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GUIDING-CENTER SIMULATION OF TOROIDAL PLASMAS

Thomas H. Johnson

December 18, 1974

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(Ph.D. Thesis)

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"You don't know what you're talking about!" cried Humpty Dumpty. "How many days are there in a year?"

"Three hundred and sixty-five," said Alice.

"And how many birthdays have you?"

"One."

"And if you take one from three hundred and sixty-five what remains?"

"Three hundred and sixty-four, of course."

Humpty Dumpty looked doubtful. "I'd rather see that done on paper," he said.

GUIDING-CENTER SIMULATION OF TOROIDAL PLASMAS

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ABSTRACT

A new computer simulation method for axisymmetric toroidal plasmas is developed. The method comprises two separate physical models representing different components of an experimental plasma. The first model is an MHD fluid equilibrium representation; the technique devised for the solution of this equilibrium is several times faster than any currently in common use. The second model is a macroparticle representation which uses guiding-center formalism for the particles' velocities and positions; consequently, the timestep used by this model may be many times longer than those possible for conventional macroparticle simulations. A new computer code, GUIDON, which incorporates these models with solution for self-consistent magnetic fields, is presented. Special features of the guiding-center model are discussed in detail.

Results of simulations using GUIDON are also presented. The major effort involves design simulation of the proposed Two-Component Torus (TCT) prototype reactor. Further results describing the SPIRE toroidal electron pinch accelerator, the Ware drift in Tokamaks, and a Physics International Corp. electron beam drift torus are presented.

I. INTRODUCTION

1.1 Motivation

Most numerical models used in the study of high-temperature plasmas on digital computers have heretofore fallen into two distinct classes: fluid and particle simulation models. Fluid models deal with sets of equations, such as the magnetohydrodynamic (MHD) equations, defining the macroscopic thermodynamic properties of the plasma--pressure, temperature, density--as functions of time and space, including external forces and (usually) the evolution of magnetic fields affected by the plasma motion. Particle simulation models^{*} attempt to represent the microscopic processes of the plasma more accurately, by assigning a representative charge and mass to each of the large number of points (or areas) within a computational grid, then allowing each to evolve in time and space according to proper (Newtonian or relativistic) equations of motion; the state of this collection of representative particles at any time thus simulates in some detail the plasma's state at a corresponding time. Usually at least some components of electromagnetic fields are computed in self-consistent fashion for this motion.

Fluid models have the advantages of following computation of

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^{*}Often these models are referred to simply as "simulation models", so we have been careful to make a distinction here: in fact, the term simulation should pertain to both methods when applied to time-dependent problems on a computer.

problems involving large and complex geometries, of producing results which may easily be correlated with experimental data, and of simulating well phenomena which occur over relatively large time and length scales (many configuration-space instabilities fall into this category). Particle simulation permits the study of extremely fast processes (such as many velocity-space instabilities), the highly specific manipulation of and understanding of microscopic physical parameters (increasingly important as the plasma distribution becomes locally more non-LTE), and the investigation of complex interactive processes on very short length scales.

Unfortunately, these advantages are to a large extent mutually exclusive, and an advantage of one method appears as a lack, and hence a disadvantage, in the other. In the work to be described here, an attempt has been made to create a model (and the means of employing it) which combines certain of the advantages of both existing classes. Specifically, the method sought would be able to deal with problems both large and complex in configuration space and occupying times at least in the intermediate range (many hundreds of gyro-periods, if not many milliseconds) while at the same time permitting some detailed study of microscopic motions of the plasma and local development of its dominant self-consistent fields.

The design of this scheme and the code, which is called GUIDON, which employs it, was motivated by the desire to perform detailed numerical analyses of the proposed SPIRE machine (see Section 4.1 for details and results). As work progressed, the

advent of a large-scale theoretical effort to study the proposed Two-Component Torus (TCT) made it clear that GUIDON could most usefully be applied simulating the TCT, and some changes in the code's design were effected; the principal results presented here are thus from TCT simulations (Chapter III). Several other versions of the code were applied in less ambitious simulations of other devices; these results are collected in Chapter IV.

GUIDON itself (and its modifications) is described in detail in Chapter II, following some discussion of background physics and general remarks on the scope of the problem. Chapter V contains interpretations of the validity of results and reconsiderations of the entire methodology.

Gaussian cgs units are employed throughout, except where noted. Magnitudes of vectors are written simply as the same quantity without the vectorial symbol, thus: $E = |\underline{B}|$.

1.2 Background

GUIDON is designed to simulate conditions which can be generally described as having axisymmetric toroidal geometry: this includes devices such as TOKAMAKs, toroidal cusps and most electron ring accelerators and toroidal pinches; it excludes linear devices, toroidal geometries (such as STELLARATOR) which are not axisymmetric, and more complex 3-dimensional geometries (most mirror machines, for instance).

1.2.1 Geometry

Figures 1a,b show the essentials of this geometry as exemplified by a TOKAMAK. The (R, θ, Z) coordinates used throughout this

dissertation are shown in Fig. 1b. It can easily be demonstrated¹ that no plasma confinement can result from fields which are purely in the ϕ (toroidal) direction or from those arising from a toroidal plasma current. Confinement of the plasma within the torus is attained by a combination of two magnetic fields. First, a toroidal field is generated by a line current running vertically through the center of the torus, shown as I_z in Fig. 1a (this field is thus purely in the ϕ direction and falls off as $1/R$). Second, a field is generated by the toroidal plasma current (indicated by the large arrow). This field lies entirely in the plane of the torus's cross section and is known as the poloidal field (all phenomena, or components of phenomena, limited to this plane are described as poloidal); it is labelled B_p in Fig. 1a; the resultant of these two fields is shown simply as B .

This resultant field twists around the torus in a helical path which never closes upon itself. The angle in the poloidal plane passed through by a particular field line in circumventing the torus once is called the rotational transform. The purpose of shaping the field lines in this manner is to destroy the high degree of symmetry which either toroidal or poloidal fields alone would have; these symmetries permit the growth of a family of hydromagnetic instabilities.² The rotational transform must therefore be less than 2π , since in making one complete poloidal turn per toroidal turn the fields would once again close upon themselves, restoring damaging symmetry. It is a bit disheartening to discover that symmetries, which one is accustomed to think-

FIGURE 1a: Toroidal, Poloidal and Total Fields in a Tokamak

FIGURE 1b: Cylindrical Geometry and Illustrative Flux Surfaces

ing of as beneficial both for analyses of and for actual construction of experiments, can actually be harmful. We should recall, however, that what we are attempting to maintain is essentially an energetically unnatural state, and a collisionless plasma is capable of making--and certain to make--a great many adiabatic transitions. Symmetries multiply the states available to such transitions and hence are dangerous, even though some configurational symmetry is required.

Since stronger plasma currents cause stronger poloidal fields and hence raise the value of the rotational transform, the maximum of 2π can be seen as setting a limiting value on the toroidal current, called the Kruskal-Shafranov limit.³ This same limit can also be expressed geometrically as

$$q \equiv \frac{r_0 B_\theta}{R_0 B_p} \geq 1 \quad (I.1)$$

where R_0 and r_0 are the major and minor radii, respectively, as shown in Fig. 1b.

The $1/R$ dependence of the toroidal field causes the field lines of the total field to have varying pitch angles as a function of radius; in other words, the line-of-force vectors at different radii are not quite parallel. This phenomenon is known as "magnetic shear," and contributes to the damping of hydromagnetic instabilities.⁴

1.2.2 Particle Motion

It is assumed that the reader is familiar with the guiding-center formulation of the motion of charged particles in electric

and magnetic fields.⁵ The essential facts are these: the motion of a particle can be decoupled into several components. The first is a circular motion about an imaginary point, the guiding center, perpendicular to $\underline{B}$ and at an angular frequency

$$\omega_G = \frac{eB}{mc} \quad (I.2)$$

called the gyro-frequency (or cyclotron frequency). The second is a motion of the guiding center parallel to the magnetic field, uniformly accelerated by any component of $\underline{E}$ parallel to $\underline{B}$. The third is a motion of the guiding center not parallel to $\underline{B}$, due to the superposition of a number of drift velocities arising from other components of $\underline{E}$ (and perpendicular to $\underline{E}$ and $\underline{B}$), from spatial gradients of $\underline{B}$ in both poloidal and toroidal directions, and from the curvature of the field lines (centrifugal forces on the particles). The formulation of these drifts actually used in GUIDON is derived in Appendix C.

Because of the overall symmetry in the ϕ -direction, that component of the canonical angular momentum,

$$P_\phi \equiv mrv_\phi + \frac{e}{c} rA_\phi, \quad (I.3)$$

is a conserved quantity (A_ϕ is the component of vector potential; note that rA_ϕ is the poloidal magnetic flux, Φ_p). Two other quantities defined by the motion are conserved to whatever extent the plasma motion is fully adiabatic,* and hence are known as

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*For practical purposes, when the plasma is not radiating strongly it is generally sufficient to require that B does not change significantly during one gyro-period.

"adiabatic invariants." The first is the magnetic moment μ , defined as

$$\mu = \frac{W_{\perp}}{B} = \frac{1/2 m v_{\perp}^2}{B}, \quad (I.4a)$$

where $W_{\perp}$ is the particle's energy perpendicular to the magnetic field (and so $v_{\perp} = v_G$, the gyro velocity). Conservation of μ requires that $v_{\perp}^2$ rise in proportion to increases in B ; obviously there is a limit to this increase, the point at which all the particle's energy is in gyro motion and $|v_{\perp}| = |v|$. At this point the particle's $v_{\parallel}$ changes sign and the particle heads back into the region of smaller B ; it is said to have been reflected at its "magnetic mirror" point. Knowing the value of B at the mirror allows one to write (I.4a) also as

$$\mu = \frac{W}{B_{\text{mirror}}}. \quad (I.4b)$$

The second adiabatic invariant is the action variable of the periodic motion,

$$J \equiv \oint m v_{\parallel} ds \quad (I.5)$$

where ds is a differential line element along the particle's path and $v_{\parallel}$ is the component parallel to $\underline{B}$. The integral is taken between mirror points.

The strength of a particular field line varies along its length, as it moves from the inside to the outside of the torus while traveling around, because of the $1/R$ toroidal field. This has the effect of creating a series of magnetic mirrors at different

points along the field lines within each shear layer. Charged particles whose energies are low enough may be "trapped" in these mirrors, oscillating back and forth; their trajectories when projected in the poloidal plane have the characteristic shape shown in Fig. 2 and are known as "banana orbits." The time between turning points is known as the bounce time; bananas drift in the toroidal direction with a frequency inversely proportional to their bounce time.⁶

1.2.3 Stability and Flux Surfaces

One can demonstrate⁷ (by examining the magnetic flux and pressure balance equations) that confinement of the plasma is provided by the poloidal magnetic field; the toroidal field is necessary primarily for stability. The value of the toroidal field, as can easily be seen in Eq. (I.1), must be much greater than the value of the poloidal field. Hence it is the smaller field which is energetically the critical one. The amount of plasma energy contained within a given poloidal field is usually expressed in the non-dimensional ratio

$$\beta_p = \frac{8\pi P}{B_p^2} \quad (I.7)$$

where P is the total plasma pressure, a scalar for simplicity. The critical value for determining the efficiency of an eventual reactor is, of course, β_T , where B is substituted for B_p in (I.7), since all energy stored in magnetic fields must be supplied from without somehow.

Fig. 2. A banana orbit. The example is a deuteron trapped in the Princeton FMT device; the contours are surfaces of constant magnetic flux.

Clearly β_p is restricted by the Kruskal-Shafranov limit on total current, Eq. (I.1). In the absence of any further applied fields, β_p is also limited⁸ by the aspect ratio ρ ,

$$\rho \equiv R_0/r_0. \quad (1.8)$$

As β_p increases the plasma tends to drift outward to the wall of the torus; it reaches the wall at $\beta_p \approx \rho$. This limit is a more stringent one than the Kruskal-Shafranov but it can easily be overcome by the application of a further "vacuum field" from outside coils. This new static field is entirely in the tz -direction and is known as the centering field (b_z^0). The Lorentz Force $\underline{J} \times \underline{B}$ arising from the interaction of the centering field with the toroidal plasma current can be directed inward by proper choice of sign on $\underline{B}_z^0$, stabilizing the plasma at any desired radius.

Figure 1b shows a representative set of nested surfaces of constant magnetic flux in the poloidal plane. A $\underline{B}_z^0$ which is constant as a function of R has been applied to center them within the torus. As a result of the centering field, the outer flux surfaces are no longer completely closed: the flux due to $\underline{B}_z^0$ is larger near the axis since it varies as $1/R$. Numerical calculations of ideal MHD equilibria similar to this one have previously been performed by Callen and Dory⁹ (among others); their results show that β_p could theoretically approach ρ^2 , with a resulting β_{total} of 1, an ideal limit. However, they conclude that several other considerations limit β_p to about 2 ($\rho^2 > 6.25$ for all existing TOKAMAKS), and β_{total} to approximately 0.1.

It is not required for confinement in toroidal geometry that the "magnetic well" in B in which the plasma rests be an absolute one (that is, that the curvature of field lines in every direction from each point be "good"--directed so as to prevent the MHD flute instability). One cannot, in fact, design such a perfect-well machine in general. The continuous geometry permits of a somewhat less stringent condition, of which certain devices such as toroidal multipoles (SPIRE, for instance) take full advantage. The explanation is simple: consider a flux tube of volume V and containing flux Φ traveling around the torus; distort the tube to a new volume V_1 (Φ remains constant). The distortion will be energetically stable if $V_1 < V$ (work cannot be done on the surrounding plasma or fields); since $\Phi \equiv BA$ (where A is the cross-sectional area of the flux tube),

$$V = \int A dl = \int \frac{d\Phi}{B} = \Phi \int \frac{dl}{B}.$$

Thus the crucial factor which must be minimized along the path of the plasma is not the local value of B ; it is rather an integral along that path, $\int dl/B$. This is known as the Minimum Average- B criterion.

1.3 Scope

We have said nothing of collisional phenomena or the important related topic of radial diffusion because plasmas simulated by GUIDON are presumed to be effectively collisionless. This raises the particles above certain minimum energies. To prevent the necessity of an absolute maximum energy (and because simulation of

SPIRE requires it), the equations of motion of the guiding-center formalism which is used to advance simulation particles are written in fully relativistic form.

GUIDON is designed to simulate plasmas in toroidal geometries, but any two-dimensional axisymmetric geometry will do. A two-dimensional representation of a device whose fields have a rotational transform is sufficient since, although a particular field line may have changed position in the poloidal plane in a distance $\Delta\phi$, a similar one will be in its place: all cross sections are geometrically identical. However, it should be obvious that particle motion cannot be restricted to two dimensions if the effects of applied accelerating voltages are to be taken account of. Thus particle positions and velocities are calculated in 3 D, but the particles always appear at the calculational poloidal plane: they are, in effect, rings of charge. For this reason the code falls into a class labelled " $2\frac{1}{2}$ -dimensional."

Two further important assumptions govern the fundamental geometry of the code: (1) Self-consistent magnetic fields dominate self-consistent electric fields in influence over large-scale motions, and (2) only those magnetic fields induced by toroidal motion of the plasma are calculated, induced poloidal fields being presumed either small or averaged to zero. Attempts to take into account exceptions to (1) are discussed in Chapter II; (2) is applied throughout except for one version of the code which makes an approximation of the effects of poloidal plasma currents on B_ϕ (also described in Chapter II). The justifications for these

assumptions is hardly at all. They were chosen because they greatly simplify the computational problem, and because they are appropriate for the devices for which the code was designed. Thus they must be carefully verified for any other experimental condition to which GUIDON is applied.

Several other important physical assumptions are made by the model; these and their effects are discussed in the following chapter.

II. GUIDON

2.1 General Description

2.1.1 Computational Parameters

GUIDON is written for use on the CDC 6600 and 7600 computers. Thus far most production has been run on 6600's in the LRLTRAN version of the FORTRAN IV language and using the LLL CHAT compiler. The code occupies approximately $(250\ 000)_8$ core locations on the 6600 for a standard TCT simulation version employing 2500 particles. All versions run fully in core, but a triple-buffering routine is available which will permit any number of particles to be run using disc storage. Different versions of the code change the 6600 storage requirement in the range $(225\ 000)_8$ to $(272\ 000)_8$. The 7600 TCT version occupies about $(55\ 000)_8$ of small core memory, the remainder being relegated to large core.

Running time on the 6600 averages about 1.5 seconds per cycle; on the 7600 about 0.55 seconds per cycle. Besides being a relatively sensitive function of the number of particles used, the timing depends rather strongly on which plotting options are used and how often they are called. Because a great variety of diagnostics have been used and continually modified, the programming of the plotting routines has never been optimized; a significant saving could be realized for any particular version by performing this optimization. In addition, the large core storage blocks used in the 7600 have not been rearranged for more efficient reading and writing; this would involve rather extensive (though straightforward) reprogramming of the code itself, but might lower

the 7600 cycle time as far as 0.25 seconds.

2.1.2 Logical Flow

Two separate physical models are employed by GUIDON, each of which involves techniques novel in computational physics. The first is a single-fluid MHD representation of toroidal background plasma current; the second is a particle model for a single species (either deuterons or electrons) which may appear as part of the confined plasma or as an injected beam. Both of these models are magnetically self-consistent in their poloidal fields, and both use the same method of solving for new $\underline{B}$ at each timestep. This solution is described in Section 2.2, followed respectively by descriptions of the plasma and particle models.

Either of the two models may, of course, be used by itself. Use of the fluid model alone simply provides a calculation of the MHD equilibrium of the specified plasma current (this calculation is always the first step of a full two-model simulation). The particle model may be used by itself either when no other plasma current is present (as in a SPIRE startup or other electron ring accelerator) or when the self-consistent motion of the plasma current is not of interest, in which case it may be represented as though it were another vacuum field coil. When both models are employed simultaneously, an effective new MHD equilibrium for the plasma current (as affected by the motion of the particles) is solved for at each timestep.

Figure 3 displays the sequence in which a full calculation proceeds. For details of time-centering difficulties involved in

FIGURE 3:
Overall Logic
of GUIDON

the principal loop, see section 2.5.3.

2.2 Magnetic Fields

2.2.1 Vacuum Fields

The toroidal field is calculated as

$$B_{\phi} = \frac{I_z}{5R} \quad (\text{II.1})$$

where I_z is the vertical current at $R = 0$ in amperes. The value of I_z is read in as an input quantity and never changed throughout the run. In most versions of the code B_{ϕ} cannot be affected by plasma currents, and since (II.1) has such a simple algebraic form B_{ϕ} is merely computed exactly whenever it is needed. In the one version in which effects of poloidal plasma currents are approximated, B_{ϕ} must be saved on a mesh as a function of R and Z ; see Section 2.3.2 for the appropriate revision to (II.1).

B_r and B_z due to all sources are stored on a computational mesh as illustrated in Fig. 4. For all forms of GUIDON now in use, the mesh size is 64×64 .

The vacuum components of B_r and B_z may be supplied by any number of toroidal coils, arbitrarily located (sample locations are shown in Fig. 4). The currents in these coils and their R and Z coordinates are input data and are presumed fixed. The calculation of these components for each coil is performed by an integration of the Biot-Savart Law exactly as in Anderson and Fuss's GUTHE code, from which it is taken (see Appendix A1). That is, the vector potential at a point (R, Z) due to a coil at radius a ($z = 0$) and with current I is given by

FIGURE 4: Computational Mesh for B

$$A_{\phi} = \frac{I}{\pi k} \sqrt{\frac{a}{r}} \left[\left(1 - \frac{1}{2} k^2\right) K - E \right] \quad (\text{II.2})$$

where $k^2 \equiv 4ar[(a+r)^2 + z^2]^{-1}$ and K and E are complete elliptic integrals of the first and second kind, respectively. The A_{ϕ} can be used to calculate a vacuum flux $\Phi_v \equiv RA_{\phi}$. The vacuum field is then computed as¹

$$B_r = \frac{I}{2\pi} \frac{z}{r\sqrt{(a+r)^2 + z^2}} \left[-K + \frac{a^2 + r^2 + z^2}{(a-r)^2 + z^2} E \right]$$

$$B_z = \frac{I}{2\pi} \frac{1}{\sqrt{(a+r)^2 + z^2}} \left[K + \frac{a^2 - r^2 - z^2}{(a-r)^2 + z^2} E \right] \quad (\text{II.3})$$

The total vacuum values of B_r and B_z at any mesh point are simply the sums of the contributions from each of the coils (by superposition). The totals are calculated at the beginning of any problem and stored permanently.

It is not necessary to have any vacuum coils at all if the problem (see Section 4.3) does not require them.

One of the vacuum coils (the first one read in) may be used to simulate a plasma current, in the following fashion: the R and Z coordinates read in for this coil are presumed to be the center of the (circular) plasma current; a radius for the current is also entered. The B_r and B_z due to this current are first calculated using (II.3) as though it were a point-source coil. The values obtained are then modified at all points within the desired plasma radius; they are multiplied by a function designed to lower them

the proper amount for internal fields in a current profile of the shape chosen.* Three current profiles are available:

Fig. 5. Current profiles for mock plasma current.

This mock plasma current saves calculation time when one is not interested in the details of the plasma current motion and distortion. It is also calculated as the trial flux in the first step of solving for a proper current equilibrium.

One must take care when placing any vacuum coils not to put one exactly at a mesh point: the exact Biot-Savart Law calculation will result in an infinite vector potential and the computer will become quite unhappy.

---

*Actually, the test is performed not on radius but on flux; all points with ϕ greater than ϕ_0 , the flux at the plasma boundary, are modified by a multiplicative factor. For profile (b) in Fig. 5, this factor is $G = F(2 - 1)$ where $F \equiv \exp(10\eta/R_p \cdot 1)$, R_p is the radius of the center of the current and $\eta \equiv \phi_0 - \phi$.

