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STUDIES OF RESISTIVE INSTABILITIES IN DIFFUSE PINCHES

James A. Dibiase

May 1974

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James A. Dibiase

Ph.D. Thesis

MS. date: May 1974

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NUMERICAL STUDIES OF RESISTIVE INSTABILITIES
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ABSTRACT

A hydromagnetic model has been developed to study resistive instabilities in cylindrical geometry, and the model is applied to study specific diffuse pinch configurations. The MHD equations include the effects of compressibility, finite resistivity, viscosity, and thermal conductivity. The plasma equilibrium configuration is assumed known and is specified by $B_{\theta 0}(r)$, $B_{z0}(r)$, $\eta_0(r)$, $\rho_0(r)$, $T_0(r)$ and these functions can be chosen to describe a particular experiment. Perturbations of the form

$$f_1(r, t) e^{i(m\theta + k_z z)}$$

are used for all plasma and field variables, and the resulting linear partial differential equations are solved numerically as an initial value problem using an implicit difference scheme. A set of six equations for B_{r1} , $B_{\theta 1}$, T_1 , v_{r1} , $v_{\theta 1}$, v_{z1} is obtained and the calculation is started by specifying an initial perturbation. An incompressible resistive model without dissipation is also developed, resulting in a set of four equations in B_{r1} , $B_{\theta 1}$, v_{r1} , $v_{\theta 1}$. For a particular problem, the parameters m , k_z , and $S = \tau_R/\tau_H$, the ratio of resistive diffusion time to hydromagnetic

transit time must also be given. For certain cases an exponential growth occurs and the growth rate $p(m, k_z, S)$ is calculated. The model can be used to test equilibrium configurations against both tearing and rippling modes. Extensive parametric studies have been done for the tearing mode using a force-free Bessel function model, a high-beta reversed field model, and 3 tokamak-type models in which the axial field does not reverse direction. Results for the tearing mode using the Bessel function model agree with analytic theory, and the outer conducting wall R_w must be placed approximately 2% closer to the plasma to insure stability against resistive tearing modes as well as ideal MHD modes. For the high-beta reversed field model, complete stability against resistive tearing modes for positive k_z values requires R_w approximately 22% closer to the plasma than the MHD stability criterion, while complete stability for negative k_z values requires R_w about 50% closer. For both reversed field models, resistive tearing instabilities are found where no ideal MHD instabilities exist. The tokamak models are found to be resistively unstable for large values of m ; the stability windows of the infinite conductivity case are replaced by regions of low growth rate, and there is considerable overlap of the azimuthal modes. For all configurations tested, true resistive tearing modes seem to occur only for $S \gtrsim 10^4 - 10^5$. The infinite conductivity case is also tested and the results agree well with analytic theory. The model can thus be applied to study both the MHD and resistive instability of a wide class of equilibrium configurations in cylindrical geometry.

I. INTRODUCTION

1.1 Motivation

In order to achieve the high densities and temperatures required for a successful thermonuclear reactor, a plasma must be confined by a magnetic field for a sufficiently long time. In the attempts to achieve this confinement, the problem of stability has emerged as one of the most important. A plasma confined by a magnetic field is potentially able to break out of the confinement system by a large variety of instabilities, and a number of field configurations have been proposed which could confine a hot plasma in equilibrium, if such configurations were stable.

The most dangerous type of instabilities are the magneto-hydrodynamic (MHD) or hydromagnetic instabilities in which the plasma is assumed to behave as a conducting fluid and the instabilities involve displacement of macroscopic portions of the plasma. General summaries of both theoretical and experimental studies of plasma instabilities have been performed by many authors,¹⁻⁵ and it is a particular macroscopic MHD instability, the resistive instability, which is addressed in this report.

One of the earliest plasma confinement schemes tested was that of the pinch effect in which an axial current in a cylindrical plasma produces an azimuthal field that constricts or pinches the plasma. It has been found both theoretically and experimentally that such pinch configurations are unstable to MHD type instabilities, and the four lowest order distortions are shown schematically in Fig. 1.1. The $m = 1$ kink instability is usually

Figure 1.1 The four lowest order distortions for a cylindrical plasma. The dotted lines represent the equilibrium cross section.

the most dangerous since it allows the plasma to break through the confining magnetic field to reach the walls.

The pinch configuration has been extensively used for the study of plasmas with high betas. In a high-beta system the plasma pressure is not negligible compared with the magnetic pressure, and the magnetic field distribution depends on plasma currents. High-beta systems usually have large currents flowing in the plasma, and experimental pinch systems have been used for the study of plasmas with beta in the range 0.1-1. One of the earliest attempts to confine a plasma in a toroidal system using the pinch effect was carried out by Cousins and Ware⁶ who found that the system was unstable to the kink instability. Their work was followed by the historic paper of Kruskal and Schwarzschild⁷ who showed theoretically that a plasma confined by the pinch effect is unstable to MHD-type instabilities.

The pinch can be made more stable if a magnetic field parallel to the axis is trapped within the cylinder and if there is an azimuthal external field generated by a current at the surface of the plasma column. For this configuration, the sausage ($m = 0$) and kink ($m = 1$) instabilities bend and compress the field lines, bringing into play forces which oppose the growth of the instability. Unfortunately, it has not proven possible to stabilize the pinch completely, since in reality the current flows in a layer of finite thickness and a localized instability can develop. In the present analysis, a pinch configuration is established in which the plasma currents are assumed to be diffuse, i.e., the axial and

azimuthal currents are not restricted to a thin plasma sheath but are uniformly distributed throughout the plasma. The plasma is assumed to completely fill an infinitely long plasma cylinder which could correspond to the limiting case of a very thin torus or a very long plasma column with electrodes at the ends.

1.2 Description of Resistive Instabilities

In ideal MHD theory the plasma is tied to the magnetic field lines because of the assumed infinite conductivity; if this assumption is relaxed very slightly, new modes, known as MHD resistive instabilities, can arise. With finite resistivity the plasma can now slip across the magnetic field lines, and systems which were stable on ideal MHD can now become unstable. Following Rostoker,¹ resistive type instabilities, which are current-driven instabilities, can be visualized by considering a stabilized pinch. If a kink mode develops without carrying the B_z field with it (see Fig. 1.2 (a)), the original J_z current now has a θ component and the $J_\theta \times B_z$ force will make the perturbation grow. In the case of a sausage instability the resultant $J_r \times B_z$ force will cause the plasma to spin, which is destabilizing [Fig. 1.2 (b)].

Resistivity can also destroy the stabilization achieved by the shearing of the lines of force. In the case of a magnetic field which has shear or which changes direction as illustrated in Fig. 1.2 (c), the magnetic energy can be reduced by allowing the fields to mix and annihilate. This is prevented by a perfectly conducting plasma. However, for finite conductivity an instability can develop in which the magnetic lines of force, as

Figure 1.2 Resistive instabilities: (a) stabilized pinch with kink in J ($m=1$); (b) stabilized pinch with sausage in J ($m=0$); (c) reversed field layer; (d) resistive tearing instability of reversed field layer.

shown in Fig. 1.2 (a), are torn into loops or rings, as shown in Fig. 1.2 (d). This type of resistive instability is known as a resistive tearing mode.^{1,8}

In order to understand more completely the role played by resistivity in this destabilization process, consider the simplified form of Ohm's law:

$$\eta \bar{J} = \bar{E} + \frac{\bar{v} \times \bar{B}}{c} . \quad (1.1)$$

The right-hand side is equal to the electric field in the coordinate system moving with the plasma and is equal to zero if η is zero; that is, the first term, $\bar{E}$, which is the electric field in the laboratory system, changes so as to cancel out the induced field given by the second term. This electric field $\bar{E}$ results from the induction of the magnetic field in such a manner that the lines of force are frozen into the plasma. If η is small but finite, $\bar{E}$ no longer exactly cancels out the second term and a slight defreezing of the lines of force occurs. The degree to which $\bar{E}$ then compensates the second term depends on the size of η . In the other extreme, when η is very large, $\bar{E}$ will not change and the motions of the plasma and the lines of force become detached. Hence Ohm's law reduces to

$$\eta \bar{J}_1 \approx \frac{\bar{v} \times \bar{B}_0}{c} , \quad (1.2)$$

where $\bar{B}_0$ is the equilibrium magnetic field and $\bar{J}_1$ is the first order perturbation of the current $\bar{J}$. This then produces a Lorentz

force $\bar{\mathbf{F}} = (\bar{\mathbf{J}}_1 \times \bar{\mathbf{B}}_0)/c$ which acts on the plasma, and using (1,2) this Lorentz force becomes

$$\bar{\mathbf{F}} = \frac{\bar{\mathbf{J}}_1 \times \bar{\mathbf{B}}_0}{c} = \frac{1}{\eta c^2} [\bar{\mathbf{B}}_0 (\bar{\mathbf{v}} \cdot \bar{\mathbf{B}}_0) - \bar{\mathbf{v}} B_0^2] . \quad (1.3)$$

Except for fluid motion along $\bar{\mathbf{B}}_0$, for which $\bar{\mathbf{F}}$ vanishes, this force opposes the fluid motion and it becomes stronger as $\eta \rightarrow 0$. This means that for small η a strong force is necessary if the plasma is to be detached from the lines of force so that $\bar{\mathbf{E}}$ and consequently $\bar{\mathbf{B}}$ do not change. However, if $\bar{\mathbf{B}}_0$ has a null point, this restoring force becomes arbitrarily weak in the neighborhood of that point and may be dominated there by other forces acting to accelerate the fluid motion. Even for a more general twisted magnetic field which does not have a null point, a wave vector $\bar{\mathbf{k}}$ can always be chosen for the perturbation such that $(\bar{\mathbf{k}} \cdot \bar{\mathbf{B}})$ becomes zero at any desired point.

There are three types of resistive modes: (1) the rippling mode, which is driven by a gradient in the resistivity and is usually not important at high-beta where large temperature gradients are unlikely; (2) the gravitational mode (g-mode) which is the resistive equivalent of the interchange instability and is important in sheared systems; and (3) the tearing mode, which is the resistive equivalent of the kink mode and involves displacement of the whole plasma. The last two modes are important at high-beta, and both are usually found to be unstable using resistive MHD theory.

The tearing mode differs from the other two modes in that it is typically a long-wave rather than a short-wave mode relative to the dimension of the current layer, and consequently it is expected that it would be stabilized by conducting walls. The driving force is due to the structure of the magnetic field outside the region of decoupled flow, i.e., the tendency of a sheet current to break up into a set of parallel pinches. The modes grow on a time scale intermediate between the resistive diffusion time

$$\tau_R = 4\pi a^2 / \eta c^2 \quad (1.4)$$

and the hydromagnetic transit time

$$\tau_H = \frac{a \sqrt{4\pi \rho}}{B}, \quad (1.4)$$

where 'a' is a characteristic dimension of the plasma layer, η is the resistivity, ρ is the mass density of the plasma, B is the magnetic field, and c is the speed of light. For theory to apply, it is required that $S = \tau_R / \tau_H \gg 1$.

Thus it is seen that the existence of the three "resistive" instabilities depends on the local relaxation of the constraint that the fluid must remain attached to the magnetic field. For a zero order field that is not a vacuum field, possibilities of lowering the potential energy are always present, and the introduction of finite conductivity makes some energetically possible modes accessible. These new modes are analogous to the modes which occur in hydrodynamics when the constraint of conservation

of vorticity is removed by the presence of finite viscosity, and the viscosity provides a mechanism for vortex lines to slip across the fluid.

1.3 Previous Results and Scope of the Present Work

1.3.1 Experimental Observations

Resistive instabilities have been observed in various systems in which oppositely directed magnetic fields are separated by a plasma sheet or annulus. The tearing mode has been observed in reversed-field theta pinches^{9,10} and in an annular plasma sheet produced between two coaxial theta pinch coils.¹¹ In the reversed field theta pinch the plasma is confined in a thin annulus by oppositely directed magnetic fields. In the middle regions where the magnetic field lines are straight the configuration is predicted to be neutrally stable using ideal MHD theory, but experimental evidence shows this system to be unstable, and the instability is usually attributed to the resistive tearing mode which causes the annulus to split into rings with wavelengths of a few centimeters. Such rings have been observed to grow⁹ and move along the axis of the system, frequently coalescing with each other. In short systems these rings are compressed axially into a single ring by the closed magnetic field lines encircling them. The formation of these rings in a time considerably shorter than the resistive diffusion time, followed by motions which showed that the plasma behaved hydromagnetically and not purely resistively, suggests that they are due to the resistive tearing mode. The formation time is in semiquantitative agreement with a simple

expression for the growth of the resistive tearing mode given by Furth, Killeen, and Rosenbluth⁸ (hereafter referred to as FKR).

Additional experimental evidence of the tearing mode has been obtained by several experimenters for linear pinches¹²⁻¹⁶ and hard-core pinches,^{13,17,18} and some instabilities observed in Levitron may possibly be attributed to the tearing mode.¹⁹ Resistive instabilities in sheared toroidal systems are more complex but have also been extensively studied.^{20,21} Experimental evidence of the resistive rippling mode has been achieved in only a few cases. In the Levitron, Birdsall, et al.²² have measured a magnetic flutter which appears to agree with the current distribution from a rippling mode, and in a hardcore discharge Aitken, et al.²³ have found some residual fluctuations which can be explained by a rippling mode.

1.3.2 Theoretical Results

Since experimental observations in a variety of pinch experiments have shown instabilities for equilibrium magnetic field configurations that the infinite conductivity hydromagnetic theory would predict to be stable, finite conductivity has been included in the hydromagnetic analysis in an attempt to identify these instabilities. The basic paper on resistive MHD instabilities is that by FKR⁸ who analytically treat all of the possible linear resistive instabilities of a plane plasma sheet in a sheared magnetic field configuration, and describe in detail the driving forces of the instabilities. In their analysis, FKR derive the general equations for a plane resistive current layer using the

incompressible hydromagnetic approximation. Applying perturbations of the form

$$f_1(y)e^{i(k_x x + k_z z) + i\omega t}, \quad (1.6)$$

a dispersion relation is obtained and an eigenvalue problem is solved for ω , the growth rate of the instability. To solve the problem the plasma is divided into two regions, a narrow resistive inner region about the plane for which the wave vector is perpendicular to the equilibrium magnetic field ($\vec{k} \cdot \vec{B}_0 = 0$) and an outer region where the infinite conductivity equations hold, and the solutions are joined at the boundaries. Using this approach, FKR identify purely growing tearing, rippling, and gravitational modes.

Using the same sheet pinch model, similar studies have been carried out by other authors²⁴⁻²⁷ for several static equilibrium magnetic field and resistivity configurations. Van Hoven and Cross,²⁸ allowing for the decay of the background fields, also obtain exponentially growing tearing modes with long wavelength. Similar analyses have been done in cylindrical geometry by Rebut²⁹ and Coppi, et al.³⁰ who examined resistive instabilities for a diffuse linear pinch, and Johnson, et al.³¹ who studied the resistive instabilities in a low-pressure stellerator. In addition, Coppi, et al.³⁰ considered the effects of viscosity and thermal conductivity and found that viscosity can have a stabilizing effect on resistive modes.

In addition to the application to laboratory plasmas, it has

long been speculated that resistive instabilities might be of prime importance in solar flares and in the geomagnetic tail of the earth, and Van Hoven and Cross³²⁻³⁴ have examined both the linear and non-linear tearing modes for these astrophysical applications.

1.3.3 Scope of the Present Work

Due to the many possible equilibrium configurations and the many approximations necessary to make the problem analytically tractable, it is usually not possible to analytically describe the general parameter dependence of the growth rates of the resistive instabilities. Also, the analytic treatments are usually done in the limits of high and low S . In order to supplement the analyses for intermediate values of S and to be able to obtain results for specific and wide choices of equilibrium magnetic fields and boundary conditions, numerical models have been developed to study these resistive instabilities.

The numerical models are based on the linearized equations which assume an arbitrary time dependence, i.e., the hydromagnetic equations are Fourier analyzed in space but not in time, and the problem becomes an initial value problem for the perturbed quantities. Also, two regions are not used, instead the same resistive equations are assumed to hold throughout the plasma. The parameters, initial conditions, boundary conditions, and type of modes to be examined can all be varied.

Based on the incompressible hydromagnetic equations, a time-dependent resistive instability computational model for both

Cartesian and cylindrical geometry was developed by Killeen³⁵ and has been applied to the Triax experiment,^{36,37} hardcore pinch configurations,^{13,29} the thetatron,³⁸ and to a force-free Bessel function model,^{39,40} all of which were tearing mode calculations. The general computational model used included resistive and gravitational effects due to convection. The dependent variables B_{r1} and v_{r1} are computed at integral time steps, the pair $B_{\theta 1}$ and $v_{\theta 1}$ are computed at half-integral time steps and the system is solved as two sets of two equations each.

In this report, two cylindrical models are developed: a general model which includes the effects of compressibility, finite resistivity, viscosity, and thermal conductivity, and a simpler incompressible model which neglects the dissipative effects but includes finite resistivity. The incompressible model is similar to the one developed by Killeen,³⁵ but in the present model gravitational effects are neglected, and instead of solving the equations in pairs, all of the dependent variables are solved for simultaneously at the same time step. This same numerical technique is also used to solve the set of equations for the general model, and both computational models are applied to diffuse pinch geometries to study the resistive tearing mode.

In Chapter II the specific equations used in the models are presented, along with a description of the approximations that are made, and in Chapter III the finite difference equations are presented and the method of solution is described. In Chapters IV, V, and VI, the computational models are applied to three

different types of equilibrium magnetic field configurations:

(1) a force-free Bessel function model^{39,40} which has an analytic solution with which the numerical results⁴¹ are compared and which also contains characteristics of actual pinch fields, namely diffuse currents and a reversed B_z ; (2) a high-beta reversed field configuration⁴² which can be set up experimentally by programming;⁴³ and (3) three magnetic field profiles⁴⁴ which represent toroidal or tokamak pinch discharges in which the axial field does not reverse direction. The equilibrium configurations are tested for resistive tearing instabilities in the parameter space about which $\bar{\mathbf{k}} \cdot \bar{\mathbf{B}} = 0$. Growth rates are calculated for various (m, k_z) modes and the results are presented in the form of stability diagrams. Additional parametric studies are also performed to determine the specific dependence of the growth rates on the position of outer conducting wall R_w , the axial wave number k_z , and the magnetic Reynolds number S .

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II. MATHEMATICAL MODEL

2.1 Basic Equations and Assumptions

The mathematical model used in this report is a hydromagnetic (or magnetohydrodynamic, MHD) model in which the ions and electrons are treated as a single electrically conducting fluid. In this section, the specific equations used in the model are presented, along with a description of the approximations that are made.

In Gaussian units, the electromagnetic field is described by the Maxwell equations:

$$\nabla \times \vec{B} = \frac{4\pi}{c} \vec{J} + \frac{1}{c} \frac{\partial \vec{E}}{\partial t} \quad (2.1)$$

$$\nabla \times \vec{E} + \frac{1}{c} \frac{\partial \vec{B}}{\partial t} = 0 \quad (2.2)$$

$$\nabla \cdot \vec{B} = 0 \quad (2.3)$$

$$\nabla \cdot \vec{E} = 4\pi\rho_e \quad (2.4)$$

where $\vec{B}$ is the magnetic field, $\vec{E}$ the electric field, $\vec{J}$ the total current density, ρ_e the volume charge density, and c the speed of light. In order that the plasma be validly described by a fluid model for the study of resistive instabilities, the characteristic length L and time τ over which the fluid properties change appreciably must be such that

$$\begin{aligned} L &\gg \lambda_c \\ \tau &\gg \tau_c \end{aligned} \quad (2.5)$$

where λ_c represents the mean free path between appreciable particle deflections, and τ_c the mean time between appreciable deflections or the time during which the particle distribution function relaxes to a Maxwellian. For the magnetohydrodynamic model to be used in this study, the following assumptions are made:^{1,2,3}

(a) The flow is nonrelativistic, i.e., $(v/c)^2 \ll 1$, where v is a characteristic fluid velocity. Taking the characteristic frequency $\omega \sim 1/\tau$, then if the hydromagnetic model is to represent the interaction between the electromagnetic field and the flow field, v can be expressed as

$$v \sim \omega L \quad \text{or} \quad \frac{\omega L}{c} \sim \frac{v}{c}. \quad (2.6)$$

(b) The electric fields are of the order of magnitude of the induced effects, i.e., of the order $(\vec{v} \times \vec{B})/c$. This assumption implies that the induced magnetic fields are much smaller than any externally applied magnetic fields. Thus, if $\vec{E}$ varies linearly over some small region, a dimensional analysis of Faraday's law (2.2) gives

$$\frac{E}{B} \sim \frac{\omega L}{c} \sim \frac{v}{c}. \quad (2.7)$$

(c) Since the resistive instability mechanism represents low frequency phenomena over macroscopic spatial scales, the displacement current is neglected in Maxwell's equation (2.1). Using (2.7) to compare terms in (2.1),

$$\left| \frac{1}{c} \frac{\partial \vec{E}}{\partial t} \right| / |\nabla \times \vec{B}| \sim \frac{E}{B} \frac{\omega L}{c} \sim \left(\frac{v}{c} \right)^2 \ll 1. \quad (2.8)$$

Therefore, Maxwell's equation (2.1) reduces to

$$\nabla \times \vec{B} = \frac{4\pi}{c} \vec{J}. \quad (2.9)$$

(d) The electric energy is negligible compared to the magnetic energy, i.e., the predominant interaction is between the fluid and the magnetic field.

