinitely conducting vacuum chamber.⁵ In other words, it is primarily a single beam phenomenon in the sense that the interaction of the two colliding beams does not produce the resonance excitation. Note the significant distinction. The beam-beam interaction is rich in azimuthal harmonics and rich in nonlinear field components. One does not need the β -function variation to provide the azimuthal harmonics. The beam-beam interaction, occurring essentially at one point in the azimuth provides this itself. But its most fundamental and significant characteristic is the falling off of the force field outside the beam⁶, which means that for particles moving outside the interaction domain, there will be a diminishing influence of the resonance interaction. From the classical view of the resonance phenomenon, the situation is intrinsically stable for amplitudes larger than the beam size. With this resonance mechanism it is difficult to see how beam loss results, since the loss means amplitude growth to the vacuum chamber boundary, which is many times the distance to the beam boundary. The same is true for the single beam space charge force, which also falls off outside the beam. It is, however, not the case for space charge image fields, which have a spatial extent on the order of the image boundary distance, i.e. the vacuum chamber radius. Thus, contrary to the case of beam space charge forces, image induced forces could, for particles moving within the vacuum chamber, excite classical resonance phenomena.² In general, the image forces are dominated by the 0th azimuthal harmonic. Thus, to induce nonlinear resonance behavior, it is the β -function variation which is required to produce the azimuthal harmonic content. It is this that is the basic substance of our model of resonance excitation: large β variations of low periodicity coupled with an intensity induced space charge image field.

In Section 2, we present the basic theory of resonance excitation as applied to our model. We discuss in detail the implications of the range of the nonlinear force, the image induced electric field, as well as the influence of low periodicity, large magnitude variations in the β -function. In Section 3, we consider how the model might be applied to the design of storage rings with low β insertions.

2. Model of Resonance Excitation

Space Charge Image Force

To be specific, we consider a parallel plate geometry. We restrict ourselves to an infinitely conducting metallic boundary, causing an image component of the space charge electric field. We do not include image contributions to the magnetic field, although we admit that they are not a priori negligible. It is presumed that such magnetic forces will not significantly affect the conclusions arising out of our resonance model. For beams not close to the boundary, the image fields for the assumed parallel plate geometry are somewhat insensitive to the transverse density distribution of the beam. We can therefore approximate the beam by an "infinitesimal wire". This simplification would not be possible with a circular geometry where the image field would vanish for an "infinitesimal wire" at the center. In this case, the image field is only non-zero for a beam displaced from the center (even if it is infinitesimal in extent) or for a centered beam with finite size. In the latter instance, the image field is, of course, sensitive to the transverse density distribution. To elucidate the principles implicit in the model proposed here, we will consider the simplified example of a symmetrically placed beam (with respect to the image boundary) of infinitesimal transverse size. In this limiting case, the circular geometry leads to no effect and we are left with the parallel plate geometry.

It can be shown⁵ that for an infinitesimal wire beam symmetrically placed between two infinitely conducting parallel plates placed in the horizontal-longitudinal plane a distance $2h$ apart vertically, the vertical force at the horizontal position of the beam and within the plates, written as F_y , is given by,

$$F_y = \left(\frac{e\lambda}{4\pi\epsilon_0} \right) \left(\frac{1}{\sin(\pi y/2h)} - \frac{2h}{\pi y} \right), \quad (2.1)$$

where y is the vertical coordinate with respect to the beam position at the center of the two plates,

h is the half-distance between the plates,

$\lambda = eN/C$ is the average linear charge density along the beam axis,

N is the total number of particles in the beam,

$C = 2\pi R$ is the ring circumference, R is the average radius,

and ϵ_0 is the free space dielectric constant.

One Dimensional Equation of Motion

To obtain the one dimensional vertical equation of motion for a particle in the presence of the image force, we simply include that force, given in (2.1), in the equation for vertical betatron motion characterized by the lattice structure for the ring. Thus, we have¹

$$y'' + K(s)y = F_y / m\gamma c^2, \quad (2.2)$$

where $K(s)$ is the gradient forcing function for the lattice,

m is the particle rest mass,

γ is the total energy of the particle in units of its rest mass,

we have taken the particle velocity to be close to c , the velocity of light,

s is the distance measured along the lattice equilibrium orbit from some reference position,

and y is the vertical particle displacement from the equilibrium orbit.