2.2.2 Self-Consistent Fields

As has already been mentioned, these fields are purely poloidal and arise from current sources which have only ϕ components; the symbol $\mathbf{j}$ will therefore be used to represent a current vector whose only component is j_ϕ . Since we are ignoring induced electric fields, the displacement current term in the Maxwell-Ampere Law is zero, recovering Ampere's Law,

$$\nabla \times \mathbf{B} = \frac{4\pi}{c} \mathbf{j} \quad (\text{II.4a})$$

whose only surviving term is

$$\frac{dB_r}{dz} - \frac{B_z}{r} = \frac{4\pi}{c} j. \quad (\text{II.4b})$$

The solution to such an equation is most easily found by casting both components of $\mathbf{B}$ in terms of a single scalar potential function. In this case, we can define a potential function ψ such that

$$B_r = -\frac{1}{r} \frac{\partial \psi}{\partial z} \quad B_z = \frac{1}{r} \frac{\partial \psi}{\partial r} \quad (\text{II.5})$$

The function ψ is then identical to the magnetic flux Φ , for substitution of (II.5) into (II.4b) yields

$$\nabla^2 \psi = -\frac{1}{r} \frac{\partial^2 \psi}{\partial z^2} - \frac{\partial}{\partial r} \left(\frac{1}{r} \frac{\partial \psi}{\partial r} \right) = \frac{4\pi}{c} j. \quad (\text{II.6})$$

Now Ampere's Law written in terms of vector potential $\mathbf{A}$ ($\mathbf{B} = \nabla \times \mathbf{A}$) reduces to

$$\nabla^2 \mathbf{A} = -\frac{4\pi}{c} \mathbf{j} \quad (\text{II.7})$$

in both Lorentz and Coulomb gauges because we have ignored the electric potential; and applying the definition $\Phi = rA_\phi$, (II.6) and (II.7) are seen to be identical equations (For the algebra to show this, see Appendix B.2.) Henceforward ψ will be used to denote flux.

Solution of the crucial flux equation (II.6) requires that the sources j be defined over the computational mesh and that appropriate boundary conditions be established. Because we shall eventually want to solve for magnetic field components using the simple difference approximations to Eqs. (II.5), it is convenient to have the flux values centered between the field values; but (II.6) requires that ψ and j have equivalent centering. So a new computational mesh is established, offset by $\Delta R/2$ and $\Delta z/2$ in R and z directions from the original mesh; obviously, for all differences to be defined, this new mesh must be one cell larger in both directions than the old one, or 65×65 . The interrelation of the two meshes is illustrated in Fig. 6 for a $5 \times 5/6 \times 6$ system.

Perfectly conducting boundary conditions are assumed in GUIDON. We have thus a Dirichlet problem; specifically, $\psi = 0$ everywhere on the boundaries.* There can be several justifications for this assumption. First, the actual machine may have conducting walls (most Tokamaks do), in which case it represents an appropriate simulation. Second, the boundaries may be far enough removed from the critical region of the problem as essentially to represent

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*Alternate choices are described in Section 2.3.5.

Fig. 6. Illustration of interlocking computational meshes.

infinity, where the free-space fields (and the vacuum fields, which were calculated with free-space boundaries) go to zero. It may be necessary to increase ΔR and ΔZ to justify this assumption; only repeated testing can guarantee that it is valid.

An additional difficulty is introduced if the real device has conducting walls: the boundaries of the code are square, those of the device curved (if not circular). The volume of extra coding and increased running time required to supply a properly curved boundary in the code are prohibitive; in addition, it is not clear that the approximations necessary for such a representation would be any better than the relatively exact solution for the square representation. Analytic estimates for the fields near the center of a Tokamak were compared with code results for the same conditions and agree within 0.05% in the worst case (where the radius of container is smallest). Very close to the walls (within a cell or two) the code's representation will be poorer (except at and close to $Z = 0$), but behavior in this region is strongly affected by other phenomena (such as sheaths) not included in the code's

physics, so no additional harm is done. Finally, one may simply ignore phenomena occurring in the corners, provided particles are not allowed to re-enter the problem after being there.

The two sections which follow describe how currents are calculated from the fluid and particle models and distributed over the Δ mesh as sources for Eq. (II.6). Once the sources have been properly defined, solution of (II.6) proceeds in three steps:

(1) Fourier transform the equation in the Z-direction. In the operation of the code, of course, only the sources are plugged through the transform algorithms. A Fast Fourier Transform² algorithm is used; since the function is identically zero at both boundaries, only the Sine Transform is necessary. (A brief description of the Fast Fourier Transform is given in Appendix B4.)

(2) the resulting 1 D difference equation in the R direction can then easily be solved for the transformed values of ψ . The equation is tridiagonal when written in operator form, permitting use of the familiar "tridiagonal algorithm."

(3) The results of step (2) are inverse-transformed and normalized, providing the solution for ψ .

Derivation of the difference equations and application of the tridiagonal algorithm are shown in detail in Appendix B1. Because of the use of the Fast Sine Transform, the number of cells in the Z direction must always be a power of 2. The number of cells in the R direction is not limited, but calculation is more efficient if it is an even number. With the new values for ψ known, new values for B are computed from (II.5), centering the differences

properly by averaging in the direction perpendicular to the difference:

$$\begin{aligned}(B_r)_{i,j} &= - \left(\frac{1}{R_{i,j}} \right) \frac{1}{2\Delta Z} (\psi_{i+1/2,j+1/2} - \psi_{i+1/2,j-1/2} \\ &\quad + \psi_{i-1/2,j+1/2} - \psi_{i-1/2,j-1/2}) \\ (B_z)_{i,j} &= \left(\frac{1}{R_{i,j}} \right) \frac{1}{2\Delta R} (\psi_{i+1/2,j+1/2} - \psi_{i-1/2,j+1/2} \\ &\quad + \psi_{i+1/2,j-1/2} - \psi_{i-1/2,j-1/2})\end{aligned}\quad (II.8a)$$

where (i,j) represent increasing indices of mesh points in the R and Z directions, respectively. As actually used in the code, of course, the R and Z meshes have separate whole-number indexing. Let (k,ℓ) be the indices in the ψ mesh, which runs from 0 to 64 instead of from 1 to 64 , so that

$$\psi_{k,\ell} = \psi_{i+1/2,j+1/2}, \text{ etc.}$$

when $i = k$ and $j = \ell$. Then (II.8a) may be more succinctly rewritten

$$\begin{aligned}(B_r)_{i,j} &= - \left(\frac{1}{R_{i,j}} \right) \frac{1}{2\Delta Z} (\psi_{k,\ell} - \psi_{k,\ell-1} + \psi_{k-1,\ell} - \psi_{k-1,\ell-1}) \\ (B_z)_{i,j} &= \left(\frac{1}{R_{i,j}} \right) \frac{1}{2\Delta R} (\psi_{k,\ell} - \psi_{k-1,\ell} + \psi_{k,\ell-1} - \psi_{k-1,\ell-1})\end{aligned}\quad (II.8b)$$

As a final step, vacuum fields (if any) must be added to the values computed by (II.8b) to obtain the total field, $\underline{B}$.

2.2.3 Timing

The total time for executing these processes for a complete update of ψ and $\underline{B}$, given particle current sources (but including

calculation of plasma current sources; see 2.3.2) is somewhat less than 300 milliseconds of 6600 time, for the standard (64 x 64) grid. How much less depends principally on adjustments made in the plasma current algorithm and pressure calculation; the fastest result has been about 250 milliseconds.

2.3 Fluid Plasma Model

2.3.1 Pressure Balance

The fluid plasma model follows the procedure of Callen and Dory³ and Anderson, Killeen, and Rensink⁴ in presuming that the most important forces in the establishment of the MHD equilibrium are the plasma's pressure gradient and the Lorentz Force. Thus if $\underline{J}$ is the total plasma current, the balance of these forces is expressed by

$$\nabla P = \frac{1}{c} \underline{J} \times \underline{B} \quad (\text{II.9})$$

Substituting (II.5) into (II.9) and equating the surviving R and Z components (since $\underline{J} = J_{\phi}$)

$$\frac{\partial P}{\partial r} = J \frac{1}{r} \frac{\partial \psi}{\partial r} \quad \text{and} \quad \frac{\partial P}{\partial z} = J \frac{1}{r} \frac{\partial \psi}{\partial z}$$

which can be written as a single equation

$$J = r \frac{\partial P}{\partial \psi} \quad (\text{II.10})$$

The choice of the functional dependence of P on ψ is mathematically arbitrary. A choice which can be adjusted to fit most pressure profiles is

$$\begin{aligned} P &= \bar{c}(\psi - \psi_0)^M & \text{for } \psi \geq \psi_0 \\ P &= 0 & \text{otherwise} \end{aligned} \quad (\text{II.11})$$

where $\bar{c}$ and M are arbitrary constants and ψ_0 is a reference flux defining the boundary of the plasma current. For a given $\bar{c}$, each integral value of M above zero will produce an equilibrium solution. Using (II.11) in (II.10) the source term for the flux is defined

$$J = \bar{c}M(\psi - \psi_0)^{M-1} \quad \psi \geq \psi_0 \quad (\text{II.12})$$

It is clear that for a fixed ψ_0 , the constant $\bar{c}$ determines the amount of total current present in the problem. Since J is a current density, the total plasma current I in the computational problem is

$$I = \sum_{\text{all cells}} J \Delta R \Delta Z = \bar{c}M \Delta R \Delta Z \sum_{\substack{\text{all cells} \\ \psi \geq \psi_0}} R(\psi - \psi_0)^{M-1} \quad (\text{II.13})$$

For the plasma current equilibrium then, (II.6) takes the form

$$\nabla^* \psi = \left\{ \frac{4\pi}{c} \right\} (\bar{c}M) (\psi - \psi_0)^{M-1} \quad (\text{II.14})$$

It is important to realize, however, that since the plasma must come to equilibrium with all fields--self-consistent and vacuum--we should not consider (II.14) to be of the form

$$\nabla^* \psi_{\text{plasma}} = f(\psi_{\text{plasma}}) \quad (\text{II.15})$$

Rather consider that

$$\psi_{\text{total}} = \psi_{\text{plasma}} + \psi_{\text{vacuum}}$$

and rewrite (II.15) in terms of total flux. The left-hand side, however, will produce only a solution for ψ_{plasma} :

$$\nabla^* \psi_{\text{total}} = \nabla^* (\psi_{\text{plasma}} + \psi_{\text{vacuum}}) = f(\psi_{\text{total}})$$

$$\text{but } \nabla^* \psi_{\text{vacuum}} = 0$$

$$\text{so } \nabla^* \psi_{\text{plasma}} = f(\psi_{\text{total}}) \quad (\text{II.16})$$

In summary, total flux must be used in evaluating current sources, but only flux arising from plasma currents will emerge as the solution of (II.14).

2.3.2 Corrections to the Toroidal Field

A provision is also included in this model to allow, in relatively arbitrary fashion, for the effects of small poloidal currents in the fluid plasma in augmenting or degrading B_ϕ . Since total current is conserved by the model, this correction is made at the expense of the toroidal current. The form of the adjustment is^{3,4}

$$\hat{B}_\phi = \frac{B_\phi}{1 + bP} \quad (\text{II.17})$$

where $\hat{B}_\phi$ is the new toroidal field, $B_\phi = I_z/5R$ is the vacuum toroidal field, P is the pressure [Eq. (II.11)], and b is an arbitrary constant specifying the fraction of current presumed to be flowing in the poloidal direction. The degradation of toroidal current is effected by rewriting (II.12) as

$$J = \left[1 - \frac{bs^2}{(1 + bs)^3} \right] RMc(\psi - \psi_0)^{M-1} \quad (II.18)$$

$$[\psi \geq \psi_0]$$

where $s \equiv I_z/5$, I_z still in amperes. Derivation of (II.18) appears in Appendix B3. This form is particularly convenient since the choice of $b = 0$ automatically recovers the original equation (II.12).

Although the option to use this correction appears in the coded versions of GUIDON and has been checked successfully for consistency, it has never been used in a long-running calculation simply because no realistic, accurate value of the constant b is known for any of the devices thus far simulated. (In all production versions of GUIDON the unnecessary storage blocks for B_ϕ have been removed to economize on core requirements, but the rest of the formalism remains. Its removal could provide a small saving of computational time in updating fields.)

2.3.3 Solution for the Equilibrium

Since the source term in (II.14) is a function of the flux, a trial solution of flux values is necessary to begin computation of the equilibrium. In addition, the following quantities are input to the code to describe the plasma current: total current flowing, I ; the constant M in (II.14), which defines the shape of the equilibrium current and pressure distributions; coordinates of the center of the plasma, (R_c, Z_c) (though normally, $Z_c = 0$ initially, and that will be assumed henceforth); and an estimate of the radius of the plasma, r_p . Using this information a vacuum

field solution for the flux is obtained as described in 2.2.1. It makes no difference which of the current profiles of Fig. 5 is chosen for this trial solution, since the final equilibrium shapes will depend only upon M (and interactions with the boundaries, if they occur).

With the flux now defined at each of the mesh points, ψ_0 is chosen as the flux at point (R_c, r_p) . The constant $\bar{c}$ can then be calculated by inverting Eq. (II.13):

$$\bar{c} = \frac{I}{\sum_{\psi > \psi_0} R(\psi - \psi_0)^{M-1}} \quad (\text{II.19})$$

And with all these values available, the method of 2.2.2, which we shall refer to as BPFI, may be applied to find new values of ψ and $\underline{B}$.

In fact, the solutions for $\underline{B}$ are not used; all iteration is performed using ψ . The flux surfaces will have changed drastically from the trial solution; R_c will have changed also as the plasma begins to move to its equilibrium position (call the new value R'_c). Now ψ_0 is redefined as the flux at (R'_c, r_p) ; $\bar{c}$ is recalculated from (II.19); and BPFI is employed once again.

This basic cycle of operations is repeated until R_c and the flux surfaces cease to change:

- (a) redefine ψ_0
- (b) recalculate $\bar{c}$
- (c) use BPFI

Once this point is reached, however, the equilibrium has not yet been found, because we have been limiting the radius of the plasma. Close to the equilibrium, though, we fix ψ_0 and cycle through steps (b) and (c), allowing the plasma to expand or contract and adjust R_c slightly if necessary.

Typically it takes between 5 and 10 complete cycles of (a)-(c), followed by only 2 or 3 of (b)-(c), to arrive at an equilibrium. The number of cycles required depends obviously on the distance from equilibrium of the trial function. Presuming one already knows the correct value of centering field B_z^0 needed, this number is most sensitively a function of the discrepancy between initial and final values of R_c . None of the test problems run, including those where the initial position was as far as possible from equilibrium (on the far side of an enlarged mesh), required more than 20 cycles.

Figures 7a-d illustrate the changes in flux surfaces from the trial to the first, second and tenth (final) iterations of an equilibrium study or an early proposal of the TCT (see Chapter III) very similar to the Princeton FLT; Figs. 7e-f show the plasma current density as a function of radius at $z = 0$.

The advantages of this entire scheme when compared to more familiar methods of equilibrium calculation (such as the successive over-relaxation scheme used by Callen and Dory) are its speed and convergence properties. Other methods require not only the several cycles to allow the plasma to move approaching its equilibrium position, but many cycles within each cycle to allow their schemes

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Fig. 7a. PLT vacuum field trial function. The solid line is ψ_0 , the plasma boundary.

FLUX 1900= 5.688E+07 N11= -3.5228E+07

Fig. 7b. First iteration on the trial function.

FLUX MIN= 4.9622E+07 MIN= -3.4914E+07

Fig. 7c. Second iteration.

FLUX MIN= 4.7598E-07 MIN= -3.5952E-07

Fig. 7d. Equilibrium solution (10th iteration).

Fig. 7e. Plasma current density versus radius at $Z = 0$ for the vacuum trial function.

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Fig. 7f. Plasma current density versus radius at $Z = 0$ for the equilibrium.

to converge mathematically. Speed and accuracy of this method are discussed in more detail in the following section.

A further significant advantage of the method is its generality --the ease with which it may be coupled interactively with other computation, such as in GUIDON. After the equilibrium has been reached and the particles initialized, the particle currents and fluid plasma current can be defined on the same mesh and a single algorithm (BPFI) used to update both in completely self-consistent fashion. The particles are then advanced and the entire process repeated at every timestep with each new equilibrium found with remarkably little extra effort (beyond what the particle model alone would require). $\bar{c}$ continues to be recalculated at every cycle, but ψ_0 is fixed permanently.

This generality is partly due to the mutual applicability of BPFI and partly a result of the speed of the process; a method requiring iteration would probably render the entire code's running time unacceptably long. The equilibrium solution is applied only once each timestep, on the theory that particle motion in one Δt will not have greatly altered the equilibrium. This hypothesis has been rather stringently tested: in runs where the current in simulation particles was actually equal to the current in the "background" fluid plasma (and therefore strongly affecting its equilibrium at each timestep), tests were made by sub-cycling the equilibrium solver as many as five times per timestep. In no case were the results after 500 cycles or more significantly different from those in which the equilibrium solver was only used once per

timestep.

2.5.4 Convergence and Speed

As was demonstrated by Marler and Weitzner,⁵ the nonlinear nature of (II.14) can in many situations lead not to a unique solution but to a finite number of solutions, and convergence is not guaranteed. In particular, for potential field problems such as this one there tend to be two "bifurcated" solutions, a "shallow" one, in which the flux gradients are very small, and a "deep" one, in which the magnetic well is much more pronounced. Convergence, once guaranteed, may lead to the physically less desirable shallow well, depending upon the initial trial function.

The form of the source term (II.12) we have chosen is a particularly dangerous one in this context. A slight lowering of the maximum flux at any cycle without renormalization of $\bar{c}$ [defined by (II.19)] leads to disastrous consequences: the flux contained within the fixed ψ_0 being less, there is less current in the problem; less current will lead on the next cycle to a further lowering of $\psi_{\max}$, initiating an unstable decrease. Before long $\psi_{\max}$ will be less than ψ_0 and the plasma current will disappear entirely: the result will have converged to $\psi = 0$ everywhere. An example of this disaster is illustrated in Fig. 8a-d.

Thus the renormalization of $\bar{c}$ is necessary and provides for the convergence of the algorithm. This dictates a constant plasma current but since the model has no dissipative mechanisms built in it is difficult to see how any other assumption would be physically justifiable. It should be possible to choose or rather to maintain

Fig. 8. Onset of the disastrous convergence to $\psi = 0$ everywhere.

Fig. 8a.

Fig. 8b.

Fig. 8c.

the maximum plasma pressure as a constant by fixing $\bar{c}$ and renormalizing ψ_0 at each step using (II.11), but this method does not appear to conserve current sensibly and tests of it showed convergence only to the shallow (or unstable $\rightarrow \psi = 0$) solution.

The question remains, of course, of whether the GUIDON algorithm converges to the correct solution at all. That it converges to something is clear since, as already mentioned, fluxes cease to change after a few cycles. To prove that this convergence is a proper one, comparisons were made using the first of the two methods proposed by Marder and Weitzner, whose guaranteed convergence properties they demonstrated.

Their method works like this: rewriting (II.14) in operator form $L\psi = f(\psi)$ where $L = \nabla^*$, denote successive applications of the solution method--BPHI as here, or ADI techniques as in Anderson, Killeen and Rensink^{6,7}--by superscripts. Then perform these iterations in sets of three subcycles, according to the following rule:

$$\begin{aligned} (1) \quad L\psi^{n+1/3} &= f(\psi^n) \\ (2) \quad L\psi^{n+2/3} &= f(\psi^{n+1/3}) \\ (3) \quad \psi^{n+1} &= (1 - \alpha)\psi^n + 2\alpha\psi^{n+1/3} - \alpha\psi^{n+2/3} \end{aligned} \quad (\text{II.20})$$

where α is an arbitrary constant (convergence being demonstrated for sufficiently low α , where sufficiency is a parameter of each specific problem).

Thus the Marder technique can be used in GUIDON by changing each single application of BPHI to two applications plus an

averaging step; renormalization of $\bar{c}$ should no longer be necessary. Precisely this technique was used, and the equilibria compared with those produced by GUIDON's normal algorithm. Results were identical in all of three different simulations run. Figure 9a-d shows the comparisons for the compressed phase of the Princeton ATC device.

GUIDON's algorithm is on the average five times faster than the Marder technique. Marder's requires three cycles instead of one to advance ψ^n to ψ^{n+1} ; it also requires about twice as many iterations $n \rightarrow n+1$ to reach equilibrium. But recall that we used the BPFI solution at each step for ψ ; BPFI is at least three times as fast as ADI or successive over-relaxation in finding a convergence of each of the subcycles. In sum then, the complete method employed by GUIDON in equilibrium solution is an order of magnitude (or more) faster than more conventional schemes whose only object is equilibrium.

2.3.5 Other Boundary Conditions

Callen and Dory⁸ use a slightly different perfectly conducting boundary condition, namely $\psi = \text{constant}$ but not necessarily zero. To do this compatibly with their over-relaxation method of obtaining convergence they maintain an extra artificial boundary and engage in an extrapolation procedure to readjust these boundary points and as a test of convergence. The justification for all this extra effort is that the boundary fluxes thus used = those generated by the trial solution. If the "bifurcation" problem is a severe one, choice of a good trial solution can force convergence to the desired physical result when this procedure is employed.

FLUX MAX= 1.0652E+06 MIN= -1.2702E+07

Fig. 9a. Equilibrium for the compressed phase of the Princeton
ATC computed using the Marder algorithm in 25 complete
iterations. $\alpha = 0.24$.

CURRENT DENSITY VERSUS RADIUS

Fig. 9b. Current density versus radius for the Marder solution.

FLUX MAX= 1.8652E+06 MIN= -1.2782E+97

Fig. 9c. Compressed-ATC equilibrium calculated with GUIDON's
own algorithm in 12 cycles.

CURRENT DENSITY VERSUS RADIUS

Fig. 9d. GUIDON solution.