(e) The plasma is assumed to be hydrogenic ($Z = 1$), fully ionized and electrically neutral, so the ion and electron number densities are equal, i.e.

$$n_i = n_e \equiv n, \quad (2.10)$$

and only slow motions of the fluid elements of size greater than the Debye length λ_D are permitted. The Debye length is given by

$$\lambda_D = \sqrt{\frac{k_B T}{8\pi n e^2}} \quad (2.11)$$

where T is the temperature in degrees Kelvin, k_B the Boltzmann constant (1.38×10^{-16} erg/ $^{\circ}$ K) and e the electronic charge (4.803×10^{-10} statcoulomb).

(f) The mass density of the fluid is approximated by

$$\rho = n(m_i + m_e) \approx nm_i, \quad (2.12)$$

where m_i and m_e are the ion and electron masses, respectively.

(g) The force density can be adequately represented by

$$\vec{F} = \rho_e \vec{E} + \frac{\vec{J} \times \vec{B}}{c} \approx \frac{\vec{J} \times \vec{B}}{c}. \quad (2.13)$$

This approximation can be justified by considering a dimensional form of the Maxwell equation (2.4), i.e.

$$\frac{E}{L} \sim \rho_e \quad (2.14)$$

in conjunction with a simplified form of Ohm's law, $\eta \vec{J} = \vec{E}$.

Using (2.7), a comparison of the force density terms gives

$$\left| \frac{\rho_e \vec{E}}{(\vec{J} \times \vec{B})/c} \right| \sim \eta \frac{v}{L}. \quad (2.15)$$

Taking the representative plasma parameters $v \sim 10^7$ cm/sec, $L \sim 100$ cm, $\eta \sim 10^{-12}$ sec, then

$$\eta \frac{v}{L} \sim 10^{-7} \ll 1, \quad (2.16)$$

so that $\rho_e \vec{E}$ can be neglected compared with $(\vec{J} \times \vec{B})/c$. This same analysis also shows that any current flow due to the transport of excess charge $\rho_e \vec{v}$ can be neglected compared with the conduction current $\vec{J}$, i.e., using (2.14),

$$\left| \frac{\rho_e \vec{v}}{\vec{J}} \right| \sim \eta \frac{v}{L} \ll 1$$

as in (2.16).

(h) It is also assumed that the fluid is isotropic and that the pressure tensor $\vec{p}$ can be expressed as the sum of a scalar

pressure p plus a viscous term, i.e.

$$\bar{\bar{p}} = -p\bar{\bar{I}} + \bar{\bar{\tau}}_{ij} \quad (2.17)$$

$$\text{and} \quad \nabla \cdot \bar{\bar{p}} = -\nabla \cdot p\bar{\bar{I}} + \nabla \cdot \bar{\bar{\tau}}, \quad (2.18)$$

where $\bar{\bar{I}}$ is the unit tensor and $\bar{\bar{\tau}}_{ik}$ is the viscous stress tensor defined as

$$\bar{\bar{\tau}}_{ik} = \mu_F \left(\frac{\partial v_i}{\partial x_k} + \frac{\partial v_k}{\partial x_i} - \frac{2}{3} \delta_{ik} \nabla \cdot \bar{\bar{v}} \right) + \zeta \delta_{ik} \nabla \cdot \bar{\bar{v}}. \quad (2.19)$$

Here, μ_F is the coefficient of dynamic viscosity, ζ the second (or bulk) coefficient of viscosity, and δ_{ik} is the Kronecker delta. Assuming that the fluid can be described by the properties of a monatomic gas, then $\zeta = 0$, i.e., p is the thermodynamic pressure, the strain rates are not too fast, and the concept of thermodynamic equilibrium does not break down. Taking the viscosity to be constant throughout the fluid, the divergence of the pressure tensor reduces to^{4,5}

$$\nabla \cdot \bar{\bar{p}} = -\nabla p + \mu_F \left[\nabla^2 \bar{\bar{v}} + \frac{1}{3} \nabla (\nabla \cdot \bar{\bar{v}}) \right]. \quad (2.20)$$

In addition to the electromagnetic equations, the complete description of a fluid model of a plasma requires that the macroscopic variables such as density, temperature, velocity, and pressure be related through the conservation laws and through the dynamical equations for momentum and energy transport. Under the foregoing assumptions, a complete set of one-fluid equations

can be developed.

The conservation of mass of the fluid is independent of electromagnetic effects and is given by the continuity equation

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \bar{v}) = S_M, \quad (2.21)$$

where S_M is a mass source term.

By making a generalization of the Navier-Stokes equation of classical fluid dynamics,^{6,7} the equation of motion of a viscous fluid in an electromagnetic field can be written in the form^{2,3,4,5}

$$\begin{aligned} \rho \frac{d\bar{v}}{dt} &= -\nabla p + \frac{\bar{J} \times \bar{B}}{c} + \rho \nu \left[\nabla^2 \bar{v} + \frac{1}{3} \nabla (\nabla \cdot \bar{v}) \right] + \bar{S}_p \\ &= -\nabla p + \frac{\bar{J} \times \bar{B}}{c} + \rho \nu \left[\frac{4}{3} \nabla (\nabla \cdot \bar{v}) - \nabla \times (\nabla \times \bar{v}) \right] + \bar{S}_p. \end{aligned} \quad (2.22)$$

where $\nu = \mu_p / \rho$ is the kinematic viscosity, $\bar{S}_p$ a momentum source term and d/dt the convective derivative

$$\frac{d}{dt} = \frac{\partial}{\partial t} + \bar{v} \cdot \nabla.$$

The other conservation equation required to describe the mathematical model of the conducting fluid is the energy equation, which can be written in the form^{2,3,4,5}

$$\rho \frac{de}{dt} = -p(\nabla \cdot \bar{v}) + \nabla \cdot (K_T \nabla \theta) + \eta \bar{J}^2 + \phi + S_E, \quad (2.23)$$

where θ is the plasma temperature in $^{\circ}\text{K}$, e the internal energy

per unit mass of the fluid, S_E an energy source term, and K_T the coefficient of thermal conductivity for an isotropic fluid. The $\eta \omega^2$ term represents the Joule heating loss per unit volume per unit time, while the $\nabla \cdot (K_T \nabla \theta)$ term is the heat acquired by conduction per unit volume per unit time. The dissipation function Φ represents the heat loss due to hydrodynamic viscous forces and in cylindrical coordinates is given by^{2,8}

$$\begin{aligned} \Phi = \mu_f \left\{ 2 \left[\left(\frac{\partial v_r}{\partial r} \right)^2 + \left(\frac{1}{r} \frac{\partial v_\theta}{\partial \theta} + \frac{v_r}{r} \right)^2 + \left(\frac{\partial v_z}{\partial z} \right)^2 \right] \right. \\ + \left(\frac{1}{r} \frac{\partial v_z}{\partial \theta} + \frac{\partial v_\theta}{\partial z} \right)^2 + \left(\frac{\partial v_r}{\partial z} + \frac{\partial v_z}{\partial r} \right)^2 \\ + \left. \left(\frac{1}{r} \frac{\partial v_r}{\partial \theta} + \frac{\partial v_\theta}{\partial r} - \frac{v_\theta}{r} \right)^2 \right\} \\ + \left(\zeta - \frac{2}{3} \mu_f \right) \left[\frac{\partial v_r}{\partial r} + \frac{1}{r} \frac{\partial v_\theta}{\partial \theta} + \frac{v_r}{r} + \frac{\partial v_z}{\partial z} \right]^2. \end{aligned} \quad (2.24)$$

In a hydromagnetic model, the conservation equations are supplemented by an equation of state $p = p(\rho, \theta)$. For the model used here it is assumed that the fluid behaves like an ideal gas, hence the equation of state is

$$p = \frac{\rho R \theta}{M}$$

where R is the universal gas constant and M is the mass of the fluid. For an ideal fluid, the internal energy e is a function of temperature only, and when the energy is assumed to be pro-

portional to the temperature, the resulting polytropic fluid can be described by

$$\rho e = \frac{p}{\gamma - 1} \quad (2.25)$$

where $\gamma = C_p/C_v = (f + 2)/f$ is the ratio of specific heats of the fluid and f is the number of degrees of freedom. Since it will prove convenient to express the temperature in energy units, the equation of state is put in the form

$$p = \rho T \quad (2.26)$$

where T has the units of energy per unit mass. Using (2.25) and (2.26), and assuming the thermal conductivity to be constant throughout the fluid, the energy equation becomes

$$\begin{aligned} \rho \frac{dT}{dt} = & -(\gamma - 1)\rho T(\nabla \cdot \vec{v}) + K \nabla^2 T + (\gamma - 1)\eta |\vec{J}|^2 \\ & + (\gamma - 1)\Phi + (\gamma - 1)S_E \end{aligned} \quad (2.27)$$

where a new coefficient of thermal conductivity has been defined such that⁹

$$K \equiv (\gamma - 1) \frac{m_i}{k_B} K_T \quad (2.28)$$

with K_T the usual coefficient of thermal conductivity in ergs/cm-sec-°K.

In order to complete the system of hydromagnetic equations for a one-fluid model, an equation relating the current to the electric field is required. This relation is obtained by adding the ion and electron momentum transport equations to obtain a

linearized equation for $\partial \bar{J} / \partial t$. In performing this derivation, it is assumed that the momentum exchanged between positive ions and electrons is proportional to the relative velocity of the two types of particles. The resulting equation is known as the generalized Ohm's law and takes the form^{1,10,11}

$$\frac{m_e}{ne^2} \frac{\partial \bar{J}}{\partial t} = \bar{E} + \frac{\bar{v} \times \bar{B}}{c} + \frac{\nabla p_e}{ne} - \frac{\bar{J} \times \bar{B}}{nec} - \eta \bar{J}, \quad (2.29)$$

where terms of order m_e/m_i have been neglected. However, this equation can be reduced further by considering a dimensional analysis of the individual terms.^{1,12} The results of the dimensional analysis will be expressed explicitly in terms of the characteristic length L , velocity v , and frequency ω , which are of interest in the problem. Using (2.6) and (2.7), the first two terms on the right hand side of Ohm's law are of the same order, and the other terms in the equation will be compared with these two terms. Using (2.6), (2.7), and a dimensional form of (2.9), the $\partial \bar{J} / \partial t$ term can be neglected if

$$\left(\frac{\omega}{\omega_p} \right)^2 \ll \left(\frac{v}{c} \right)^2 \quad \text{or} \quad \frac{L^2 \omega_p^2}{c^2} \gg 1, \quad (2.30)$$

where $\omega_p^2 = 4\pi n e^2 / m_e$ is the electron plasma frequency. Similarly, the $\bar{J} \times \bar{B}$ term can be neglected provided

$$\left(\frac{\omega}{\omega_p} \right) \left(\frac{\Omega_e}{\omega_p} \right) \ll \left(\frac{v}{c} \right)^2 \quad \text{or} \quad \frac{L \omega_p^2 v}{c^2 \Omega_e} \gg 1, \quad (2.31)$$

where $\Omega_e = eB/m_e c$ is the electron cyclotron frequency. The pressure term in Ohm's law can be neglected if

$$\frac{\omega}{\Omega_i} \ll \left(\frac{v}{c_s} \right)^2 \quad \text{or} \quad \frac{Lv\Omega_e}{k_B T_e / m_e} \gg 1, \quad (2.32)$$

where $\Omega_i = eB/m_i c$ is the ion cyclotron frequency and $c_s \sim \sqrt{p/\rho}$ is the speed of sound in the fluid. The conditions derived in Eqs. (2.30)-(2.32) can be tested numerically by taking the following characteristic values which represent the regimes of interest for the study of hydromagnetic instabilities, namely^{13,14}

$$\begin{aligned} n &\sim 10^{15} / \text{cm}^3 \\ B &\sim 10^4 \text{ G} \\ T &\sim 10^5 \text{ }^\circ\text{K} \\ L &\sim 100 \text{ cm} \\ v &\sim 10^7 \text{ cm/sec.} \end{aligned}$$

Using these values, conditions (2.30)-(2.32) become

$$\frac{L^2 \omega^2}{c^2} \sim 3 \times 10^7 \gg 1, \quad (2.33)$$

$$\frac{Lv^2}{c^2 \Omega_e} \sim 20 > 1, \quad (2.34)$$

$$\frac{Lv\Omega_e}{k_B T_e / m_e} \sim 10^4 \gg 1, \quad (2.35)$$

and the generalized form of Ohm's law reduces to

$$\eta \vec{J} = \vec{E} + \frac{\vec{v} \times \vec{B}}{c}. \quad (2.36)$$

In summary, the hydromagnetic model used here to describe resistive instabilities includes the Maxwell equations (2.2) and (2.9), the mass continuity equation (2.21), the equation of motion (2.22), the equation of state (2.26), the energy equation (2.27), and the simplified form of Ohm's law (2.36). This system of seven equations in the seven dependent variables $\bar{E}$, $\bar{B}$, $\bar{J}$, $\bar{v}$, p , ρ , T can be solved in principle provided that the transport coefficients η , ν , and K are known. Note that Eq. (2.5) is redundant and is actually an initial condition since from (2.2),

$$\frac{\partial}{\partial t} (\nabla \cdot \bar{B}) = 0, \quad (2.37)$$

and Eq. (2.4) provides a check on the validity of the charge neutrality assumption (e).

The complete system of one-fluid hydromagnetic equations is summarized in Table 2.1. This system of equations, however, can be reduced by eliminating $\bar{E}$, $\bar{J}$, and p . Combining Maxwell's equations (2.2) and (2.9) with Ohm's law (2.36) yields an equation describing the interaction of the magnetic field with the flow field:

$$\frac{\partial \bar{B}}{\partial t} = \nabla \times (\bar{v} \times \bar{B}) - \left(\frac{c^2}{4\pi} \right) \nabla \times (\eta \nabla \times \bar{B}). \quad (2.38)$$

Using the Maxwell equation (2.9) and the equation of state (2.26) in the equation of motion (2.22) results in

$$\rho \frac{d\bar{v}}{dt} = \frac{1}{4\pi} (\nabla \times \bar{B}) \times \bar{B} - \nabla(\rho T) + \rho \nu \left[\nabla^2 \bar{v} + \frac{1}{3} \nabla(\nabla \cdot \bar{v}) \right] + \bar{S}_p, \quad (2.39)$$

Table 2.1. One-fluid MHD equations (vector form).

$$(1) \quad \nabla \times \bar{E} + \frac{1}{c} \frac{\partial \bar{B}}{\partial t} = 0$$

$$(2) \quad \nabla \times \bar{B} = \frac{4\pi}{c} \bar{J}$$

$$(3) \quad \eta \bar{J} = \bar{E} + \frac{\bar{v} \times \bar{B}}{c}$$

$$(4) \quad \rho \frac{\partial \bar{v}}{\partial t} + \nabla \cdot (\rho \bar{v}) = S_M$$

$$\begin{aligned} (5) \quad \rho \frac{\partial \bar{v}}{\partial t} + \rho \bar{v} \cdot \nabla \bar{v} &= - \nabla p + \frac{\bar{J} \times \bar{B}}{c} \\ &+ \rho v \left[\nabla^2 \bar{v} + \frac{1}{3} \nabla (\nabla \cdot \bar{v}) \right] + \bar{S}_p \\ &= - \nabla p + \frac{\bar{J} \times \bar{B}}{c} \\ &+ \rho v \left[\frac{4}{3} \nabla (\nabla \cdot \bar{v}) - \nabla \times (\nabla \times \bar{v}) \right] + \bar{S}_p \end{aligned}$$

$$\begin{aligned} (6) \quad \rho \frac{\partial T}{\partial t} + \bar{v} \cdot \nabla T &= - (\gamma - 1) \rho T (\nabla \cdot \bar{v}) + K \nabla^2 T + (\gamma - 1) \eta \bar{J}^2 \\ &+ (\gamma - 1) \Phi + (\gamma - 1) S_E \end{aligned}$$

$$(7) \quad p = \rho T$$

$$(8) \quad \nabla \cdot \bar{B} = 0$$

and using Maxwell's equation (2.9) in the energy equation gives

$$\rho \frac{dT}{dt} = -(\gamma - 1)\rho T(\nabla \cdot \vec{v}) + K\nabla^2 T + \left(\frac{c}{4\pi}\right)^2 (\gamma - 1)\eta(\nabla \times \vec{B})^2 + (\gamma - 1)\Phi + (\gamma - 1)S_E. \quad (2.40)$$

These equations are supplemented by the mass continuity equation

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \vec{v}) = S_M$$

and the Maxwell equation $\nabla \cdot \vec{B} = 0$. This reduced system of equations in the dependent variables $\vec{B}$, $\vec{v}$, ρ , and T is summarized in Table 2.2.

2.2 First Order Equations

The plasma equilibrium configuration is assumed to be known and to be given by $\vec{B}_0$, $\vec{v}_0$, T_0 , ρ_0 , η_0 , and p_0 . It is assumed that the equilibrium velocity $\vec{v}_0$ is zero, but that there exists throughout the fluid a known static magnetic field $\vec{B}_0$ given in cylindrical geometry by

$$\vec{B}_0(r) = \hat{\theta} B_{\theta 0}(r) + \hat{z} B_{z 0}(r). \quad (2.41)$$

The reduced system of MHD equations listed in Table 2.2 are linearized by assuming that the resistive instabilities represent only a small perturbation compared with the known plasma equilibrium configuration, and all of the macroscopic fluid variables are assumed to have the following functional dependence, where the equilibrium quantities are denoted by the subscript 0, and the small amplitude perturbed quantities by the subscript 1:

Table 2.2. Reduced system of one-fluid MHD equations (vector form)

$$(1) \quad \frac{\partial \bar{\mathbf{B}}}{\partial t} = \nabla \times (\bar{\mathbf{v}} \times \bar{\mathbf{B}}) - \frac{c^2}{4\pi} \nabla \times (\eta \nabla \times \bar{\mathbf{B}})$$

$$(2) \quad \rho \frac{\partial \bar{\mathbf{v}}}{\partial t} + \rho \bar{\mathbf{v}} \cdot \nabla \bar{\mathbf{v}} = \frac{1}{4\pi} (\nabla \times \bar{\mathbf{B}}) \times \bar{\mathbf{B}} - \nabla(\rho T) \\ + \rho \nu \left[\nabla^2 \bar{\mathbf{v}} + \frac{1}{3} \nabla(\nabla \cdot \bar{\mathbf{v}}) \right] + \bar{\mathbf{S}}_p \\ = \frac{1}{4\pi} \left[(\bar{\mathbf{B}} \cdot \nabla) \bar{\mathbf{B}} - \frac{1}{2} \nabla(\bar{\mathbf{B}} \cdot \bar{\mathbf{B}}) \right] - \nabla(\rho T) \\ + \rho \nu \left[\frac{4}{3} \nabla(\nabla \cdot \bar{\mathbf{v}}) - \nabla \times (\nabla \times \bar{\mathbf{v}}) \right] + \bar{\mathbf{S}}_p$$

$$(3) \quad \rho \frac{\partial T}{\partial t} + \rho \bar{\mathbf{v}} \cdot \nabla T = -(\gamma - 1) \rho T (\nabla \cdot \bar{\mathbf{v}}) + K \nabla^2 T + \left(\frac{c}{4\pi} \right)^2 (\gamma - 1) \eta (\nabla \times \bar{\mathbf{B}})^2 \\ + (\gamma - 1) \Phi + (\gamma - 1) S_E$$

$$(4) \quad \frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \bar{\mathbf{v}}) = S_M$$

$$(5) \quad \nabla \cdot \bar{\mathbf{B}} = 0$$

$$\begin{aligned}
 \bar{\mathbf{B}}(\bar{\mathbf{r}}, t) &= \bar{\mathbf{B}}_0(r) + \bar{\mathbf{B}}_1(r, \theta, z, t) \\
 \bar{\mathbf{v}}(\bar{\mathbf{r}}, t) &= \bar{\mathbf{v}}_1(r, \theta, z, t) \\
 T(\bar{\mathbf{r}}, t) &= T_0(r) + T_1(r, \theta, z, t) \\
 p(\bar{\mathbf{r}}, t) &= p_0(r) + p_1(r, \theta, z, t) \\
 \rho(\bar{\mathbf{r}}, t) &= \rho_0(r) + \rho_1(r, \theta, z, t) \\
 \eta(\bar{\mathbf{r}}, t) &= \eta_0(r) + \eta_1(r, \theta, z, t).
 \end{aligned} \tag{2.42}$$

In the following linearization, the source terms are also assumed to have this same functional dependence. These functions are chosen to describe a particular experiment; in a toroidal device $B_{\theta 0}$ is called the poloidal field, and B_{z0} is called the toroidal field.