To describe the unperturbed motion, we introduce¹ an "amplitude function", $\beta(s)$. Introducing a tune,

$$\nu = \frac{1}{2\pi} \int_0^C \frac{ds}{\beta(s)}, \quad (2.3)$$

a phase for the independent variable,

$$\theta(s) = \frac{1}{\nu} \int_0^s \frac{ds}{\beta(s)}, \quad (2.4)$$

and a new displacement variable

$$t = y / h\nu^{\frac{1}{2}}, \quad (2.5)$$

where we have defined

$$\omega(s) = \beta(s) / \beta_{av}, \quad (2.6)$$

with

$$\beta_{av} = R / \nu, \quad (2.7)$$

then the Eq. (2.2), using (2.1) for the force, F_y , transforms to

$$\ddot{t} + \nu^2 t = - 2\nu \Delta\nu_{IM} \omega^2 t H(\omega^{\frac{1}{2}} t). \quad (2.8)$$

Here, $\Delta\nu_{IM}$ is just the tune shift caused by the image space charge force⁷,

$$\Delta\nu_{IM} = - \frac{\pi N r_0 R}{48 \gamma v h^2}, \quad (2.9)$$

where r_0 is the classical radius of the particle ($= e^2/4\pi\epsilon_0 mc^2$); H is given by

$$H(\omega^{\frac{1}{2}}t) = \frac{12}{\pi\omega^{\frac{1}{2}}t} \left[\frac{1}{\sin \frac{\pi}{2} \omega^{\frac{1}{2}}t} - \frac{2}{\pi\omega^{\frac{1}{2}}t} \right]; \quad (2.10)$$

differentiation is with respect to the betatron phase angle θ , which is similar but not identical to the azimuth, θ_z ; and ω is considered as a function of θ , related to s by (2.4).

Fourier Decomposition of $\beta(\theta)$

In an alternating gradient structure with many cells, the amplitude function modulates with a high periodicity. The harmonic structure is therefore widely spaced. In looking for resonance effects, this rapid modulation is of little consequence and the β -function in these parts of the azimuth can be replaced by its average value, given in (2.7). In the insertions, however, the variation of $\beta(\theta)$ is of a much larger magnitude. In particular, in low β insertions for storage rings⁴, β may reach values many times larger than β_{av} . However, these regions where β rises to large values occur in only short azimuthal extents. We therefore can approximate this effect with a δ -function in azimuth.

We will require various powers of the β -function. With the above discussion in mind, we write for the m th power of $\omega(\theta) = B(\theta)/\beta_{av}$, in the case of M identical large fluctuations,

$$\omega^m(\theta) \simeq 1 + \frac{2\pi\Gamma}{M} \sum_{i=1}^M \delta(\theta - \theta_i) \quad (2.11)$$

We obtain Γ_m by integrating over θ , resulting in

$$\Gamma_m \simeq \frac{2M}{C} \int_0^{\Delta/2} [\omega(s)]^{m-1} ds, \quad (2.12)$$

where Δ is the total length over which the β -function modulation extends: $\Delta/C \ll 1$.

Notice that the power of ω is reduced by one. This arises from the fact that in terms of the betatron phase, which is the relevant "time" variable, the effective "length" over which the β -function rises and falls shrinks. This contraction in the effective azimuth results in a diminution of the factor Γ_m , i.e. the strength of resonance excitation, by one power of

$$\omega_{\max} = \beta_{\max} / \beta_{av} . \quad (2.13)$$

In fact, it is clear that for $\omega_{\max} \gg 1$, we will have an estimate for Γ_m of the order of $\Gamma_m \sim \frac{M\Delta}{C} \frac{(\omega_{\max})^{m-1}}{m}$, roughly independent of the details of β -function shape. For $\omega(s)$ linear in s , Γ_m approaches precisely this value, while for a quadratic dependence on s , we have

$$\Gamma_m \sim \frac{M\Delta}{C} \frac{\omega_{\max}^{m-1}}{2m-1} . \quad (2.14)$$