Another motivation for boundary conditions in which $\psi \neq 0$ is an attempt to simulate free-field, rather than perfectly conducting, boundary conditions. This is done by redefining the boundary fluxes at each timestep using a Biot-Savart solution (similar to that of 2.2.1) applied to the whole plasma current. Solution of (II.14) with these ψ as boundary conditions provides a good approximation to a case in which all fields behave as vacuum fields.

Both of these nonzero boundary conditions have been used in GUIDON successfully. Employment of the first one did not require the extensive apparatus used by Callen and Dory. It was found that the simplest method was merely to set the ψ 's on the boundary equal to the ψ 's at the nearest inside mesh points at each cycle of RPPI, then fix them permanently during the time-dependent calculation. Results of equilibria calculated in this manner agreed with those calculated using $\psi = 0$ boundaries by 1 part in 10^6 at all mesh points. The method requires, of course, more coding and slightly longer running times for equilibria than does $\psi = 0$; it was therefore abandoned when it was discovered that renormalization of $\bar{c}$ (required by both kinds of boundary conditions) was sufficient to force convergence to the physically correct answer with the Biot-Savart trial functions.

The free-field boundary condition approximation was never used in production runs because one of the perfectly-conducting infinite-wall assumptions was always found to be sufficient as an approximation to the devices studied. Use of the free-field approximation greatly increases running time of the equilibrium calculation.

tion (roughly a factor 1.5).

The employment of any nonzero boundary condition in GUIDON causes one special problem, the clever solution to which described here is reputedly due to Oscar Buneman.⁹ The difficulty is quite simply that the sine transform used to reduce solution of (II.6) in the Z direction requires that $\psi = 0$ on both endpoints, the rows of points at $Z = Z_{\min}$ and $Z = Z_{\max}$ represented by $(\psi_{k,0})$ and $(\psi_{k,\ell_{\max}})$.^{*} The solution effectively places an artificial row of currents at $(\psi_{k,1})$ and $(\psi_{k,\ell_{\max}-1})$ which produce exactly the effect of countering the eddy currents represented by the boundary row, which is then permitted to be zero. This can be done quite simply.

Consider only the top ($Z_{\max}$) boundary. The difference equation for (II.6) is shown in Appendix B1. At the row $\ell_{\max}-1$, the Z-derivative term looks like this:

$$\frac{\psi_{\ell_{\max}} - 2\psi_{\ell_{\max}-1} + \psi_{\ell_{\max}-2}}{\Delta Z^2} \quad (\text{II.21})$$

Since these derivatives are not taken at the boundary itself, it is sufficient for the transform to have set $\psi_{\ell_{\max}} = 0$ in (II.21). And to introduce exactly the correct artificial current, we need only add the discarded term $\psi_{\ell_{\max}}/\Delta Z^2$ as a source term on the right-hand side. The resultant form of (II.6) looks like this (compare with Appendix B1):

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*No such difficulty exists in the R direction

$$\frac{1}{\Delta z^2} (-2\psi_{lmax-1} + \psi_{lmax-2}) + [\text{terms in } R] = \frac{4\pi}{c} R_1 J_{1, lmax-1} - \frac{\psi_{lmax}}{\Delta z^2} \quad (\text{II.22})$$

This small subterfuge is mathematically clean and introduces no real physical difficulties (except in the outermost cell, where the particles must be removed anyhow). It produces the correct fields and can be programmed into EPHI with negligible loss of efficiency.

2.4 Discrete Particle Model

2.4.1 Assumptions

GUIDON permits the use of an arbitrary number of discrete simulation particles, which may represent the entire plasma, a designated portion of the plasma, or a beam of injected deuterons or electrons. So that motions requiring relatively long time scales may be studied, the particles are advanced using the guiding-center formalism. In other words, the details of gyro-motion are ignored; thus, where a normal particle model might require as many as 30 timesteps to allow the particle to execute a single gyro orbit, GUIDON may have timesteps as long as several gyro-periods: the limitations on the timestep depend on other criteria (see 2.5.2). As has been mentioned, the particle dynamics is cast in fully-relativistic form to permit maximum applicability of the code, so all the discussion that follows will contain those terms necessary to relativistic dynamics.

Strict application of the guiding-center equations involves an assumption of invariance of the magnetic moment, μ , of the

particles. The definition of μ given in (I.4a) was, however, the standard nonrelativistic approximation to

$$\mu = \frac{p^2}{2m_0 B} \quad (\text{II.23a})$$

where p is the relativistic momentum and m_0 the rest mass. Thus since $m = \gamma m_0$, the correct invariant in terms of (I.4a) is not μ but $\gamma\mu$; henceforward reference to μ will use the form of (II.23a), so that any relation of μ to v_G ($= v_1$) will assume

$$\mu = \frac{\gamma^2 m_0 v_G^2}{2B} \quad (\text{II.23b})$$

Two other significant physical assumptions are made by this model. First, since electrostatic interactions are not considered, the plasma is effectively assumed to be neutralized both macroscopically and microscopically by a uniform background.

Second, it is assumed that the kinetic energy of each particle is conserved since no radiative or collisional processes are included. One significant exception is made; a key feature of many devices is an applied toroidal electric field used to accelerate the plasma current. Presumably this field is supplied inductively by the rapidly changing flux of an additional vacuum coil with time-dependent current; the plasma then may be seen as the secondary winding of a transformer. The details of this process are not treated. The code merely presumes that a constant electric potential V (in volts per toroidal turn) exists within the torus, supplied, if you like, by magic. The correct terms for electric

drift (Hall current) are included in the guiding-center equations, and kinetic energy is supplied to (or removed from) the particles by the potential at each timestep according to

$$\Delta W^n = - (qRV)\Delta\phi^n \quad (\text{II.24})$$

where $\Delta\phi^n$ is the toroidal angle the particle has moved through from time $n-1$ to time n .

2.4.2 Particle Pusher

At each timestep, then, the following quantities are known for each particle: position in three dimensions, magnetic moment and kinetic energy. The values of all components of $\underline{B}$ can be found by interpolation (though in most versions B_ϕ is known exactly). A straightforward bilinear interpolation is used. The component of the particle's velocity perpendicular to the magnetic field may be calculated directly from (II.23b). By definition,

$$\gamma \equiv \left(1 - \frac{v^2}{c^2}\right)^{-1/2} \approx \left(1 - \frac{v_\parallel^2 + v_G^2}{c^2}\right)^{-1/2} \quad (\text{II.25})$$

Equation (II.25) may be used with the known value of v_G to calculate the velocity parallel to the magnetic field. Equating equivalent expressions for $1/\gamma^2$,

$$\frac{1}{\gamma^2} = \frac{m_0 v_G^2}{2\mu B} \approx 1 - \frac{v_\parallel^2}{c^2} - \frac{v_G^2}{c^2} \quad (\text{II.26})$$

hence

$$v_\parallel^2 = c^2 - v_G^2 \left(1 + \frac{m_0 c^2}{2\mu B}\right) \quad (\text{II.27})$$

The form of (II.27) is particularly handy, since it may be used to check for the particle's arrival at a mirror point (the end of a banana). As the particle passes through a mirror point in the course of one timestep, $v_{\parallel}^2$ in (II.27) will go negative, indicating that the particle has entered a region forbidden by its laws of motion. It will not have entered very far, however, for as it approached $v_{\parallel} \rightarrow 0$ and the error incurred by not turning the particle exactly at its mirror point is very small (see Appendix A1). Mirroring is very simply accomplished in the code by keeping track of the sign of $v_{\parallel}$ for every particle and always applying that sign to $\sqrt{|v_{\parallel}^2|}$ (it is stored conveniently as the sign of μ , which we know to be positive definite). When $v_{\parallel}^2$ is calculated as being negative, the sign of $v_{\parallel}$ is changed.

Finally, using the resulting $v_{\parallel}$ and the appropriate derivatives of $\underline{B}$ at the particle's position, the drift velocity v_D of its guiding center due to all drifts is calculated using¹⁰

$$v_D = c \frac{(\underline{E} \times \underline{B})}{B^2} + \frac{v_{\perp}^2}{2\omega_G} \frac{\underline{B} \times \underline{\nabla} B}{B^2} + \frac{v_{\parallel}^2}{\omega_G} \frac{\underline{B} \times \underline{B}}{R^2 B}$$

$$= \frac{c}{B^2} \underline{E} \times \underline{B} + \frac{mc}{2e} \frac{v_{\perp}^2}{B^2} \underline{b} \times \underline{\nabla} B + \frac{2cv_{\parallel}}{eB^2} \underline{B} \times (\underline{b} \cdot \underline{\nabla}) \underline{b} \quad (\text{II.28a})$$

$$= \frac{b}{B} \times \left[-\underline{E} + \frac{\mu c}{\gamma e} \underline{\nabla} F \right] + \frac{c \gamma m_0 v_{\perp}^2}{eB} \underline{b} \times (\underline{b} \cdot \underline{\nabla}) \underline{b} \quad (\text{II.28b})$$

where ω_G is the gyro-frequency (I.2), $\underline{b} \equiv \underline{B}/B$, and as usual $B \equiv |\underline{B}|$. The term-by-term form of (II.28) actually used in GUIDON for each of the three components is derived in Appendix C. The total

velocity of the guiding center may now be calculated simply as the sum

$$\underline{v} = \underline{v}_{||} + \underline{v}_D \quad (\text{II.29})$$

Knowing the particle's velocity (since the particle and its guiding center are in the model identical), we may update its position from time n to time $n+1$ (actually, some time-centering difficulties exist: see 2.5.2). This is accomplished by a "leap-frog-corrector" technique, in which a trial value for the position at $n+1$ is generated by a standard leapfrog procedure; using that new position and new local fields, (II.28-29) are used to find a trial $\underline{v}$. Finally, position is updated by a simple, centered difference where velocity is an average of $\underline{v}^n$ and the trial value at $n+1$. $\underline{v}^{n+1}$ can then be evaluated. Denoting position as $\underline{r}$ and the trial values with asterisks, the four-step procedure is:

$$\begin{aligned} \text{Step 1 } (\underline{r}^{n+1})^* &= \underline{r}^{n-1} + 2\Delta t \underline{v}^n \\ \text{Step 2 } (\underline{r}^{n+1})^* &\rightarrow [\text{guiding-center formalism}] \rightarrow (\underline{v}^{n+1})^* \\ \text{Step 3 } \underline{r}^{n+1} &= \underline{r}^n + \frac{\Delta t}{2} \left[\underline{v}^n + (\underline{v}^{n+1})^* \right] \\ \text{Step 4 } \underline{r}^{n+1} &\rightarrow [\text{guiding-center formalism}] \rightarrow \underline{v}^{n+1} \end{aligned} \quad (\text{II.30})$$

With the particles at new positions at time $n+1$, it is now necessary to solve for new values of the self-consistent fields over the whole mesh, after which this process is repeated.

2.4.3 Current Algorithm: Charge Sharing

Although the particles are moved about as though each were a single deuteron or electron (that is, each given the charge and

mass of one deuteron or electron), we want each particle to represent many within the plasma. Thus its motion is regarded as the motion of a single sample deuteron or electron, but when self-consistent fields are generated we expect its charge to be the charge of whatever fraction of the total plasma it represents.

Local current density in statamperes/cm² is defined as

$$j = \frac{qv}{\Delta R \Delta Z} \quad (\text{II.31})$$

for each particle. We consider only the toroidal contribution to current, as already explained. In addition, the charge on any particle Q is actually defined as the total charge on a ring traveling completely around the torus. For a given value of the input parameter N , the total number of electrons or deuterons per particle, these variables are defined:

$$Q_{\text{ring}} = 2\pi e N \quad (\text{II.32})$$

$$Q = \frac{Q_{\text{ring}}}{2\pi e \Delta R \Delta Z} \quad (\text{II.33})$$

and thus

$$j_{\phi} = eQ \frac{d\phi}{dt} \quad (\text{II.34})$$

It may seem at first that a factor of R is missing from (II.34), since $v_{\phi} = R \frac{d\phi}{dt}$. This factor has already been canceled by dividing out a multiplicative R which would appear on the right-hand side of (II.32). This particular set of definitions is chosen because it conserves charge logically. If Q_{ring} were defined initially as $2\pi e R_c N$, then charge in the problem would change

whenever R changed; consequently current would increase or decrease anomalously with radius.

The charge of each particle must now be assigned a location in the grid for the solution of (II.6). We cannot simply define all the particle's charge as being at its instantaneous position because the grid-point nature of the definition of $\underline{B}$ would permit a single particle to move under the influence of its own field (unless it were precisely in the center of a cell). Some form of averaging the charge over adjacent cells so that the self-force is always zero is necessary. The simplest, most common of such techniques is used in GUIDON; it is called "area weighting," after the explanation which shows its logic most intuitively.

In Fig. 20, the particle--referred to in common vernacular as a "macro-particle" because of its now-macroscopic cross section--is at point $\otimes$. A new cell, exactly centered at point $\otimes$, is imagined as drawn about it. Portions of the particle's total charge are then assigned among the four neighboring cells, at points ① - ④, according to the corresponding ratios of areas ① - ④ to the total area of a cell.

In codes which calculate all the electromagnetic interactions among particles, the sharp square edge of such a macro-particle must often be shaped (as, for instance, a Gaussian) to prevent anomalous high-frequency components from being introduced into the radiated fields. Since GUIDON has no radiating fields and is essentially quasi-static in nature, no such correction to the simple bilinearity of area-weighting is necessary.

FIGURE 10: Area Weighting

2.5 Difficulties and Limitations of the Particle Model

This section discusses some matters which are purely numerical, and some which arise from both numerical and physical models and represent further underlying assumptions.

2.5.1 Number of Particles

A particularly felicitous result of ignoring radiation fields and using the guiding-center particle pusher is that far fewer particles are required to perform a simulation than are required by more conventional codes. The statistics problems they encounter simply do not exist for GUIDON's quasi-static time dependence. The number of particles required is defined by the dynamics to be simulated--on how small a scale motions which actually affect the overall distribution actually occur. Typically, the number is an order of magnitude or more less than what one commonly thinks of for full-simulation codes (~ 100 000).

Studies must be performed during each simulation to guarantee that enough particles are being used. These may most straightforwardly (but often expensively) be accomplished by raising the number of particles and repeating runs until no differences in spatial and velocity distributions appear among the runs after long times. Fortunately, such studies of disparate experimental devices do not reveal a great variance in the particle requirements. 2000-3000 particles usually prove to be sufficient. 2500 particles were used in the TCT simulations described in Chapter III and were carefully demonstrated to be sufficient.

2.5.2 Orbit Integration

Several comments should be made concerning the integration of particle orbits in general, and the leapfrog-corrector method (II.30) specifically.

First, a necessary condition on accuracy of GUIDON's procedure is satisfied impressively: behavior at small timesteps. As is described in Appendix A1, the GUIDE code calculates particle orbits in exact vacuum fields using a highly precise Hamming predictor-corrector method. The influence of two of GUIDON's most fundamental techniques--discretizing the fields onto a grid from which they are calculated locally by bilinear interpolation, and use of the leapfrog-corrector--were tested by using them in GUIDE itself in place of its own methods. These tests were run by replacing each of the methods individually, then both together. The three runs were then compared with the same test case run using the original GUIDE, and all four results agreed to six significant figures.

Second, the accuracy of the leapfrog-corrector algorithm itself can be analyzed mathematically. Evers¹¹ shows that the amplification factor λ , defined for a difference solution of $u(n\Delta t)$ as

$$u^n = \lambda^n u^0, \quad (\text{II.35})$$

can be written for the leapfrog-corrector scheme as

$$|\lambda| = 1 - \alpha b^4 \quad (\text{II.36})$$

where $b \equiv \omega \Delta t$, ω being the characteristic frequency of the motion studied, and α is a parameter depending on the particular scheme

and how many problems are run. The negative sign results from the fact that the leapfrog correctors act to damp, rather than amplify, "computational modes." For a problem which has executed 8 cycles, Byers shows (by different means) the square of the total error for the leapfrog corrector to be 0.83 (for $b = 0.2$). Thus, using the approximation (also used by Byers)

$$|\lambda|^N \sim 1 - \alpha N b^4, \quad (\text{II.37})$$

we may solve for α from

$$|\lambda|^{2N} = 1 - 2N\alpha b^4 = 0.83 \quad (\text{II.38a})$$

or more simply

$$\alpha 2N b^4 = 0.17 \quad (\text{II.38b})$$

Since the total number of timesteps $N = (2\pi/b) \times (\text{number of cycles})$ we have

$$\alpha(2)(2\pi)(8)b^3 = 0.17$$

and since $b = 0.2$,

$$\alpha = 0.212. \quad (\text{II.39})$$

An estimate of the effective value of α in GUIDON was made by examining a TCT simulation which was run with Δt slightly too high (parametric studies of Δt effects were conducted during TCT simulation). The reduced (damped) orbits thus produced were compared with correct ones (using "passing" particles whose poloidal orbits were approximately circular) to measure $|\lambda|^2$, and hence compute the effective α using (II.38a). The resulting "measured" value was $\alpha = 0.24$.

Third, although the orbit integration involves only an ordinary differential equation and is not subject to the customary sort of Courant condition limiting the timestep, a sort of effective Courant condition does exist. When the interactive motion is very complex (especially if involving turning points or locally strong fields), it is possible to have a timestep small enough for overall accuracy (according to II.38a) but too large at certain crucially curved sections of the particle's orbit. In such regions the straight-line segments of each timestep may simply be too long to approximate the curvature properly, and the entire subsequent motion will be grossly incorrect. This motion is illustrated in Fig. 11.

Fig. 11. Effective Courant condition.

The practical timestep limitation introduced by this difficulty is

$$\Delta t = \frac{\hat{r}}{2\hat{v}} \quad (\text{II.40})$$

where $\hat{r}$ is the smallest radius of curvature involved in the motion and $\hat{v}$ is an average velocity in that region. This expression is

chosen since it provides roughly 30 points in the definition of a circular orbit, a standard acceptable figure. It should be obvious from the sample stability requirement placed on solution of the ordinary differential equation,¹² that this limitation is a restrictive one only when $\hat{r} \lesssim \Delta r$, where Δr is the average distance moved by a particle in a timestep. Thus the criterion is not terribly stringent and can be easily guarded against with some practical knowledge of the simulation's parameters.

Finally, a more serious difficulty arises from the fact that the basic calculation (as shown schematically in Fig. 3) is not properly time-centered. The use of the guiding-center equations implies that a time integral has already been performed over one gyro-period, the result being that we have no equation of motion for dv/dt ; we have merely an algebraic expression for $\underline{v}$ at each timestep. Proper centering would have the velocities, currents and fields defined half a timestep off from the positions, leaving a simple $d\underline{r}/dt = \underline{v}\Delta t$ for the particle pusher. Unfortunately, $\underline{v}$, $\underline{r}$ and $\underline{B}$ are all known at even timesteps and are functions of one another. The inconsistencies which result are obvious when one considers that both $(v^{n+1})^*$ and v^{n+1} in (II-30) are evaluated using $\underline{B}^n$.

There is no well-closed solution to this problem since one must always be evaluating either $\underline{v}$ or $\underline{B}$ using improper values of $\underline{B}$ or $\underline{r}$. How well the scheme that is used represents the evolution of the fields is a critical test of the fundamental assumption of infinitesimal Δt upon which all approximations to the differential

quantities depends. Clearly the quasi-static nature of the solution helps. But we still must require somehow that things do not change too much in one timestep. A method of gauging how severe the changes are is to see whether the single application of $\underline{v}^{n+1} = f(\underline{x}^{n+1}, \underline{E}^n)$ followed by BPHI is enough to produce fields which have properly converged.

This can be done by a sort of intermediate relaxation process. Having received a $\underline{E}^{n+1}$ based upon $\underline{v}^{n+1}$ from (II.30), we may compute $(\underline{v}^{n+1})_{\text{corrected}}$ using $\underline{v} = f(\underline{x}^{n+1}, \underline{E}^{n+1})$, and then apply BPHI to get $(\underline{E}^{n+1})_{\text{corrected}}$. This relaxation process may be applied as many times as one likes between timesteps. It's correction to the logic flow of Fig. 3 is shown in Fig. 12, with n relaxations.

An important point is that in the velocity update in the relaxation cycle only the drift components are recalculated; v_G and $v_{||}$ remain as calculated in the first (ordinary pass). If v_G and $v_{||}$ are also recalculated, the result is simply a continuous re-partitioning of energy among all the particles' perpendicular and parallel motion which never converges. In effect, the fixed spatial inhomogeneities allow the particles to trade re-partitioning effects back and forth by raising and lowering local fields.

Relaxation performed between the drift velocities and the fields typically converges to within 5 significant figures in one or two cycles. Fortunately (for the code's running time), extensive tests performed with versions of GUIDON containing 1, 2, 3 and 5 relaxation cycles per timestep showed no significant differences in results, so the relaxation cycles were removed from production

FIGURE 12: Relaxation Cycle (n times)

versions of the code. Figure 13 shows the agreement between versions in a relatively extreme case (where particle current was high and many particles were trapped); some individual differences occur but the macroscopic dynamics is unaffected. The difference, even in this extreme case, between 3 and 5 relaxation cycles, was negligible.

A further motivation for using this relaxation scheme is discussed in 2.5.5.

2.5.3 Applicability of the Guiding-Center Model

Clearly the assumptions discussed in 2.4.1 are not met by every experimental device with the proper geometry. Obvious exceptions are those for which radiative or collisional processes may significantly alter the energy distribution of the plasma. One must also, however, be careful to ensure that the adiabaticity criterion is satisfied by the ratio of particle energy to the strength of field gradients.