2.2.1 General Case

The complete system of hydromagnetic equations takes into account the effects of compressibility, viscosity, and thermal conductivity. Using (2.41) and (2.42), the linearized equation for the magnetic field (2.38) is

$$\frac{\partial \bar{\mathbf{B}}_1}{\partial t} = \nabla \times (\bar{\mathbf{v}}_1 \times \bar{\mathbf{B}}_0) - \frac{c^2}{4\pi} \nabla \times (\eta_0 \nabla \times \bar{\mathbf{B}}_1 + \eta_1 \nabla \times \bar{\mathbf{B}}_0), \tag{2.43}$$

where the assumption $\bar{\mathbf{v}}_0 = 0$ implies the equilibrium condition

$$\nabla \times (\eta_0 \nabla \times \bar{\mathbf{B}}_0) = 0.$$

As pointed out by FKR,¹⁵ for resistive instability modes with sufficiently large growth rates and wavelengths, the assumption of $\bar{\mathbf{v}}_0 = 0$ is valid even if the above equilibrium condition is

not strictly satisfied, since the values of $\bar{v}_0$ to be expected are those of ordinary resistive diffusion. Similarly, the linearized equation of motion (2.39) is

$$\begin{aligned} \rho_0 \frac{\partial \bar{v}_1}{\partial t} = & \frac{1}{4\pi} \left[(\nabla \times \bar{B}_0) \times \bar{B}_1 + (\nabla \times \bar{B}_1) \times \bar{B}_0 \right] \\ & - \nabla(\rho_0 T_1) - \nabla(\rho_1 T_0) + (\bar{S}_p)_1 \\ & + \rho_0 v \left[\nabla^2 \bar{v}_1 + \frac{1}{3} \nabla(\nabla \cdot \bar{v}_1) \right] \end{aligned} \quad (2.44)$$

with the equilibrium condition

$$\frac{1}{4\pi} (\nabla \times \bar{B}_0) \times \bar{B}_0 - \nabla(\rho_0 T_0) + (\bar{S}_p)_0 = 0,$$

and the linearized energy equation (2.40) is

$$\begin{aligned} \rho_0 \frac{\partial T_1}{\partial t} + \rho_0 \bar{v}_1 \cdot \nabla T_0 = & -(\gamma - 1) \rho_0 T_0 (\nabla \cdot \bar{v}_1) + K \nabla^2 T_1 \\ & + 2(\gamma - 1) \left(\frac{c}{4\pi} \right)^2 \eta_0 (\nabla \times \bar{B}_0) \cdot (\nabla \times \bar{B}_1) \\ & + (\gamma - 1) \left(\frac{c}{4\pi} \right)^2 \eta_1 (\nabla \times \bar{B}_0) \cdot (\nabla \times \bar{B}_0) \\ & + (S_E)_1 (\gamma - 1), \end{aligned} \quad (2.45)$$

with the equilibrium condition

$$K \nabla^2 T_0 + (\gamma - 1) \eta_0 \left(\frac{c}{4\pi} \right)^2 (\nabla \times \bar{B}_0) \cdot (\nabla \times \bar{B}_0) + (S_E)_0 = 0.$$

Note that for the energy equation, the viscous terms do not appear in the linearized form since $\Phi \sim O(\bar{v}_1^2)$. The linearized continuity equation, equation of state, and Maxwell's equation

are, respectively,

$$\frac{\partial \rho_1}{\partial t} + \nabla \cdot (\rho_0 \bar{v}_1) = (S_M)_1, \quad (2.46)$$

$$P_1 = \rho_0 T_1 + \rho_1 T_0, \quad (2.47)$$

$$\nabla \cdot \bar{B}_1 = 0. \quad (2.48)$$

Provided that the equilibrium functions B_0 , T_0 , ρ_0 , η_0 are known, the system of linearized equations (2.43)-(2.46) represents four independent vector equations in the four dependent variables $\bar{B}_1$, $\bar{v}_1$, T_1 , and ρ_1 . The perturbed resistivity η_1 can be determined by assuming that it satisfies some functional relation; for example, FKR¹⁵ assume that perturbations in the plasma resistivity result only from convection, so that

$$\frac{\partial \eta_1}{\partial t} + \bar{v}_1 \cdot \nabla \eta_0 = 0. \quad (2.49)$$

However, any arbitrary function can be assumed for η (and thus for η_0) and η_1 can then be determined using (2.42), i.e.

$$\eta_1 = \eta - \eta_0.$$

This procedure is described in more detail in Section 2.3 and Section 3.4. Once ρ_1 and T_1 are known, p_1 can be determined from the linearized equation of state (2.47). The complete set of linearized hydromagnetic equations for the general case is listed in Table 2.3.

Table 2.3. Linearized equations for the general case.

$$\begin{aligned}
 (1) \quad \frac{\partial \bar{\mathbf{E}}_1}{\partial t} &= \nabla \times (\bar{\mathbf{v}}_1 \times \bar{\mathbf{E}}_0) - \frac{c^2}{4\pi} \nabla \times (\eta_0 \nabla \times \bar{\mathbf{E}}_1 + \eta_1 \nabla \times \bar{\mathbf{E}}_0) \\
 &= (\bar{\mathbf{E}}_0 \cdot \nabla) \bar{\mathbf{v}}_1 - (\bar{\mathbf{v}}_1 \cdot \nabla) \bar{\mathbf{E}}_0 - \bar{\mathbf{E}}_0 (\nabla \cdot \bar{\mathbf{v}}_1) \\
 &\quad + \frac{c^2}{4\pi} \left[(\nabla \times \bar{\mathbf{E}}_0) \times \nabla \eta_1 + (\nabla \times \bar{\mathbf{E}}_1) \times \nabla \eta_0 + \eta_0 \nabla^2 \bar{\mathbf{E}}_1 + \eta_1 \nabla^2 \bar{\mathbf{E}}_0 \right]
 \end{aligned}$$

$$\begin{aligned}
 (2) \quad \rho_0 \frac{\partial \bar{\mathbf{v}}_1}{\partial t} &= \frac{1}{4\pi} \left[(\nabla \times \bar{\mathbf{E}}_0) \times \bar{\mathbf{B}}_1 + (\nabla \times \bar{\mathbf{E}}_1) \times \bar{\mathbf{B}}_0 \right] - \nabla p_1 \\
 &\quad + \rho_0 \nu \left[\frac{4}{3} \nabla (\nabla \cdot \bar{\mathbf{v}}_1) - \nabla \times (\nabla \times \bar{\mathbf{v}}_1) \right] + (\bar{\mathbf{S}}_p)_1 \\
 &= \frac{1}{4\pi} \left[(\bar{\mathbf{E}}_0 \cdot \nabla) \bar{\mathbf{B}}_1 + (\bar{\mathbf{B}}_1 \cdot \nabla) \bar{\mathbf{E}}_0 - \nabla (\bar{\mathbf{E}}_0 \cdot \bar{\mathbf{B}}_1) \right] \\
 &\quad - \nabla (\rho_0 T_1) - \nabla (\rho_1 T_0) + (\bar{\mathbf{S}}_p)_1 + \rho_0 \nu \left[\frac{1}{3} \nabla (\nabla \cdot \bar{\mathbf{v}}_1) + \nabla^2 \bar{\mathbf{v}}_1 \right]
 \end{aligned}$$

$$\begin{aligned}
 (3) \quad \rho_0 \frac{\partial T_1}{\partial t} + \rho_0 \bar{\mathbf{v}}_1 \cdot \nabla T_0 &= -(\gamma - 1) \rho_0 T_0 (\nabla \cdot \bar{\mathbf{v}}_1) + \kappa \nabla^2 T_1 \\
 &\quad + 2(\gamma - 1) \left(\frac{c}{4\pi} \right)^2 \eta_0 (\nabla \times \bar{\mathbf{E}}_0) \cdot (\nabla \times \bar{\mathbf{E}}_1) \\
 &\quad + (\gamma - 1) \left(\frac{c}{4\pi} \right)^2 \eta_1 (\nabla \times \bar{\mathbf{E}}_0) \cdot (\nabla \times \bar{\mathbf{E}}_0) \\
 &\quad + (\gamma - 1) (\bar{\mathbf{S}}_E)_1
 \end{aligned}$$

$$(4) \quad \frac{\partial \rho_1}{\partial t} + \nabla \cdot (\rho_0 \bar{\mathbf{v}}_1) = (\bar{\mathbf{S}}_M)_1$$

$$(5) \quad p_1 = \rho_0 T_1 + \rho_1 T_0$$

$$(6) \quad \nabla \cdot \bar{\mathbf{B}}_1 = 0$$

2.2.2 Incompressible Case

A simpler hydromagnetic model can be developed by assuming that the fluid is incompressible ($\rho = \rho_0 = \text{const}$) and by neglecting the dissipative effects of viscosity and thermal conductivity.¹⁶ Under these conditions, and neglecting the source terms, the mass continuity equation (2.46) reduces to the incompressibility condition

$$\nabla \cdot \bar{v}_1 = 0, \quad (2.50)$$

and the linearized equation for the magnetic field (2.43) becomes

$$\begin{aligned} \frac{\partial \bar{B}_1}{\partial t} &= \nabla \times (\bar{v}_1 \times \bar{B}_0) - \frac{c^2}{4\pi} \nabla \times (\eta_0 \nabla \times \bar{B}_1 + \eta_1 \nabla \times \bar{B}_0) \\ &= (\bar{B}_0 \cdot \nabla) \bar{v}_1 - (\bar{v}_1 \cdot \nabla) \bar{B}_0 + \frac{c^2}{4\pi} \left[(\nabla \times \bar{B}_0) \times \nabla \eta_1 \right. \\ &\quad \left. + (\nabla \times \bar{B}_1) \times \nabla \eta_0 + \eta_0 \nabla^2 \bar{B}_1 + \eta_1 \nabla^2 \bar{B}_0 \right]. \end{aligned} \quad (2.51)$$

The first form of (2.51) is the same as (2.43), but the second form has made use of (2.50).

Taking the double curl of the equation of motion (2.44) and using (2.50) yields

$$-\rho_0 \frac{\partial}{\partial t} (\nabla^2 \bar{v}_1) = \frac{1}{4\pi} \nabla \times \nabla \times \left[(\bar{B}_0 \cdot \nabla) \bar{B}_1 + (\bar{B}_1 \cdot \nabla) \bar{B}_0 \right]. \quad (2.52)$$

This particular form has been used so that the equation of motion is second order in the spatial independent variable, and this property will prove convenient when the equation is put into finite difference form as described in Chapter III. This simplified set

of hydromagnetic equations is shown in Table 2.4.

2.3 Fourier Analyzed Equations--Cylindrical Model

The hydromagnetic model developed in the previous sections is applied in cylindrical geometry in order to study resistive instabilities in diffuse pinches. The plasma equilibrium configuration is assumed to be known, and the fluid variables are assumed to have perturbations of the form¹⁶

$$f_1(r,t)e^{i(m\theta + k_z z)}, \quad (2.53)$$

where m is the azimuthal wave number and k_z the axial wave number. This form of perturbation, when applied to the linearized hydro-magnetic model, results in a consistent set of component equations which can be solved for both the general case including compressibility, viscosity, and thermal conductivity, as well as the simpler incompressible case without dissipation.

2.3.1 General Case

The linearized system of equations listed in Table 2.3 represents nine component equations in the nine dependent variables B_{r1} , $B_{\theta 1}$, B_{z1} , v_{r1} , $v_{\theta 1}$, v_{z1} , T_1 , p_1 , ρ_1 , with η_1 assumed to be known. It is desired to reduce this system of nine equations to a basic set of six equations in the six dependent variables B_{r1} , $B_{\theta 1}$, T_1 , v_{r1} , $v_{\theta 1}$, v_{z1} , where the remaining perturbed quantities ρ_1 , η_1 , and p_1 are to be evaluated separately. The system of nine equations is reduced to six equations as follows:

- (a) Using $\nabla \cdot \vec{B}_1 = 0$ to eliminate B_{z1} in both the r and θ

Table 2.4. Linearized equations for the incompressible case.

$$(1) \quad \frac{\partial \bar{\mathbf{E}}_1}{\partial t} = \nabla \times (\bar{\mathbf{v}}_1 \times \bar{\mathbf{E}}_0) - \frac{c^2}{4\pi} \nabla \times (\eta_0 \nabla \times \bar{\mathbf{E}}_1 + \eta_1 \nabla \times \bar{\mathbf{E}}_0)$$

$$= (\bar{\mathbf{E}}_0 \cdot \nabla) \bar{\mathbf{v}}_1 - (\bar{\mathbf{v}}_1 \cdot \nabla) \bar{\mathbf{E}}_0 + \frac{c^2}{4\pi} \left[(\nabla \times \bar{\mathbf{E}}_0) \times \nabla \eta_1 \right. \\ \left. + (\nabla \times \bar{\mathbf{E}}_1) \times \nabla \eta_0 + \eta_0 \nabla^2 \bar{\mathbf{E}}_1 + \eta_1 \nabla^2 \bar{\mathbf{E}}_0 \right]$$

$$(2) \quad -\rho_0 \frac{\partial}{\partial t} (\nabla^2 \bar{\mathbf{v}}_1) = \frac{1}{4\pi} \nabla \times \nabla \times [(\bar{\mathbf{E}}_0 \cdot \nabla) \bar{\mathbf{E}}_1 + (\bar{\mathbf{E}}_1 \cdot \nabla) \bar{\mathbf{E}}_0]$$

$$(3) \quad \nabla \cdot \bar{\mathbf{v}}_1 = 0$$

$$(4) \quad \nabla \cdot \bar{\mathbf{E}}_1 = 0$$

components of the magnetic field equation (2.43) results in

2 equations (r, θ) in the unknowns $B_{r1}, B_{\theta1}, v_{r1}, v_{\theta1}, v_{z1}$;

(b) Using $\nabla \cdot \vec{B}_1 = 0$ to eliminate B_{z1} in all 3 components of the equation of motion (2.44) results in

3 equations (r, θ, z) in the unknowns $B_{r1}, B_{\theta1}, T_1, v_{r1}, v_{\theta1}, v_{z1}$;

(c) Using $\nabla \cdot \vec{B}_1 = 0$ to eliminate B_{z1} in the energy equation (2.45) results in

1 equation in the unknowns $B_{r1}, B_{\theta1}, T_1, v_{r1}, v_{\theta1}, v_{z1}$.

In performing the above reduction, it has been assumed that η_1 and ρ_1 are known functions. Thus, combining results (a)-(c) yields the six equations

- (1) $\hat{r}$ component of the magnetic field equation,
- (2) $\hat{\theta}$ component of the magnetic field equation,
- (3) $\hat{r}$ component of the equation of motion,
- (4) $\hat{\theta}$ component of the equation of motion,
- (5) $\hat{z}$ component of the equation of motion,
- (6) energy equation,

in the six dependent variables

$B_{r1}, B_{\theta1}, T_1, v_{r1}, v_{\theta1}, v_{z1}$.

Now, using the current value of $\vec{v}_1$ as determined from the system of six equations, $\rho_1(\vec{r}, t)$ can be determined at each time step using the mass continuity equation (2.46). To determine η_1 , a similar procedure can be followed, i.e., given some function g where

$$\eta(r, t) = g(\rho, T, \vec{B}),$$

then $\eta_0(r) = g(\rho_0, T_0, \bar{B}_0)$,

where ρ_0 , T_0 , and $\bar{B}_0$ are known, thus

$$\eta(\bar{r}, t) = g[(\rho_0 + \rho_1), (T_0 + T_1), (\bar{B}_0 + \bar{B}_1)] .$$

So using the current values of T_1 and $\bar{B}_1$ as determined from the system of six equations, and using the current value of ρ_1 as determined from the continuity equation, then η can be calculated at each time step, and η_1 can be determined at each time step from (2.42), i.e.

$$\eta_1(\bar{r}, t) = \eta(\bar{r}, t) - \eta_0(r).$$

With ρ_1 and T_1 determined at each time step, the pressure p_1 can then be determined at each time step using the equation of state (2.47).

Therefore, the general system of linearized equations which includes the effects of compressibility, viscosity, and thermal conductivity can be completely determined by solving a set of six "basic" equations for the dependent variables

$$B_{r1}, B_{\theta 1}, T_1, v_{r1}, v_{\theta 1}, v_{z1},$$

in conjunction with three "auxiliary" equations for the variables ρ_1 , η_1 , and p_1 . Applying the form of the perturbation given in (2.53), the component equations for the general case are listed in Table 2.5. The parameter I in these equations indicates those terms which would vanish if the fluid were assumed to be incompressible.

2.3.2 Incompressible Case

The incompressible model without dissipation, as summarized

Table 2.5. Component equations for the general case.

$$(1) \quad \frac{\partial B_{r1}}{\partial t} = i v_{r1} \left(\frac{m}{r} B_{\theta 0} + k_z B_{z0} \right) + \frac{c^2}{4\pi} \eta_0 \left[\frac{\partial^2 B_{r1}}{\partial r^2} + \frac{1}{r} \frac{\partial B_{r1}}{\partial r} \right. \\ \left. - B_{r1} \left(\frac{m^2 + 1}{r^2} + k_z^2 \right) - \frac{2im}{r^2} B_{\theta 1} \right] \\ - \frac{c^2}{4\pi} i \eta_1 \left[\frac{m}{r} \left(\frac{dB_{\theta 0}}{dr} + \frac{B_{\theta 0}}{r} \right) + k_z \frac{dB_{z0}}{dr} \right]$$

$$(2) \quad \frac{\partial B_{\theta 1}}{\partial t} = v_{r1} \left(\frac{B_{\theta 0}}{r} - \frac{dB_{\theta 0}}{dr} \right) + i v_{\theta 1} \left(\frac{m}{r} B_{\theta 0} + k_z B_{z0} \right) \\ - i B_{\theta 0} \left(\frac{\partial v_{r1}}{\partial r} + \frac{v_{r1}}{r} + \frac{im}{r} v_{\theta 1} + i k_z v_{z1} \right) \\ + \frac{c^2}{4\pi} \eta_0 \left[\frac{\partial^2 B_{\theta 1}}{\partial r^2} + \frac{1}{r} \frac{\partial B_{\theta 1}}{\partial r} - B_{\theta 1} \left(\frac{m^2 + 1}{r^2} + k_z^2 \right) + \frac{2im}{r^2} B_{r1} \right] \\ + \frac{c^2}{4\pi} \eta_1 \left(\frac{d^2 B_{\theta 0}}{dr^2} + \frac{1}{r} \frac{dB_{\theta 0}}{dr} - \frac{B_{\theta 0}}{r^2} \right) \\ + \frac{c^2}{4\pi} \frac{d\eta_0}{dr} \left(\frac{\partial B_{\theta 1}}{\partial r} + \frac{B_{\theta 1}}{r} - \frac{im}{r} B_{r1} \right) + \frac{c^2}{4\pi} \frac{\partial \eta_1}{\partial r} \left(\frac{dB_{\theta 0}}{dr} + \frac{B_{\theta 0}}{r} \right)$$

$$(3) \quad i k_z \rho_0 \frac{\partial T_1}{\partial t} = - i k_z \rho_0 v_{r1} \frac{dT_0}{dr} - I(\gamma - 1) i k_z \rho_0 \eta_0 \left(\frac{\partial v_{r1}}{\partial r} + \frac{v_{r1}}{r} \right. \\ \left. + \frac{im}{r} v_{\theta 1} + i k_z v_{z1} \right) + i k_z K \left[\frac{\partial^2 T_1}{\partial r^2} + \frac{1}{r} \frac{\partial T_1}{\partial r} \right]$$