We will use this latter value as an approximation to the actual value of Γ_m , which can be obtained from the complete expression, in (2.12), for any given s dependence of $\omega(s)$. A fourier expansion of (2.11) in terms of the "effective azimuthal variable", θ , results in

$$\omega^m(\theta) = 1 + \Gamma_m + 2\Gamma_m \sum_{\ell=1}^{\infty} \cos \ell M \theta, \quad (2.15)$$

where we have chosen the coordinate zero such that

$$\theta_i = \frac{2\pi(i-1)}{M} . \quad (2.16)$$

For $M\Gamma_m \gg 1$, (2.15) can be written

$$\omega^m(\theta) = \Gamma_m u(\theta) , \quad (2.17)$$

with

$$u(\theta) = 1 + 2 \sum_{\ell=1}^{\infty} \cos \ell M \theta . \quad (2.18)$$

Resonance Equations and Invariant

To obtain the equations of motion under resonance conditions as well as the resonance invariant, we transform to amplitude and phase variables, I and ϕ respectively, related to t and $\dot{t}$ by

$$\begin{aligned} t &= \sqrt{I} \cos \phi , \\ \dot{t} &= -\omega \sqrt{I} \sin \phi . \end{aligned} \quad (2.19)$$

The resulting equations for I and ϕ are

$$\dot{I} = \frac{2\dot{t}}{v} (\ddot{t} + v^2 t) \quad , \quad (2.20)$$

and

$$\dot{\phi} = v - \frac{\cos \phi}{w/I} (\ddot{t} + v^2 t) \quad ;$$

or, using (2.8) and (2.10),

$$\dot{I} = 2I \Delta v_{IM} \omega^2 \sin 2\phi H(\omega^{\frac{1}{2}} t) \quad (2.21)$$

$$\dot{\phi} = v + 2 \Delta v_{IM} \omega^2 \cos^2 \phi H(\omega^{\frac{1}{2}} t) \quad . \quad (2.22)$$

Now we must find the Taylor series for H .

Writing $z = \frac{\pi}{2} \omega^{\frac{1}{2}} t$, we have

$$H = \frac{6}{z} \left[\frac{1}{\sin z} - \frac{1}{z} \right] \quad , \quad (2.23)$$

which, when expanded in powers of z , yields

$$H = 24 \sum_{n=1}^{\infty} \frac{(2^{2n-1} - 1)}{(2\pi)^{2n}} \left(\sum_{k=1}^{\infty} \frac{1}{k^{2n}} \right) z^{2n-2} \quad . \quad (2.24)$$

The reciprocal power sum is the Riemann zeta function

$$\zeta(2n) = \sum_{k=1}^{\infty} \frac{1}{k^{2n}} \quad . \quad (2.25)$$

In particular, $\zeta(2) = \pi^2/6$, $\zeta(4) = \pi^4/90$ and $\zeta(2n) \rightarrow 1$ rapidly as n increases.

In fact, with only a small error, we may replace the reciprocal power sum by 1 except for the first term. In fact, we can even make a further simplification by recognizing that

$$\frac{2^{2n-1} - 1}{2^{2n-1}} \zeta(2n) \approx 1 \text{ for } n \geq 2 \quad .$$

Making this replacement in (2.24), except for the first term in the sum, and using (2.19), we have

$$H = 1 + \frac{12}{\pi} \sum_{n=2}^{\infty} \omega^{n-1} \left(\frac{1}{4}\right)^{n-1} \cos^{2n-2} \phi . \quad (2.26)$$