Usually this resolves as a test of the size of a gyro-radius, since the most fundamental assumption is that the particle sees the same total field strength throughout one gyro orbit. Three types of field gradients may be responsible for difficulties-- poloidal, toroidal, or time-dependent. Since the first two are static field gradients they can easily be tested by a single comparison of gyro-radius to the largest gradient at any one time. A good rule of thumb is that the gyro-radius should be smaller than the cell size, but with discrete particles we have not the hydrodynamicist's comfortable guarantee that motions on scales

Fig. 13a. SPIRE-like Electron Ring Accelerator with strong seed current of "passing" particles. Configuration after 400 cycles with a version of the code in which no "relaxation" of B is performed. The contours are surfaces of constant $|B|$.

Fig. 13b. Same as 13a, run with code in which B relaxation cycle is performed 3 times per timestep. The result for relaxing 5 times per timestep is identical to this one.

smaller than the cell size are negligible (in terms of the model). One should require at least as a necessary condition that no fields vary as much as 5% over a gyro-radius.

A similar criterion must be applied with regard to the local time-dependent changes in the field, $\partial B/\partial t$. The problem is more difficult to analyze for complex motions because even after one has completed a calculation and checked the largest value of $\partial B/\partial t$ within a gyro-period, one still has no guarantee that the variation was not anomalously low due to finite gyro-radius effects. This question can be partly dealt with by running duplicate problems with Δt significantly below a gyro-period and comparing results. For the rest, it is usually sufficient to estimate the manner in which finite gyro-radius effects would appear; if they would be such as to lower $\partial B/\partial t$, the question is answered. If they would tend to raise it, an estimate of the rate can be compared with initial cycles of a calculation.

For most problems (and all those reported here), the problem does not seem serious and the final method was applied only heuristically; $\partial B/\partial t$ was simply not that large a term (see 2.9). Careful comparisons of $\partial B/\partial t$ in a gyro-period and $\partial B/\partial r$ within a gyro-radius were made for all production runs and resulted in high confidence in the model.

2.5.4 Renormalization of Magnetic Moment

The interdependence of γ , $\underline{A}$ and $\underline{B}$ causes a difficulty in initializing problems. The first step in preparing for particles' motion once they have been located in the vacuum and plasma current

fields is to calculate their magnetic moment. We may not use the value of μ resulting as the permanently conserved quantity, however, because the self-consistent fields of the particle currents were not taken into account in its calculation.

This difficulty is readily disposed of using a simple relaxation or renormalization procedure in which the particles at fixed positions in the most recently defined $\underline{B}$ have the value of μ recalculated. Total velocities and $\underline{B}$ are then redefined on the basis of these μ 's, and the procedure repeated. It converges quite rapidly, usually in three and thus far never in more than five cycles; ten are still provided in the code.

The suitability of this convergence was tested using conditions in which the desired total $\underline{B}$ (vacuum plus self-consistent) was known.* Results agreed to five significant figures.

2.5.5 Particle "Sticking"

This was the most severe difficulty encountered in the development of a workable guiding-center particle pusher. The visible manifestation is that many particles may spend long periods of time--even hundreds of cycles--apparently not moving in the poloidal plane, or tracing out minute fractions of what their orbits should be.

The explanation is simple enough. When a particle has just

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* The particles were put in a strong beam and results for $\underline{B}$ outside the beam compared to those of a Biot-Savart solution for equivalent current.

mirrored its velocity is still quite small for at least two or three, and often as many as five or six timesteps. During this otherwise uninteresting part of its orbit, * the particle is prey to the time-dependent part of B . Quite small changes in local magnetic field caused by the collective effects of the other particles may suddenly make it appear that the particle in question has passed a mirror point again. For example: at cycle n particle p mirrors, changing the sign of $v_{||}$, but because of its small velocity it travels less than a cell width, to a point where $|B|$ is only 1% less than B_{mirror} . Between cycles n and $n+1$, B_{poloidal} is updated, and changes at the four neighboring mesh points raise $|B|$ at the location of p by 2%. At cycle $n+1$ p will again change the sign of $v_{||}$, heading actually toward a region of higher $|B|$. The particle will have become "stuck" in an artificially created magnetic well.

The effect is an anomalous result of the discretization of the time scale imposed by the numerical solution. In this form, it is peculiar to the guiding-center equations of motion and is essentially unavoidable. In fact, it exhibits all the perfidy of a real physical effect, for it often seems that any particle which might find a way to become stuck will do so. A catalogue of

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* Its parallel velocity being several orders of magnitude lower than usual, the solution of (II.27) becomes complicated by the machine's difficulty in taking differences of very large, nearly equal numbers; see Appendix A1.

unsuccessful attempts to solve the problem will serve to illustrate its tenacity:

(a) Lower Δt so that changes in $\underline{B}$ per timestep are smaller. Unfortunately the particles then spend more timesteps near their turning points and are equally vulnerable.

(b) Redefine the turning point as being the point at which $v_{||}$ falls below a minimum value. This method requires more time-consuming coding per particle per timestep and destroys much of the accuracy of the orbit if the minimum value of $v_{||}$ is chosen large enough to do any real good. (Recall that the decrease in $v_{||}$ generally covers 7, 8 or 9 orders of magnitude.)

(c) Recast the guiding-center equations or reprogram them. Unfortunately the difficulty is inherent in the equations themselves and not a function of their particular expression.

(d) Use the relaxation method of 2.5.2 (Fig. 12) in hopes that the slightly-more-accurate fields will not cause particles to stick. They do. The point is that local raising of the fields may be a real physical effect of the behavior of the other particles; the un-physical thing is the approximation that the particle sees (and must deal with) all this change at once.

(e) Smooth the fields out at each timestep; that failing, smooth the currents as well; that failing, try smoothing in conjunction with relaxation, as above. The smoothing algorithm actually applied to ψ (and to $\underline{j}$, though with no noticeable effect) is

$$\psi_{i,j} = \frac{1}{16} \left[4\psi_{i,j} + \sum_{\substack{8 \\ \text{neighboring} \\ \text{cells}}} (\beta\psi)_{i',j'} \right] \quad (\text{II.41})$$

where the multiplicative constants β are defined for the 8 cells surrounding (i,j) according to the numbers displayed in Fig. 14:

$j+1$	1	2	1
j	2	4	2
$j-1$	1	2	1
	$i-1$	i	$i+1$

Fig. 14. Constants for smoothing algorithm (II.41).

This algorithm is the proper two-dimensional generalization of quadratic smoothing. Unfortunately, none of these attempts was fruitful, for precisely the reason explained in (d).

(f) Provide a specific limitation on mirroring: keep track of the number of cycles since each particle last turned and forbid any particle to turn with x cycles of its last turn. There is a danger here, in the artificiality of the method, particularly of preventing turnings which really ought to occur. It was found that for sufficiently turbulent problems (SPIRE startups), x had to be at least 7, raising the danger significantly. Further, the method is expensive both in terms of added core requirements and length-

ened running time.

(g) Combinations of the above. None of these worked significantly better than the individual attempts, though the most expensive--(d) plus (f)--produced the best results of any.

The sufficient solution finally arrived at is actually simpler than any of these attempts: recalculate μ for each particle at its turning point. The results are beautifully smooth trajectories with precise turning points. The approach was avoided for a time as tampering too seriously with the physics--constancy of μ being, after all, fundamental to the formulation of the model. Exhaustive studies of the effects of the technique show that $\Delta\mu$ is usually on the order of 0.5%, and rarely as much as 2%. Some changes in two turnings amounting to as much as 4% were observed occasionally in turbulent problems, but subsequent changes always brought the total $\Delta\mu$ back within 2%.

Figure 15 illustrates the phenomenon of sticking, the various effects of some of the attempted solutions, and a correct trajectory.

Buneman¹³ has kindly suggested that the technique may actually be physically beneficial by taking into account small deviations from adiabaticity ignored by the rigid model. In the author's view, the temporally discrete guiding-center formalism introduces a sort of artificial non-adiabaticity into the problem, in much the same manner in which spatially discrete meshes introduce artificial diffusion into hydrodynamics codes.* The final attitude towards the effects is similar: minimize it, then put up with it.

Fig. 15a. A particle becoming "stuck" attempting to turn near the symmetry plane. A SPIRE run with moderate seed current.

Fig. 15b. A small banana created by sticking: the correct banana should be five times as long, drifting outward in R.

Fig. 15c. Correct trajectory of the particle in Fig. 15a, showing drift of banana.

Fig. 15d. The ... produced by combining methods (d) and (f).

FLUX MAX= 5.0336E+06 MIN= 2.5535E+06

Fig. 15e. Combination of methods (d) and (e).

Fig. 15f. Correct trajectory of a SPIRE startup run, with
violently fluctuating fields near the center.

Fig. 15g. Continuation of Fig. 15e, with banana forming as particle drifts into more stable fields.

Fig. 15h. Same as fig. 15f, using methods (d) + (e) + (f) --
relaxation and smoothing of fields plus turn limitation.

Fig. 15i. Continuation of Fig. 15h--compare with Fig. 15g:

The particle has become stuck attempting the first turn
in forming the banana.

Fig. 15j. Same as Fig. 15f, showing the effects of smoothing method (e) alone: The particle misses a turn and crosses a saddle in the field.

2.6 Input and Output

2.6.1 Initializations

Spatial initialization of the particles is generally specified as a circular cross section whose center coordinates and radius are input quantities.[†] One may also choose a profile of density versus minor radius. All three profiles shown in Fig. 5--constant, $1/R^2$, and skin current--are available; in addition one may choose a $1/R$ profile or a profile defined

$$n = n_0 \left[1 - \frac{3}{4} \left(\frac{r}{r_0} \right)^3 \right] \quad (\text{II.42})$$

which is the projected beam profile for the Princeton TCT proposal.

In all these cases the plasma radius is divided into a number of zones (also available as input) whose final density is assigned according to the choice of profile. Particles are then distributed randomly within each zone in numbers appropriate to produce the correct profile and use up all the particles. The charge-per-particle N in (II.32) may be read as input or adjusted by the initializer to produce a desired physical density for the profile (II.42). In addition, the charge-per-particle may be increased from 1 to any final value from $t = 0$ to a specified final time t_{inj} as a linear function of time. This process simulates the injection of a beam and is used in the TCT studies (Chapter III).

Velocity initialization follows the method of GUIDE. The

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* In many cases anomalous diffusion is viewed as a salutary effect, too.

† Previous versions also include an option for a square.

kinetic energy per particle is input, and also a fraction $T_{\perp}$ of the kinetic energy which is in motion perpendicular to B (gyro-motion).

This fraction is defined as follows:

$$p_{\perp}^2 = T^2/c^2 + 2m_0T \quad (\text{II.43})$$

so define

$$p_{\perp}^2 \equiv T_{\perp}^2/c^2 + 2m_0T_{\perp} \quad (\text{II.44})$$

and using the definition of magnetic moment (II.23a), calculate

$$\mu = \frac{T_{\perp}}{B} \left(\frac{T_{\perp}}{2m_0c^2} + 1 \right) \quad (\text{II.45})$$

The fraction KPF is

$$\text{KPF} \equiv T_{\perp}/T \quad (\text{II.46})$$

KPF is read in for all particles as a single number, but it may be randomized over any specified range between KPF and zero if desired. The value of T per particle may also be randomized over a specified range. Obviously, any particle assigned KPF = 1 is by definition sitting at its mirror point; consequently, values of KPF differing from 1 effectively specify the particle's distance from its mirror point. If it is on a closed flux contour, it may indeed have none and be a passing particle; if the machine has open flux contours (such as in a toroidal quadrupole), the turning point may well lie outside the machine (hence the particle would be in the loss cone in velocity space).

For beam injection problems, the actual values of KPF must be calculated for each particle by the code, since they will be specified by the injection parameters. The injection angle β of a beam is shown in Fig. 15.

Fig. 15. Beam injection angle viewed from above the torus.

The components of the injection velocity $\underline{v}$ are

$$\begin{aligned} v_R &= v \sin \beta \\ v_\phi &= v \cos \beta \end{aligned} \quad (\text{II.47})$$

where v was derived from the input kinetic energy using

$$\begin{aligned} T &\equiv 1 + \frac{W}{m_0 c^2} \\ T^2 &= \left(1 - \frac{v^2}{c^2} \right)^{-1} \\ v^2 &= c^2 \left[1 - \frac{1}{T^2} \right] \end{aligned} \quad (\text{II.48})$$

What we need to get KPF are the components of $\underline{v}$ perpendicular and parallel to the magnetic field. The angle between $\underline{v}$ and $\underline{B}$ is α , defined by

$$\cos \alpha = \frac{\underline{v} \cdot \underline{B}}{vB} \quad (\text{II.49a})$$

$$\text{or} \quad \cos \alpha = \frac{v_{R/R} + v_{\theta/B}}{vB} \quad (\text{II.49b})$$

And by further obvious trigonometry,

$$v_{\perp}^2 = v^2(1 - \cos^2 \alpha) \quad (\text{II.50})$$

Using $\cos \alpha$ from (II.49b) we may calculate $v_{\perp}^2$ from (II.50) and use it in (II.23b) to calculate μ .

Once μ has been found, KPF must still be calculated by reversing (II.45) for use in the renormalization of μ .

2.6.2 Outputs Available

Several examples of outputs normally produced by GUIDON have already been shown. In summary, the following plots are available for the trial and equilibrium solutions (and at intermediate steps, if desired): contour plots of B and ψ , and plasma current plotted as a function of radius at the value of Z on which ψ_{max} lies (presumably $Z = 0$ for the equilibrium). Once particles have been introduced, the following are normally produced: physical positions of all particles superimposed on the E plot; the trajectory of a sample particle showing that portion of its orbit since the last call to the plotting routine, superimposed on the contour of

the latest flux surfaces; a scatter-plot showing the positions of all particles in the v_R versus v_z velocity space. These plots may also be called for: the particle current versus radius for that value of Z containing the largest single current density; the energy distribution function of the particles; the velocity distribution function for $\underline{v} = v_{||} + v_{\perp}$ of the particles (different from the energy distribution function because it does not contain the v_G contribution); a plot of q , as defined by (I.1), versus radius for the same Z as the plasma current plot. Some examples are shown in Figs. 16a-f.

Virtually all the information generated by the code is immediately available as printed output, including complete displays of $\underline{R}$ and ψ and $\underline{r}$, $\underline{v}$, μ and T for every particle.

A recent version of the code makes a 15-mm movie showing the trajectories of all particles against the changing flux surfaces.

2.7 Additional Tests of Accuracy

Several relatively independent checks on the accuracy (and convergence) of the two basic models have been made.

The fluid plasma model has been checked by comparing the pressure profiles and peak pressure p^* resulting from (II.11) against experimental measurements for equilibrium solutions of the Princeton ATC devices and analyses for the Princeton PLT device (under construction). In both cases the agreement was within the estimated experimental error of the measurements.

In addition, convergence of the fluid plasma model was checked by using the pressures to calculate the total energy in the com-

Fig. 16.. Velocity phase space scatter plot for a TCT simulation.

The "hole" at low velocities is a result of the fact that
all the particles are passing.

Fig. 16b. Velocity phase space scatter plot for a toroidal quadrupole, in whose fields motion is relatively highly ordered. The high density near $v = 0$ results from particles near their turning points (nearly all these particles are trapped).

Fig. 16c. Particle current density plot, J vs radius at $Z = 0$.

This is a TCT injected beam of counterstreaming ions, hence all velocities are negative.

Fig. 16d. $q \equiv rB_p / RB_p$ as a function of radius, plotted only across the width of the plasma current, for a TCT. The apparent discontinuity is caused by the presence of a strong particle current.

Fig. 16e. An initial velocity distribution function for a SPiRE run (showing particles distant from mirror points).

Fig. 16f. Same SPIRE run after 200 cycles, showing effective
"thermalization" of drifts.

tained plasma, as

$$\text{Plasma Energy} = \sum_{\substack{\text{all cells} \\ \text{containing} \\ \text{plasma}}} P(2\pi R) \Delta R \Delta z \quad (\text{II.51})$$

This value compared easily and well with total energy in the flowing plasma for PLT and ATC and the planned values for TCT.

The second adiabatic invariant J ,

$$J \equiv \int m \mathbf{v}_{\parallel} \cdot d\mathbf{s} = m_0 \int \gamma \mathbf{v}_{\parallel} \cdot d\mathbf{s} \quad (\text{I.5})$$

is calculated for trapped particles in GUIDON just as in GUIDE (see Appendix A1). Though the conservation is not quite as phenomenally good in the larger code (presumably due to the inherent non-adiabaticity already discussed), it does remain generally within 1 part in 10^4 .

On the other hand, the ϕ component of canonical angular momentum

$$P_{\phi} \equiv \gamma m_c R v_{\phi} + \frac{e}{c} R A_{\phi}, \quad (\text{I.5})$$

which is an exact rather than an approximate invariant of the motion, has been calculated as being conserved in GUIDON to better than 1 part in 10^6 for both trapped and passing particles.

2.8 Induced Electric Fields

One important contribution of electric fields may be included in the calculations without violating the initial assumption ignoring electrostatic interactions among the particles. Poloidal motion of the particles makes the toroidal current at any one

point, and hence the poloidal fields generated by it, a function of time; this time variation of poloidal fields induces a toroidal electric field E_ϕ . This field can be of interest because, in accelerating or decelerating the particles in the toroidal direction, it can change the location of the turning points and hence revise the orbits of all trapped particles. Further, in these devices (such as SPIRE) in which an accelerating potential is crucial to the function, this induced E acts to modify that potential.

Beginning with the Faraday Law,

$$\nabla \times \underline{E} = - \frac{1}{c} \frac{\partial \underline{B}}{\partial t}$$

we may retain either of the components of $\nabla \times \underline{E}$ containing the non-zero E_ϕ term; using the R component,

$$- \frac{\partial E_\phi}{\partial z} = - \frac{1}{c} \frac{\partial B_r}{\partial t} \quad (\text{II.52})$$

And using (II.5) for B_r ,

$$\frac{\partial E_\phi}{\partial z} = \frac{1}{c} \frac{\partial}{\partial t} \left[- \frac{1}{R} \frac{\partial \psi}{\partial z} \right] \quad (\text{II.53})$$

or

$$\frac{\partial E_\phi}{\partial z} = - \frac{1}{cR} \frac{\partial}{\partial z} \left[\frac{\partial \psi}{\partial t} \right]$$

and finally, integrating in z ,

$$E_\phi = - \frac{1}{cR} \left(\frac{\partial \psi}{\partial t} \right) \quad (\text{II.54})$$

Note that as a result of the spatial centering of (II.54),

values of E_ϕ are stored on the same mesh as ψ and $\underline{u}$, rather than with $\underline{B}$. The difference expression (where n denotes the time index) is

$$E_{k,\ell}^{n+1/2} = - \frac{1}{cR_{k,\ell}\Delta t} (\psi_{k,\ell}^{n+1} - \psi_{k,\ell}^n) \quad (\text{II.55})$$

There is an additional time-centering difficulty involved here, since, as has been discussed, the code defines quantities only at integral values of n . The inaccuracy arising from this discrepancy is felt to be a small one, and the $E^{n+1/2}$ of (II.55) is treated as E^{n+1} by the code. Approximations of this sort are common to many particle codes; at any rate, sensible methods for centering this term properly would essentially double the code's running time and it seems ludicrous to go to that much trouble for what is at most a second-order correction.

Core space is saved by computing the E_ϕ within BPFI, after the ψ^{n+1} have been calculated but before they have been stored in the ψ mesh; thus, old values of ψ need not be saved.

For simulations in which the great majority of particles are passing ones the effects of E_ϕ are scarcely noticed. In the TCT simulations, for example, they were neglected entirely. They were first included in the code at the suggestion of Charles Hartman, who believed them important for SPIKE initialization. Though the effects were notable, they did not affect the results (see 4.1).

III. TCT SIMULATIONS

3.1 The Two-Component Torus

In a 1971 letter Dawson, Furth and Tenney¹ revived the early concept of a fusion reactor which uses two separate plasma components, a relatively cold one and a hot beam injected into it (in some earlier versions, the idea is referred to as the Wet-Wood Burner). Multiplication of the beam energy through charged-particle fusion products as the beam thermalizes makes the scheme especially attractive; this process is described in detail by Post.²

The fundamental idea of the TCT is to begin with a cold tritium plasma well-confined within a Tokamak. This plasma can be heated somewhat by injection of a preliminary beam, then more by adiabatic compression.³ The confined "target" plasma will then consist of warm ions and hot electrons. An energetic beam of deuterons is then injected into the Tokamak, and heating through collisions (and micro-instabilities with the hot electrons) takes place.

The closed magnetic geometry of the system requires that the beam actually be injected as neutral deuterium, with ionization occurring quickly after injection into the tritium target. It is predicted that hydromagnetic stability conditions should be improved over a conventional D-T Tokamak since most of the plasma energy is concentrated in the energetic deuterons, rather than being spread among three components (counting electrons), and the total plasma pressure required for containment may be lowered. That is, presuming there are no disastrous interactions between injected deuterons and confined tritons, a major question which the

present study addresses.

The big advantage of the TCT system over a conventional D-T Tokamak is that the effective $n\tau$ requirement for operation is only $10^{13} \text{ cm}^{-3} \text{ sec}$,⁴ whereas the normal Lawson criterion requires $n\tau \gtrsim 10^{14} \text{ cm}^{-3} \text{ sec}$. Present Tokamaks are already in the regime $\sim 10^{12} \text{ cm}^{-3} \text{ sec}$ and the PLT should soon be operating in the range $\gtrsim 10^{13} \text{ cm}^{-3} \text{ sec}$. This incipient practicability (coupled with the fact that very high current neutral beams have recently become available with favorable results of neutral beam injections in Culham, Princeton and Oak Ridge Tokamaks) spurred the AEC to initiate a large design program for a prototype TCT, the first stage of which culminated in theoretical studies presented in June of 1974.

GUIDON simulations contributed to these studies⁵ in three ways: (1) calculations of ideal MHD equilibria of the target plasma; (2) single-particle orbits of hot deuterons injected into the target plasma; and (3) injection of pulsed neutral beams into the target plasma, including self-fields of the beam particles and modifications of the target plasma's equilibrium state, for beams injected parallel (co-injected) and anti-parallel (counter-injected) to the sign of the toroidal field and triton plasma current, and for beams injected both ways simultaneously.