Table 2.5 (continued)

$$\begin{aligned}
 (3) \quad & -T_1 \left(\frac{m^2 + 1}{r^2} + k_z^2 \right) + \frac{T_1}{r^2} + 2 \left(\frac{c}{4\pi} \right)^2 (\gamma - 1) \eta_0 \\
 & \left\{ -\frac{m}{r} \frac{dB_{z0}}{dr} \left(-\frac{B_{r1}}{r} - \frac{iB_{\theta 1}}{r} + i \frac{\partial B_{\theta 1}}{\partial r} \right) \right. \\
 & - \frac{dB_{z0}}{dr} \left[\frac{\partial^2 B_{r1}}{\partial r^2} + \frac{1}{r} \frac{\partial B_{r1}}{\partial r} - B_{r1} \left(\frac{m^2 + 1}{r^2} + k_z^2 \right) \right] \\
 & + k_z \left(\frac{dB_{\theta 0}}{dr} + \frac{B_{\theta 0}}{r} \right) \left(-\frac{B_{r1}}{r} + \frac{iB_{\theta 1}}{r} + i \frac{\partial B_{\theta 1}}{\partial r} \right) \Bigg\} \\
 & + k_z \left(\frac{c}{4\pi} \right)^2 (\gamma - 1) i \eta_1 \left[\left(\frac{dB_{z0}}{dr} \right)^2 + \left(\frac{dB_{\theta 0}}{dr} + \frac{B_{\theta 0}}{r} \right)^2 \right] \\
 & + i k_z (S_E)_1 (\gamma - 1)
 \end{aligned}$$

$$\begin{aligned}
 (4) \quad -i k_z \rho_0 \frac{\partial v_{r1}}{\partial t} = & -\frac{B_{z0}}{4\pi} \left[\frac{\partial^2 B_{r1}}{\partial r^2} + \frac{1}{r} \frac{\partial B_{r1}}{\partial r} - B_{r1} \left(\frac{m^2 + 1}{r^2} + k_z^2 \right) \right] \\
 & + \frac{B_{r1}}{4\pi} \left(\frac{m}{r} k_z B_{\theta 0} - \frac{1}{r} \frac{dB_{z0}}{dr} - \frac{m^2}{r^2} B_{z0} \right) \\
 & - \frac{\partial B_{r1}}{\partial r} \left(\frac{1}{4\pi} \frac{dB_{z0}}{dr} \right) + \frac{1}{4\pi} \frac{\partial B_{\theta 1}}{\partial r} \left(k_z B_{\theta 0} - \frac{m}{r} B_{z0} \right) \\
 & + \frac{iB_{\theta 1}}{4\pi} \left[k_z \left(\frac{dB_{\theta 0}}{dr} + 2 \frac{B_{\theta 0}}{r} \right) + \frac{m}{r} \left(\frac{B_{z0}}{r} - \frac{dB_{z0}}{dr} \right) \right] \\
 & + i k_z \left(\rho_0 \frac{\partial T_1}{\partial r} + T_0 \frac{\partial \rho_1}{\partial r} + T_1 \frac{d\rho_0}{dr} + \rho_1 \frac{dT_0}{dr} \right)
 \end{aligned}$$

$$\begin{aligned}
 & - ik_z \rho_0 v \left\{ \left(1 + \frac{I}{3} \right) \left[\frac{\partial^2 v_{r1}}{\partial r^2} + \frac{1}{r} \frac{\partial v_{r1}}{\partial r} \right. \right. \\
 & \left. \left. - v_{r1} \left(\frac{m^2}{r^2} + k_z^2 \right) \right] + \frac{I}{3} v_{r1} \left(\frac{m^2}{r^2} + k_z^2 \right) \right. \\
 & \left. - \left(2 + \frac{I}{3} \right) \frac{im}{r^2} v_{\theta 1} + \frac{I}{3} \frac{im}{r} \frac{\partial v_{\theta 1}}{\partial r} + \frac{I}{3} ik_z \frac{\partial v_{z1}}{\partial r} \right\} \\
 & - ik_z (S_{p_1})_r
 \end{aligned}$$

$$\begin{aligned}
 (5) \quad k_z \rho_0 \frac{\partial v_{\theta 1}}{\partial r} &= \frac{\partial B_{r1}}{\partial r} \left(\frac{m}{4\pi r} B_{z0} \right) + \frac{B_{r1}}{4\pi} \left[\frac{m}{r^2} B_{z0} + k_z \left(\frac{dB_{\theta 0}}{dr} + \frac{B_{\theta 0}}{r} \right) \right] \\
 &+ i B_{\theta 1} \left[\frac{B_{z0}}{4\pi} \left(\frac{m^2}{r^2} + k_z^2 \right) \right] - ik_z \frac{m}{r} (\rho_0 T_1 + \rho_1 T_0) \\
 &+ \rho_0 k_z v \left[\frac{\partial^2 v_{\theta 1}}{\partial r^2} + \frac{1}{r} \frac{\partial v_{\theta 1}}{\partial r} - v_{\theta 1} \left(\frac{m^2}{r^2} + k_z^2 \right) \right. \\
 &\left. - \frac{I}{3} \frac{m^2}{r^2} v_{\theta 1} - \frac{I}{3} \frac{m}{r} k_z v_{z1} + \frac{I}{3} \frac{im}{r} \frac{\partial v_{r1}}{\partial r} \right. \\
 &\left. + \left(2 + \frac{I}{3} \right) \frac{im}{r^2} v_{r1} \right] + k_z (S_{p_1})_{\theta}
 \end{aligned}$$

$$\begin{aligned}
 (6) \quad k_z \rho_0 \frac{\partial v_{z1}}{\partial t} &= \frac{B_{r1}}{4\pi} \left(k_z \frac{dB_{z0}}{dr} - \frac{m}{r^2} B_{\theta 0} \right) - \frac{\partial B_{r1}}{\partial r} \left(\frac{m}{4\pi r} B_{\theta 0} \right) \\
 &- i B_{\theta 1} \left[\frac{B_{\theta 0}}{4\pi} \left(\frac{m^2}{r^2} + k_z^2 \right) \right] - ik_z^2 (\rho_0 T_1 + \rho_1 T_0) \\
 &+ \rho_0 k_z v \left[\frac{\partial^2 v_{z1}}{\partial r^2} + \frac{1}{r} \frac{\partial v_{z1}}{\partial r} - v_{z1} \left(\frac{m^2}{r^2} + k_z^2 \right) \right]
 \end{aligned}$$

$$+ v_{z1} \left(\frac{1}{r^2} - \frac{I}{3} k_z^2 \right) - \frac{I}{3} \frac{m}{r} k_z v_{\theta 1} \\ + \frac{I}{3} i k_z \left(\frac{\partial v_{r1}}{\partial r} + \frac{v_{r1}}{r} \right) \Bigg] + k_z (S_{p_1})_z$$

$$(7) \quad \frac{\partial \rho_1}{\partial t} + \rho_0 I \left(\frac{\partial v_{r1}}{\partial r} + \frac{v_{r1}}{r} + \frac{im}{r} v_{\theta 1} + i k_z v_{z1} \right) + v_{r1} \frac{d\rho_0}{dr} = (S_M)_1$$

$$(8) \quad p_1 = \rho_0 T_1 + \rho_1 T_0$$

in Table 2.4, represents a linearized system of six component equations in the six dependent variables B_{r1} , $B_{\theta1}$, B_{z1} , v_{r1} , $v_{\theta1}$, v_{z1} . This system of equations can be reduced to a set of four "basic" equations by eliminating B_{z1} using $\nabla \cdot \vec{B}_1 = 0$ and by eliminating v_{z1} using $\nabla \cdot \vec{v}_1 = 0$. The resulting system of four equations is a consistent set involving the components B_{r1} , $B_{\theta1}$, v_{r1} , $v_{\theta1}$, and these four component equations are shown in Table 2.6.¹⁶ The resistivity η_1 is computed using a standard resistivity expression in the same manner as described in the previous section.

2.4 Boundary and Initial Conditions

In order to solve the systems of equations for both the general case (Table 2.5) and the incompressible case (Table 2.6) on the domain $0 \leq r \leq R_W$, where R_W is the radius of the outer conducting wall, boundary conditions must be specified at both $r = 0$ and $r = R_W$. For a numerical solution, a boundedness condition at $r = 0$ is not sufficient, therefore the following symmetry conditions are used at $r = 0$ for all t :

$$\underline{m = 0}$$

$$B_{r1} = B_{\theta1} = v_{r1} = v_{\theta1} = 0 \quad (2.54)$$

$$\frac{\partial v_{z1}}{\partial r} = \frac{\partial T_1}{\partial r} = \frac{\partial \rho_1}{\partial r} = 0$$

$$\underline{m = 1}$$

$$\frac{\partial B_{r1}}{\partial r} = \frac{\partial B_{\theta1}}{\partial r} = \frac{\partial v_{r1}}{\partial r} = \frac{\partial v_{\theta1}}{\partial r} = 0 \quad (2.55)$$

$$v_{z1} = T_1 = \rho_1 = 0$$

Table 2.6. Component equations for the incompressible case.

$$\begin{aligned}
 (1) \quad \frac{\partial B_{r1}}{\partial t} &= i v_{r1} \left(\frac{m}{r} B_{\theta 0} + k_z B_{z0} \right) + \frac{c^2}{4\pi} \eta_0 \left[\frac{\partial^2 B_{r1}}{\partial r^2} + \frac{1}{r} \frac{\partial B_{r1}}{\partial r} \right. \\
 &\quad \left. - B_{r1} \left(\frac{m^2}{r^2} + k_z^2 \right) - \frac{2im}{r^2} B_{\theta 1} \right] \\
 &\quad - \frac{c^2}{4\pi} i \eta_1 \left[\frac{m}{r} \left(\frac{dB_{\theta 0}}{dr} + \frac{B_{\theta 0}}{r} \right) + k_z \frac{dB_{z0}}{dr} \right] \\
 (2) \quad -4\pi \rho_0 i \frac{\partial}{\partial t} &\left[\frac{\partial^2 v_{r1}}{\partial r^2} + \frac{1}{r} \frac{\partial v_{r1}}{\partial r} - \left(\frac{m^2}{r^2} + k_z^2 \right) v_{r1} - \frac{2im}{r^2} v_{\theta 1} \right] \\
 &= \left(\frac{m}{r} B_{\theta 0} + k_z B_{z0} \right) \left[\frac{\partial^2 B_{r1}}{\partial r^2} + \frac{1}{r} \frac{\partial B_{r1}}{\partial r} - \left(\frac{m^2}{r^2} + k_z^2 \right) B_{r1} \right. \\
 &\quad \left. - \frac{2im}{r^2} B_{\theta 1} \right] + B_{r1} \left(-\frac{m}{r} \frac{d^2 B_{\theta 0}}{dr^2} - \frac{m}{r^2} \frac{dB_{\theta 0}}{dr} - \frac{m}{r^3} B_{\theta 0} \right. \\
 &\quad \left. - k_z \frac{d^2 B_{z0}}{dr^2} + \frac{k_z}{r} \frac{dB_{z0}}{dr} \right) + \frac{\partial B_{r1}}{\partial r} \left(-\frac{2m}{r^2} B_{\theta 0} \right) \\
 &\quad + i B_{\theta 1} \left[-\frac{2}{r} B_{\theta 0} \left(\frac{m^2}{r^2} + k_z^2 \right) \right] \\
 (3) \quad \frac{\partial \theta_1}{\partial t} &= i v_{\theta 1} \left(\frac{m}{r} B_{\theta 0} + k_z B_{z0} \right) + v_{r1} \left(\frac{B_{\theta 0}}{r} - \frac{dB_{\theta 0}}{dr} \right) \\
 &\quad + \frac{c^2}{4\pi} \eta_0 \left[\frac{\partial^2 B_{\theta 1}}{\partial r^2} + \frac{1}{r} \frac{\partial B_{\theta 1}}{\partial r} - B_{\theta 1} \left(\frac{m^2}{r^2} + k_z^2 \right) \right. \\
 &\quad \left. + \frac{2im}{r^2} B_{r1} \right] + \frac{c^2}{4\pi} \eta_1 \left(\frac{d^2 B_{\theta 0}}{dr^2} + \frac{1}{r} \frac{dB_{\theta 0}}{dr} - \frac{B_{\theta 0}}{r^2} \right)
 \end{aligned}$$

$$\begin{aligned}
 & + \frac{c^2}{4\pi} \frac{d\eta_0}{dr} \left(\frac{\partial B_{\theta 1}}{\partial r} + \frac{B_{\theta 1}}{r} - \frac{im}{r} B_{r1} \right) \\
 & + \frac{c^2}{4\pi} \frac{\partial \eta_1}{\partial r} \left(\frac{dB_{\theta 0}}{dr} + \frac{B_{\theta 0}}{r} \right) \\
 (4) \quad & -4\pi\rho_0 \frac{\partial}{\partial t} \left[\frac{\partial^2 v_{\theta 1}}{\partial r^2} + \frac{1}{r} \frac{\partial v_{\theta 1}}{\partial r} - \left(\frac{m^2 + 1}{r^2} + k_z^2 \right) v_{\theta 1} + \frac{2im}{r^2} v_{r1} \right] \\
 & = -i \left(\frac{m}{r} B_{\theta 0} + k_z B_{z0} \right) \left[\frac{\partial^2 B_{\theta 1}}{\partial r^2} + \frac{1}{r} \frac{\partial B_{\theta 1}}{\partial r} - \left(\frac{m^2 + 1}{r^2} + k_z^2 \right) B_{\theta 1} \right. \\
 & \quad \left. + \frac{2im}{r^2} B_{r1} \right] - \left[\frac{1}{r} \frac{\partial}{\partial r} (r B_{\theta 0}) \right] \left[\frac{\partial^2 B_{r1}}{\partial r^2} + \frac{1}{r} \frac{\partial B_{r1}}{\partial r} \right. \\
 & \quad \left. - \left(\frac{m^2 + 1}{r^2} + k_z^2 \right) B_{r1} \right] - B_{r1} \left[\frac{d^3 B_{\theta 0}}{dr^3} + \frac{2}{r} \frac{d^2 B_{\theta 0}}{dr^2} \right. \\
 & \quad \left. + \left(\frac{2m^2 - 1}{r^2} \right) \frac{dB_{\theta 0}}{dr} + \frac{B_{\theta 0}}{r^3} + \frac{2mk_z}{r} \frac{dB_{z0}}{dr} \right] \\
 & \quad - \frac{\partial B_{r1}}{\partial r} \left[2 \left(\frac{d^2 B_{\theta 0}}{dr^2} + \frac{1}{r} \frac{dB_{\theta 0}}{dr} - \frac{B_{\theta 0}}{r^2} \right) \right] \\
 & \quad - i \frac{\partial B_{\theta 1}}{\partial r} \left[2 \left(\frac{m}{r} \frac{dB_{\theta 0}}{dr} + k_z \frac{dB_{z0}}{dr} \right) \right] + i B_{\theta 1} \left[-\frac{m}{r} \frac{d^2 B_{\theta 0}}{dr^2} \right. \\
 & \quad \left. - \frac{m}{r^2} \frac{dB_{\theta 0}}{dr} + \frac{3m}{r^3} B_{\theta 0} - k_z \left(\frac{d^2 B_{z0}}{dr^2} + \frac{1}{r} \frac{dB_{z0}}{dr} \right) \right]
 \end{aligned}$$

$$\frac{m}{2}$$

$$B_{r1} = B_{\theta 1} = T_1 = v_{r1} = v_{\theta 1} = v_{z1} = \rho_1 = 0. \quad (2.56)$$

Using the premise that the perturbed radial current vanishes at the outer wall R_W , D. C. Robinson¹⁷ has used infinite conductivity MHD theory to derive the following set of boundary conditions at R_W :

$$B_{r1}(R_W, t) = v_{r1}(R_W, t) = T_1(R_W, t) = \rho_1(R_W, t) = 0 \quad (2.57)$$

$$B_{\theta 1}(R_W, t) = \frac{imR_W}{m^2 + k_z^2 R_W^2} \frac{\partial B_{r1}}{\partial r} \Big|_{r=R_W} \quad (2.58)$$

$$v_{\theta 1}(R_W, t) = \frac{imR_W}{m^2 + k_z^2 R_W^2} \frac{\partial v_{r1}}{\partial r} \Big|_{r=R_W} \quad (2.59)$$

$$v_{z1}(R_W, t) = \frac{ik_z R_W^2}{m^2 + k_z^2 R_W^2} \frac{\partial v_{r1}}{\partial r} \Big|_{r=R_W}. \quad (2.60)$$

These conditions are valid for both the general case and the incompressible case provided $\bar{k} \cdot \bar{B} \neq 0$ at the wall and that the density ρ_1 , temperature T_1 , and pressure p_1 tend to zero as $r \rightarrow R_W$. Note that the systems of equations are solved throughout the domain $0 \leq r \leq R_W$, i.e., there is not a plasma region surrounded by a vacuum region nor is there a plasma region of finite conductivity surrounded by a region of infinite conductivity.

In addition to the boundary conditions at $r = 0$ and $r = R_W$, an initial perturbation must be specified for at least one of the

dependent variables, usually $v_{r1}(r,0)$, and the specific forms used for the initial perturbation are described in detail in Section 3.5.2.

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III. FINITE DIFFERENCE EQUATIONS AND METHOD OF SOLUTION

3.1 Dimensionless Equations

It is convenient to put the systems of hydromagnetic equations into dimensionless form. The two relevant times for the problem are the decay time of the equilibrium magnetic field or the resistive diffusion time

$$\tau_R = \frac{4\pi a^2}{c^2 \langle \eta \rangle}, \quad (3.1)$$

and the hydromagnetic (or Alfvén-wave) transit time

$$\tau_H = \frac{a(4\pi \langle \rho \rangle)^{1/2}}{B}, \quad (3.2)$$

where 'a' is a characteristic length which is usually a measure of the thickness of the current layer, and B, $\langle \eta \rangle$, and $\langle \rho \rangle$ are characteristic values for the magnetic field, resistivity, and mass density, respectively. The two dimensionless independent variables are defined as

$$\mu = r/a, \quad \tau = t/\tau_R. \quad (3.3)$$

If R is the value of r such that

$$\bar{k} \cdot \bar{B} = \frac{m}{r} B_{\theta 0} + k_z B_{z0} = 0, \quad (3.4)$$

then R is the radius about which the instability will develop if the equilibrium configuration is resistively unstable for the particular mode specified by the pair of mode numbers m and $k_z^{1,2}$

In the cylindrical geometry used here, these two mode numbers are specified independently, while in the planar sheet pinch model^{1,2} only one number is required to describe a particular mode.

The dimensionless dependent variables are defined as

$$\begin{aligned} \psi &= \frac{B_{r1}}{B} & \phi &= i \frac{B_{\theta 1}}{B} & T &= \frac{i T_1}{\langle T \rangle} \\ W &= -ik\tau_R^v r_1 & U &= k\tau_R^v \theta_1 & V &= k\tau_R^v z_1 & (3.5) \\ R &= \frac{i\rho_1}{\langle \rho \rangle} & \delta &= -i \frac{\eta_1}{\langle \eta \rangle} & P &= \frac{i p_1}{\langle \rho \rangle \langle T \rangle} \end{aligned}$$

where $k = (m^2/R^2 + k_z^2)^{1/2}$ and $\langle T \rangle$ is a characteristic value for the (normalized) temperature in units of energy/mass. The following parameters are also required:

$$\alpha = ka \quad (3.6)$$

$$\kappa_z = k_z a \quad (3.7)$$

$$S \equiv \frac{\tau_R}{\tau_H} = \frac{(4\pi)^{1/2} a B}{c^2 \langle \eta \rangle \langle \rho \rangle^{1/2}} \quad (3.8)$$

$$\Gamma \equiv \frac{4\pi v}{c^2 \langle \eta \rangle} = \frac{\tau_R^v}{a^2} \quad (3.9)$$

$$\Lambda \equiv \frac{4\pi K}{c^2 \langle \eta \rangle \langle \rho \rangle} = \frac{\tau_R^K}{a^2 \langle \rho \rangle} \quad (3.10)$$

$$\beta = \frac{8\pi \langle \rho \rangle \langle T \rangle}{B^2} \quad (3.11)$$

where S is the magnetic Reynolds number at the Alfvén speed,

Γ is a dimensionless viscosity, Λ is a dimensionless thermal conductivity, and β is the plasma beta which is the usual ratio of the fluid energy density to the energy density in the magnetic field.³

The following functions are given by the equilibrium configuration:

$$\begin{aligned}
 H &= \frac{B_{\theta 0}}{B} & M &= \frac{B_{z 0}}{B} \\
 F &= \frac{1}{kB} \left(\frac{m}{r} B_{\theta 0} + k_z B_{z 0} \right) = \frac{1}{\alpha} \left(\frac{m}{\mu} H + \kappa_z M \right) \\
 F' &= \frac{dF}{d\mu} & F'' &= \frac{d^2 F}{d\mu^2} \\
 M' &= \frac{dM}{d\mu} & M'' &= \frac{d^2 M}{d\mu^2} \\
 H' &= \frac{dH}{d\mu} & H'' &= \frac{d^2 H}{d\mu^2} & H''' &= \frac{d^3 H}{d\mu^3} & (3.12) \\
 N &= \frac{1}{\mu} \frac{\partial}{\partial \mu} (\mu H) & N' &= \frac{dN}{d\mu} & N'' &= \frac{d^2 N}{d\mu^2} \\
 \tilde{\rho} &= \frac{\rho_0}{\langle \rho \rangle} & \tilde{T} &= \frac{T_0}{\langle T \rangle} & \tilde{\eta} &= \frac{\eta_0}{\langle \eta \rangle} \\
 \tilde{\rho}' &= \frac{d\rho}{d\mu} & \tilde{T}' &= \frac{dT}{d\mu} & \tilde{\eta}' &= \frac{d\eta}{d\mu} \\
 \tilde{p} &= \frac{p_0}{\langle \rho \rangle \langle T \rangle}
 \end{aligned}$$

and the dimensionless source terms are

$$\begin{aligned}
 \mathcal{J}_M &= \frac{i\tau_R}{\langle \rho \rangle} S_{M_1} \\
 \mathcal{J}_E &= \frac{i\tau_R}{\langle \rho \rangle \langle T \rangle} S_{E_1} \\
 \mathcal{J}_{P_r} &= -i \frac{4\pi a}{B^2} (S_{P_1})_r \\
 \mathcal{J}_{P_\theta} &= \frac{4\pi a}{B^2} (S_{P_1})_\theta \\
 \mathcal{J}_{P_z} &= \frac{4\pi a}{B^2} (S_{P_1})_z.
 \end{aligned} \tag{3.13}$$

Defining the dimensionless operator²

$$\mathcal{L} \equiv \frac{\partial^2}{\partial \mu^2} + \frac{1}{\mu} \frac{\partial}{\partial \mu} - \left(\frac{m^2 + 1}{\mu^2} + \kappa_z^2 \right), \tag{3.14}$$

the systems of dimensionless equations can be put into a more compact form. The dimensionless equations for both the general case and the incompressible case are given in the following sections.

3.1.1 General Case

Using the dimensionless parameters described in the previous section, the system of linearized hydromagnetic equations for the general case with compressibility, viscosity, and thermal conductivity (Table 2.5) is presented in Table 3.1. The parameter I in these dimensionless equations indicates those terms which would vanish if the fluid were assumed to be incompressible.