Substituting (2.26) into (2.21) and (2.22) and using (2.17), the phase and amplitude equations become

$$\dot{\phi} = v + 2\Delta v_{IM} u(\theta) \left\{ \Gamma_2 \cos^2 \phi + \frac{12}{\pi} \sum_{n=2}^{\infty} \Gamma_{n+1} \left(\frac{1}{4}\right)^{n-1} \cos^{2n} \phi \right\} , \quad (2.27)$$

$$\dot{I} = 2I \Delta v_{IM} u(\theta) \left\{ \Gamma_2 \sin^2 \phi + \frac{12}{\pi} \sum_{n=2}^{\infty} \Gamma_{n+1} \left(\frac{1}{4}\right)^{n-1} \cos^{2n-2} \phi \sin^2 \phi \right\} . \quad (2.28)$$

Defining a new strength parameter

$$\Gamma_{IM} = \Delta v_{IM} \Gamma_2 , \quad (2.29)$$

noting from (2.14) that

$$\frac{\Gamma_{n+1}}{\Gamma_2} = \frac{3}{2n+1} \omega_{\max}^{n-1} , \quad (2.30)$$

and introducing a new amplitude variable,

$$\alpha = \frac{1}{4} \omega_{\max} I , \quad (2.31)$$

we can write the phase and amplitude equations as

$$\dot{\phi} = v + 2\Gamma_{IM} u(\theta) \left\{ \cos^2 \phi + \frac{36}{\pi} \sum_{n=2}^{\infty} \frac{1}{2n+1} \alpha^{n-1} \cos^{2n} \phi \right\} , \quad (2.32)$$

$$\dot{\alpha} = 2\alpha \Gamma_{IM} u(\theta) \left\{ \sin^2 \phi + \frac{36}{\pi} \sum_{n=2}^{\infty} \frac{1}{2n+1} \alpha^{n-1} \cos^{2n-2} \phi \sin^2 \phi \right\} . \quad (2.33)$$

The detuning term is obtained by averaging over phase and azimuth, while the resonant excitation term is obtained by neglecting rapidly oscillating terms. If we introduce the slowly varying phase variable

$$\psi = \phi - \frac{\omega_M}{p} \theta , \quad (2.34)$$

where p is an integer such that

$$\delta = \nu + \Gamma_{IM} - \frac{\omega_M}{p} \text{ is small,} \quad (2.35)$$

then we can write the resonance equations,

$$\dot{\Psi} = \delta + \Gamma_{IM} [D(\alpha) + \cos p\Psi R_p(\alpha)] \quad , \quad (2.36)$$

and

$$\dot{\alpha} = \alpha \Gamma_{IM} \sin p\Psi R_a(\alpha) \quad , \quad (2.37)$$

where the detuning function, $D(\alpha)$, is

$$D(\alpha) = \frac{72}{\pi^2} \sum_{n=2}^{\infty} \frac{\alpha^{n-1}}{2n+1} \frac{1}{2\pi} \int_0^{2\pi} \cos^{2n} \phi d\phi \quad , \quad (2.38)$$

and the resonance functions, R_p and R_a , are

$$R_p(\alpha) = \frac{72}{\pi^2} \sum_{n=2}^{\infty} \frac{\alpha^{n-1}}{2n+1} \frac{1}{\pi} \int_0^{2\pi} d\phi \cos p\phi \cos^{2n} \phi \quad , \quad (2.39)$$

and

$$R_a(\alpha) = \frac{72}{\pi^2} \sum_{n=2}^{\infty} \frac{\alpha^{n-1}}{2n+1} \frac{1}{\pi} \int_0^{2\pi} d\phi \sin p\phi \cos^{2n-2} \phi \sin 2\phi \quad . \quad (2.40)$$