Specific results for several cases are presented in the next section. The most important qualitative contribution of the study is a conclusion that on a short time scale counter-injected beams significantly distort the equilibrium plasma due to the mutual repulsion between opposing currents; beam particle losses may also

occur. Co-injected beams do not exhibit this behavior.

3.2 Simulation Results

3.2.1 Scenarios

Two general proposed TCT scenarios were studied in these simulations; the parameters were suggested by preliminary design studies performed jointly by Princeton and Westinghouse.⁶ The scenarios are shown schematically in Fig. 17a, b.*

Scenario I:

- (a) Ohmic heating is assumed to produce plasma temperatures in the confined plasma of about 1 keV.
- (b) Heating by neutral injection of a beam at 80 keV raises this to about 4 keV.
- (c) Thermalization of heating beam and target.
- (d) Major radius compression (by a factor 1.25) raises this to a final temperature of 6 keV.
- (e) Injection of 150 keV fusion beam to achieve a "break-even experiment."

Scenario II: (It is assumed in this Scenario that efficient high-current neutral beams are available only at energies less than about 80 keV.)

- (a) Tritium target plasma warmed by ohmic heating.
- (b) 80-keV deuterons injected with sufficient current to raise the target plasma energy to 6.3 keV after
- (c) Thermalization.

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* They appear as Scenarios III and IV in Ref. 5.

Fig. 17a. Scenario I.

Fig. 17b. Scenario II.

- (d) Injection of 80-keV fusion beam.
- (e) Entire system compressed to raise the fusion beam energy to about 120 keV. The "breakeven experiment" is then defined to occur during the subsequent thermalization of the fusion beam and plasma when the plasma energy is being maintained by input from the hot deuterons.

The figures illustrating the results described in Sections 3.2.2-3.2.4 are gathered at the end of the chapter to facilitate comparisons among them.

3.2.2 Equilibria

An important result of the equilibrium calculations, in addition to the details of the flux surfaces and plasma radius, is the value of the centering field B_c (usually in the -Z direction) required to center the plasma at the desired radius. As mentioned in Chapter II, the equilibrium code is capable of handling poloidal currents, but in this simulation they are considered to be negligible.

For Scenario I there are two equilibria of interest: (A) the ohmic-heated plasma target for the heating beams, and (B) the compressed plasma target for the 150-keV fusion beams.

A. Equilibrium Target for Heating Beams

For the ohmic-heated plasma target the constant M in (II.11) is presumed equal to 1, giving a scalar pressure model

$$\begin{aligned}
 P(\psi) &= \bar{a}(\psi - \psi_0) & \psi > \psi_0 & \quad \text{(III.1a)} \\
 &= 0 & \text{otherwise} &
 \end{aligned}$$

and resulting in an approximately uniform toroidal current density

$$J_{\phi} = -cR \quad (\psi > \psi_0) \quad (\text{III.1b})$$

A set of relevant plasma parameters (taken from the Princeton/Westinghouse TOT studies) is:

$$I = 0.77 \text{ MA}$$

$$R_0 = 325 \text{ cm}$$

$$a = 64 \text{ cm}$$

and $B_T = 37.5 \text{ kG}$

where a is the approximate value of the equilibrium plasma radius.

The self-consistent plasma equilibrium results for these parameters are given in Fig. 18a, b, c. The poloidal flux surfaces in the (R, Z) domain are shown in Fig. 18a, including the effect of a uniform vertical centering field, $B_c = -350 \text{ G}$, needed to position the plasma at R_0 . The plasma boundary is indicated by the solid line in this figure. For other values of the vertical field strength the major radius of the plasma varies as in Fig. 18b. The perfectly conducting walls of the computational domain are responsible for limiting the radial excursions of the plasma as $B_c \rightarrow 0, \infty$. The toroidal plasma current density as a function of radius is shown in Fig. 18c. The radial pressure profile (not shown) has a peak value of $P^* = 4.08 \times 10^5 \text{ dynes/cm}^2$ which is consistent with a density and temperature given by

$$n^* = 10^{14} \text{ cm}^{-3}$$

$$T = 1 \text{ keV}$$

which are the proposed values.

B. Equilibrium Compressed Target for Fusion Beams

For the beam-heated plasma target, the constant M is set

equal to 2, giving a scalar pressure model

$$\begin{aligned} P(\psi) &= \bar{c}(\psi - \psi_0)^2 & \psi > \psi_0 \\ &= 0 & \text{otherwise} \end{aligned} \quad (\text{III.2a})$$

which yields a toroidal current density

$$J_\theta = -\bar{c}2R(\psi - \psi_0) \quad (\text{III.2b})$$

with a roughly parabolic profile (since ψ is a function of R). The relevant parameters used here are:

$$I = 1.0 \text{ MA}$$

$$R_0 = 260 \text{ cm}$$

$$a = 58 \text{ cm}$$

$$B_T = 46.8 \text{ kG}$$

The poloidal flux surfaces for this equilibrium plasma are shown in Fig. 19a; the plasma boundary is again represented by the solid line. The vertical field required to position the plasma at R_0 was $B_c = -600 \text{ G}$. The toroidal plasma current density as a function of radius is shown in Fig. 19b. The peak value of pressure in the radial profile at $Z = 0$ is

$$P^* = 1.629 \times 10^6 \text{ dynes/cm}^2,$$

which is consistent with a density and temperature given by

$$n^* = 10^{14} \text{ cm}^{-3}$$

$$T^* = 5 \text{ keV.}$$

For Scenario II there are three equilibria of interest:

(C) ohmic-heated uncompressed plasma--the same as case A (for the

flux surfaces of this equilibrium presuming $M = 2$, see Fig. 24a);
(D) beam-heated target plasma; and (E) compressed plasma after
fusion beam injection.

D. Equilibrium Heated Target for Fusion Beams

The uncompressed target for the 80-keV fusion beams is represented by the parabolic pressure model, Eqs. (III.2a,b). The relevant plasma parameters are:

$$I = 0.77 \text{ MA}$$

$$R_0 = 325 \text{ cm}$$

$$a = 64 \text{ cm}$$

$$B_T = 37.5 \text{ kG.}$$

The poloidal flux and plasma boundary (solid line) are shown in Fig. 20a. The necessary vertical field is $B_c = -425 \text{ G}$. The toroidal plasma current density as a function of radius is shown in Fig. 19b. The peak pressure is

$$P^* = 8.36 \times 10^5 \text{ dynes/cm}^2$$

which is consistent with density and temperature

$$n^* = 10^{14} \text{ cm}^{-3}$$

$$T^* = 3 \text{ keV.}$$

F. Equilibrium Compressed Plasma for Breakeven Experiment

After the 80-keV fusion beams have been injected and the system is compressed, the equilibrium tritium plasma has parameters:

$$I = 1.0 \text{ MA}$$

$$R_0 = 250 \text{ cm}$$

$$a = 56 \text{ cm}$$

$$B_T = 46.8 \text{ kG.}$$

The poloidal flux surfaces and plasma boundary (assuming no net residual current due to the fusion beams) are given in Fig. 21a. The vertical field is $B_c = -900$ G. The toroidal plasma current density as a function of radius is shown in Fig. 21b. Peak value in the $Z = 0$ radial pressure profile is

$$P^* = 1.647 \times 10^6 \text{ dynes/cm}^2$$

which is consistent with

$$n^* = 10^{14} \text{ cm}^{-3}$$

$$T^* = 5 \text{ keV.}$$

3.2.3 Single Particle Orbits

The guiding-center orbits of individual particles in both vacuum and plasma fields have been checked against approximate, analytically calculable orbits and the agreement is excellent. Theory predicts⁷ the existence of "stagnation" points along the $Z = 0$ axis; the orbits of passing particles when projected in the poloidal plane are circles whose centers are the stagnation points. There are two stagnation points: one for co-injected particles at $R > R_0$ and one for counter-injected particles at $R < R_0$.

Figures 22 and 23 demonstrate this phenomenon for a co-injected and a counter-injected particle, respectively. (The orbit is identified among the background flux surfaces by an arrow.) The equilibrium used is similar to Scenario 1A, the ohmic heated target plasma, except that a value of $M = 2$ is used in the pressure-defining equation (II.11) for a steeper profile (and higher value of P^*). The test particles followed are 150-keV neutrons. The center of the plasma current is at $R_0 = 333$ cm; the predicted

stagnation points are $R_+ = 335$ cm for co-injected particles and $R_- = 331$ cm for counter-injected particles.

Both Figs. 22a, b, c and 23a, b, c show deuterons injected at $R = 330, 335$ and 345 cm. The co-injected deuteron of Fig. 22 is exactly on its stagnation point in 22b, at 335 cm; on either side is it circling properly. The counter-injected deuteron of Fig. 23 is almost on its stagnation point in 23a, at 330 cm; as it moves outward the radius of its orbit increases.

Figures 24a and b show graphically the significant difference between the trajectories of co-injected and counter-injected particles at large radii, a phenomenon which will be seen to be important in beam separation.

Single-particle studies of trapped particles and of particles injected at angles other than $\beta = 0$ (see Fig. 15) have also been performed with equal success.

3.2.4 Beam Injection

There are three primary beam injection problems associated with the TCT Scenarios I and II already described: (A) Injection of the 80-keV heating beams into a 1-keV target plasma, and (b) injection of either 80-keV or (C) 150-keV fusion beams into a hot target plasma for the breakeven experiment. For each of these situations the appropriate equilibrium target plasma is the starting point for the beam injection run.

Each of the 2500 "macro-particles" injected into this equilibrium system represents a very large number (typically $\sim 10^{15}$) of deuterons, uniformly distributed around the torus in a real

system. In the model all 2500 particles are injected simultaneously at the initial timestep. However, the toroidal current associated with each particle is initially set equal to zero and is defined to increase linearly with time over a finite interval, τ_b . This has the effect of modeling the decay of the induced plasma current due to an instantaneously injected beam. At time $t = 0$ the induced current is assumed to cancel the beam particle current exactly.

The typical initial distribution of the particles is in a circular beam with a density profile

$$n_b(R) \sim \frac{1}{R} - \frac{1}{r_0} \quad (\text{III.3})$$

where r_0 is the minor radius of the beam distribution, or with the profile of Eq. (II.42). Other profiles have also been used, but the results are similar. In all of the cases to be described, the particle velocities were assumed to be tangential to the torus at time $t = 0$ (that is, $\beta = 0$); the initial sign of v_ϕ would then distinguish co-injection from counter-injection.

It should be noted that the time-scales for the computer runs are typically on the order of 100 microseconds whereas the actual beam injection time in the TCT is likely to be on the order of 100 milliseconds. At present it is not feasible to simulate the real injection times or decay times for induced plasma currents. However, by choosing the beam current buildup time, τ_b , long compared to the time for the guiding centers to circulate around the torus, a somewhat realistic picture of the true situation should be obtained.

A. 80-keV Heating Beams

The injection of 80-keV heating beams is examined for the case of neutral beam sources supplying a total of 442 A in a 30 msec pulse. These deuterons are assumed to be injected into the ohmic-heated plasma, Equilibrium (A). The magnitude of the source current is chosen large enough to yield a plasma temperature of 4 keV after thermalization of the heating beam. The mean radius of injection is 325 cm with a peaked beam deposition profile having a minor radius $r_0 = 50$ cm. Three cases were run:

(1) Both co- and counter-injected beams are used (one-half of the total current in each direction); the particle current build-up time is $\tau_b = 25$ microseconds; each beam is centered on its appropriate stagnation point;

(2) same as (1) above, except $\tau_b = 150$ microseconds;

(3) only co-injection is used; $\tau_b = 150$ microseconds.

The results for case (1) are shown in Figs. 25a-5. The toroidal particle current flowing in each direction after full buildup is about 0.90 MA whereas the plasma current is 0.77 MA. The initial spatial distribution of the beam particles is shown in Fig. 25a and the flux surfaces and plasma boundary (solid line) at the injection are shown in Fig. 25b. After 10 μ sec, when the beam currents are about ± 0.36 MA, the situation is as shown in Figs. 25c and d. On the particle position plot 25c, one sees a slight concentration of particles to the upper right of center and this is verified in the flux surface plot. The flux surface deformation one sees is due to the counter-injected beam. The plasma boundary

surface has not yet been disturbed. At 16.5 μsec , however, the distortions have become more obvious. In Fig. 25e two concentrations of particles can be identified as the co- and counter-injected beams, now spatially separated by a few centimeters. The corresponding flux plots (Fig. 25f) reveal that the plasma has been pushed toward the upper-right of the domain and its surface is no longer circular. What is apparently happening is that the co-injected and counter-injected beams are repelling one another. Any small transverse perturbation, in many cases even the one introduced by the normal differences in orbits illustrated in Fig. 24, is sufficient to start this process. Centering the two beams initially at the same point rather than at their stagnation points does not prevent this phenomenon; it enhances it. Meanwhile, the plasma tends to stay near the co-injected beam (parallel currents attract). As time proceeds, the gross motion of the system in the R-Z domain is basically an orbital motion of the oppositely directed currents about one another. The time scale for this orbital motion is observed to be less than or on the order of 1 μsec in this run.

The results for case (2) are shown in Figs. 26a-h. In this instance the beam-particle current is allowed to build up more slowly than in case (1): $\tau_b = 150 \mu\text{sec}$. Initial distributions are the same as case (1) (Figs. 26a and b). After 36 μsec , when the beam current in each direction has built up to 0.22 MA, the particles and flux surfaces are as shown in Figs. 26c and d; although gross distortions are not yet evident, the larger buildup time has allowed some of the counter-injected particles to drift

outside the plasma boundary and eventually be lost on the limiter. By 49.5 μ sec (Figs. 26e and f), the beams can clearly be seen to be separating; and by 75 μ sec with the beam current ± 0.478 MA (Figs. 26g and h), the separation of beam centers is about 30 cm and many counter-injected particles are being lost on the limiter.

Results for case (3) are shown in Figs. 27a-3. To avoid the possible undesirable effects associated with a counter-injected beam, only co-injection is used. The particle current buildup time is again 150 μ sec. Initial position of beam particles and flux surfaces are shown in Figs. 27a and b. After 40.5 μ sec, with the circulating beam current at 0.470 MA, the situation is as shown in Figs. 27c and d. There are no noticeable changes in the spatial distribution of the beam particles, and the plasma surface (solid line in 27d) has expanded only slightly further. Things remain this way all the way to 150 μ sec.

B. 80-keV Fusion Beams

It is assumed that neutral beam sources produce 60 amperes of 80-keV deuterons in a 30 msec pulse. This source current is injected into the equilibrium (C) of 3.2.3 at a mean radius of 325 cm and with a peaked deposition profile having a minor radius $r_0 = 50$ cm. Half of the source particles are assumed to be co-injected, the other half counter-injected. The toroidal beam current flowing in each direction is ultimately about 0.125 MA while the equilibrium plasma current is 0.77 MA. The buildup time for particle current in this run is taken as 25 μ sec. The initial spatial and flux surface distributions are shown in Figs. 28a and b.

After 25 μ sec, when the particle currents have assumed their full values, the situation is as shown in Figs. 28c and d. A few counter-injected particles have begun to separate from the main dual-beam; the plasma remains relatively undisturbed. After 50, 100 and 150 μ sec the beam particle distributions and flux surfaces are as shown in Figs. 28e-j, respectively. A significant number of counter-injected particles appears to be leaving the plasma and main beam, to be lost on the limiter. But the plasma has remained essentially undisturbed throughout the run; this was to be expected since the toroidal particle currents are only a fraction of the plasma current.

C. 150 keV Fusion Beams

The 150-keV deuterons are assumed to be injected continuously for 160 msec during the breakeven experiment in Scenario I, described earlier. An injection of 60 amperes is sufficient to maintain a constant plasma energy during the experiment if the energy confinement time is 200 msec. The appropriate equilibrium plasma for this injection problem is Equilibrium (B). The mean radius of injection for the particles is 260 cm with a peaked beam deposition profile having a minor radius $r_0 = 50$ cm.

Results of this case are similar to those of case (A) (heating beam injection) since the total circulating beam current after buildup is comparable to the plasma current.

(1) With both co- and counter-injected beams (one-half the total current in each direction) and a buildup time of 25 μ sec, the results after 10 μ sec are as shown in Figs. 29a and b. The

opposing beams are well separated in space and the plasma has been pushed toward the domain boundary. A few particles from the counter-injected beam have been lost at the upper edge of the domain. A limiter near the original plasma boundary would have increased the number of lost particles. Rotation of the beams about one another can be seen in the trajectory of the sample particle (arrow in Fig. 29b); it is a counter-injected particle, here moving upward and left.

(2) By increasing the buildup time to $\tau_b = 150 \mu\text{sec}$, the distortion of the plasma by the beams can be made less severe. The situation after 50 μsec is as shown in Fig. 30a and b. The longer buildup time has allowed some of the counter-injected beam particles to drift outside the plasma and eventually be lost on the limiter; however, the number of such particles is still small. By 70 μsec --Figs. 30c and d--an apparently significant fraction of the counter-injected beam has been lost, and things are getting worse. The longer injection time has, however, considerably slowed the loss rate while maintaining the stable equilibrium.

(3) If only a co-injected beam is used (with buildup time $\tau_b = 150 \mu\text{sec}$), results are considerably more stable. The situation at 65 μsec is shown in Figs. 31a and b; at this time the toroidal beam current is 0.917 MA (almost equal to the plasma current). The beam particles have not spread appreciably. A possible disadvantage of this mode of injection is illustrated by the fact that the plasma surface has expanded its minor radius from the original equilibrium 58 cm to 85 cm.

Fig. 18a. Equilibrium flux surfaces for Scenario 1A.

MAJOR RADIUS VS. VERTICAL FIELD

Fig. 18b. Major radius as a function of applied centering field
for Scenario 1A.

Fig. 13c. Plasma current density for Scenario 1A (as a function of q at $Z = 0$).

Fig. 10a. Equilibrium flux surfaces for Scenario 1P.

Fig. 19B. Plasma current profile ($z = 0$) for Scenario IB.

Fig. 20a. Flux surfaces for equilibrium Scenario IID.

Fig. 20b. Plasma current density for equilibrium Scenario IID.

Fig. 21a. Equilibrium flux surfaces for Scenario III.

Fig. 21b. Equilibrium current density versus radius for
Scenario IIE.

Fig. 22a. Trajectory of deuteron co-injected at $R = 330$ cm
(an arrow indicates the trajectory).

Fig. 2' Trajectory of a deuteron co-injected at $R = 335$ cm.

Fig. 22c. Trajectory of a deuteron co-injected at $R = 345$ cm.

Fig. 23a. Trajectory of a deuteron counter-injected at $R = 330$ cm.

Fig. 23b. Trajectory of a deuteron counter-injected at $R = 335$ cm.

Fig. 23c. Trajectory of a deuteron counter-injected at $R = 345$ cm.

Fig. 24a. Deuteron co-injected at 370 cm.

Fig. 24b. Deuteron counter-injected at 370 cm.

Fig. 25a. Initial beam configuration for case A1. The beam particles are shown against plots of constant $|B|$.

Fig. 25b. Initial flux surfaces for case A1 including contributions from the beams.

Fig. 25c. Particle locations at 10 μ sec.

Fig. 25d. Flux surfaces at 10 msec.

Fig. 2%. Particle locations at 16.5 usec.

Fig. 25f. Flux surfaces at 16.5 μsec .

Fig. 26a. Initial beam distribution for case A2.

Fig. 26b. Initial flux surfaces (including beams) for case A2.

Fig. 26d. Flux surfaces at 36 sec for case A2.

Fig. 26e. Particle positions at 49.5 μ sec for case A2. The
 beams are separating.

Fig. 26. Flux surfaces at 49.5 μsec for case A2.

Fig. 26g. Particle positions at 75 μ sec for case A2. Beams
have separated.

Fig. 26h. Flux surfaces at 75 μ sec for case A2.

Fig. 27a. Initial particle distribution for case A3.

Fig. 27b. Initial flux surfaces for case A3.

Fig. 27c. Particle distribution for case A3 at 40.5 μ sec.

Fig. 27d. Flux surfaces for case A3 at 40.5 μ sec.

Fig. 28b. Initial flux surfaces for case B.

Fig. 28c. Beams after full injection ($t = 25 \mu\text{sec}$) for case B.

Fig. 28d. Flux surfaces at 25 μ sec for case B.

Fig. 28e. Beams for case B at 50 μ sec.

Fig. 28r. Flux surfaces for case B at 50 μ sec.

Fig. 88g. Means for case B at 100 μ sec.

Fig. 26h. Flux surfaces for case B at 100 msec.

Fig. 28J. Flux surfaces for case B at 150 μsec .

Fig. 29a. Separated beams at 14 μ sec for case C1.

Fig. 29b. Flux surfaces for case C1 at 14 μ sec.

Fig. 3a. Beam position for case 02 at 50 msec.

Fig. 30b. Flux surfaces for case C2 at 50 μ sec.

Fig. 30d. Flux surfaces for case C2 at 70 μ sec.

Fig. 31a. Beam position for case C3 at 65 msec.

Fig. 31b. Flux surfaces for case C3 at 65 μ sec.

IV. OTHER APPLICATIONS

Despite the apparently strong restrictions on GUIDON's usefulness imposed by the limitations of its assumptions, the code can yield valuable information for a great variety of toroidal plasma experiments. Some specific results of simulations of three different devices are presented here.

4.1 SPIRE

4.1.1 Description of the Problem

SPIRE (an acronym for Stabilized Pinch of Relativistic Electrons) is an idea, proposed by Charles Hartman,^{1,2} for a device which was never fully constructed.* The research which culminated in the design of the GUIDON code was first undertaken as an attempt to provide a theoretical evaluation of SPIRE which, if favorable, might lead to the experiment's resumption elsewhere. The combination of unpromising predictions and strong impetus behind the TCT program forced a redirection of the major effort toward the TCT.