Table 3.1. Component equations for the general case (dimensionless form).

$$(1) \quad \frac{\partial \psi}{\partial \tau} = -FW + \tilde{\eta}(L\psi - \frac{2m}{\mu} \phi) + (\alpha F' + \frac{2m}{\mu^2} H)\delta$$

$$(2) \quad \frac{\partial \phi}{\partial \tau} = \left(N - \frac{2H}{\mu} \right) \frac{W}{\alpha} - \left(\frac{m}{\mu} H + \kappa_z M \right) \frac{U}{\alpha} + I \frac{H}{\alpha} \left(\frac{\partial W}{\partial \mu} + \frac{W}{\mu} + \frac{m}{\mu} U + \kappa_z V \right) \\ + \tilde{\eta}(L\phi - \frac{2m}{\mu^2} \psi) + \tilde{\eta}' \left(\frac{\partial \phi}{\partial \mu} + \frac{\phi}{\mu} + \frac{m}{\mu} \psi \right) - N'\delta - N \frac{\partial \delta}{\partial \mu}$$

$$(3) \quad \kappa_z \rho \frac{\partial T}{\partial \tau} = \frac{\kappa_z \tilde{\rho} \tilde{T}'}{\alpha} W + \kappa_z \Lambda \left(L T + \frac{T}{\mu^2} \right) \\ + I(\gamma - 1) \frac{\kappa_z \tilde{\rho} \tilde{T}'}{\alpha} \left(\frac{\partial W}{\partial \mu} + \frac{W}{\mu} + \frac{m}{\mu} U + \kappa_z V \right) \\ + 4(\gamma - 1) \frac{\tilde{\eta}}{\beta} \left[\left(\kappa_z N + \frac{m}{\mu} M' \right) \frac{\phi}{\mu} - M'(L\psi) \right. \\ \left. + \left(\kappa_z N - \frac{m}{\mu} M' \right) \left(\frac{m}{\mu} \psi + \frac{\partial \phi}{\partial \mu} \right) \right] \\ - 2(\gamma - 1) \frac{\kappa_z}{\beta} (M'^2 + N^2)\delta + (\gamma - 1) \kappa_z \Delta_E$$

$$(4) \quad \frac{\kappa_z \rho}{\alpha^2} \frac{\partial W}{\partial \tau} = \left[\frac{m}{\mu} (\kappa_z H - \frac{m}{\mu} M) - \frac{M'}{\mu} \right] \psi - M(L\psi) - M' \frac{\partial \psi}{\partial \mu} \\ + \left[\frac{m}{\mu} \left(\frac{M}{\mu} - M' \right) + \kappa_z \left(N + \frac{H}{\mu} \right) \right] \phi + (\kappa_z H - \frac{m}{\mu} M) \frac{\partial \phi}{\partial \mu} \\ + \frac{\beta \kappa_z}{2} \left(\tilde{\rho} \frac{\partial T}{\partial \mu} + \tilde{T} \frac{\partial \rho}{\partial \mu} + T \tilde{\rho}' + \rho \tilde{T}' \right) \\ + \frac{\kappa_z \tilde{\rho} \tilde{T}'}{\alpha^2} \left[\left(1 + \frac{I}{3} \right) (LW) + \frac{I}{3} \left(\frac{m^2}{\mu^2} + \kappa_z^2 \right) W - \left(2 + \frac{I}{3} \right) \frac{m}{\mu^2} U \right. \\ \left. + \frac{Im}{3\mu} \frac{\partial U}{\partial \mu} + \frac{Ik_z}{3} \frac{\partial V}{\partial \mu} \right] + \kappa_z \left(\frac{\Delta}{p} \right)_r$$

Table 3.1 (continued)

$$(5) \quad \frac{\kappa_z \tilde{\rho}}{\alpha S^2} \frac{\partial U}{\partial \tau} = \left(\frac{m}{\mu^2} M + \kappa_z N \right) \psi + \left[M \left(\frac{m^2}{\mu^2} + \kappa_z^2 \right) \right] \phi$$

$$+ \left(\frac{m}{\mu} M \right) \frac{\partial \psi}{\partial \mu} - \frac{\beta \kappa_z m}{2\mu} (\tilde{\rho} T + \tilde{T} R) + \frac{\kappa_z \tilde{\rho} \Gamma}{\alpha S^2} \left[LU - \frac{2m}{\mu^2} W \right.$$

$$\left. - \frac{I m}{3\mu} \left(\frac{\partial W}{\partial \mu} + \frac{W}{\mu} + \frac{m}{\mu} U + \kappa_z V \right) \right] + \kappa_z (\mathcal{A}_P)_\theta$$

$$(6) \quad \frac{\kappa_z \tilde{\rho}}{\alpha S^2} \frac{\partial V}{\partial \tau} = (\kappa_z M' - \frac{m}{\mu^2} H) \psi - \left(\frac{m}{\mu} H \right) \frac{\partial \psi}{\partial \mu} - \left[H \left(\frac{m^2}{\mu^2} + \kappa_z^2 \right) \right] \phi$$

$$- \frac{\beta \kappa_z^2}{2} (\tilde{\rho} T + \tilde{T} R) + \frac{\kappa_z \tilde{\rho} \Gamma}{\alpha S^2} \left[LV + \frac{V}{\mu^2} \right.$$

$$\left. - \frac{I \kappa_z}{3} \left(\frac{\partial W}{\partial \mu} + \frac{W}{\mu} + \frac{m}{\mu} U + \kappa_z V \right) \right] + \kappa_z (\mathcal{A}_P)_z$$

$$(7) \quad \frac{\partial R}{\partial \tau} = \frac{\tilde{\rho}'}{\alpha} W + I \frac{\tilde{\rho}}{\alpha} \left(\frac{\partial W}{\partial \mu} + \frac{W}{\mu} + \frac{m}{\mu} U + \kappa_z V \right) + \mathcal{A}_M$$

$$(8) \quad P = \tilde{\rho} T + \tilde{T} R$$

3.1.2 Incompressible Case

For the incompressible model without viscosity and thermal conductivity, the dimensionless form of the linearized equations (Table 2.6) is presented in Table 3.2. Note that the equation for B_{r1} (i.e., ψ) is the same as the general case, while the equation for $B_{\theta 1}$ (i.e., ϕ) is similar to the general case except that it has made use of the incompressibility condition $\nabla \cdot \vec{v}_1 = 0$.

3.2 Difference Equations

In this section the finite difference equations for the hydromagnetic models are described. The equations are presented in Eulerian form in which the independent spatial variable refers to a fixed coordinate system through which the fluid moves. The flow is characterized by a time-dependent velocity field which is determined by solving an initial value problem.

In order to solve the systems of equations, the continuous space-time domain on which the equations hold is replaced by a discrete space-time mesh. For the space-time mesh to be used, the superscript n denotes the time level for the independent variable τ , and the subscript j denotes the spatial mesh point for the independent variable μ . The finite difference grid is shown in Fig. 3.1, where

$$\begin{aligned} \tau &\geq 0, & \mu &\geq 0 \\ \tau^n &= n\Delta\tau, & n &= 0, 1, 2, \dots \\ \mu_j &= j\Delta\mu, & j &= 0, 1, 2, \dots, J \\ \mu_J &= J\Delta\mu = \mu_W. \end{aligned} \tag{3.15}$$

Table 3.2. Component equations for the incompressible case (dimensionless form).

$$(1) \quad \frac{\partial \psi}{\partial \tau} = -FW + \tilde{\eta} \left(L\psi - \frac{2m}{\mu} \phi \right) + \left(\alpha F' + \frac{2m}{\mu} H \right) \delta$$

$$(2) \quad \frac{\tilde{\rho}}{\alpha^2 S^2} \frac{\partial}{\partial \tau} \left(LW - \frac{2m}{\mu^2} U \right) = F \left(L\psi - \frac{2m}{\mu^2} \phi \right) - \left(\frac{2mH}{\alpha \mu^2} \right) \frac{\partial \psi}{\partial \mu} - \left(F'' - \frac{F'}{\mu} + \frac{4m}{\alpha \mu^2} N - \frac{6m}{\alpha \mu^3} H \right) \psi - \left[\frac{2H}{\alpha \mu} \left(\frac{m^2}{\mu^2} + \kappa_z^2 \right) \right] \phi$$

$$(3) \quad \frac{\partial \phi}{\partial \tau} = -FU + \left(N - \frac{2H}{\mu} \right) \frac{W}{\alpha} + \tilde{\eta} \left(L\phi - \frac{2m}{\mu^2} \psi \right) + \tilde{\eta}' \left(\frac{\partial \phi}{\partial \mu} + \frac{\phi}{\mu} + \frac{m}{\mu} \psi \right) - N' \delta - N \frac{\partial \delta}{\partial \mu}$$

$$(4) \quad \frac{\tilde{\rho}}{\alpha^2 S^2} \frac{\partial}{\partial \tau} \left(LU - \frac{2m}{\mu^2} W \right) = F \left(L\phi - \frac{2m}{\mu^2} \psi \right) + \frac{N}{\alpha} (L\psi) + \left(-\frac{2m}{\mu} F' + \frac{2m^2}{\alpha \mu^3} H + \frac{N'}{\alpha \mu} + \frac{N''}{\alpha} \right) \psi + \left(\frac{2N'}{\alpha} \right) \frac{\partial \psi}{\partial \mu} + \left(F'' + \frac{F'}{\mu} + \frac{2m}{\alpha \mu^2} N - \frac{6m}{\alpha \mu^3} H \right) \phi + \left[2 \left(F' + \frac{m}{\alpha \mu^2} H \right) \right] \frac{\partial \phi}{\partial \mu}$$

It is assumed that values of all the dependent variables are known at all spatial mesh points at a time τ^n , and the problem is to determine their values at a later time τ^{n+1} . To solve the problem, the differential equations are approximated at each spatial point by finite difference equations which are algebraic in form and which relate the dependent variables at time τ^{n+1} to the dependent variables at time τ^n . To numerically solve a system of m differential equations in m dependent variables on a mesh consisting of J points, a set of $m \times J$ algebraic equations in $m \times J$ unknowns must be solved at time τ^{n+1} .

For the hydromagnetic models to be examined here, it is desired to find $\psi(\mu, \tau)$, $\phi(\mu, \tau)$, $T(\mu, \tau)$, $W(\mu, \tau)$, $U(\mu, \tau)$, $V(\mu, \tau)$, $R(\mu, \tau)$, and $\delta(\mu, \tau)$ on the domain $0 \leq \mu \leq \mu_W$, and where the initial distributions of these functions are given. The notation for the dependent variables is such that $\psi_j^n = \psi(\mu_j, \tau^n)$, $\phi_j^n = \phi(\mu_j, \tau^n)$, $T_j^n = T(\mu_j, \tau^n)$, etc. It is assumed that the dependent variables and their derivatives are single-valued, finite and continuous functions of the independent variables μ and τ . Then using a Taylor expansion, the derivatives of the dependent variables are approximated by^{4,5}

$$\frac{\partial x}{\partial \tau} = \frac{x_j^{n+1} - x_j^n}{\Delta \tau}, \quad (3.16)$$

$$\frac{\partial x}{\partial \mu} = \frac{x_{j+1}^n - x_{j-1}^n}{2\Delta \mu}, \quad (3.17)$$

$$\frac{\partial^2 X}{\partial \mu^2} = \frac{X_{j+1}^n - 2X_j^n + X_{j-1}^n}{(\Delta \mu)^2}, \quad (3.18)$$

$$\frac{\partial}{\partial \tau} \left(\frac{\partial X}{\partial \mu} \right) = \frac{(\partial X / \partial \mu)^{n+1} - (\partial X / \partial \mu)^n}{\Delta \tau} \quad (3.19)$$

$$\frac{\partial}{\partial \tau} \left(\frac{\partial^2 X}{\partial \mu^2} \right) = \frac{(\partial^2 X / \partial \mu^2)^{n+1} - (\partial^2 X / \partial \mu^2)^n}{\Delta \tau} \quad (3.20)$$

where X represents any of the dependent variables. Equations (3.16), (3.19), and (3.20) have truncation errors of order $\Delta \tau$, while the central difference approximations used in (3.17) and (3.18) have truncation errors of order $(\Delta \mu)^2$, where the truncation error ϵ is defined⁶ as the difference between the differential equation and its corresponding finite difference equation with both evaluated at the same space-time point. Thus, the truncation error for the systems of hydromagnetic equations is

$$\epsilon = \mathcal{O}[\Delta \tau] + \mathcal{O}[(\Delta \mu)^2].$$

Using (3.17) and (3.18), the operator L , as defined in Eq. (3.14), is given in finite difference form as

$$(LX)_j^n = \frac{X_{j+1}^n - 2X_j^n + X_{j-1}^n}{(\Delta \mu)^2} + \frac{1}{\mu_j} \frac{X_{j+1}^n - X_{j-1}^n}{2\Delta \mu} - \left(\frac{m^2 + 1}{\mu_j^2} + \kappa_z^2 \right) X_j^n, \quad (3.21)$$

$$\begin{aligned}
 (LX)_j^n = & X_{j+1}^n \left[\frac{1}{(\Delta\mu)^2} + \frac{1}{2\mu_j \Delta\mu} \right] \\
 & - X_j^n \left[\frac{2}{(\Delta\mu)^2} + \frac{m^2 + 1}{\mu_j^2} + \kappa_z^2 \right] \\
 & + X_{j-1}^n \left[\frac{1}{(\Delta\mu)^2} - \frac{1}{2\mu_j \Delta\mu} \right]. \quad (3.21a)
 \end{aligned}$$

The systems of hydromagnetic equations (Tables 3.1 and 3.2) are differenced implicitly with a weighting factor ℓ , where $0 \leq \ell \leq 1$. For $\ell = 0$, the differencing scheme becomes fully explicit, i.e., the finite difference equations are forward differenced and are uncoupled in the unknowns, each equation giving one of the unknowns X_j^{n+1} directly in terms of the known quantities X_j^n . For $\ell = 1$, the differencing scheme becomes fully implicit, i.e., the finite difference equations are backward differenced and are coupled in the unknowns, and a system of simultaneous equations must be solved to obtain X_j^{n+1} . For stability,^{6,7} $1/2 \leq \ell \leq 1$, and for all of the results to be discussed in this report, a value of $\ell = 1/2$ will be used, i.e., the equations will be completely centered in space and time. The six space-time points required for the differencing scheme are shown schematically in Fig. 3.2.⁴

3.2.1 General Case

For the general case with compressibility, viscosity, and thermal conductivity (Table 3.1), the finite difference equations for $\psi(\mu, \tau)$ and $\phi(\mu, \tau)$, i.e., $B_{r1}(r, t)$ and $B_{\theta 1}(r, t)$, are

Figure 3.1. Finite difference grid.

Figure 3.2. Six point implicit difference scheme for hydromagnetic equations.

$$\begin{aligned}
 \frac{\psi_j^{n+1} - \psi_j^n}{\Delta\tau} &= \tilde{\eta}_j \left[s(L\psi)_j^{n+1} + (1-s)(L\psi)_j^n \right] \\
 &- \tilde{\eta}_j \frac{2m}{\mu_j^2} \left[s\phi_j^{n+1} + (1-s)\phi_j^n \right] \\
 &- F_j \left[sW_j^{n+1} + (1-s)W_j^n \right] \\
 &+ (\alpha F'_j + \frac{2m}{u_j^2} H_j) \left[s\phi_j^{n+1} + (1-s)\phi_j^n \right], \quad (3.22)
 \end{aligned}$$

$$\begin{aligned}
 \frac{\phi_j^{n+1} - \phi_j^n}{\Delta\tau} &= \tilde{\eta}_j \left[s(L\phi)_j^{n+1} + (1-s)(L\phi)_j^n \right] \\
 &- \tilde{\eta}_j \frac{2m}{\mu_j^2} \left[s\psi_j^{n+1} + (1-s)\psi_j^n \right] \\
 &+ \tilde{\eta}_j \left[\frac{\phi_{j+1}^{n+1} - \phi_{j-1}^{n+1}}{2\Delta u} + \frac{\phi_j^{n+1}}{\mu_j} + \frac{m}{\mu_j} \psi_j^{n+1} \right] \\
 &+ (1-s)\tilde{\eta}_j \left[\frac{\phi_{j+1}^n - \phi_{j-1}^n}{2\Delta u} + \frac{\phi_j^n}{\mu_j} + \frac{m}{\mu_j} \psi_j^n \right] \\
 &+ \frac{1}{\alpha} \left(H_j - \frac{H_j}{\mu_j} \right) \left[sW_j^{n+1} + (1-s)W_j^n \right] \\
 &+ \frac{H_j}{2\alpha\Delta u} \left[s(W_{j+1}^{n+1} - W_{j-1}^{n+1}) + (1-s)(W_{j+1}^n - W_{j-1}^n) \right] \\
 &- \frac{\kappa_{EM}}{\alpha} \left[sU_j^{n+1} + (1-s)U_j^n \right] + \frac{\kappa_{EM}}{\alpha} \left[sV_j^{n+1} + (1-s)V_j^n \right] \\
 &- \frac{H_j}{2\Delta u} \left[s(\phi_{j+1}^{n+1} - \phi_{j-1}^{n+1}) + (1-s)(\phi_{j+1}^n - \phi_{j-1}^n) \right] \\
 &- H_j \left[s\phi_j^{n+1} + (1-s)\phi_j^n \right]. \quad (3.23)
 \end{aligned}$$

The finite difference form of the energy equation is

$$\begin{aligned}
 \frac{\kappa_z \tilde{\rho}_j}{\Delta \tau} (\mathbf{T}_j^{n+1} - \mathbf{T}_j^n) = & \kappa_z \Lambda \left\{ \ell \left[(\mathbf{L}\mathbf{T})_j^{n+1} + \frac{\mathbf{T}_j^{n+1}}{\mu_j^2} \right] + (1 - \ell) \left[(\mathbf{L}\mathbf{T})_j^n + \frac{\mathbf{T}_j^n}{\mu_j^2} \right] \right\} \\
 & + 4(\gamma - 1) \frac{\tilde{\eta}_j}{\beta} \ell \left[-\mathbf{M}'_j (\mathbf{L}\psi)_j^{n+1} + \left(\kappa_z \mathbf{N}_j + \frac{m}{\mu_j} \mathbf{M}'_j \right) \frac{\phi_j^{n+1}}{\mu_j} \right. \\
 & \quad \left. + \left(\kappa_z \mathbf{N}_j - \frac{m}{\mu_j} \mathbf{M}'_j \right) \left(\frac{m}{\mu_j} \psi_j^{n+1} + \frac{\phi_{j+1}^{n+1} - \phi_{j-1}^{n+1}}{2\Delta u} \right) \right] \\
 & + 4(\gamma - 1) \frac{\tilde{\eta}_j}{\beta} (1 - \ell) \left[-\mathbf{M}'_j (\mathbf{L}\psi)_j^n + \left(\kappa_z \mathbf{N}_j + \frac{m}{\mu_j} \mathbf{M}'_j \right) \frac{\phi_j^n}{\mu_j} \right. \\
 & \quad \left. + \left(\kappa_z \mathbf{N}_j - \frac{m}{\mu_j} \mathbf{M}'_j \right) \left(\frac{m}{\mu_j} \psi_j^n + \frac{\phi_{j+1}^n - \phi_{j-1}^n}{2\Delta u} \right) \right] \\
 & + 1(\gamma - 1) \frac{\kappa_z \tilde{\rho}_j \tilde{\eta}_j}{\alpha} \left\{ \ell \left[\frac{w_{j+1}^{n+1} - w_{j-1}^{n+1}}{2\Delta u} + \frac{w_j^{n+1}}{\mu_j} + \frac{m}{\mu_j} u_j^{n+1} + \kappa_z v_j^{n+1} \right] \right. \\
 & \quad \left. + (1 - \ell) \left[\frac{w_{j+1}^n - w_{j-1}^n}{2\Delta u} + \frac{w_j^n}{\mu_j} + \frac{m}{\mu_j} u_j^n + \kappa_z v_j^n \right] \right\} \\
 & + \frac{\kappa_z \tilde{\rho}_j \tilde{\eta}_j}{\alpha} \left[\ell w_j^{n+1} + (1 - \ell) w_j^n \right] \\
 & - 2(\gamma - 1) \frac{\kappa_z}{\beta} (\mathbf{M}_j^2 + \mathbf{N}_j^2) \left[\ell \delta_j^{n+1} + (1 - \ell) \delta_j^n \right] \\
 & + (\gamma - 1) \kappa_z (\mathcal{A}_E)_j. \tag{3.24}
 \end{aligned}$$