It is clear that R_p and R_a are non-zero only if p is even, and furthermore, the non-zero terms in the sum are those for which $n \geq p/2$. Using the integrals

$$\frac{1}{2\pi} \int_0^{2\pi} \cos^{2n} \phi d\phi = \frac{1 \cdot 3 \cdot 5 \cdot \dots \cdot (2n-1)}{2 \cdot 4 \cdot 6 \cdot \dots \cdot 2n} \quad , \quad (2.41)$$

and

$$\frac{1}{2\pi} \int_0^{2\pi} d\phi \cos p\phi \cos^{2n} \phi = \begin{cases} 0 & \text{if } p \text{ odd or } n < p/2 \\ \frac{1}{2^{2n}} {}^{2n}C_{n-p/2} & \text{if } p \text{ even and } n \geq p/2, \end{cases} \quad (2.42)$$

where the quantity rC_q is the usual binomial coefficient,

$$r_c^q = \frac{r(r-1) \dots (r-q+1)}{q!} ,$$

$$r_c^0 = 1 ,$$

we can evaluate the detuning and resonance functions in (2.38), (2.39) and (2.40).

However, for our purposes here, this would be an unnecessary complication. The important point is that these functions do not manifest a small amplitude leveling off, characteristic of direct space charge forces. This type of behavior would be reflected by an oscillating series (i.e. "small" α cancellation), which is evidently not the case here, as can be seen from the fact that the terms of the series which represent $D(\alpha)$ and $R(\alpha)$ are all positive. This is simply a consequence of the nature of the image force, (2.1), which is an increasing function of amplitude within the aperture of the parallel plates. Recognizing this, we here simplify the problem by taking only the lowest terms in α . From (2.38), (2.39) and (2.40) we thus obtain,

$$D(\alpha) = \frac{27}{5\pi^2} \alpha + O(\alpha^2) , \quad (2.43)$$

$$R_p(\alpha) = \frac{144}{2^p \pi^2 (p+1)} \alpha^{\frac{(p-1)}{2}} + O(\alpha^{p/2}) , \quad (2.44)$$

and

$$R_a(\alpha) = \frac{288}{2^p \pi^2 (p+1)} \alpha^{\frac{(p-1)}{2}} + O(\alpha^{p/2}) . \quad (2.45)$$

To obtain resonance widths, we use the standard emittance parameter, ϵ , which is related to α by

$$\alpha = \frac{1}{4} \frac{\beta_{\max}}{h} \epsilon . \quad (2.46)$$

Defining γ by

$$\gamma = \epsilon / \epsilon_0 , \quad (2.47)$$

where ϵ_0 is a reference emittance, for example, the beam emittance, we have that α and γ are related by

$$\alpha = \frac{1}{4} \frac{\beta_{\max} \epsilon_0}{h^2} \gamma . \quad (2.48)$$

We thus have for the resonance equations, from (2.36) and (2.37),

$$\dot{\Psi} = \delta + \Delta_{NL} \gamma + \Delta_e \cos p\Psi \gamma^{(p/2-1)} , \quad (2.49)$$

$$\dot{\gamma} = 2 \Delta_e \sin p\Psi \gamma^{p/2} , \quad (2.50)$$

where

$$\Delta_{NL} = \frac{27}{20\pi^2} \Gamma_{IM} \frac{\beta_{\max}^2 \epsilon_0}{h^2} , \quad (2.51)$$

and

$$\Delta_e = \frac{576}{4\pi^2(p+1)} \Gamma_{IM} \left(\frac{\beta_{\max}^2 \epsilon_0}{h^2} \right)^{\frac{p}{2}-1} , \quad (2.52)$$

with Γ_{IM} given by (2.29),

$$\Gamma_{IM} = \frac{M\Delta}{3C} \left(\frac{\beta_{\max}}{\beta_{av}} \right) \Delta v_{IM} . \quad (2.53)$$

The resonance invariant, H , satisfying

$$\dot{\Psi} = \frac{\partial H}{\partial \gamma} , \quad \dot{\gamma} = - \frac{\partial H}{\partial \Psi} ,$$

is easily seen to be

$$H = \delta \gamma + \frac{1}{2} \Delta_{NL} \gamma^2 + \frac{2}{p} \Delta_e \gamma^{p/2} \cos p\Psi . \quad (2.54)$$

Beam Behavior in the Presence of a Resonance

Nonlinear resonances become manifest in two essential ways.