In its simplest form, the formation of the stabilized pinch follows a two-stage process. Consider that initially a plasma is contained within a minimum-B toroidal quadrupole mirror, whose flux surfaces are shown in Fig. 52a. This arrangement is produced by four vacuum magnetic coils located as shown in the figure; it is an open-ended mirror field since particles of the right magnetic

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* An AEC Headquarters directive restricting toroidal containment research at LLL caused the SPIRE to be discontinued in the initial phases of construction.

moment will be able to escape along the diagonal separatrices. Note the existence of an absolute null of the poloidal field at the center. In the first stage of operation a hot electron plasma is produced by heating this confined plasma with microwaves at the electron-cyclotron frequency, "possibly augmented by off-resonance heating. It has been experimentally shown that a hot electron plasma with $n \approx 10^{12} \text{ cm}^{-3}$, $T_{e, \text{hot}} \sim 0.1\text{-}1 \text{ MeV}$ can be produced by this technique, thus providing a starting plasma with a large β and an energy density approaching that required for ignition in a toroidal system."³

The second stage, actual formation of the pinch, is achieved by applying a strong toroidal electric field. Electrons will be accelerated by this field and those near the stagnation point, where the magnetic field is purely toroidal, can "pass" and form a toroidal runaway current. Other electrons will experience a strong $\underline{E} \times \underline{B}$ drift velocity and their banana will drift (depending on which quadrant of the poloidal plane they are in) either toward the stagnation point or away from it and out of the machine.

If a strong enough current can be induced at the stagnation point, a closed flux surface can be created, inside which all particles will be passing ones. The machine will then have two stagnation points, as shown in Fig. 32b, above and below the old one (or left and right of it, depending on how the separatrices happen to divide). Bananas will drift toward and away from these stagnation points as before. As a banana reaches a stagnation point, it forms a new complex figure called a "hyperbanana", as

FIGURE 32b: Final Flux Configuration for SPIRE

FIGURE 32c: A Superbanana Formed Around a Stagnation Point. The electron begins at point (A) and moves to point (B), where the banana is now drifting downward.

illustrated in Fig. 32c: the particle bounces around the stagnation point until its bounce into a quadrant where the drift velocity is directed away occurs sufficiently far down the separatrix for it to escape.

The difference between the configurations of Figs. 32a and 32b can now be appreciated. In the fields of 32a nearly all electrons convected toward the stagnation point will eventually drift away from it and out of the machine without contributing to the pinch current. In the fields of 32b, however, half of those electrons convected toward each stagnation point will, in drifting away from the stagnation point, drift into the region of closed flux surfaces; the ends of the bananas of these particles will lengthen until they touch and the electrons pass, adding to the growing current of runaway electrons. This process is illustrated in Fig. 32d for the upper stagnation point, from which downward-drifting electrons are favorable. Figure 32e shows an electron beginning at the same point with slightly different energy; it is lost when it drifts upward from the stagnation point.

The crucial questions concerning the SPIRE device, then become (a) how much "seed" current at the initial stagnation point is necessary to produce the favorable surfaces of Fig. 32b, which lead to buildup of the runaway pinch current? and (b) will sufficient seed current be induced normally by the beginning of stage two? Some very preliminary experimental observations indicated that a current might be generated, but no solid analytical prediction can be made for this manifestly non-linear interactive

Fig. 32d. Electron swept into a passing orbit in SPIRE. The electron is initially at upper right; the plots show successive positions of the banana (each contains two bananas) and the hyperbanana as the electron turns downward. The electron's kinetic energy is 0.52 MeV.^b

Fig. 32e. Same case as Fig. 32d, except the electron's kinetic energy is 0.48 MeV. The slight difference in trajectory at the hyperbana allows it to turn upward and escape.

process.

4.1.2 Simulation Results

The answers produced by the GUIDON simulations to the two questions posed above are discouraging. Qualitatively, they are: (a) a relatively quite large seed current is necessary to pass the desired fraction of trapped hot electrons, and (b) it does not appear that any simple process such as the switching of the accelerating potential will create this large current.

Figure 33 illustrates a simulation of SPIRE in which there is initially no seed current. The initial spatial distribution of hot electrons is shown in Fig. 33a; the coils have been adjusted so that the inner surfaces of $|B|$ are nearly circular and the 0.5 MeV electrons have been placed so that their density is a function of $|B|$ only, the condition for pressure equilibrium. It is assumed that all the particles begin at their turning points. This has proved to be the best of many possible initialization conditions since it helps minimize highly fluctuating fields early in the run and guarantees that all the simulation particles represent electrons actually trapped within the quadrupole mirror (a difficult condition to guarantee otherwise). In response to Harman's suggestion, this run contains the effects of induced E_{θ} ; these effects do make differences in some detailed motion of the electrons (when compared with the same run with no induced E_{θ}), but do not influence the final outcome or its time scale. The initial flux surfaces and velocity phase space are shown in Figs. 33b and c. After 100 timesteps the spatial distribution of electrons is as

Fig. 33a. Initial electron distribution $n = f(B)$ for SPIRE simulation with no seed current.

Fig. 33b. Initial flux surfaces to accompany Fig. 33a.

Fig. 33c. Initial velocity phase space to accompany Fig. 33a.

shown in Fig. 33d; Fig. 33e shows that enough electrons have remained within the center region to create a small passing current and close the total flux surfaces. The dual stagnation points in this case are to the left and right of the closed area; the short solid line is the trajectory of the sample particle for the first 100 timesteps. Figure 33f illustrates the rapid ordering in the velocity phase space of the trapped particles. After 300 timesteps (Figs. 33g and h) a few of the electrons have begun to escape. Acceleration of trapped particles has also produced such strong currents outside the central region that the total flux only contains a very small region of closed surfaces (none is actually plotted in 33h). After nearly 600 timesteps (Figs. 33i and j), large numbers of particles have begun to escape into the lower-left and upper-right corners, and there are no longer any closed flux surfaces in the center (the sample particle itself is about to escape). At later times things continue to get worse; accelerating trapped particles drift rapidly out of the machine and the flux surfaces return to the initial condition.

A simulation in which a seed current of 10 kA is assumed already to exist (provided by the plasma current model) is illustrated by Fig. 34. The initial conditions for electron spatial distribution and $|B|$ are shown in Fig. 34a, those for the flux surfaces in Fig. 34b. The total pinch current that could possibly be formed by this configuration if fully half the electrons were passed is about 400 kA; thus the presumed seed current represents about 5% of the final current. The flux surfaces after 500 time-

Fig. 33d. Spatial distribution of electrons after 100 timesteps
(2 nanoseconds).

Fig. 33e. Flux surfaces after 100 timesteps. The short solid line is the trajectory of the sample particle.

Fig. 33f. Velocity phase space at 100 timesteps.

Fig. 33g. Spatial distribution of electrons after 300 timesteps.

Fig. 33h. Flux surfaces after 300 timesteps.

Fig. 331. Spatial distribution of electrons after ~ 600 timesteps.

Fig. 33j. Flux surfaces after ~ 600 timesteps.

Fig. 34a. Initial spatial distribution of electrons and $|B|$ contours for SPIRE simulation with 10 kA seed current.

Fig. 34b. Initial flux surfaces accompanying Fig. 34a.

steps are shown in Fig. 34c; a good region of closed contours exists. By coincidence, the sample particle is one of those swept into the closed region and onto a passing trajectory. Unfortunately, the seed current is still not quite strong enough. As shown by Figs. 34d and e, after 700 timesteps (14 nanoseconds), many of the electrons have managed to escape and, though several have entered passing orbits, the flux surfaces are no longer quite closed. From here on things continue downhill just as in the previous case.

Figures 35 and 36 illustrate a transition and an asymptotic case. The seed current used for the former run was 100 kA, or about 30% of the total expected. After 21 nanoseconds (Fig. 35) most of the particles had either passed or escaped; the total passing current was only about 15 kA: more electrons escaped than were passed. The peak value reached by the passed current at later times was 20 kA. In the run illustrated by Fig. 36, a seed current essentially equivalent to the desired pinch current was used, with the not unexpected result that nearly all the electrons were passed in the first 50 nanoseconds. Passing current eventually approached the desired level after a further period of acceleration.

4.2 The Wave Drift

Construction of the circulating plasma current in a Tokamak would seem in a magnetohydrodynamic model to be limited by the Hall current (or $\underline{E} \times \underline{B}$ drift),

$$v = c \left| \frac{\underline{E} \times \underline{B}}{B^2} \right| \approx c \frac{E_{\theta} r_p}{B_{\theta} r_p} \quad (\text{IV.1})$$

Fig. 34c. Flux surfaces after 500 timesteps. The sample particle is entering a passing trajectory.

Fig. 34d. Spatial distribution of electrons after 700 timesteps.

Fig. 34e. Flux surfaces after 700 timesteps.

Fig. 35. SPIRE with a seed current of 100 kA after 1100 timesteps.

Fig. 36. SPIRE with a seed current of 350 kA after 1500 timesteps.

where E_ϕ is the accelerating electric field, B_p is the magnitude of the poloidal field and B_ϕ the magnitude of the toroidal field. Ware⁴ has shown that the inward drift velocity of trapped particles in a Tokamak can effectively be much higher; the bananas may drift at

$$v = c \frac{E_\phi}{B_p} \quad (\text{IV.2})$$

This drift rate applies to particles trapped by VB and to those trapped by electrostatic potential. The velocity is typically much greater than that given by (IV.1) since $B_\phi > B_p$, usually by a factor of 5 or more. The higher rate was actually observed experimentally before it was explained by Ware.

It is possible using GUIDON to simulate the details of this process. The configuration used for this particular simulation was that of the Princeton PLT device (under construction). Figure 37a shows the banana orbit of a typical trapped deuteron, produced with the GUIDE code. Figure 37b shows the same particle also under the influence of an accelerating potential $E_\phi = 1500$ volts/turn. The measured velocity of similar points on the banana over the duration of the run is 6×10^4 cm/sec, exactly the value found by substitution of the appropriate values into (IV.2). Note that the ends of the banana are also pushed farther apart by the E_ϕ ; Fig. 37c shows that for an applied potential $E_\phi = 4500$ volts the same particle is no longer trapped but executes a typical passing orbit.

Figures 38a-p illustrate the detailed nonlinear contribution of the Ware drift to the circulating plasma current. 2500 simula-

Fig. 37a. Trapped particle in PLT with no accelerating potential.

Fig. 37b. Trapped particle in FLT with accelerating potential

$$E_{\phi} = 500 \text{ volts.}$$

Fig. 37c. Trapped particle in FLT with accelerating potential

$E_0 = 1500$ volts.

tion particles are loaded according to the skin-current distribution of Fig. 5c (see Fig. 38a) to represent the trapped-ion component of the plasma. The component already passing is represented by the fluid model and creates flux surfaces as shown in Fig. 38b (the solid line is the plasma boundary). As before, each particle is loaded at or near a mirror point. The accelerating potential is chosen to be $E_{\phi} = 1500$ volts/turn because at this value very few of the deuterons will pass immediately. The Ware drift will then cause the "hole" in the spatial distribution to fill in; as ions drift inward their lengthened bananas will connect into circles, the particles will pass and contribute to the total plasma current. Figure 38c shows the initial distribution of the particles in velocity space. Figure 38d shows the initial current density due to the particles plotted versus R near $Z = 0$; particles on the left side of the plasma's center are starting on bananas out of the page, those on the right-hand side into the page.

The spatial distribution of ions after 12 μ sec is as shown in Fig. 38e. They seem to be congregating since ion turning points are on the outside of the torus (electron turning points are on the inside). The velocity phase space at 12 μ sec is shown in Fig. 38f; it has the characteristic appearance of two overlapping distributions (for particles entering and leaving the turning-point region). Evolution of the spatial distribution at 15, 18 and 32 μ sec is shown in Figs. 38g, h and i, respectively. Inward motion of the particles is already apparent in Fig. 38f, and this impression is confirmed by the velocity space plot, Fig. 38j: the

Fig. 38a. Initial loading of trapped particles for the Ware drift simulation.

Fig. 38b. FLT flux surfaces for the Ware drift simulation.

Fig. 38c. Initial velocity phase space for trapped particles
in the Ware drift simulation.

Fig. 38d. Current density due to particle motion in the Ware
drift simulation near $z = 0$ at $t = 0$.

Fig. 38e. Spatial distribution of ions in the PLT after 12 μ sec.

Fig. 38f. Velocity phase space at 12 μ sec.

Fig. 38g. Spatial distribution of ions at 15 μsec.

Fig. 38h. Spatial distribution of ions at 18 μsec.

Fig. 381. Spatial distribution of ions at 39 μ sec.

Fig. 38j. Velocity-space distribution at 39 μ sec.

accumulation of particles in the region of small v_R and v_z indicates that more particles are passing and have converted most of their motion into v_ϕ .

Similar further evolution of the spatial distribution at 48 and 84 μsec can be seen in Figs. 38k and l; the increase in passing particles is apparent in the velocity phase space plot, Fig. 38m, for 85 μsec .

At 198 μsec nearly all the trapped ions have passed. The hole in the spatial distribution has filled in, as shown in Fig. 38n. The preponderance of passing particles is evident in the velocity phase-space plot, Fig. 38o; it can also be seen in the plot of particle current density, Fig. 38p, in which all the current is parallel to B_ϕ . The buildup of current due to the passing of trapped particles is shown as a function of time in Fig. 38q; at the end of the run the particles are contributing 10% of the total toroidal current.

4.3 Electron Beam Propagation

For circumstances in which the electrostatic forces within the beam are not dominant, GUIDON can provide useful information concerning the propagation of intense electron beams in toroidal geometries. As an example, some straightforward early studies of a toroidal beam injection apparatus at the Physics International Company will be presented.

Benford, et al.^{5,6} have reported results of experiments in which an intense relativistic electron beam is injected into an evacuated half torus with a purely toroidal magnetic field.

Fig. 58k. Spatial distribution of ions at 48 μsec.

Fig. 381. Spatial distribution of ions at 84 μ sec.

Fig. 32a. Velocity phase space at 84 μ sec.

Fig. 38n. Spatial distribution at the end of the Ware drift
simulation, $t = 198 \mu\text{sec.}$

Fig. 38o. Velocity phase space at 198 μ sec showing most particles passing.

Fig. 38p. Particle current density at 198 μ sec showing parallel currents.

Figures 39a-c show the initialization of a GUIDON simulation using the parameters of one of their experiments with a 0.5 MeV beam of circular cross section; the figures illustrate the initial location of the beam (3000 simulation particles), the flux surfaces for the beam's self-fields, and the usual velocity phase-space scatter plot, respectively. In the purely toroidal vacuum field, drift velocities for both ΔB and curvature drifts are in the $-Z$ direction; the strong self-fields can be expected to cause some rotation of the beam since their flux surfaces are concentric. Figures 39c-f shown the location of the beam and the flux surfaces at 4 and 6 nanoseconds after injection, respectively. The downward drift velocity of the beam is 7.5×10^8 cm/sec, in exact agreement with the theoretical predictions of Ref. 5 and with experimental results reported in Ref. 6. The rotation of the beam which is evident in Fig. 39e can also be seen in the folding-over of the distribution of particles in velocity phase space, Fig. 39g. After 10 nanoseconds the beam is being turned more abruptly because of stronger forces from the self-fields beneath it, which are compressed against the perfectly conducting boundary (Figs. 39h and i).

To test the distorting effects of the self-fields on non-circular beam cross sections, this experiment was also run in a configuration in which the injected beam was directed through a rectangular aperture. The initial conditions for a simulation of this version of the experiment are shown in Figs. 40a and b. Figures 40c and d show evolution of the beam shape as it drifts downward at 2 and 4 nanoseconds, respectively; Fig. 40e shows the

Fig. 39a. Initial beam location for simulation of the Physics International experiment. The beam appears to be slightly elliptical because grid spacing on the two axes is not the same.

Fig. 39b. Initial flux surfaces for the PI simulation.

Fig. 39c. Initial velocity phase space for the PI simulation.

Fig. 39c. PI beam after 4 nanoseconds.

Fig. 99d. Flux surfaces after 4 nanoseconds.

Fig. 39e. PI beam after 6 nanoseconds.

Fig. 39f. Flux surfaces after 6 nanoseconds. The arrow indicates the trajectory of the sample particle.

Fig. 39g. Velocity phase-space after 6 nanoseconds showing folded distribution.

Fig. 39h. PI beam after 10 nanoseconds.

Fig. 391. Flux surfaces after 10 nanoseconds.

Fig. 40a. Initial beam distribution for the PI experiment with rectangular beam.

Fig. 40b. Initial flux surfaces for the rectangular beam version.

Fig. 40c. Rectangular beam after 2 nanoseconds.

Fig. 40d. Rectangular beam after 4 nanoseconds.

Fig. 40e. Flux surfaces after 4 nanoseconds. The trajectory of the sample particle is indicated by an arrow.

flux surfaces at 4 nanoseconds, with the slightly diagonal trajectory of the sample particle indicated. Figure 40d is virtually identical to Fig. 11 of Ref. 6, which shows the expected distortion after an equivalent time and the final damage pattern made by the beam's deposition on a target plate. Figures 40f-h show further evolution of the beam shape at 6, 8 and 10 nanoseconds, where the compression of self-fields again becomes an important factor.

Among the further objects of the Physics International experimentation is the addition of vacuum poloidal fields which can halt the downward drift of the beam at a desired value of Z after several passes around a full torus. GUIDON could easily and profitably be used to design such an experiment.

Fig. 40f. Rectangular beam after 6 nanoseconds.

Fig. 40g. Rectangular beam after 8 nanoseconds.

Fig. 40h. Rectangular beam after 10 nanoseconds.

V. RECONSIDERATIONS

"What then can we look for from this confusion of groundless and extraordinary opinions: but error and falsehood? And how can we justify to ourselves any belief we repose in them?"

--David Hume, A Treatise of Human Nature.

It is appropriate now to ask how much confidence we may put in the results of this new computational technique. The question is particularly pressing since of the four different classes of results presented in Chapters III and IV, the two most important (for TCT and SPIRE) were simulations of experiments that have never been run. Further, experimental evidence of the Ware drift effect is largely inferential, and the Physics International machine is relatively trivial when compared to the complex interactive processes in the other three. Therefore we must conclude that experimental verification of the most significant results is essentially nonexistent. If we are to believe these results, we must have some other criteria for evaluating the code's effectiveness, the verisimilitude between what it produces and physical reality.

It would be convenient simply to judge GUIDON by the standards appropriate to computational simulations. Unfortunately, such standards do not exist; no one has ever taken the trouble to define them. In what follows I shall discuss factors in the nature of computational physics which make the establishment of standards difficult; I shall then propose a general approach toward standards, and suggest how they might be applied to the present case, and with

what results.

5.1 Is Computational Physics Physics?

The question is admittedly primarily a matter of semantics, but an attempt to answer it can lead to discussion of some of the principal and peculiar aspects of computational physics. First we need a working definition of physics. Let us begin with a very simple categorization, under which: pure physics concerns itself with the discovery of new principles, and with the facts and procedures leading to them; applied physics attempts to discover new manifestations of known principles; and engineering makes practical use of the results of both.

Within this scheme we can definitely say that computational physics is not pure physics since it is capable of operating only on principles supplied ab initio: known. Whereas in a true experiment the physicist is "querying" nature in the sense of testing his hypotheses for validity, in a computer experiment he is in effect querying the hypotheses themselves--testing them for logical consistency.

Applied physics, perhaps. One can, in the first place, test new hypotheses for conformity with existing theory, something of a grey region in our definitions, which we shall investigate presently. Further, one can uncover new effects within the framework of existing theory; clearly this is a possibility. But it may also be a questionable methodology, for the (mathematical or logical) creation of specious physical effects is often a far more complicated and difficult question than the customarily knotty problem

of experimental error. For instance, the nonlinear behavior of the electron-cyclotron drift instability, inaccessible to analytical mathematical techniques and probably beyond sufficiently controlled laboratory procedures, was finally settled through computational studies. But the demonstrated solution mode depended upon destruction of the integrity of cyclotron orbits caused by extreme fluctuations in electrostatic fields; and the severest problem in simulations of the kind used to investigate this instability is high-frequency electric noise fundamental to the computational approximation. Settling of the dispute over the role played by such "artificial" effects was the occasion of a long and bitter dispute among highly competent colleagues. Thus difficulties extraneous to the physical processes can not only mask but actually masquerade as precisely the phenomenon under investigation.

Certainly we may classify the results of computer simulation as engineering, since they are manifestly useful. But it should still strike us as peculiar that one may reach the solution to a physical problem by completely mathematical means. In arriving at what he can verify as a correct solution to a control problem, the physicist may appeal to mathematical logic whose effect and nuances he is perfectly acquainted with, yet still for a time not understand fully the real physics underlying the appropriateness of his mathematical choices. This is partially a result of the fact that one is restricted to operation within a small, restrictive set of assumptions about the physics; it is far easier for the results of mathematical manipulations to be clear in the artificial

system, where only a few equations pertain, than for the effects of excluded physics (and their interactions with the included physics) to be evident in the same system, where everything that is known is by definition approximate. Thus the real world is more complex first than any conception of it, and second than any logical structure. This is why nature can be self-consistent in spite of the fact that we cannot demonstrate that any system of mathematics is, for mathematics is but one further construct.

The score at this point is pure physics, no; applied physics and engineering, yes. We must now return to our definitions to deal with the most fundamental issues. The full import of Hume's devastating criticism of the value of inductive logic has only come to light in our century. Through a more accurate evaluation of the inductive (that is subjective) aspects of the development of scientific theories, philosophers and scientists have cast doubt upon the autonomy, the rational authority of science itself. For some comments on the applicability of this debate in its broader sense, see Appendix D. An unavoidable conclusion is that what physics is, and how valuable it is, depends very much on the way it is done. Hume's criticism applies with a vengeance to the methods of computational physics.