The finite difference equations for the r , θ , and z components of the equation of motion are, respectively,

$$\begin{aligned}
 \frac{\kappa_z \tilde{\rho}_j}{\alpha S^2 \Delta \tau} (W_j^{n+1} - W_j^n) = & -M_j \left[\ell(L\psi)_j^{n+1} + (1 - \ell)(L\psi)_j^n \right] \\
 & - \frac{M'_j}{2\Delta u} \left[\ell(\psi_{j+1}^{n+1} - \psi_{j-1}^{n+1}) + (1 - \ell)(\psi_{j+1}^n - \psi_{j-1}^n) \right] \\
 & + \left[\frac{m}{\mu_j} (\kappa_z H_j - \frac{m}{\mu_j} M_j) - \frac{M'_j}{\mu_j} \right] \left[\ell\psi_j^{n+1} + (1 - \ell)\psi_j^n \right] \\
 & + \frac{1}{2\Delta u} (\kappa_z H_j - \frac{m}{\mu_j} M_j) \left[\ell(\phi_{j+1}^{n+1} - \phi_{j-1}^{n+1}) + (1 - \ell)(\phi_{j+1}^n - \phi_{j-1}^n) \right] \\
 & + \left[\kappa_z \left(N_j + \frac{H_j}{\mu_j} \right) + \frac{m}{\mu_j} \left(\frac{M_j}{\mu_j} - M'_j \right) \right] \left[\ell\phi_j^{n+1} + (1 - \ell)\phi_j^n \right] \\
 & + \frac{\beta \kappa_z \tilde{\rho}_j}{4\Delta u} \left[\ell(\tau_{j+1}^{n+1} - \tau_{j-1}^{n+1}) + (1 - \ell)(\tau_{j+1}^n - \tau_{j-1}^n) \right] \\
 & + \frac{\beta \kappa_z \tilde{\rho}'_j}{2} \left[\ell\tau_j^{n+1} + (1 - \ell)\tau_j^n \right] \\
 & + \frac{\kappa_z \tilde{\rho}_j \Gamma}{\alpha S^2} \left\{ \left(1 + \frac{I}{3} \right) \left[\ell(LW)_j^{n+1} + (1 - \ell)(LW)_j^n \right] \right. \\
 & + \frac{I}{3} \left(\frac{m^2}{\mu_j^2} + \kappa_z^2 \right) \left[\ell W_j^{n+1} + (1 - \ell)W_j^n \right] \\
 & - \left(2 + \frac{I}{3} \right) \frac{m}{\mu_j^2} \left[\ell U_j^{n+1} + (1 - \ell)U_j^n \right] \\
 & + \frac{I}{6\Delta u \mu_j} \left[\ell(U_{j+1}^{n+1} - U_{j-1}^{n+1}) + (1 - \ell)(U_{j+1}^n - U_{j-1}^n) \right] \\
 & \left. + \frac{I\kappa_z}{6\Delta u} \left[\ell(V_{j+1}^{n+1} - V_{j-1}^{n+1}) + (1 - \ell)(V_{j+1}^n - V_{j-1}^n) \right] \right\}
 \end{aligned}$$

$$\begin{aligned}
 & + \frac{\beta \kappa_z \tilde{\rho}_j}{4\Delta\mu} \left[\ell (\mathcal{R}_{j+1}^{n+1} - \mathcal{R}_{j-1}^{n+1}) + (1 - \ell) (\mathcal{R}_{j+1}^n - \mathcal{R}_{j-1}^n) \right] \\
 & + \frac{\beta \kappa_z \tilde{\rho}_j}{2} \left[\ell \mathcal{R}_j^{n+1} + (1 - \ell) \mathcal{R}_j^n \right] + \kappa_z (\mathcal{L}_{p_r})_j, \quad (3.25)
 \end{aligned}$$

$$\begin{aligned}
 \frac{\kappa_z \tilde{\rho}_j}{\alpha S^2 \Delta\tau} (U_j^{n+1} - U_j^n) &= \left(\frac{m}{\mu_j^2} M_j + \kappa_z N_j \right) \left[\ell \psi_j^{n+1} + (1 - \ell) \psi_j^n \right] \\
 &+ M_j \left(\frac{m^2}{\mu_j^2} + \kappa_z^2 \right) \left[\ell \phi_j^{n+1} + (1 - \ell) \phi_j^n \right] \\
 &+ \frac{m M_j}{2\mu_j \Delta\mu} \left[\ell (\psi_{j+1}^{n+1} - \psi_{j-1}^{n+1}) + (1 - \ell) (\psi_{j+1}^n - \psi_{j-1}^n) \right] \\
 &- \frac{\beta \kappa_z m}{2\mu_j} \left[\ell (\tilde{\rho}_j \mathcal{R}_j^{n+1} + \tilde{\tau}_j \mathcal{R}_j^{n+1}) + (1 - \ell) (\tilde{\rho}_j \mathcal{R}_j^n + \tilde{\tau}_j \mathcal{R}_j^n) \right] \\
 &+ \frac{\kappa_z \tilde{\rho}_j \Gamma}{\alpha S^2} \left\{ \ell (LU)_j^{n+1} + (1 - \ell) (LU)_j^n - 2 \frac{m}{\mu_j} \left[\ell w_j^{n+1} + (1 - \ell) w_j^n \right] \right. \\
 &- \frac{Im}{6\mu_j \Delta\mu} \left[\ell (w_{j+1}^{n+1} - w_{j-1}^{n+1}) + (1 - \ell) (w_{j+1}^n - w_{j-1}^n) \right] \\
 &- \frac{Im}{3\mu_j} \left[\ell \left(\frac{w_j^{n+1}}{\mu_j} + \frac{m}{\mu_j} U_j^{n+1} + \kappa_z v_j^{n+1} \right) + (1 - \ell) \left(\frac{w_j^n}{\mu_j} + \frac{m}{\mu_j} U_j^n + \kappa_z v_j^n \right) \right] \Bigg\} \\
 &+ \kappa_z (\mathcal{L}_{p_6})_j, \quad (3.26)
 \end{aligned}$$

$$\frac{\kappa_z \tilde{\rho}_j}{\alpha S^2 \Delta\tau} (V_j^{n+1} - V_j^n) = \left(\kappa_z M_j - \frac{m}{\mu_j} H_j \right) \left[\ell \psi_j^{n+1} + (1 - \ell) \psi_j^n \right]$$

$$\begin{aligned}
 & - \frac{mH_j}{2\mu_j \Delta u} \left[\ell(\psi_{j+1}^{n+1} - \psi_{j-1}^{n+1}) + (1 - \ell)(\psi_{j+1}^n - \psi_{j-1}^n) \right] \\
 & - H_j \left(\frac{m^2}{\mu_j^2} + \kappa_z^2 \right) \left[\ell \rho_j^{n+1} + (1 - \ell) \rho_j^n \right] \\
 & - \frac{\beta \kappa_z^2}{2} \left[\ell(\tilde{\rho}_j^{n+1} + \tilde{\tau}_j \mathcal{R}_j^{n+1}) + (1 - \ell)(\tilde{\rho}_j^n + \tilde{\tau}_j \mathcal{R}_j^n) \right] \\
 & + \frac{\kappa_z \tilde{\rho}_j \Gamma}{\alpha S^2} \left\{ \ell(LV)_j^{n+1} + (1 - \ell)(LV)_j^n + \frac{1}{\mu_j^2} \left[\ell v_j^{n+1} + (1 - \ell) v_j^n \right] \right. \\
 & - \frac{I \kappa_z}{6 \Delta u} \left[\ell(w_{j+1}^{n+1} - w_{j-1}^{n+1}) + (1 - \ell)(w_{j+1}^n - w_{j-1}^n) \right] \\
 & - \frac{I \kappa_z}{3} \left[\ell \left(\frac{w_j^{n+1}}{\mu_j} + \frac{m}{\mu_j} u_j^{n+1} + \kappa_z v_j^{n+1} \right) + (1 - \ell) \left(\frac{w_j^n}{\mu_j} + \frac{m}{\mu_j} u_j^n \right) \right. \\
 & \left. \left. + \kappa_z v_j^n \right] \right\} + \kappa_z (\mathcal{A}_{p_z})_j. \tag{3.27}
 \end{aligned}$$

Finally, the finite difference approximations to the continuity equation and the equation of state are

$$\begin{aligned}
 \frac{\mathcal{R}_j^{n+1} - \mathcal{R}_j^n}{\Delta \tau} &= \frac{I \tilde{\rho}_j}{2 \alpha \Delta u} \left[\ell(w_{j+1}^{n+1} - w_{j-1}^{n+1}) + (1 - \ell)(w_{j+1}^n - w_{j-1}^n) \right] \\
 &+ \frac{1}{\alpha} \left(I \frac{\tilde{\rho}_j}{\mu_j} + \rho_j^i \right) \left[\ell w_j^{n+1} + (1 - \ell) w_j^n \right] \\
 &+ \frac{I m \tilde{\rho}_j}{\alpha \mu_j} \left[\ell u_j^{n+1} + (1 - \ell) u_j^n \right]
 \end{aligned}$$

$$+ \frac{I \kappa_z \tilde{\rho}_j}{\alpha} \left[\ell v_j^{n+1} + (1 - \ell) v_j^n \right] + (\mathcal{L}_M)_j, \quad (3.28)$$

$$\begin{aligned} \ell p_j^{n+1} + (1 - \ell) p_j^n &= \tilde{\rho}_j \left[\ell T_j^{n+1} + (1 - \ell) T_j^n \right] \\ &+ \tilde{T}_j \left[\ell \mathcal{R}_j^{n+1} + (1 - \ell) \mathcal{R}_j^n \right]. \end{aligned} \quad (3.29)$$

3.2.2 Incompressible Case

For the incompressible case without viscosity and thermal conductivity (Table 3.2), the finite difference equations for $\psi(\mu, \tau)$ and $W(\mu, \tau)$, i.e., for $B_{r1}(r, t)$ and $v_{r1}(r, t)$ are

$$\begin{aligned} \frac{\psi_j^{n+1} - \psi_j^n}{\Delta \tau} &= \tilde{\eta}_j \left[\ell (L\psi)_j^{n+1} + (1 - \ell) (L\psi)_j^n \right] \\ &- \tilde{\eta}_j \frac{2m}{\mu_j} \left[\ell \phi_j^{n+1} + (1 - \ell) \phi_j^n \right] \\ &- F_j \left[\ell W_j^{n+1} + (1 - \ell) W_j^n \right] \\ &+ (\alpha F'_j + \frac{2m}{\mu_j} H_j) \left[\ell \delta_j^{n+1} + (1 - \ell) \delta_j^n \right], \end{aligned} \quad (3.30)$$

$$\begin{aligned} \frac{\tilde{\rho}_j}{\alpha^2 s^2 \Delta \tau} \left[(LW)_j^{n+1} - (LW)_j^n - \frac{2m}{\mu_j} (U_j^{n+1} - U_j^n) \right] \\ = F_j \left\{ \ell (L\psi)_j^{n+1} + (1 - \ell) (L\psi)_j^n - \frac{2m}{\mu_j} \left[\ell \phi_j^{n+1} + (1 - \ell) \phi_j^n \right] \right\} \\ - \frac{mH_j}{\alpha \mu_j^2 \Delta u} \left[\ell (\psi_{j+1}^{n+1} - \psi_{j-1}^{n+1}) + (1 - \ell) (\psi_{j+1}^n - \psi_{j-1}^n) \right] \end{aligned}$$

$$\begin{aligned}
 & - \left(F_j' - \frac{F_j'}{\mu_j} + \frac{4m}{\alpha \mu_j^2} N_j - \frac{6m}{\alpha \mu_j^3} H_j \right) \left[\ell \psi_j^{n+1} + (1 - \ell) \psi_j^n \right] \\
 & - \left[\frac{2H_j}{\alpha \mu_j} \left(\frac{m^2}{\mu_j^2} + \kappa_z^2 \right) \right] \left[\ell \phi_j^{n+1} + (1 - \ell) \phi_j^n \right]. \quad (3.31)
 \end{aligned}$$

For $\phi(\mu, \tau)$ and $U(\mu, \tau)$, i.e., $B_{\theta 1}(r, t)$ and $v_{\theta 1}(r, t)$, the finite difference equations are

$$\begin{aligned}
 \frac{\phi_j^{n+1} - \phi_j^n}{\Delta \tau} = & \tilde{\eta}_j \left\{ \ell (L\phi)_j^{n+1} + (1 - \ell) (L\phi)_j^n - \frac{2m}{\mu_j^2} \left[\ell \psi_j^{n+1} + (1 - \ell) \psi_j^n \right] \right\} \\
 & + \ell \tilde{\eta}_j' \left[\frac{\phi_{j+1}^{n+1} - \phi_{j-1}^{n+1}}{2\Delta \mu} + \frac{\phi_j^{n+1}}{\mu_j} + \frac{m}{\mu_j} \psi_j^{n+1} \right] \\
 & + (1 - \ell) \tilde{\eta}_j' \left[\frac{\phi_{j+1}^n - \phi_{j-1}^n}{2\Delta \mu} + \frac{\phi_j^n}{\mu_j} + \frac{m}{\mu_j} \psi_j^n \right] \\
 & + \frac{1}{\alpha} \left(N_j - \frac{2H_j}{\mu_j} \right) \left[\ell w_j^{n+1} + (1 - \ell) w_j^n \right] \\
 & - F_j \left[\ell U_j^{n+1} + (1 - \ell) U_j^n \right] \\
 & - \frac{N_j}{2\Delta \mu} \left[\ell (\delta_{j+1}^{n+1} - \delta_{j-1}^{n+1}) + (1 - \ell) (\delta_{j+1}^n - \delta_{j-1}^n) \right] \\
 & - N_j' \left[\ell \delta_j^{n+1} + (1 - \ell) \delta_j^n \right], \quad (3.32)
 \end{aligned}$$

$$\frac{\tilde{\rho}_j}{\alpha^2 S^2 \Delta \tau} \left[(LU)_j^{n+1} - (LU)_j^n - \frac{2m}{\mu_j^2} (w_j^{n+1} - w_j^n) \right] =$$

$$\begin{aligned}
 = & F_j \left\{ \ell(L\phi)_j^{n+1} + (1 - \ell)(L\phi)_j^n - \frac{2m}{\mu_j^2} \left[\ell\psi_j^{n+1} + (1 - \ell)\psi_j^n \right] \right\} \\
 & + \frac{N_j}{\alpha} \left[\ell(L\psi)_j^{n+1} + (1 - \ell)(L\psi)_j^n \right] \\
 & + \frac{N'_j}{\alpha\Delta\mu_j} \left[\ell(\psi_{j+1}^{n+1} - \psi_{j-1}^{n+1}) + (1 - \ell)(\psi_{j+1}^n - \psi_{j-1}^n) \right] \\
 & + \left(\frac{2m}{\mu_j} F'_j + \frac{2m^2}{\alpha\mu_j^3} H_j + \frac{N'_j}{\alpha\mu_j} + \frac{N''_j}{\alpha} \right) \left[\ell\psi_j^{n+1} + (1 - \ell)\psi_j^n \right] \\
 & + \left(F''_j + \frac{F'_j}{\mu_j} + \frac{2m}{\alpha\mu_j^2} N_j - \frac{6m}{\alpha\mu_j^3} H_j \right) \left[\ell\phi_j^{n+1} + (1 - \ell)\phi_j^n \right] \\
 & + \frac{1}{\Delta\mu_j} \left(F'_j + \frac{m}{\alpha\mu_j^2} H_j \right) \left[\ell(\phi_{j+1}^{n+1} - \phi_{j-1}^{n+1}) + (1 - \ell)(\phi_{j+1}^n - \phi_{j-1}^n) \right].
 \end{aligned}
 \tag{3.33}$$

As pointed out earlier, Eq. (3.30) for the incompressible case is identical to Eq. (3.22) of the general case, but it has been included again here for completeness.

For both the general case and the incompressible case, a separate equation is required to determine the resistivity δ . The method of evaluating the resistivity simultaneously with the other dependent variables is given in detail in the following sections.

3.3 Normalization and Characteristic Plasma Parameters

In order to determine realistic values for the equilibrium functions and the dissipative coefficients, characteristic values are chosen for the plasma number density n_1 , temperature T_1 , and

magnetic field B. These characteristic values are then used in standard relations to get characteristic values for the resistivity, viscosity, and thermal conductivity. The appropriate characteristic values are then combined to determine the normalization for the dimensionless viscosity Γ , the dimensionless thermal conductivity Λ , the magnetic Reynolds number S , and the dimensionless parameter β . The details of this procedure are discussed in the following paragraphs.

For a fully ionized hydrogen plasma, the ion resistivity transverse to a strong magnetic field is given by⁸

$$\eta_{\perp} = 1.435 \times 10^{-8} \frac{\ln \lambda_c}{T_i^{3/2}} \text{ statohm-cm} \quad (3.34)$$

where T_i is the ion temperature in $^{\circ}\text{K}$ and $\ln \lambda_c$ is the Coulomb logarithm in which λ_c is taken to be⁸

$$\lambda_c = \frac{3}{2e^3} \left(\frac{k_B^3 T_i^3}{m_i} \right)^{1/2} = 1.239 \times 10^4 \frac{T_i^{3/2}}{n_i^{1/2}}, \quad (3.35)$$

with n_i the plasma number density in $\text{\#}/\text{cm}^3$ and e the electronic charge in statcoulombs.

The dynamic viscosity transverse to a strong magnetic field is given by⁸

$$\mu_{\perp} = \frac{2}{5} \left(\frac{\pi}{m_i k_B T_i} \right)^{1/2} \frac{Z^4 e^4 n_i^2 \ln \lambda_c}{\Omega_i^2} \text{ gm/cm sec} \quad (3.36)$$

where the ion cyclotron frequency $\Omega_i = ZeB/m_i c$, so

$$\mu_1 = \frac{2}{5} \sqrt{\pi} \frac{Z^2 e^2 c^2 n_1^2 m_i^{3/2} \ln \lambda_c}{B^2 (k_B T_1)^{1/2}} \text{ gm/cm-sec,} \quad (3.37)$$

where B is in gauss, c is the speed of light in cm/sec, and m_i is the ion mass in grams. For the kinematic viscosity⁹ ν_1 ,

$$\nu_1 = \frac{\mu_1}{\rho_1} = \frac{\mu_1}{m_1 n_1},$$

and using (3.37) for μ_1 ,

$$\nu_1 = \frac{2}{5} \sqrt{\pi} \frac{Z^2 e^2 c^2 n_1 m_i^{1/2} \ln \lambda_c}{B^2 (k_B T_1)^{1/2}} \text{ cm}^2/\text{sec,} \quad (3.38)$$

$$\nu_1 = 1.618 \times 10^{-2} \frac{n_1 \ln \lambda_c}{B^2 T_1^{1/2}} \text{ cm}^2/\text{sec.} \quad (3.39)$$

So in dimensionless form the viscosity is

$$\Gamma \equiv \frac{4\pi \nu_1}{c^2 \langle \eta \rangle} = 1.576 \times 10^{-14} \left(\frac{n_1 T_1}{B^2} \right), \quad (3.40)$$

where (3.34) has been used for $\langle \eta \rangle$.

The thermal conductivity transverse to a strong magnetic field can be written in the form^{8,10,11}

$$K_T = \frac{8}{3} \frac{(\pi k_B m_i)^{1/2} e^2 c^2 n_1^2 \ln \lambda_c}{B^2 T_1^{1/2}} \text{ ergs/sec-deg-cm.} \quad (3.41)$$

But, as discussed in Section 2.1, K_T is related to the form of the thermal conductivity K as used in the energy equation by (2.28),⁵

i.e.

$$K = (\gamma - 1) \frac{m_i}{k_B} K_T ,$$

so

$$K = \frac{8}{3} (\gamma - 1) \sqrt{\pi} \frac{e^2 c^2 n_i^2 m_i^{3/2} \ln \lambda_c}{B^2 (k_B T_i)^{1/2}} , \quad (3.42)$$

or

$$K = 1.804 \times 10^{-25} \frac{n_i^2 \ln \lambda_c}{B^2 T_i^{1/2}} . \quad (3.43)$$

So in dimensionless form the thermal conductivity is

$$\Lambda = \frac{4\pi K}{c^2 \langle \rho \rangle \langle \eta \rangle} = 1.051 \times 10^{-13} (\gamma - 1) \left(\frac{n_i T_i}{B^2} \right) , \quad (3.44)$$

where $\langle \rho \rangle = m_i n_i$ and (3.34) has been used for $\langle \eta \rangle$.

Similarly, the magnetic Reynolds number S as defined in Eq.

(3.8) becomes

$$S = 0.2122 \frac{a B T_i^{3/2}}{n_i^{1/2} \ln \lambda_c} . \quad (3.45)$$

For the results to be presented here, it is further assumed that the equilibrium profiles for the temperature, density, and resistivity have magnitudes equal to their characteristic values; then in dimensionless form,

$$\bar{T}(\mu) = \frac{T_0(r)}{\langle T \rangle} = 1, \quad (3.46)$$

$$\tilde{\rho}(\mu) = \frac{\rho_0(r)}{\langle \rho \rangle} = 1, \quad (3.47)$$

$$\tilde{\eta}(\mu) = \frac{\eta_0(r)}{\langle \eta \rangle} = 1. \quad (3.48)$$

Note that the dimensionless parameters Γ and Λ have the same functional dependence on the number density, temperature, and magnetic field, i.e.

$$\Gamma, \Lambda \sim \frac{n_1 T_1}{B^2}$$

which is the same functional dependence as for the β^3 of a plasma, i.e.

$$\beta = 3.469 \times 10^{-15} \left(\frac{n_1 T_1}{B^2} \right). \quad (3.49)$$

Using $\langle \rho \rangle = m_1 n_1$ and using (3.34) for $\langle \eta \rangle$, the following relationships among Γ , Λ , and β can be established:

$$\frac{\Gamma}{\beta} = 4.543 \quad (3.50)$$

$$\frac{\Lambda}{\beta} = 30.3 (\gamma - 1) \quad (3.51)$$

$$\frac{\Lambda}{\Gamma} = \frac{20}{3} (\gamma - 1). \quad (3.52)$$

The characteristic values and dimensionless plasma parameters which are used in this report are summarized in Table 3.3.

Table 3.3. Characteristic plasma parameters.

Resistivity	$\eta = 1.435 \times 10^{-8} \frac{\ln \lambda_c}{T_1^{3/2}} \text{ statohm-cm}$
Coulomb logarithm collision parameter	$\lambda_c = 1.239 \times 10^4 \frac{T_1^{3/2}}{n_1^{1/2}}$
Viscosity	$\Gamma = 1.576 \times 10^{-14} \frac{n_1 T_1}{B^2}$
Thermal conductivity	$\Lambda = 1.051 \times 10^{-13} (\gamma - 1) \frac{n_1 T_1}{B^2}$
Plasma beta	$\beta = 3.469 \times 10^{-15} \frac{n_1 T_1}{B^2}$
Magnetic Reynolds number	$S = 0.2122 \frac{a B T_1^{3/2}}{n_1^{1/2} \ln \lambda_c}$
Equilibrium temperature profile	$\tilde{T}(\mu) = 1$
Equilibrium density profile	$\tilde{\rho}(\mu) = 1$
Equilibrium resistivity profile	$\tilde{\eta}(\mu) = 1$
$n_1 = \#/\text{cm}^3, T_1 = ^\circ\text{K}, B = \text{gauss}, a = \text{cm}$	

3.4 Method of Solution

It is desired to solve the initial value problem described by the systems of component equations as given in Table 2.5 for the general case and in Table 2.6 for the incompressible case. These same equations are presented in dimensionless form in Table 3.1 and Table 3.2, respectively, and in finite difference form in Eqs. (3.22)-(3.29) for the general case and in Eqs. (3.30)-(3.33) for the incompressible case.