Firstly, in the case where there is large resonance excitation, relative to the nonlinear detuning, the result is explosive growth at some amplitude, which is referred to as the stability limit. From (2.54), we can estimate the stability limit for a p th order resonance, with $p > 4$:

$$\gamma_{S.L.} \approx \left| \frac{\Delta_{NL}}{\Delta_e} \right|^{2/(p-4)} . \quad (2.55)$$

If the stability limit is within the aperture and particles reach this amplitude, we expect that they will be quickly lost to the chamber wall. In this sense, the stability limit represents the amplitude for a particle sink. The

stability limit is, of course, nothing more than position of unstable fixed points in the betatron phase plane--specifically, the amplitude of these unstable fixed points. A design allowing the presence of such unstable fixed points within the aperture is clearly untenable. Thus, in the model case we describe here, we have a limiting condition for the resonance excitation parameters. If we assume that a design criterion should be that the stability limit be q times the beam size, then we can write the design criterion

$$\gamma_{S.L.}^{\frac{1}{2}} \approx \left| \frac{\Delta_{NL}}{\Delta_e} \right|^{[1/(p-4)]} > q ,$$

or

$$\left| \frac{\Delta_{NL}}{\Delta_e} \right| > q^{(p-4)} . \quad (2.56)$$

Using (2.51) and (2.52), (2.56) can be written as a limit on β_{max} :

$$\beta_{max} < \left[\frac{3(p+1)}{5} \right]^{2/(p-4)} \frac{16h^2}{\epsilon_0 q^2} . \quad (2.57)$$

It should be recognized that the unstable fixed points can indeed move closer than the value given in (2.55). As the tune changes, $\gamma_{S.L.}$ may in fact decrease by a factor of about 2 to 3, thus decreasing the limiting value of β_{max} in (2.57) by this amount. However, in the spirit of our approach, it is unnecessary to introduce these complications. It should also be noted that to this order of approximation, the condition is independent of the image tune shift, $\Delta\nu_{IM}$. This cancellation is reasonable in that the distance of these unstable fixed from the beam depends on two competing sources, the resonance excitation tending to bring them closer to the beam and the nonlinear detuning tending to move them away, with each of these effects depending equally on the magnitude of the image tune shift.

A second manner in which nonlinear resonances may become manifest is through the "lock-in" process. Under dynamic conditions, where the linear tune, δ , is changing with time, the particle linear tune may combine with the nonlinear tune such that the particle remains in the resonant state. Thus, we have "lock-in". Looked at another way, particles can be trapped in islands moving away from the beam and be drawn to large amplitudes. This type of behavior tends

to be characteristic of nonlinear systems where the resonance excitation width and the nonlinear detuning are both large. Normally, these systems are referred to as "slow crossing" systems, i.e. where the crossing parameter³

$$u = \frac{\delta_{\text{rev}}}{4\pi|\Delta_e\Delta_{\text{NL}}|} < 1, \quad (2.58)$$

with δ_{rev} the rate of change of tune, δ , per revolution. The two quantities defining the behavior of nonlinearities under these circumstances are (1) the trapping efficiency of the "lock-in" process, i.e. the probability of a particle being trapped in one crossing; and (2) the amplitude to which the particles are drawn. In other words, for particle loss, the particle must be trapped, and further it must be transported to a sink, whether the sink is the chamber wall, or some other absorbing boundary, such as the unstable fixed points described above. It has been demonstrated that the trapping efficiency, P_T , can be estimated from³

$$P_T = \frac{\sqrt{2}\pi}{\sqrt{P}} \left| \frac{\Delta_e}{\Delta_{\text{NL}}} \right|^{\frac{1}{2}} \exp(-u^{2/P}); \quad (2.59)$$

while the amplitude to which trapped particles are drawn is approximately given by³