Hume demonstrated, in effect, that inductive logic is not logic at all, in the rigorous sense in which we understand logical proof. In short, no matter how many times we observe a particular result, we can never be justified in concluding solely on the grounds of that previous knowledge that apparently identical

circumstances will produce that result once again. This conclusion does not depend upon a presumption of the possibility of unknown forces at work, but applies to the logical process of surmising based upon past events. We may experience a greatly enhanced confidence in our generalizations, but a certainty is impossible.

The application to computational physics is direct: no matter how many well-understood physical phenomena (test problems) a particular code may have simulated successfully, we can have no guarantee whatsoever that when we apply it to an imperfectly-understood problem (that is, employ it to a useful rather than a trivial purpose) the result will have anything to do with the physical reality in question.

It is hardly a defense to reply that the same may be said of all physical laws, that being a manifestly more nebulous issue and our confidence level in them being, at any rate, immeasurable higher (a meaningful distinction to either side of the larger controversy). Further, in applying analytical equations we may remain constantly aware of the limitations of our "model"--what physics is or is not represented by it. We may be at some pains to learn whether this model is or is not equivalent to a particular phenomenon, and in fact we may not be able to say for certain without further knowledge--but we can at the very least make an assignment of class (does represent, does not represent, cannot know) on the basis of information at hand. Such an assignment gives us valuable advice on which direction to proceed in search of further information.

A difficulty of computational physics is that often we cannot

make such an assignment with confidence: a sort of Gödel's Law is in play. In some instances we cannot be certain that unknown artificial (numerical, logical or modeling) effects are dominant, and so cannot say whether or not the code is adequate or whether we cannot say if it is: Hume's dictum indicates that in some classes this uncertainty may be absolute, and the only thing we can say is that we cannot say. If the analogy with Gödel's Proof holds fully, of course, we shall have the added disadvantage of never being able to demonstrate that for many particular difficult problems our solution is adequate even when we are quite certain that it is. Thus a "second degree" of uncertainties is established, in which even an assignment into a class is formally meaningless. Fortunately, we seldom care to follow the analysis this far. Attempting to formulate such a dictum rigorously would lead us to the creation of a frustrating third-degree of uncertainty and the waste of much time far from the physical problem; it would also bring us up against the paradox that such a determination should also be forbidden by Gödel's Proof (an absolute test for nonapplicability would lead to an absolute criterion of applicability).

5.2 The Elusive Trustworthy Code

A computer simulation is thus more than a complicated gedanken-experiment, and considerably more troublesome. Two questions are raised by the difficulties discussed in the previous section:

- (a) Do these difficulties render the results of computational studies either meaningless or useless?
- (b) If the answer is "No", we must still admit that the

results are to some extent suspect; how must our procedures and interpretations be altered by the realization that the difficulties exist?

I propose that the answer to (a) is No; to some extent this is merely an opinion resting on the faith that all physicists *must* eventually appeal to in their work, that logical approaches will open a pathway to physical truth. The rather abstract argument among the philosophers (Appendix D), and the criterion of high confidence which is one of its results, applies with some real force here. For though any physicist may be aware that logical difficulties in the philosophical groundwork exist, this awareness does not in the least shake his confidence in applying the laws of mechanics or optics or thermodynamics in his daily work, nor even give him pause over his results. But for the computational physicist these difficulties lie much closer to his working and thinking environment, and must never be ignored. Confidence *must* always remain an open question in a field in which one single parameter chosen one-tenth of a percent too large or too small may produce an apparently valid simulation for a thousand cycles and garbage at cycle 1001.

Yet there is an important compensation, one which permits the establishment of confidence levels on, as it were, an individual basis. The experimentalist knows that, though unknown forces may be at work, he is operating within a reliably self-consistent logical system (at least he has this belief). The computational physicist, struggling without such a guarantee, has something

valuable which his colleague does not: a system in which all experimental parameters are readily available for adjustment. This has long been understood to be one of the most useful aspects of numerical simulation; it can serve as a valuable source of confidence. The analogy of Poppers' confidence-through-repeated-failure-of refutation is one gained by having varied all the physical parameters in the simulation to guarantee that their behavior conforms in each case to one's predictions based on an understanding of the dominant principles, as elucidated by a combination of theory, experiment, and the principal simulation itself. That is, everything--including especially those parameters which exist in the simulation but not in the real experiment, such as the timestep--may be tested for verifiability and to measure its actual role in the interactive processes. We may, in effect, run experiments of a secondary nature to question the reality and self-consistency of our modeled world. The results of this process do not guarantee a logically rigorous solution any more than a physical experiment may in Popper's philosophical construct, but they approach statistical confidence in predictive capability and verisimilitude in precisely the same fashion. Furthermore, it is not necessary to investigate in detail every parameter of the simulated world on each run or even with the creation of each code, for there are certainly mathematically rigorous conclusions we may draw about many aspects, such as timestep size and minimum noise generated in Fourier transforms, and many more for which experience and theory can set combined limits, such as the potential noisiness of certain

combinations of grid sizes and particle densities.

What I am suggesting is more than the straightforward testing of various "clean" aspects of the physics in a code as that procedure is already understood. I am suggesting first that parametric tests must be conducted on all the crucial aspects of any new simulation to demonstrate as fully as possible that interactive and numerical effects are well understood and that the model represents, for that particular simulation, at the very least itself. Second and more important, both general and specific testing of the physics must be conducted beyond the "failure point," that is, run in regimes predictably forbidden to prove that the failure mode is the one expected and that its quantitative behavior is well understood. This crucial procedure is seldom actually performed, and virtually never with anything like organization and precision. Still, it is the closest one may come to a true refutability criterion, and the more failure-testing one does the more one may rely on his results and know how to interpret them. Finally, interpretation must be performed with a critical eye as to which variables should produce ordered behavior and which should not. There is, of course, a limit to how much data one can produce and analyze and still get anything done; yet virtually every computational physicist has had the (often unpleasant) experience of plotting or printing a hitherto unexamined variable in a well-trusted code, only to discover a symmetry or asymmetry that he cannot explain. The extent of output and post-analysis performed on output should, therefore, be controlled actually by limits of

working time and expense, and not by what simply seems reasonable beforehand or as results begin to appear. An early corroboration of expectations can prove a most dangerous enemy if it curtails critical analysis.

Obviously we have passed on to question (b). Common parlance holds that computing is no substitute for thinking, and indeed the two activities often seem to be mutually exclusive; a properly critical attitude must include a great deal of thought about one's computing, and no information supplied by the computer should be neglected. One must be prepared to defend all his choices of elements and procedures in his numerical models with specific proofs of verifiability, whether original to the problem or not. Since this condition includes all elements neglected by the model, it is a very difficult one indeed and requires not only ingenuity but a tireless re-examination of fundamental interactions by the computational physicist, both as the model is applied to new physical conditions and as it grows in sophistication.

It must be left to the reader himself to judge how well these criteria have been fulfilled by GUIDON. The tests for verisimilitude of each physical aspect of the models described in Chapter II have been augmented by failure tests run parametrically for test problems and for all of the simulations described in Chapters III and IV. Many such tests have not been reported individually because their enumeration would prove merely timesome in a general reporting { for example, extensive tests of the effects of grid size on the SPIRE simulation revealed that, within reasonable

limites, the grid size was immaterial; extensive failure testing of timestep limitations in the same simulation essentially verified the wisdom of the limitation of Eq. (II.40)].

Two important limiting findings should be reported. Self-consistent electrostatic fields are important for more accurate simulation of the Ware drift, since an ambipolar field is generated among trapped particles which can enhance the drift velocities significantly. And finally, there remains some lingering question about the SPIRE start-up even within the range of GUIDON's physics, since the strength of transient magnetic fields at the start of each simulation may be anomalously dominant. The results presented are those for the "quietest" start yet achieved for a well-randomized, very hot plasma. Further study is being given to the possibility of unstable initial conditions.

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Margaret Thomas typed the entire awesomemanuscript.

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APPENDICES

A. The GUIDE and ACTION Codes

I. GUIDE

GUIDE is a compact, highly accurate code for solving for the trajectories of the guiding centers of single relativistic electrons in axisymmetric toroidal vacuum fields; it was written by Oscar Anderson and Dieter Fuss. GUIDE calculates the exact field at the particle's location at each timestep (using the Biot-Savart solution described in 2.2.1), solves for the guiding-center drift velocities (as in Appendix C), and advances the electron's position using a fourth-order Hamming¹ integrator. Its fine accuracy is demonstrated by the fact that the longitudinal adiabatic invariant J , Eq. (I.5), is conserved by trapped electrons in various simulated experimental configurations to better than one part in 10^6 . In a study performed jointly by Dieter Fuss and this author, it was further demonstrated that such accuracy is difficult to obtain. A modified version of a standard 3D LLL trajectory code MAFCO-TIBRO was constructed using Anderson's more complete, relativistic guiding-center equations; but the new MAFCO often failed to conserve J by 100% or more. It was shown that coordinate transformations inherent in MAFCO (from 3D Cartesian to 3D cylindrical and back again) were responsible for the destruction of accuracy.* Since these

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* Particle trajectories, it should be noted, remained qualitatively quite correct. The check on J investigates the accuracy of the details of the orbit.

transformations were performed on a 60-bit computer (CDC 6600), the incredible sensitivity of calculation of the details of relativistic orbits should be apparent.

GUIDON has borrowed liberally from GUIDE, most importantly in the complete and correct relativistic guiding-center drift velocities written by Anderson (and in the efficient calculation of vacuum fields).

Examples of results of GUIDE have already been shown in Figs. 3 and 3d and e. Figure 41 shows the drifting banana orbits of an electron in the flux surfaces of the LEVITRON^{2,3} machine, in which a floating superconducting coil is used to establish a magnetic well. The coil is located at $R \approx 40$ in the figure.

2. ACTION

ACTION⁴ is a design and diagnostic code written by Anderson, Fuss and Johnson. It provides a wealth of useful information about the vacuum fields in toroidal devices and about the orbits of particles in them--without ever calculating individual orbits. The code first calculates and plots, by methods already discussed, both the vacuum $|B|$ and B_{poloidal} . Examples of these for the same LEVITRON configuration of Fig. 41 are shown in Figs. 42a and b. Surfaces of constant flux are then calculated and plotted over the $|B|$ surfaces. The occurrence of different values of J (for a particle with a specified energy) along the constant flux lines are indicated by corresponding numbers written at the appropriate points on the lines. Figure 42c shows this superimposition for the LEVITRON, and Fig. 42d shows a more easily readable blowup of

Fig. 41. A further example of GUIDE results: the drifting banana orbit of a trapped electron in Levitron.

Fig. 42a. $|B|$ surfaces in the Levitron.

Fig. 42b. B_p surfaces in the Levitron.

Fig. 42c. $|B|$ surfaces (dotted lines), constant flux surfaces (solid lines) and points of constant values of longitudinal adiabatic invariant, J (numbers on flux lines) in the Levitron.

Fig. 42d. Blowup of the magnetic-well region of Fig. 42c.

the critical magnetic-well region of the machine.

The values of J are calculated (nonrelativistically) as follows:
using

$$v^2 = v_{\parallel}^2 + v_G^2 \quad (\text{A.1})$$

and hence

$$\frac{1}{2} m v_{\parallel}^2 = \frac{1}{2} m v^2 - \frac{1}{2} m v_G^2 = T - \mu B \quad (\text{A.2})$$

where T is the kinetic energy; the parallel velocity can be solved for as

$$v_{\parallel} = \sqrt{\frac{2\mu}{m}} \sqrt{\frac{T}{\mu} - B} \quad (\text{A.3})$$

At the mirror points, $v_{\parallel} = 0$, $B = B_M$ and $T = \mu B_M$ [from (A.3)]. We can rewrite (A.3) using these mirror conditions as

$$\begin{aligned} v_{\parallel} &= \sqrt{\frac{v_{GM}^2}{B_M}} \sqrt{B_M - B} \\ &= v_{GM} \sqrt{\frac{B_M - B}{B_M}} \end{aligned} \quad (\text{A.4})$$

where v_{GM} is the gyro-velocity at the mirror point,

$$\frac{1}{2} m v_{GM}^2 = T = \mu B_M \quad (\text{A.5})$$

(Thus we may specify either the particle's energy or its magnetic moment). Then using the definition of J , substitute (A.4):

$$\begin{aligned}
 J &\equiv m \int v_{\parallel} \cdot ds = m v_{GM} \int \sqrt{\frac{B_M - B}{B_M}} \cdot ds \\
 &= \sqrt{2m\mu} \int \sqrt{B_M - B} \, ds
 \end{aligned}
 \tag{A.6}$$

And since the integration is taken along a poloidal flux line,

$$ds = d\ell \frac{B}{B_p}$$

and (A.6) may finally be written

$$J = \sqrt{2m\mu} \int \frac{B}{B_p} \sqrt{B_M - B} \, d\ell \tag{A.7}$$

the form in which it is solved (as a sum with constant $\Delta\ell$ along lines of constant ψ) by ACTION.

These numbers can be connected by hand (a line, for instance, through all the ℓ 's) to form contours of constant J . These are valuable since the ends of drifting bananas are constrained to move along constant- J lines, and hence these contours show the drift motion qualitatively at a glance. In another version, ACTION plots constant ψ and J over contours of B_p , from which the actual poloidal drift velocity of the bananas, $v_p = cE/B_p$, can be quickly estimated everywhere.

The toroidal drift frequency of trapped particles is also calculated by ACTION. The bounce time is

$$\tau = \frac{\partial J}{\partial T} = \frac{1}{\mu} \frac{\partial J}{\partial B_M}, \tag{A.8}$$

and the toroidal drift frequency is then written⁵

$$w_D = - \frac{c}{e} \left(\frac{1}{2} \right) \frac{\partial J}{\partial \psi} \quad (A.9)$$

(A.9) is used to calculate w_D over the mesh of field points; unfortunately it is inadequate for use in making contours of constant w_D because the differences must be taken at constant B and (as can easily be seen in Fig. 42d) B is often nearly parallel to ψ . An alternate formula can be found by combining (A.8) and (A.9):

$$w_D = - \frac{c}{e} \mu \left(\frac{\partial B}{\partial \psi} \right)_J \quad (A.10)$$

The form of (A.10) is much more convenient since B and ψ are typically at small angles with respect to one another in the region of interest and both have large angles with respect to J. Figure 42e shows the same blowup region of LEVITRON with numbers representing values of constant w_D (for bananas whose ends rest at these points).

As a final example, Fig. 43 shows the B and ψ contours for the upper half of a SPIRE machine with a strong beam already built up. The heavy dashed line is the envelope of motion (constant J) for a banana quite similar to the one whose detailed motion is illustrated by Fig. 32d.

Fig. 42c. Values of constant toroidal drift frequency ω_D plotted on lines of constant flux for Levitron.

Fig 43. Envelope of motion for a banana in SPIRE.⁵

B. Magnetic Field Algebra

1. Difference Equations for ψ

Ampere's Law is

$$\nabla \times \underline{B} = \frac{4\pi}{c} \underline{j} = \frac{4\pi}{c} j \hat{\phi}$$

Using the definitions (II.5),

$$B_r = -\frac{1}{r} \frac{\partial \psi}{\partial z} \quad \text{and} \quad B_z = \frac{1}{r} \frac{\partial \psi}{\partial r},$$

$$\frac{\partial}{\partial z} \left(-\frac{1}{r} \frac{\partial \psi}{\partial z} \right) - \frac{\partial}{\partial r} \left(\frac{1}{r} \frac{\partial \psi}{\partial r} \right) = \frac{4\pi}{c} j \quad (\text{B.1})$$

which yields (II.6),

$$-\frac{1}{r} \frac{\partial^2 \psi}{\partial z^2} - \frac{\partial}{\partial r} \left(\frac{1}{r} \frac{\partial \psi}{\partial r} \right) = \frac{4\pi}{c} j$$

Multiply by r ,

$$-\frac{\partial^2 \psi}{\partial z^2} - r \frac{\partial}{\partial r} \left(\frac{1}{r} \frac{\partial \psi}{\partial r} \right) = \frac{4\pi}{c} r j \quad (\text{B.2})$$

Transform (B.2) in the z -direction, $\psi \sim \psi_k \exp(-ikz)$,

$$k^2 \psi_k - r \frac{\partial}{\partial r} \left(\frac{1}{r} \frac{\partial \psi_k}{\partial r} \right) = \frac{4\pi}{c} r j_k \quad (\text{B.3})$$

where j_k is defined similarly and the equation has been divided through by $\exp(-ikz)$. Now for simplicity drop the k subscript (still understood), denote grid positions with subscript m , and difference (B.3) directly, taking the second-difference as a difference of first-differences:

$$k^2 \psi_m - \frac{r_m}{\Delta r^2} \left[\frac{1}{r_{m+1/2}} (\psi_{m+1} - \psi_m) - \frac{1}{r_{m-1/2}} (\psi_m - \psi_{m-1}) \right] = \frac{4\pi}{c} r_m j_m \quad (B.4)$$

Multiply through by $(\Delta r)^2$ (denoted simply as Δr^2), separate and collect terms:

$$- \left(\frac{r_m}{r_{m+1/2}} \right) \psi_{m+1} + \left(\frac{r_m}{r_{m+1/2}} + \frac{r_m}{r_{m-1/2}} + \Delta r^2 k^2 \right) \psi_m - \left(\frac{r_m}{r_{m-1/2}} \right) \psi_{m-1} = \frac{4\pi}{c} r_m \Delta r^2 j_m \quad (B.5)$$

(B.5) correlates easily with the standard tridiagonal form,

$$-A_m \psi_{m+1} + B_m \psi_m - C_m \psi_{m-1} = D_m$$

with the coefficients defined

$$\left. \begin{aligned} A_m &= \left(\frac{r_m}{r_{m+1/2}} \right) \\ B_m &= \frac{r_m}{r_{m+1/2}} + \frac{r_m}{r_{m-1/2}} + \Delta r^2 k^2 \\ C_m &= \left(\frac{r_m}{r_{m-1/2}} \right) \\ D_m &= \frac{4\pi}{c} r_m \Delta r^2 j_m \end{aligned} \right\} \quad (B.6)$$

The solution to (B.5) may now be found, evaluating successive values of ψ from the boundary conditions using the recursion relation

$$\psi_m = E_m \psi_{m+1} + F_m$$

where the coefficients E_m and F_m are related to the coefficients (B.6) by

$$E_m = \frac{A_m}{B_m - C_m E_{m-1}} \quad m \geq 1$$

$$F_m = \frac{D_m + C_m F_{m-1}}{B_m - C_m E_{m-1}} \quad m \geq 1$$

Thus far we have not defined one important parameter: the wave number k . Equation (B.3) demonstrates the logical process of applying the transform. In practice, however, it is somewhat more accurate to write the difference expression for the z derivative term first, then transform that. The difference expression is

$$\frac{\partial^2 \psi}{\partial z^2} = \frac{1}{\Delta z^2} (\psi_{l+1} - 2\psi_l + \psi_{l-1}) \quad (B.7)$$

Transforming as before and dividing through as usual by $\exp(-ikz)$ where $z = l\Delta z$, leaves

$$\frac{1}{\Delta z^2} \psi_k (e^{-ik\Delta z} - 2 + e^{ik\Delta z})$$

or

$$\left[\frac{2}{\Delta z} \sin \left[\frac{k\Delta z}{2} \right] \right]^2 \psi_k \quad (B.8)$$

where $k = 2\pi/l\Delta z$ and $L \equiv l_{\max}$. Expression (B.8) is then substituted for k^2 in Eq. (B.3).

2. Demonstration That ψ is the Magnetic Flux

We wish to show that Eqs. (II.6) and (II.7) are equivalent if $\psi = rA_\phi$, the magnetic flux. Begin with (II.6),

$$-\frac{1}{r} \frac{\partial^2 \psi}{\partial z^2} - \frac{\partial}{\partial r} \left(\frac{1}{r} \frac{\partial \psi}{\partial r} \right) = \frac{4\pi}{c} j$$

and substitute in $\psi = rA_\phi$:

$$\begin{aligned} -\frac{1}{r} \frac{\partial^2}{\partial z^2} (rA_\phi) - \frac{\partial}{\partial r} \left[\frac{1}{r} \frac{\partial}{\partial r} (rA_\phi) \right] &= \frac{4\pi}{c} j \\ -\frac{\partial^2 A_\phi}{\partial z^2} - \frac{\partial}{\partial r} \left[\frac{\partial A_\phi}{\partial r} + \frac{A_\phi}{r} \right] &= \frac{4\pi}{c} j \\ -\frac{\partial^2 A_\phi}{\partial z^2} - \frac{\partial^2 A_\phi}{\partial r^2} + \frac{A_\phi}{r^2} - \frac{1}{r} \frac{\partial A_\phi}{\partial r} &= \frac{4\pi}{c} j \end{aligned} \quad (B.9)$$

Now expand (II.7)

$$\nabla^2 \underline{A} = \frac{4\pi}{c} \underline{j}$$

using $\underline{\nabla} \times (\underline{\nabla} \times \underline{A}) = \underline{\nabla} \underline{\nabla} \cdot \underline{A} - \nabla^2 \underline{A}$

under the Coulomb gauge condition $\underline{\nabla} \cdot \underline{A} = 0$,

$$\underline{\nabla} \times (\underline{\nabla} \times \underline{A}) = \frac{4\pi}{c} \underline{j} \quad (B.10)$$

Since $\underline{j}$ has only a ϕ component, we are interested only in the ϕ component terms of $\underline{\nabla} \times (\underline{\nabla} \times \underline{A})$ containing A_ϕ . Retaining only A_ϕ terms in the curl,

$$\underline{\nabla} \times \underline{A} = -\frac{\partial A_\phi}{\partial z} \underline{e}_r + \left[\frac{1}{r} \frac{\partial}{\partial r} (rA_\phi) \right] \underline{e}_z \quad (B.11)$$

The ϕ component of the next curl is given by

$$\underline{\nabla} \times \underline{D} = \frac{\partial D_r}{\partial z} - \frac{\partial D_z}{\partial r}$$

where $\underline{D} = \underline{\nabla} \times \underline{A}$. Applying this to (B.11)

$$\begin{aligned} [\nabla \times (\nabla \times A)]_\phi &= \frac{\partial}{\partial z} \left(-\frac{\partial A_\phi}{\partial z} \right) - \frac{\partial}{\partial r} \left[\frac{1}{r} \frac{\partial}{\partial r} (r A_\phi) \right] \\ &= -\frac{\partial^2 A_\phi}{\partial z^2} - \frac{\partial^2 A_\phi}{\partial r^2} + \frac{A_\phi}{r^2} - \frac{1}{r} \frac{\partial A_\phi}{\partial r} \end{aligned} \quad (B.12)$$

Substituting (B.12) into (B.10), we see that it is equivalent to (B.9) as desired.