The incompressible case will be discussed first because it is simpler and the code required to implement the numerical solution was developed earlier. The four equations for the incompressible case contain first-order time derivatives and first- and second-order spatial derivatives for the four dependent variables B_{r1} , $B_{\theta 1}$, v_{r1} , v_{z1} . In addition, the left hand side of the v_{r1} and $v_{\theta 1}$ equations contain mixed derivatives which are first-order in time and second-order in space. The character of these linear partial differential equations and their associated boundary conditions require an implicit finite-difference scheme as used to solve coupled difference equations.^{6,7}

Computational solutions using implicit differencing schemes^{5,7,12} are generally more difficult than solutions using explicit differencing schemes, but the implicit schemes usually have some distinct advantages. For example, in a one-dimensional explicit scheme, a perturbation can propagate only one mesh spacing Δr in time Δt ; however, if the differential equations physically allow disturbances to propagate by wave motion or convection

at a speed v , then the condition $v \leq \Delta r / \Delta t$ must be satisfied in order to get physically meaningful computational results. Similarly, if diffusion processes predominate, then for diffusion constant D , the condition $D \leq (\Delta r)^2 / 2\Delta t$ must also be satisfied to get meaningful results. For an implicit differencing scheme, however, a disturbance at some point in the mesh can propagate to any other point in the mesh during one time step Δt , so it might be expected that the above restrictions on Δr and Δt would not have to be satisfied.

In Ref. 6 it is shown for linear differential equations with constant coefficients, when using an appropriate explicit scheme, that the calculations become numerically unstable (i.e., the dependent variables become unbounded as the calculations progress in time) when conditions analogous to the above are violated. However, when using an appropriate implicit differencing scheme, the calculations remain stable independent of the size of the time step. Thus, implicit methods should allow calculations to be carried out with time steps that are larger than those possible with explicit techniques. Even though implicit schemes require more computations at each time step than explicit schemes, the use of a larger Δt in the implicit scheme should result in fewer total time steps to complete a calculation, and a net reduction in total computer time is expected.

Reference 2 describes the details of an implicit differencing technique used to solve an incompressible cylindrical model similar to the model presented here, but including resistivity

and gravitational effects due to convection. In that analysis, the pair of dependent variables B_{r1} and v_{r1} were computed at integral time steps, and the pair $B_{\theta 1}$ and $v_{\theta 1}$ were computed at half integral time steps and the system was solved as two sets of two equations each. In the present analysis, the four component equations are solved simultaneously for B_{r1} , $B_{\theta 1}$, v_{r1} , $v_{\theta 1}$ at integral time steps using the single vector difference equation⁶

$$-A_j X_{j+1}^n + B_j X_j^n - C_j X_{j-1}^n = d_j^n \quad (3.53)$$

for $j = 1, \dots, J - 1$, where A , B , and C are 4×4 matrices, d is a 4-component vector, and

$$X = \begin{bmatrix} B_{r1} \\ B_{\theta 1} \\ v_{r1} \\ v_{\theta 1} \end{bmatrix} \rightarrow \begin{bmatrix} \psi \\ \phi \\ W \\ U \end{bmatrix}. \quad (3.54)$$

The elements of matrices A , B , and C are known functions of position and the elements of the vector d_j^n are functions of X_j^n , δ_j^n , δ_j^{n+1} , δ_{j+1}^n , and δ_{j+1}^{n+1} . Equation (3.53) represents a system of simultaneous equations which has a matrix that is tridiagonal in form, i.e., a matrix which has elements only along the main diagonal and the two nearest adjacent diagonals. The four component Eqs. (3.30)-(3.33) for the incompressible case are arranged in the order

- (1) r component of the magnetic field equation--(3.30),
- (2) r component of the equation of motion--(3.31),

(3) θ component of the magnetic field equation--(3.32),

(4) θ component of the equation of motion--(3.33),

and for this arrangement of equations, the elements of the matrices A, B, C, and the vector d are given in Appendix B.

The system of equations described by the vector difference Eq. (3.53) is linear in that the time-independent matrices A, B, and C depend only on r through the zero-order or equilibrium functions and their derivatives. Consequently, the system can be solved using the familiar algorithm for tridiagonal systems,^{2,5,6,7,12} i.e., let

$$X_j^{n+1} = E_j X_{j+1}^{n+1} + F_j^n, \quad j = 0, 1, 2, \dots, J-1. \quad (3.55)$$

Substituting this algorithm evaluated at X_{j-1}^{n+1} into (3.53) yields

$$X_j^{n+1} = \left[\frac{A_j}{B_j - C_j E_{j-1}} \right] X_{j+1}^{n+1} + \left[\frac{d_j^n + C_j F_{j-1}^n}{B_j - C_j E_{j-1}} \right],$$

and comparison with (3.55) shows that E_j and F_j are determined from the recurrence relations

$$E_j = (B_j - C_j E_{j-1})^{-1} A_j, \quad (3.56)$$

$$F_j^n = (B_j - C_j E_{j-1})^{-1} (d_j^n + C_j F_{j-1}^n), \quad (3.57)$$

$j = 1, \dots, J-1$. For a general set of m equations on a mesh of size J, m x J simultaneous equations must be solved, and in principle an m x J matrix could be inverted at each time step and the inverse matrix then used to obtain the solution at each time step in turn.⁶ In addition to inverting the m x J matrix, this pro-

cedure would require multiplying a matrix by a vector and thus require J multiplications for each mesh point at each time step. However, the use of the algorithm (3.55) takes advantage of the tridiagonal nature of (3.53), and the evaluation of (3.56) and (3.57) requires the inversion of $J \times m$ matrices with a corresponding reduction in computer time. For the four component equations of the incompressible case, evaluation of E_j and F_j^n using (3.56) and (3.57) requires inverting a 4×4 matrix at each mesh point which is done numerically.^{4,13,14}

The boundary conditions (see Sections 2.4 and 3.4) are applied in finite difference form at $r = 0$ to determine E_0 and F_0^n , and at $r = R_w$ to determine X_j . For example, if $\partial X / \partial r = 0$ at $r = 0$, then $E_0 = I$, the identity matrix, and $F_0^n = 0$, the null vector; similarly, if $X = 0$ at $r = 0$, then $E_0 = F_0^n = 0$. For the boundary conditions discussed in Section 2.4, the precise forms of E_0 and F_0^n are given in Section 3.5.

With E_0 and F_0^n known, the computational procedure⁶ involves two sweeps across the mesh at each time step, sweeping first through the mesh from $r = 0$ to $r = R_w$ to calculate E_j and F_j^n using (3.56) and (3.57), then a sweep back through the mesh from $r = R_w$ to $r = 0$ to calculate X_j^{n+1} using (3.55). This cycle is then repeated a predetermined number of time steps or until a purely exponential growing mode has emerged and the growth rate settles down to a constant value.

For the general case which includes compressibility, viscosity, and thermal conductivity, the dimensionless equations are differenced

in the same implicit manner as for the incompressible case, and the finite difference equations are given in Eqs. (3.22)-(3.29). As in the incompressible model, the dependent variables for Eqs. (3.22)-(3.27) of the general case are computed simultaneously at integral time steps with a vector difference equation of the form of Eq. (3.53), but with

$$X = \begin{bmatrix} B_{r1} \\ B_{\theta 1} \\ T_1 \\ v_{r1} \\ v_{\theta 1} \\ v_{z1} \end{bmatrix} \rightarrow \begin{bmatrix} \psi \\ \phi \\ T \\ W \\ U \\ V \end{bmatrix}, \quad (3.58)$$

and where A, B, and C are now 6 x 6 matrices and d is a 6-component vector. The elements of matrices A, B, and C are again known functions of X_j^n , κ_j^n , κ_j^{n+1} , $\kappa_{j\pm 1}^n$, $\kappa_{j\pm 1}^{n+1}$, δ_j^n , δ_j^{n+1} , $\delta_{j\pm 1}^n$, and $\delta_{j\pm 1}^{n+1}$. The numerical algorithm described by Eqs. (3.55)-(3.57) is again used to solve the system of equations, but E_j is now a 6 x 6 matrix and F_j^n is now a 6-component vector. The application of the boundary conditions is performed in the same manner as for the incompressible case. For the general case, the 6 component Eqs. (3.22)-(3.27) are arranged in the order

- (1) r component of the magnetic field equation--(3.22),
- (2) θ component of the magnetic field equation--(3.23),
- (3) energy equation--(3.24),
- (4) r component of the equation of motion--(3.25),

(5) θ component of the equation of motion--(3.26),

(6) z component of the equation of motion--(3.27),

and for this arrangement of equations, the elements of the matrices A, B, C, and the vector d are given in Appendix A.

The mass continuity equation shown in dimensionless form in Table 3.1, Eq. (7) for the perturbed density $\mathcal{R}(\mu, \tau)$, i.e., $\rho_1(r, t)$, is not a diffusion equation so it is solved simultaneously but with a separate difference Eq. (3.28). That is $\mathcal{R}_j^{n+1}$ is first determined explicitly ($\ell = 0$) using

$$\begin{aligned} \mathcal{R}_j^{n+1} = \mathcal{R}_j^n + \frac{I \Delta \tau \tilde{\rho}_j}{2 \alpha \Delta \mu} (w_{j+1}^n - w_{j-1}^n) + \frac{\Delta \tau}{\alpha} (I \frac{\tilde{\rho}_j}{\mu_j} + \tilde{\rho}_j^n) w_j^n \\ + \frac{I \Delta \tau \tilde{\rho}_j}{\alpha} \left(\frac{m}{\mu_j} U_j^n + \kappa_z V_j^n \right) + \Delta \tau (\mathcal{L}_M)_j \end{aligned} \quad (3.59)$$

and this explicit value of $\mathcal{R}_j^{n+1}$ is then used in d_j^n to determine X_j^{n+1} . The new values of w_j^{n+1} , U_j^{n+1} , and V_j^{n+1} are then used with w_j^n , U_j^n , and V_j^n to recompute $\mathcal{R}_j^{n+1}$ implicitly ($\ell = 1/2$), and this implicit value of $\mathcal{R}_j^{n+1}$ is then used to determine $\mathcal{R}_j^{n+1}$ explicitly at the next time step using (3.59) with $n \rightarrow n + 1$. This same procedure is carried out at each time step of the calculation.

In order to test the sensitivity of the calculations to this particular method of computation for $\mathcal{R}_j^{n+1}$, an iteration was performed on $\mathcal{R}_j^{n+1}$ at each time step, i.e., using Eq. (3.28) with $\ell = 1/2$, the latest implicit value of $\mathcal{R}_j^{n+1}$ was used in d_j^n to determine X_j^{n+1} which was in turn used to determine a better implicit value for $\mathcal{R}_j^{n+1}$, etc. This iteration was continued at

each time step until R_j^{n+1} was no longer changing to within a previously specified tolerance.

For sufficiently small time steps it was found that such an iterative method slightly improved the density profiles at very early time steps, but that the resulting profiles for all of the dependent variables and, most importantly, the resulting growth rates were not affected. However, it was also found that performing the iteration on R allowed time steps to be used that were approximately twice as large as that possible without an iteration, i.e., reliable results could not be obtained when the larger time steps were used without performing the iteration on R . Since R is directly proportional to Δt , it was found in practice that reliable results were achieved only if Δt were less than or approximately equal to the convergence tolerance specified for R .

Since each iteration on R is approximately equivalent to adding one time step to the problem, significant computer time can be saved by not performing the iteration, and this was done wherever possible, especially for the larger growth rates which require a smaller total number of time steps to settle down to an exponential growth. For the smaller growth rates, however, it was usually necessary to perform the iteration on R in order to get reliable results.

As described in Section 2.3, the perturbed resistivity $\eta_1(r,t)$ can be computed using a standard resistivity expression. For the model to be used here, a Spitzer⁸ type resistivity expression is chosen, i.e., following (3.34), let

$$\eta_0(r) = \text{const.} [T_0(r)]^{-3/2}, \quad (3.60)$$

then

$$\eta(r,t) = \text{const.} [T_0(r) + T_1(r,t)]^{-3/2}, \quad (3.61)$$

and using (2.42), the perturbed resistivity becomes

$$\eta_1(r,t) = \eta(r,t) - \eta_0(r)$$

$$\eta_1(r,t) = \text{const.} [T_0(r)]^{-3/2} \left\{ \left[1 + \frac{T_1(r,t)}{T_0(r)} \right]^{-3/2} - 1 \right\}. \quad (3.62)$$

Since $T_1 \ll T_0$, the evaluation of η_1 using the form given in (3.62) would require the subtraction of nearly equal quantities, a hazardous procedure.^{13,14} To avoid this problem it is convenient to expand (3.62) in a Taylor series to get

$$\eta_1(r,t) = \text{const.} (T_0^{-3/2}) \left[-\frac{3}{2} \left(\frac{T_1}{T_0} \right) + \frac{15}{8} \left(\frac{T_1}{T_0} \right)^2 - \dots + \dots \right], \quad (3.63)$$

where the notation for the functional dependence on r and t has been omitted. Since $T_1 \ll T_0$, as stated above, second order and higher terms can be omitted, and in practice it is sufficient to compute η_1 using

$$\eta_1(r,t) = \text{const.} \left\{ -\frac{3}{2} \frac{T_1(r,t)}{[T_0(r)]^{5/2}} \right\}. \quad (3.64)$$

In dimensionless form, (3.64) becomes

$$\delta(\mu, \tau) = \frac{3}{2} \frac{\tau(\mu, \tau)}{[\tilde{\tau}(\mu)]^{5/2}}, \quad (3.65)$$

and in finite difference form, (3.65) becomes

$$\delta_j^n = \frac{3}{2} \frac{\tau_j^n}{\tilde{\tau}_j^{5/2}}. \quad (3.66)$$

The computational procedure used to solve for δ_j^n simultaneously with the other dependent variables is described in Section 3.6.

3.5 Boundary and Initial Conditions

The required boundary conditions for both the general case and the incompressible case were described in Section 2.4 in Eqs. (2.54)-(2.60). The boundary conditions at $r = 0$ are used to find the E_0 and F_0^n matrices of the tridiagonal algorithm (3.55), and the boundary conditions at $r = R_w$ are used to determine X_j^n . Differencing equations (2.54)-(2.60) using the points $j = 0$ and $j = 1$, and then using (3.55), the boundary conditions and the matrices E_0 and F_0^n for all t for $r = 0$ are:

$$\underline{m = 0}$$

$$(B_{r1})_0 = (B_{\theta 1})_0 = (v_{r1})_0 = (v_{\theta 1})_0 = 0$$

$$(T_1)_0 = (T_1)_1$$

$$(v_{z1})_0 = (v_{z1})_1$$

$$(\rho_1)_0 = (\rho_1)_1$$

$$\left. \begin{aligned} E_0 &= \begin{bmatrix} 0 & & & 0 \\ & 0 & & \\ & & 1 & \\ & & & 0 \\ 0 & & & & 1 \end{bmatrix} \\ F_0^n &= 0, \text{ the null vector} \end{aligned} \right\} \text{general case}$$

$$\left. \begin{array}{l} E_0 = 0, \text{ the null matrix} \\ F_0^n = 0, \text{ the null vector} \end{array} \right\} \text{incompressible case} \quad (3.67)$$

$m = 1$

$$(B_{r1})_0 = (B_{r1})_1$$

$$(B_{\theta 1})_0 = (B_{\theta 1})_1$$

$$(T_1)_0 = 0$$

$$(v_{r1})_0 = (v_{r1})_1$$

$$(v_{\theta 1})_0 = (v_{\theta 1})_1$$

$$(v_{z1})_0 = 0$$

$$(\rho_1)_0 = 0$$

$$\left. \begin{array}{l} E_0 = \begin{bmatrix} 1 & & & 0 \\ & 1 & & \\ & & 0 & 1 \\ 0 & & & 1 & \\ & & & & 1 \\ & & & & & 0 \end{bmatrix} \\ F_0^n = 0, \text{ the null vector} \end{array} \right\} \text{general case}$$

$$\left. \begin{array}{l} E_0 = \begin{bmatrix} 1 & & 0 \\ & 1 & \\ & & 1 \\ 0 & & & 1 \end{bmatrix} \text{ the identity matrix} \\ F_0^n = 0, \text{ the null vector} \end{array} \right\} \text{incompressible case} \quad (3.68)$$

$m \geq 2$

$$(B_{r1})_0 = (B_{\theta 1})_0 = (T_1)_0 = (v_{r1})_0 = (v_{\theta 1})_0 = (v_{z1})_0 = (\rho_1)_0 = 0$$

$$\left. \begin{array}{l} E_0 = 0, \text{ the null matrix} \\ F_0^n = 0, \text{ the null vector} \end{array} \right\} \begin{array}{l} \text{general case and} \\ \text{incompressible case} \end{array}$$

Note that for all values of m , $F_0^n = 0$, the null vector, for both the general case and the incompressible case.

At the outer wall R_W where $j = J$, the boundary conditions specified in Eqs. (2.57)-(2.60) can be expressed in finite difference form as

$$(B_{r1})_J^n = (v_{r1})_J^n = (T_1)_J^n = (\rho_1)_J^n = 0, \quad (3.69)$$

$$(B_{\theta 1})_J^{n+1} = \frac{-imR_W}{m^2 + k_z^2 R_W^2} \frac{(B_{r1})_{J-1}^n}{\Delta r}, \quad (3.70)$$

$$(v_{\theta 1})_J^{n+1} = \frac{-imR_W}{m^2 + k_z^2 R_W^2} \frac{(v_{r1})_{J-1}^n}{\Delta r}, \quad (3.71)$$

$$(v_{z1})_J^{n+1} = \frac{-ik_z R_W^2}{m^2 + k_z^2 R_W^2} \frac{(v_{r1})_{J-1}^n}{\Delta r}, \quad (3.72)$$

where an explicit differencing has been used by approximating B_{r1} and v_{r1} at the previous time step, and where use has been made of (3.69). In dimensionless form, (3.70)-(3.72) become

$$\phi_J^{n+1} = \frac{m\mu_J}{(m^2 + \kappa_z^2 \mu_J^2)\Delta\mu} \psi_{J-1}^n, \quad (3.73)$$

$$U_J^{n+1} = \frac{m\mu_J}{(m^2 + \kappa_z^2 \mu_J^2)\Delta\mu} W_{J-1}^n, \quad (3.74)$$

$$V_J^{n+1} = \frac{\kappa_z \mu_J^2}{(m^2 + \kappa_z^2 \mu_J^2)\Delta\mu} W_{J-1}^n. \quad (3.75)$$

For the special case of $m = 0$, the boundary conditions at R_W reduce to

$$(B_{r1})_J^n = (B_{\theta 1})_J^n = (T_1)_J^n = (v_{r1})_J^n = (v_{\theta 1})_J^n = (\rho_1)_J^n = 0, \quad (3.76)$$

$$(v_{z1})_J^{n+1} = -\frac{1}{k_z \Delta r} (v_{r1})_{J-1}^n \quad (3.77)$$

The boundary conditions for both the general case and the incompressible case at $r = 0$ and $r = R_W$ are summarized in Table 3.4.

In order to begin the calculation, an initial perturbation must be applied to one of the dependent variables, usually $v_{r1}(r, 0)$. If the method of solving the linearized systems of hydromagnetic equations is consistent with the original assumptions, then the results should be independent of this initial perturbation. A variety of forms for this initial perturbation were tested; for example

$$v_{r1}(r, 0) = - (r - r_0) e^{-C_0 |r - r_0|}, \quad (3.78)$$

where r_0 is an adjustable constant which determines the zero point of the initial profile, and C_0 is an adjustable constant which is used to vary the shape of the initial profile. Using (3.78), various $v_{r1}(r, 0)$ profiles were tested using values of $C_0 = 1, 2, 5, 10, 20$ in combination with $r_0 = R_W, R_W/2, R_W/3, R_W/4$, and some representative profiles are shown in Figs. 3.3(a)-(f). For all combinations of C_0 and r_0 , it was found that the results were independent of the initial perturbation for both $m = 0$ and

Table 3.4. Boundary conditions.

General case		Incompressible case
	$r = 0$	
$B_{r1} = B_{\theta 1} = v_{r1} = v_{\theta 1} = 0$	$m = 0$	$B_{r1} = B_{\theta 1} = v_{r1} = v_{\theta 1} = 0$
$\frac{\partial v_{z1}}{\partial r} = \frac{\partial T_1}{\partial r} = \frac{\partial \rho_1}{\partial r} = 0$		
$E_0 = \begin{bmatrix} 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}, F_0^n = 0$		$E_0 = 0, F_0^n = 0$
	$m = 1$	
$T_1 = v_{z1} = \rho_1 = 0$		
$\frac{\partial B_{r1}}{\partial r} = \frac{\partial B_{\theta 1}}{\partial r} = \frac{\partial v_{r1}}{\partial r} = \frac{\partial v_{\theta 1}}{\partial r} = 0$		$\frac{\partial B_{r1}}{\partial r} = \frac{\partial B_{\theta 1}}{\partial r} = \frac{\partial v_{r1}}{\partial r} = \frac{\partial v_{\theta 1}}{\partial r} = 0$
$E_0 = \begin{bmatrix} 1 & 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 1 & 0 \end{bmatrix}, F_0^n = 0$		$E_0 = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix}, F_0^n = 0$
	$m \geq 2$	
$B_{r1} = B_{\theta 1} = T_1 = v_{r1} = v_{\theta 1} = v_{z1} = \rho_1 = 0$		$B_{r1} = B_{\theta 1} = v_{r1} = v_{\theta 1} = 0$
$E_0 = 0, F_0^n = 0$		$E_0 = 0, F_0^n = 0$

Table 3.4 (continued).