$$v_{\text{TR}} = \frac{\delta_L}{\Delta_{\text{NL}}}, \quad (2.60)$$

where δ_L is the tune shift from the resonance value, $\delta = 0$, as the resonance is crossed. From these equations, (2.59) and (2.60), we can point out the gross dependence on the image tune shift, Δv_{IM} . As Δv_{IM} is increased, the trapping efficiency will go up, as can be seen from (2.59). But this increase is correlated only with the fact that the "crossing" parameter is reduced. The limiting value of P_T , when $u \rightarrow 0$ is independent of Δv_{IM} . This arises from the existence of two competing trapping sources. In the slow crossing limit, the resonance excitation heightens trapping, but the nonlinear detuning tends to prevent it. On the other hand, as Δv_{IM} increases, the transport to large amplitudes diminishes unless the linear tune shift is increased proportionately. [See (2.60)]. Thus, increasing Δv_{IM} increases the trapping efficiency but decreases the amplitude to which trapped particles will be transported.

In general, from a knowledge of the rate of change of tune shift per revolution and the total shifts resulting, we can, for any given low β design, predict the trapping and transport and thus the evolution of the beam size, and in particular the loss rate. In our model, i.e. for chambers of width greater than height, we can use (2.51), (2.52), and (2.53) and express these characteristics in terms of the specific design parameters, principally Δv_{IM} and β_{max} .

3. Applications and Conclusions

We have shown how a long range space charge force, such as that produced by image fields of a single beam, can combine with a lattice design having large β variations, such as occurs in machines with low β insertions, to induce characteristic resonance behavior. The significant aspect of the image force, distinguishing it from beam self forces or beam-beam forces, is its long-range nature. This means that the force field does not diminish outside the beam, but extends and increases for amplitudes approaching the image boundary. Although the image force is highly nonlinear, it will, by itself, excite no resonances since its azimuthal Fourier decomposition is essentially composed of 0th harmonic. It is pointed out that low periodicity, large β variations are harmonically rich and can provide the necessary azimuthal harmonics for resonance excitation. We have developed the resonance characteristics of this type of system. In particular, we have emphasized the influence of the two main parameters, β_{max} and Δv_{IM} , although 3 other parameters, the chamber radius, h , the beam emittance, ϵ_o , and the azimuthal extent of the β variation, $M\Delta$, enter in the detailed description of the resonance behavior.

In terms of storage ring design, our resonance analysis leads to four limiting criteria. First, we have a limit on how close the unstable resonance amplitude will be allowed to encroach, and this has led to an upper limit on β_{max} , given in (2.57). Writing the normalized emittance

$$E_o = \beta \gamma \epsilon_o, \quad (3.1)$$

and using $\beta \approx 1$, we can rewrite (2.57), showing explicitly the energy dependence:

$$\beta_{max} < \left[\frac{3(p+1)}{5} \right]^{2/(p-4)} \frac{16h^2 \gamma}{E_o q^2}. \quad (3.2)$$

Note that this criterion on $\beta_{\max}$ is a function only of the basic ring parameters, energy, emittance (normalized), and chamber radius. A similar constraint can be obtained from (2.59). Assuming slow tune changes, i.e. δ_{rev} small, a limit on the ratio $\Delta_e/\Delta_{\text{NL}}$ again results. It is different from the previous limit in that the former is a limit for explosive resonance growth. Here we are concerned with small losses by "lock-in" or "trapping" over longer time periods, i.e. slow tune change rates. In practice this would probably lead to a more stringent criterion on $\beta_{\max}$ than (3.2). If τ_{TR} is the maximum allowable trapping efficiency, then (2.59) leads to

$$\frac{\sqrt{2}}{\sqrt{p}} \pi \left| \frac{\Delta_e}{\Delta_{\text{NL}}} \right|^{\frac{1}{2}} < \tau_{\text{TR}} \quad , \quad (3.3)$$

or,

$$\beta_{\max} < \left(\frac{3p(p+1)\tau_{\text{TR}}^2}{10\pi^2} \right)^{2/(p-4)} \frac{16h^2\gamma}{E_o} \quad , \quad (3.4)$$

again dependent only on h , γ , and E_o .