3. Adjustments to the Toroidal Field

Following Callen and Dory,⁶ we define a function $f(\psi)$

$$f(\psi) \equiv R B_\phi = \frac{k}{1 + b P(\psi)} \quad (B.13)$$

where $P(\psi)$ is the pressure function, k and b are arbitrary constants. It is logical to select

$$k = I_z/5$$

since then the constant b can be interpreted as the exact fractional degradation of B_ϕ by the poloidal currents which are functions of $p(\psi)$ (since in vacuo $B_\phi = I_z/5R$). Also, if $f'(\psi) \equiv \partial f / \partial \psi$ is zero, (B.13) will return a purely vacuum dependence on $1/r$ a desirable feature.

We now rewrite Eq. (II.14), adding the new degradation term.

as

$$\Delta^* \psi = R^2 P'(\psi) + g'(\psi) \quad (\text{B.14})$$

where

$$\Delta^* \equiv -\frac{\partial^2}{\partial \xi^2} - R \frac{\partial}{\partial R} \left\{ \frac{1}{R} \frac{\partial}{\partial R} \right\}, \quad P'(\psi) \equiv \frac{\partial P}{\partial \psi},$$

and the new function $g'(\psi)$ is simply

$$g'(\psi) \equiv f(\psi)f'(\psi),$$

which gives the correct dependence on $f'(\psi)$. Inserting (B.13) into (B.14),

$$\begin{aligned} \Delta^* \psi &= R^2 P'(\psi) + \left[\frac{k}{1 + bP(\psi)} \right] \frac{\partial}{\partial \psi} \left[\frac{k}{1 + bP(\psi)} \right] \\ &= R^2 P'(\psi) + \left[\frac{k}{1 + bP(\psi)} \right] \left[\frac{-kb}{[1 + bP(\psi)]^2} \right] P'(\psi) \\ &= R^2 P'(\psi) - \left\{ \frac{bk^2}{[1 + bP(\psi)]^3} \right\} P'(\psi) \\ \Delta^* \psi &= \left\{ R^2 - \frac{bk^2}{[1 + bP(\psi)]^3} \right\} P'(\psi) \quad (\text{B.15}) \end{aligned}$$

Equation (B.15) is the one actually solved in GUIDON by the method described in Chapter II.

4. The Fast Fourier Transform (FFT)

The idea behind the FFT is to find a way to break down the summation defining the Fourier coefficients through some symmetry argument so that fewer actual additions must be performed. The specific gimmick is to split the series into symmetric pairs over and over until but a single pair of functional evaluations remains.

This is the reason that any computational grid to which the FFT is applied must have a dimension in the relevant direction that is a power of two.

The r^{th} coefficient of the discrete Fourier transform is defined

$$A_r = \sum_{k=0}^{N-1} X_k e^{-2\pi i r k / N} \quad r = 0 \dots (N-1) \quad (\text{B.16})$$

where X_k is the k^{th} sample of the time series. Rewrite (B.16) as

$$A_r = \sum_{k=0}^{N-1} X_k W^{rk}, \quad W \equiv e^{-2\pi i / N} \quad (\text{B.17})$$

and the inverse time series is

$$X_k = \left(\frac{1}{N} \right) \sum_{r=0}^{N-1} A_r W^{-rk} \quad (\text{B.18})$$

To construct the FFT, divide the X_k into Y_k and Z_k , each of size $N/2$:

$$Y_k = X_{2k} \quad Z_k = X_{2k+1}$$

Then

$$A_r = \sum_{k=0}^{(N/2)-1} Y_k e^{-4\pi i r k / N} + Z_k e^{-2\pi i r (2k+1) / N} \quad r = 0, 1, 2, \dots, (N-1)$$

or

$$A_r = \sum_{k=0}^{(N/2)-1} Y_k e^{-4\pi i r k / N} + e^{-2\pi i r / N} \sum_{k=0}^{(N/2)-1} Z_k e^{-4\pi i r k / N} \quad (\text{B.19})$$

Define for convenience

$$B_r \equiv \sum_{k=0}^{(N/2)-1} y_k e^{-4\pi i r k / N} \quad \text{and} \quad C_r \equiv \sum_{k=0}^{(N/2)-1} z_k e^{-4\pi i r k / N}$$

and rewrite (B.19) as

$$A_r = B_r + (e^{-2\pi i r / N}) C_r \quad (0 \leq r < N/2) \quad (\text{B.20})$$

But for values of $r > N/2$, the discrete Fourier transforms of B_r and C_r repeat periodically the values for $r < N/2$. Substituting $(r + N/2)$ for r in (B.20),

$$\begin{aligned} A_r &= B_r + w^r C_r \\ A_{r+N/2} &= B_r - w^r C_r \end{aligned} \quad 0 \leq r < N/2$$

Thus we need only $N/2$ y_k 's and z_k is to form all N A_r 's. Repeating this process continually can reduce the requirement to one of each.

C. Guiding-Center Algebra

The guiding-center drift velocity is defined by (II.28a) in three terms

$$\underline{v}_D = \underbrace{\frac{c}{B^2} \underline{E} \times \underline{B}}_{\text{Hall current}} + \underbrace{-\frac{mc}{2e} \frac{v_\perp^2}{B^2} \underline{b} \times \underline{\nabla} B}_{\text{VB drift}} + \underbrace{\frac{2c v_\parallel}{e B^2} \underline{E} \times (\underline{b} \cdot \underline{\nabla}) \underline{b}}_{\text{curvature drift}}$$

Each of these terms will be dealt with separately and derived in final form for the r-component only, to save needless repetition. All three components will be displayed as actually defined by GUIDE and GUIDON.

1. Hall Current

This term is trivial. The applied electric potential is defined in the code for convenience as $V = RE_\phi$. Since only E_ϕ is non-zero, one term in the R component and one in the Z are all that survive of $\underline{E} \times \underline{B}$. The R component is

$$v_{1R} = \left(\frac{c v_\perp^2}{R} \frac{B_z}{B^2} \right) \quad (C.1)$$

2. VB Drift

$$\begin{aligned} v_2 &= \frac{mc}{2e} \frac{v_\perp^2}{B^2} \underline{b} \times \underline{\nabla} B = \frac{c}{e} \left(\frac{mv_\perp^2}{2B} \right) \frac{1}{B} \underline{b} \times \underline{\nabla} B = \frac{c}{e} \frac{\mu}{\gamma} \frac{1}{R^2} (\underline{b} \times \underline{\nabla} B) \\ &= \frac{c}{e} \frac{\mu}{\gamma} \frac{1}{B^3} (\underline{B} \times \underline{\nabla} B) \\ &= \frac{c}{e} \frac{\mu}{\gamma} \frac{1}{B^3} \left[\underline{B} \times \underline{\nabla} \left(\frac{1}{2} B^2 \right) \right] \end{aligned} \quad (C.2)$$

We shall now continue to derive only the R component of (C.2):

$$v_{2R} = \frac{c}{e} \left(\frac{\mu}{\gamma} \right) \frac{1}{B^3} B_\phi \cdot \nabla \left(\frac{1}{2} B^2 \right)_z - B_z \cdot \nabla \left(\frac{1}{2} B^2 \right)_\phi$$

by expanding the cross-product. The last term on the right is zero since there are no gradients of B^2 in the ϕ direction. So, by definition of the ∇B^2 ,

$$\begin{aligned} v_{2R} &= \frac{c}{e} \left(\frac{\mu}{\gamma} \right) \frac{1}{B^3} B_\phi \left[\frac{\partial}{\partial z} \frac{1}{2} (B_R^2 + B_z^2 + B_\phi^2) \right] \\ &= - \frac{c}{e} \left(\frac{\mu}{\gamma} \right) \frac{B_\phi}{B^3} \left(\frac{1}{2} \right) \left(2B_R \frac{\partial B_R}{\partial z} + 2B_z \frac{\partial B_z}{\partial z} \right) \\ &= - \frac{c}{e} \left(\frac{\mu}{\gamma} \right) \frac{B_\phi}{B^3} \left(B_R \frac{\partial B_R}{\partial z} + B_z \frac{\partial B_z}{\partial z} \right) \end{aligned} \quad (C.3)$$

(since $\partial B_\phi / \partial z = 0$).

3. Curvature Drift

The curvature drift is somewhat more complicated.

$$v_3 = \frac{2ev\mu}{eB^2} \underline{B} \times (\underline{b} \cdot \nabla) \underline{B} = \frac{\gamma \pi_0 c}{e} \frac{v\mu^2}{B^3} \left[\underline{B} \times (\underline{B} \cdot \nabla) \underline{B} \right] \quad (C.4)$$

To evaluate the term in brackets, rewrite it as

$$\frac{1}{B^2} \underline{b} \times (\underline{B} \cdot \nabla) \underline{B} \quad (C.5)$$

since

$$\underline{b} \times (\underline{b} \cdot \nabla) \underline{B} = \frac{1}{B^2} \underline{b} \times (\underline{B} \cdot \nabla) \underline{B} - \underline{B} \cdot \nabla \underline{B}$$

and the last term on the right is identically zero. Then

$$\frac{1}{B^2} \underline{b} \times (\underline{B} \cdot \underline{\nabla} \underline{B}) = \frac{1}{B^2} \underline{b} \times \left[\underline{\nabla} \left(\frac{B^2}{2} \right) + \underline{\nabla} \times \underline{B} \right] \quad (C.6)$$

by the simple expansion of $\underline{B} \times \underline{\nabla} \times \underline{B}$. Then

$$\frac{1}{B^2} \underline{b} \times (\underline{B} \cdot \underline{\nabla} \underline{B}) = \frac{1}{B^2} \underline{b} \times \underline{\nabla} \left(\frac{B^2}{2} \right) + \frac{1}{B^2} (\underline{B} \underline{J} - \underline{B} \underline{J} \cdot \underline{b}) \quad (C.7)$$

where $\underline{J} \equiv \underline{\nabla} \times \underline{B}$ and the last term on the right has been expanded using the BAC-CAB rule (and the fact that $\underline{b} \cdot \underline{B} = B$). The first expression on the right-hand side is recognized as identical to the expression in the VP drift. For the R component of the second expression, note that only $\underline{J}_\phi$ has a non-zero component, so $(\underline{B} \underline{J})_R = 0$ and the surviving term is

$$\frac{1}{B^2} (-\underline{B} \underline{J} \cdot \underline{b}) = - \frac{1}{B} \left[\frac{\underline{B}_R}{B} \right] \left[\frac{\partial \underline{B}_R}{\partial z} - \frac{\partial \underline{B}_z}{\partial r} \right] \frac{\underline{B}_\phi}{B} \quad (C.8)$$

Combining (C.4) with (C.8) and the relevant portion of (C.3)

$$\underline{v}_R = \left(\frac{\gamma m_0 c}{e} \right) \frac{v_\parallel^2}{B^2} \left\{ \frac{\underline{B}_\phi}{B} \underline{B}_R \frac{\partial \underline{B}_R}{\partial z} + \underline{F}_z \frac{\partial \underline{B}_z}{\partial z} - \frac{\underline{B}_R \underline{B}_\phi}{B^2} \left[\frac{\partial \underline{B}_R}{\partial z} - \frac{\partial \underline{B}_z}{\partial r} \right] \right\} \quad (C.9)$$

4. Total Velocity

The results of all three of these drifts can then be combined with the appropriate components of $v_\parallel$ to give the total velocity of the guiding-center. These are the expressions used in the code:

$$v_R = v_{\parallel} \left(\frac{B_R}{B} \right) + c v \left(\frac{B_z}{RB^2} \right) - \left(\frac{c}{e} \right) \gamma m_0 v_{\parallel}^2 \frac{1}{B^3} \left[\frac{\partial B_R}{\partial z} - \frac{\partial B_z}{\partial R} \right] \frac{B_R B_{\phi}}{B} \\ + \left(\frac{c}{e} \right) \left[\frac{\mu}{\gamma} + \frac{v_{\parallel}^2 \gamma m_0}{B} \right] \frac{B_{\phi}}{B^3} \left(B_R \frac{\partial B_R}{\partial z} + B_z \frac{\partial B_z}{\partial R} \right)$$

$$R v_{\phi} = v_{\parallel} \left(\frac{B_{\phi}}{B} \right) + \left(\frac{c}{e} \right) \gamma m_0 v_{\parallel}^2 \frac{1}{B^3} \left(\frac{\partial B_R}{\partial z} - \frac{\partial B_z}{\partial R} \right) \frac{B_R^2 + B_z^2}{B} \\ + \left(\frac{c}{e} \right) \left[\frac{\mu}{\gamma} + \frac{v_{\parallel}^2 \gamma m_0}{B} \right] \frac{1}{B^3} \left[B_z P_R \left(\frac{\partial B_R}{\partial R} - \frac{\partial B_z}{\partial z} \right) + B_R^2 \frac{\partial B_z}{\partial R} \right. \\ \left. - B_R^2 \frac{\partial B_R}{\partial z} - \frac{B_{\phi}^2}{R} B_z \right]$$

$$v_z = v_{\parallel} \left(\frac{B_z}{B} \right) - c v \left(\frac{B_R}{RB^2} \right) - \left(\frac{c}{e} \right) \gamma m_0 v_{\parallel}^2 \frac{1}{B^3} \left[\frac{\partial B_R}{\partial z} - \frac{\partial B_z}{\partial R} \right] \frac{B_{\phi} B_z}{B} \\ - \left(\frac{c}{e} \right) \left[\frac{\mu}{\gamma} + \frac{v_{\parallel}^2 \gamma m_0}{B} \right] \frac{B_{\phi}}{B^3} \left(B_R \frac{\partial B_R}{\partial R} + B_z \frac{\partial B_z}{\partial R} - \frac{B_{\phi}^2}{R} \right)$$

D. The Rationality Controversy

Hume's criticism of the fallacy of inductive logic fell as a bombshell in speculative philosophy. Science, however, proceeded almost unaffected; Francis Bacon's conception of science, which relied in essence on induction, continued as the accepted notion among scientists for centuries, and is in fact still widely taught to students and undergraduates. Many competent practicing physicists are unaware that any problem exists.

The most promising explanation of how science attains rational authority was first presented in this century by philosopher Karl Popper.^{7,8,9} Conceding that science cannot attain absolute truth, Popper sought a view in which it remained autonomous, based entirely upon logic and observation. Popper views the progress of science as a series of conjectures (hypotheses) concerning the unknown which make predictions which may be refuted by observation or experiment. The greater the risk of refutation (the more easily or directly it may be refuted), the more meaningful the hypothesis. To Popper, supportive and interpretive evidence are meaningless as science, for theories constructed from them may never risk refutation, simply because categorical refutation may be impossible. His classic examples of pseudo-sciences based only on such evidence are astrology, psychiatry and Marxism.

Popper's concept admits a subjective element (the formation of conjectures) into the scientific process, but succeeds in placing all scientific authority on a completely rational base. Though a particular theory may never attain absolute truth, we may come

to place very high confidence in it through its repeated exposure to the risk of refutation.

In the last decade challenge has been raised to the rationality of even this more limited picture of science by a group of scientific philosopher-historians, whose most prominent spokesman is Thomas Kuhn. Even if Popper is correct, Kuhn maintains, science is robbed of its autonomy by the confidence criterion: it is reduced to being nothing more than what scientists think it is. But Kuhn has further argued from historical studies that science does not actually progress according to Popper's formula. According to Kuhn,¹⁰ scientists of any particular era work trapped within a rigid, universally accepted notion of scientific theory, the current "paradigm" of science; their work in these periods of "normal science" consists only of "problem solving" within the limits of the paradigm. Only during extraordinary historical moments--scientific revolutions--is the paradigm questioned meaningfully and broken by a great subjective leap; a new paradigm then forms quickly. And, says Kuhn, arguing with historical examples, during the process of revolution rationality is not strictly observed; the genius primarily responsible for the advance will often place such great faith in the beauty and power of his new concept that he will ignore apparently contradictory experimental evidence--he will hang onto his theory in spite of its refutation. This irrationality during the advances, and the irrationality of maintenance of the paradigm in spite of the accumulation of refutations during periods of normal science, lead Kuhn to contend that science has no truly

rational base. If even the great scientists (and he cites Newton and Galileo) will maintain their theories in spite of experimental contradictions, how can there be a meaningful standard for distinguishing them from the crackpots?

The conflict between Kuhn and Popper is hardly as well-formed as the contestants think, for they rely greatly on their own definitions of problems, presuming their categories to be precise and exclusive. In logical terms, Kuhn is guilty of the fallacy of the excluded middle.

Lack of firsthand scientific experience could be at fault. In his purely historical study,¹⁰ Kuhn has arbitrarily divided science into revolutions and problem solving; surely both are correct, but they are not a complete set. Most scientific knowledge accretes through a third process, one whose methods are so diverse that it is misleading to collect them under a descriptive name. The characteristic of this process is that new relationships, influences, laws and effects are discovered which do not happen to conflict with existing general theory but which nevertheless are not a direct result of it, or even contingent upon it. Thus, for instance we should classify the origins of quantum mechanics as revolutionary and might even (at our most critical) denigrate quantum electrodynamics to "problem solving". But the physics of superfluidity and superconductivity which came about through the blossoming of quantum theory (and certainly in their present formulations rely upon it) provided us with understanding of realms of reality previously completely unknown, a valuable and completely novel understanding.

It was not gained through a process of mere problem solving for it was an accretion of new physical laws, the formulation of whole new sciences. Yet it did not require a revolution and, though employing much problem solving, had other methods as well: Kuhn's simplification is simply not adequate.

That being the case (and countless examples of this third category could be advanced--plasma physics, lasers, the new astronomy), Kuhn's criticisms of the nature of science, which rely for logical validity upon the "evidence" of his interpretation (or rather-- explanation) of its methodology, cannot stand. If there is more to science than great inductive leaps and problem solving, this more needs to be explained as well in a self-consistent view. And granting that the actual methods employed by scientists in this third category are scarcely Baconian, they are not far from Popper's notion of refutation (which is even valuable as a problem-solving technique, as any computational physicist knows and as James Watson has described in connection with microbiology).

A further problem I perceive in the debate arises from having begun it too far along the logical train, with little agreement on what should have come before. The questions, it seems to me, should be asked in this fashion:

(a) Is there a fixed physical truth to which scientific laws seek to conform?

(b) Can science hope to express this truth in exact laws, or at least approach within a reasonably known ϵ of expressing it?

(c) Is there a purely objective criterion by which we can

judge whether or not one theory is better than another in its approximation?

Kuhn and Popper et al. are primarily arguing (c), while occasionally expressing private opinions on (a) and (b) (or worse, using these opinions as arguments), without having established a proper context for the debate. Kuhn, in the 1969 Appendix to Ref. 10, concludes that because his answer to (c) is No, his answer to (a) must be No as well. Surely this is nonsense. But Popper is scarcely better; by his brilliant sidestepping of (a), his qualified Yes (the qualification being that ϵ may remain only subjectively known) to (b) leaves him hanging on a logical hook perilously close to empty opinion. One does not have to be an "essentialist" to conclude that there is some Essential--it need not be Spinozistic; though Bentham and Berkeley might not be able to accept it, Mach, Bridgman and Einstein might. There are many paths round that logical corner.

One's answer to (a) is at bottom a matter of faith, and clearly it is a working physicist's faith (barring any cavil at the wording which would exclude an orderly stochasticism: I mean it in such a way that Bohr could comfortably answer Yes). One need not answer Yes even to (b) (though it would require cleverness or delusion to say No and continue working on many problems), and certainly not to (c). A Yes to (c) can even be viewed as unnecessary to the rational objectiveness of science; if the means of perceiving the better theories do turn out, as Kuhn contends, to require subjective processes and time, so what?

The historians all seem amazed at the discovery that scientists,

great and ordinary, may place such strong, often irrational faith in hypotheses, frequently to the detriment of science. But common sense and experience tell us that subjective elements will necessarily proliferate in any endeavors of human beings. In this case, one important element is the inability of the innovator to know precisely which experimental results he can or cannot accept fully at face value. Surely science cannot advance at all if he must perform them all himself. The historians suggest that their conclusions are forced by historical example in a paradoxical tacit expression or inductive logic, but fail to recognize this parallel or grasp the difficulty of evaluating experimental evidence properly. The good scientists know that very few experiments are truly decisive: most, like computer simulations, query primarily the assumptions of their designers. Since Newton there has never been a comprehensive physical theory, one that accounted for the results of all relevant experiments. The mark of scientific genius (as the historians themselves have noted) is knowing which contradictory experiments may be ignored in the evolution of hypotheses as embodying errors of assumption or of procedure. What disturbs the historians is not so much that this process violates objectivism, which is actually simply how one is disposed to look at things, but that accepting this fact of hypothesis-formulation makes it objectively difficult to distinguish between the genius and the crackpots. Except, that is, by long hindsight, which is the source of the value of the "Whig" attitude toward scientific history that they take such pains to deplore; it is in precise analogy to the evolution of

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value in art--so it is really subjectivity they are objecting to. Life is unfortunately more complicated than science can be, and science, as Kuhn, Popper and Oppenheimer would agree, is eventually, only what scientists do.

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