A third condition can be obtained from (2.59). To ease the requirement on $\beta_{\max}$, (3.4), we could insist on sufficiently fast crossing so that the damping exponential reduces the trapping efficiency. If e_{TR} is the minimum magnitude of the exponent desired, the resulting constraint

$$u^{2/p} > e_{\text{TR}} \quad (3.5)$$

can be written

$$\delta_{\text{rev}} > \frac{(27)(64)}{5\pi^3(p+1)} \left(\frac{M\Delta}{C} \frac{\beta_{\max}}{\beta_{\text{av}}} \Delta v_{\text{IM}} \right)^2 \left(\frac{\beta_{\max} E_o}{16h^2\gamma} \right)^{p/2} \quad . \quad (3.6)$$

Of course, δ_{rev} may not be freely chosen, in which case (3.6) then becomes a further limit on $\beta_{\max}$ or an upper limit on the strength of the image tune shift.

The final condition we discuss is related to (2.60). This simply says that if total linear tune shift is small compared to the nonlinear detuning, then, although trapping could occur, the transport will be to an amplitude determined by this small ratio. As it stands, (2.60) can be viewed as an upper limit on the linear tune shift, i.e. $\gamma_{\text{TR}} < t_{\text{TR}} < 1$. However, it is interesting that the larger the interaction strength, i.e. $\beta_{\max}$ and Δv_{IM} , the larger

Δ_{NL} and the more easily is such a constraint satisfied. Specifically, the condition

$$\gamma_{TR} < t_{TR} \quad , \quad (3.7)$$

as an upper limit on δ_L , is

$$\delta_L < \frac{9}{20\pi^2} \left(\frac{\beta_{\max}}{h^2} \frac{E_0}{\gamma} \right) \left(\frac{M\Delta}{C} \right) \left(\frac{\beta_{\max}}{\beta_{av}} \right) \Delta v_{IM} \quad . \quad (3.8)$$

It is important to recognize that in any particular situation, other phenomena, such as the beam-beam interaction, could contribute significantly to Δ_{NL} and Δ_e , which must then be modified. This would of course change the criteria given here. In this sense each design or operating ring must be treated separately.

As an extreme example, consider the $\beta_{\max}$ limit, (3.2), arising from stability limit (2.56). Taking a 12th order resonance, $p = 12$, $h = 2$ cm, and $E_0 = 30 \times 10^{-6}$ rad-m, we obtain, in meters,

$$\beta_{\max} < 356.5 \frac{\gamma}{q} \quad .$$

With $\gamma = 26$, $q = 10$, then we get $\beta_{\max} < 92.7$, a severe limit indeed. Of course, it is not necessary to choose such a small vacuum chamber, and it is not necessary to require the explosive amplitude to be at a radius in the phase plane 10 times the beam size. However, this computation does indicate the sensitivity of the $\beta_{\max}$ limit to the basic ring design concepts.

References

1. E.D. Courant and H.S. Snyder, Ann. of Phys. 3, 1 (1958).
2. See, for example, A. Schoch, CERN Report, CERN 57-23 (1958).
3. See, for example, A.W. Chao and M. Month, BNL Informal Report, CRISP 74-9 (1974). This paper, in a shortened version will be published in Nucl. Instrum. Methods, 1974.
4. See, for example, "A Proposal for Construction of a Proton-Proton Storage Accelerator Facility--ISABELLE", BNL Report BNL 18891, May 1974.
5. M. Month and R.L. Gluckstern, to be published in Particle Accelerators, 1974.

6. See, for example, J. LeDuff, R.T. 6-72, Anneaux de Collisions, Laboratoire de l'Accélérateur Linéaire, Orsay; or, A. Jejcic and J. LeDuff, SLAC TRANS-133; or J.E. Augustin, SLAC TRANS-116 and 117; or A.C. Ruggiero and L. Smith, in Proceedings of 1973 PEP Summer Study (1973).
7. L.J. Laslett, Proceedings of the 1963 Summer Study on Storage Rings, Accelerators and Experimentation at Super-High Energies" (J.W. Bittner, ed.), BNL 7534, p.324 (1963).

MM/lsk