General case	$r = R_w$	Incompressible case
$B_{r1} = B_{\theta1} = T_1 = v_{r1} = v_{\theta1} = \rho_1 = 0$	$m = 0$	$B_{r1} = B_{\theta1} = v_{r1} = v_{\theta1} = 0$
$v_{z1} = \frac{1}{k_z} \frac{\partial v_{r1}}{\partial r} \Big _{r=R_w}$		
$B_{r1} = v_{r1} = T_1 = \rho_1 = 0$	$m \geq 1$	$B_{r1} = v_{r1} = 0$
$B_{\theta1} = \frac{imR_w}{m^2 + k_z^2 R_w^2} \frac{\partial B_{r1}}{\partial r} \Big _{r=R_w}$		$B_{\theta1} = \frac{imR_w}{m^2 + k_z^2 R_w^2} \frac{\partial B_{r1}}{\partial r} \Big _{r=R_w}$
$v_{\theta1} = \frac{imR_w}{m^2 + k_z^2 R_w^2} \frac{\partial v_{r1}}{\partial r} \Big _{r=R_w}$		$v_{\theta1} = \frac{imR_w}{m^2 + k_z^2 R_w^2} \frac{\partial v_{r1}}{\partial r} \Big _{r=R_w}$
$v_{z1} = \frac{ik_z R_w^2}{m^2 + k_z^2 R_w^2} \frac{\partial v_{r1}}{\partial r} \Big _{r=R_w}$		

$m \geq 1$.

Other analytic forms for the initial perturbation were also tested, in particular,¹⁵

$$v_{r1}(r,0) = e^{-C_0} \left[1 - \left(\frac{r}{R_W} \right)^2 \right] \quad \text{for } m = 1 \quad (3.79)$$

$$v_{r1}(r,0) = e^{-C_0} \left(\frac{r}{R_W} \right) \left[1 - \left(\frac{r}{R_W} \right)^2 \right] \quad \text{for } m \neq 1 \quad (3.80)$$

where here C_0 is an adjustable constant used to vary the magnitude of the perturbation and where the shapes of the profiles are independent of C_0 . The shape of the initial perturbation profile for $m = 1$ using Eq. (3.79) is shown in Fig. 3.3(g), and the profile shape for $m \neq 1$ using Eq. (3.80) is shown in Fig. 3.3(h). For values of C_0 ranging from $C_0 = 1$ to $C_0 = 100$, the magnitude of the perturbation varies from approximately 10^{-1} to 10^{-44} , and for this range of values it was again found that the resultant profiles and growth rates were independent of the initial perturbation.

In general it was found that a form of initial perturbation profile as given by Eqs. (3.79) and (3.80) seemed to get the calculations started more smoothly at early time steps, and this form of perturbation is used for most of the equilibrium models discussed in this report. Since the magnitudes of the perturbed profiles grow with time, a value of $C_0 \approx 75$ was used so that for even the largest growth rates encountered, the problem settled down to an exponential growth long before the absolute magnitude

Figure 3.3. Initial perturbation profiles for V_{r1} :
 (a)-(f) using equation (3.78); (g) using
 equation (3.79) for $m = 1$, (h) using
 equation (3.80) for $m \neq 1$.

of the profiles exceeded the limits of the computer.

3.6 Computational Procedure

In order to solve for the perturbed or first order variables, the equilibrium or zero-order variables must be specified as functions of the radius. For the general case, the functions which must be specified are

$$\left(\frac{B_{\theta 0}}{B} \right), \left(\frac{B_{z0}}{B} \right), \left(\frac{T_0}{\langle T \rangle} \right), \left(\frac{\rho_0}{\langle \rho \rangle} \right), \left(\frac{\eta_0}{\langle \eta \rangle} \right), \quad (3.81)$$

and for the incompressible case the required functions are

$$\left(\frac{B_{\theta 0}}{B} \right), \left(\frac{B_{z0}}{B} \right), \left(\frac{\rho_0}{\langle \rho \rangle} \right), \left(\frac{\eta_0}{\langle \eta \rangle} \right), \quad (3.82)$$

on the domain $0 \leq r \leq R_w$. These functions then characterize the equilibrium model. To set up a particular problem for the general case, it is also necessary to specify the parameters

$$a, m, k_z, \gamma, n_1, T_1, B, \quad (3.83)$$

while for the incompressible case, only the following additional parameters are required:

$$a, m, k_z, S. \quad (3.84)$$

With the above quantities, the computer code then prepares the input needed to solve the systems of hydromagnetic equations.

The equilibrium functions and their derivatives are specified in analytic form and computed within the code. For the

general case these include

$$\left(\frac{B_{\theta 0}}{B}\right), \left(\frac{B_{\theta 0}}{B}\right)', \left(\frac{B_{\theta 0}}{B}\right)'', \left(\frac{B_{z0}}{B}\right), \left(\frac{B_{z0}}{B}\right)', \left(\frac{\rho_0}{\langle \rho \rangle}\right), \left(\frac{\rho_0}{\langle \rho \rangle}\right)',$$

$$\left(\frac{T_0}{\langle T \rangle}\right), \left(\frac{T_0}{\langle T \rangle}\right)', \left(\frac{\eta_0}{\langle \eta \rangle}\right), \left(\frac{\eta_0}{\langle \eta \rangle}\right)', \quad (3.85)$$

while the incompressible case requires

$$\left(\frac{B_{\theta 0}}{B}\right), \left(\frac{B_{\theta 0}}{B}\right)', \left(\frac{B_{\theta 0}}{B}\right)'', \left(\frac{B_{\theta 0}}{B}\right)''',$$

$$\left(\frac{B_{z0}}{B}\right), \left(\frac{B_{z0}}{B}\right)', \left(\frac{B_{z0}}{B}\right)'',$$

$$\left(\frac{\rho_0}{\langle \rho \rangle}\right), \left(\frac{\eta_0}{\langle \eta \rangle}\right), \left(\frac{\eta_0}{\langle \eta \rangle}\right)', \quad (3.86)$$

where the primes denote differentiation with respect to the independent spatial variable r . Some specific models for the equilibrium functions are discussed in subsequent chapters.

Given the equilibrium functions and their derivatives, and specifying the additional parameters required for either the general case or the incompressible case, the computational procedure is as follows:

a. Initial Procedure

- (1) Set up the fixed mesh, i.e., determine $\Delta u = \mu_J/J$ and evaluate μ_j for $j = 0, 1, \dots, J$;

(2) compute κ_z , Γ , Λ , S , β ;

(3) compute $\left[(m/\mu_j)H_j + \kappa_z M_j \right]$ for $j = 1, \dots, J-1$, and find the value $\mu_j = R$ where the function passes through zero;

(4) compute $\alpha = \left[(m^2/R^2) + \kappa_z^2 \right]^{1/2}$;

(5) compute $F_j = \frac{1}{\alpha} \left[\frac{m}{\mu_j} H_j + \kappa_z M_j \right]$ and its derivatives

$$F'_j = \frac{1}{\alpha} \left[\frac{m}{\mu_j} \left(H'_j - \frac{H_j}{\mu_j} \right) + \kappa_z M'_j \right]$$

$$F''_j = \frac{1}{\alpha} \left[\frac{m}{\mu_j} H''_j - \frac{2m}{\mu_j^2} \left(H'_j - \frac{H_j}{\mu_j} \right) + \kappa_z M''_j \right]$$

for $j = 1, \dots, J-1$;

(6) compute $N_j = H'_j + \frac{H_j}{\mu_j}$ and its derivatives

$$N'_j = H''_j + \frac{1}{\mu_j} \left(H'_j - \frac{H_j}{\mu_j} \right),$$

$$N''_j = H'''_j + \frac{H''_j}{\mu_j} - \frac{2}{\mu_j^2} \left(H'_j - \frac{H_j}{\mu_j} \right),$$

for $j = 1, \dots, J-1$;

(7) compute the elements of the time independent matrices

A_j , B_j , and C_j for $j = 1, \dots, J-1$;

(8) apply an initial perturbation to establish X_j^0 , and choose

$$R_j^0 = \delta_j^0 = 0, \quad j = 1, \dots, J-1;$$

- (9) use the boundary conditions at $r = R_W$ to set

$$\psi_J, \phi_J, T_J, W_J, U_J, V_J, R_J, \text{ and } \delta_J;$$

- (10) use the boundary conditions at $r = 0$ to set R_0^0 and δ_0^0 ,
and use the boundary conditions on ψ, ϕ, T, W, U, V at
 $r = 0$ in conjunction with the tridiagonal algorithm (3.55)
to evaluate E_0 and F_0^n for all n ;

- (11) compute the inverse matrix $(B_j - C_j E_{j-1})^{-1}$ and then compute E_j
for $j = 1, \dots, J - 1$ using (3.56);

- (12) choose as temporary starting values, $R_j^1 = R_j^0$ and $\delta_j^1 = \delta_j^0$
for $j = 0, \dots, J$.

b. Normal Procedure

- (1) Compute R_j^n explicitly for $j = 1, \dots, J - 1$ using Eq. (3.59)
and apply the boundary condition at $r = 0$ to determine R_0^n ;
- (2) compute d_j^n for $j = 1, \dots, J - 1$ using X_j^n ($j = 1$ to $J - 1$),
 $R_j^n, R_j^{n+1}, \delta_j^n, \delta_j^{n+1}$ ($j = 0$ to J), and use d_j^n to evaluate
 F_j^n ($j = 1$ to $J - 1$) using Eq. (3.57);
- (3) update the boundary conditions on ϕ, U, V , i.e., compute
 ϕ_j^{n+1} using ψ_{j-1}^n in Eq. (3.73), and compute U_j^{n+1} and V_j^{n+1}
using W_{j-1}^n in Eqs. (3.74) and (3.75);
- (4) using the tridiagonal algorithm (3.55), evaluate X_j^n for
 $j = 0, \dots, J - 1$ using E_j and F_j^n ;
- (5) using Eq. (3.28), recompute R_j^n implicitly using $W_j^n, U_j^n,$

$v_j^n, w_j^{n-1}, u_j^{n-1}, v_j^{n-1}, R_j^{n-1}$ for $j = 1, \dots, J-1$ and re-apply the boundary condition at $r = 0$ to determine R_0^n ;

$R_J^n = 0$ for all n and all m ;

- (6) recompute δ_j^n using, in general, $\psi_j^n, \phi_j^n, \tau_j^n$, and R_j^n for $j = 1, \dots, J-1$, and apply the boundary conditions at $r = 0$ and $r = R_w$ to determine δ_0 and δ_J . For the special case of a Spitzer type resistivity as given in Eq. (3.66), δ_0 and δ_J obey the same boundary conditions at the temperature T ;
- (7) if an iteration is to be performed on R , repeat steps (2)-(6) until R_j^n converges to within the specified tolerance;
- (8) using Eq. (3.29), compute P_j^n using R_j^n and τ_j^n for $j = 1, \dots, J-1$, and apply the boundary condition at $r = 0$ to determine P_0^n ; $P_J^n = 0$ for all n and all m ;

Steps (1)-(8) of the normal procedure are repeated at each time step, and a problem is run until it settles down to an exponential growth. The exponential growth rate p is then computed using²

$$p = \frac{(\partial \psi / \partial \tau)}{\psi}, \quad (3.87)$$

or in finite difference form,

$$p_j^{n+1} = \frac{\psi_j^{n+1} - \psi_j^{n-1}}{2\psi_j^n \Delta \tau}. \quad (3.88)$$

In performing the calculations it is necessary to choose Δr so that $p\Delta r \ll 1$. This is not a numerical stability condition but is necessary in order to obtain accurate values for p . In most cases, Δr was varied to determine the most optimum value, i.e., the best Δr to use to obtain the most accurate growth rates for the least amount of computer time.

The computational procedure for the incompressible case is the same as outlined above for the general case except that for the incompressible case only the procedures for the dependent variables ψ , ϕ , W , U , and δ apply.

3.7 $k_z = 0$ Case

When the systems of hydromagnetic equations are examined in detail, a special situation occurs for the case where the axial wave number k_z is zero. Recalling Eq. (3.4),

$$\bar{k} \cdot \bar{B} = \frac{m}{r} B_{\theta 0} + k_z B_{z0} = 0,$$

it can be seen that when m and k_z are both zero, $\bar{k} \cdot \bar{B}$ is identically zero regardless of the equilibrium fields. Thus, $\bar{k} \cdot \bar{B}$ does not have a unique root R , and the method of solution outlined in the preceding sections cannot be applied.

For $k_z = 0$ but $m \neq 0$, $\bar{k} \cdot \bar{B}$ will have a null point provided that $B_{\theta 0}$ is finite and has a null point, and for this situation the hydromagnetic equations for the incompressible case can be routinely solved using the method of solution described previously. For the general case, however, this method of solution cannot be applied for the case with $k_z = 0$, $m \neq 0$ since for this

case the matrix $(B_j - C_j E_{j-1})$ becomes singular and the required inverse cannot be obtained, thus the equations cannot be solved using the tridiagonal algorithm (3.55)-(3.57).

In particular, for $K_z = 0$, and neglecting the source terms, the dimensionless component equations for the general case (Table 3.1) reduce to

$$\frac{\partial \psi}{\partial \tau} = -FW + \tilde{\eta}'(\hat{L}\psi - \frac{2m}{\mu^2} \phi) + \delta(\alpha F' + \frac{2m}{\mu^2} H), \quad (3.89)$$

$$\begin{aligned} \frac{\partial \phi}{\partial \tau} = \frac{1}{\alpha} \left(WH' + H \frac{\partial W}{\partial \mu} \right) + \tilde{\eta}'(\hat{L}\phi - \frac{2m}{\mu^2} \psi) \\ - N'\delta - N \frac{\partial \delta}{\partial \mu} + \tilde{\eta}' \left(\frac{\partial \phi}{\partial \mu} + \frac{\phi}{\mu} + \frac{m}{\mu} \psi \right), \end{aligned} \quad (3.90)$$

$$M' \left[\hat{L}\psi + \frac{m^2}{\mu^2} \psi + \frac{m}{\mu} \left(\frac{\partial \phi}{\partial \mu} - \frac{\phi}{\mu} \right) \right] = 0, \quad (3.91)$$

$$\begin{aligned} M \left[\hat{L}\psi + \frac{m^2}{\mu^2} \psi + \frac{m}{\mu} \left(\frac{\partial \phi}{\partial \mu} - \frac{\phi}{\mu} \right) \right] \\ + M' \left[\frac{\partial \psi}{\partial \mu} + \frac{\psi}{\mu} + \frac{m}{\mu} \phi \right] = 0, \end{aligned} \quad (3.92)$$

$$\frac{m}{\mu} M \left[\frac{\partial \psi}{\partial \mu} + \frac{\psi}{\mu} + \frac{m}{\mu} \phi \right] = 0, \quad (3.93)$$

$$\frac{m}{\mu} H \left[\frac{\partial \psi}{\partial \mu} + \frac{\psi}{\mu} + \frac{m}{\mu} \phi \right] = 0, \quad (3.94)$$

$$\frac{\partial \mathcal{R}}{\partial \tau} = \frac{W}{\alpha} \tilde{\rho}' + \frac{\tilde{\rho}}{\alpha} \left(\frac{\partial W}{\partial \mu} + \frac{W}{\mu} + \frac{m}{\mu} U \right), \quad (3.95)$$

where $\hat{L}$ is the operator defined in Eq. (3.14) but with $\kappa_z = 0$. For non-zero values of m , H , M , and M' , Eqs. (3.93) and (3.94) are identical, and using this result in Eq. (3.92) yields Eq. (3.91). Thus, for $\kappa_z = 0$, the general system of seven equations in the seven unknowns ψ , ϕ , T , W , U , V , and $\mathcal{R}$ reduces to Eqs. (3.89), (3.90), (3.91), and (3.95) in the four unknowns ψ , ϕ , W , and $\mathcal{R}$. The general case with $\kappa_z = 0$ thus effectively eliminates the three dependent variables T , U , and V and thereby generates a singular matrix. In order to test this special case for resistive instability, a separate computer code would be required, and this special case will not be discussed further in this report.

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IV. BESSEL FUNCTION MAGNETIC FIELD MODEL

To begin the numerical study of resistive tearing instabilities in diffuse pinch geometries, an equilibrium magnetic field model is chosen which has a non-trivial analytic solution and which also contains characteristics of actual pinch fields, namely diffuse currents and a reversed B_z . The model is referred to as the Lundquist¹ fields or the Bessel Function model (BFM), and its analytic solution is compared with the numerical results in order to test the applicability of the linearized hydromagnetic equations and their associated computational solution.

Such an equilibrium magnetic field model is also of interest experimentally since similar fields can be set up in the laboratory. For example, in the ZETA device,^{2,3} which is a diffuse pinch with a reversed axial field, the axial and azimuthal currents are not restricted to a thin plasma sheath but are distributed fairly uniformly throughout the plasma. Since the average fields in ZETA can be closely represented⁴ by a force-free Bessel function model (FFBFM), it is desirable to know if such field configurations are stable.

4.1 Description of the Model

The Bessel Function model is a cylindrical force-free hydromagnetic equilibrium model with

$$\begin{aligned} B_{r0} &= 0, \\ B_{\theta 0} &= B_0 J_1(vr), \\ B_{z0} &= B_0 J_0(vr), \end{aligned} \tag{4.1}$$

where J_0 and J_1 are Bessel functions of the first kind, and B_0 and v are constants. The force-free feature of the model is characterized by the vanishing of the Lorentz force, i.e., at equilibrium

$$\frac{\bar{\rho} \times \bar{B}}{c} = \nabla p = 0. \quad (4.2)$$

Using the Maxwell equation $\nabla \times \bar{B} = 4\pi\bar{g}/c$, Eq. (4.2) becomes

$$(\nabla \times \bar{B}) \times \bar{B} = 0, \quad (4.3)$$

and this condition is satisfied by the Bessel function model as given in Eq. (4.1). [To avoid confusion with the Bessel function notation J , in this Chapter the current density will be represented by g .] The magnetic fields of a force-free configuration can also be expressed by

$$\nabla \times \bar{B} = v\bar{B}, \quad (4.4)$$

where v is in general a scalar function of space satisfying

$$\bar{B} \cdot \nabla v = 0, \quad (4.5)$$

and Eqs. (4.4) and (4.5) are satisfied by the BFM of (4.1) with $v = \text{constant}$. The equilibrium profiles for the BFM with $v = 1$ are shown in Fig. 4.1, where it can be seen that the first two zeroes of B_{z0} (i.e., J_0) are $J_{0,1} = 2.405$ and $J_{0,2} = 5.520$, and the first zero of $B_{\theta 0}$ (i.e., J_1) is $J_{1,1} = 3.832$. The conducting plasma boundary R_w is placed at various radii, but in order to keep the equilibrium fields realistic, only one reversal in B_{z0} is considered, and this implies $R_w < 5.52$.

Figure 4.1. Magnetic field profiles for the Bessel function model (BFM).

In order to numerically evaluate the equilibrium fields at all mesh points, it is convenient to use analytic forms to represent $B_{\theta 0}$, B_{z0} and their derivatives. The Bessel functions of the first kind of order β are given by

$$J_{\beta}(r) = \left(\frac{r}{2}\right)^{\beta} \sum_{j=0}^{\infty} \frac{(-1)^j}{j! \Gamma(j + \beta + 1)} \left(\frac{r}{2}\right)^{2j}, \quad (4.6)$$

where Γ is the gamma function. For integer values of β , denoted by $n \geq 0$, Eq. (4.6) becomes

$$J_n(r) = r^n \sum_{m=0}^{\infty} \frac{(-1)^m r^{2m}}{2^{2m+n} m! (n+m)!}, \quad (4.7)$$

and for $n = 0$ and $n = 1$, the first two Bessel functions are

$$J_0(r) = 1 - \frac{(r/2)^2}{(1!)^2} + \frac{(r/2)^4}{(2!)^2} - \frac{(r/2)^6}{(3!)^2} + \dots - \dots, \quad (4.8)$$

$$J_1(r) = \frac{r}{2} - \frac{(r/2)^3}{(1!)(2!)} + \frac{(r/2)^5}{(2!)(3!)} - \frac{(r/2)^7}{(3!)(4!)} + \dots - \dots. \quad (4.9)$$

The equilibrium fields $B_{\theta 0}$ and B_{z0} are determined numerically by evaluating these series for the range of r values $0 < r < R_w$.

Using the following properties of the Bessel functions,

$$J_0'(r) = -J_1(r), \quad (4.10)$$

$$J_1'(r) = J_0(r) - \frac{1}{r} J_1(r), \quad (4.11)$$

the derivatives of the equilibrium fields can be written in terms of J_0 and J_1 , i.e., in dimensionless form,

$$H = \frac{B_{\theta 0}}{B} = C_1 J_1(vr) \quad (4.12)$$

$$H' = C_1 v \left[J_0(vr) - \frac{1}{vr} J_1(vr) \right] = C_1 v J_0(vr) - \frac{H}{r} \quad (4.12a)$$

$$H'' = C_1 v^2 \left[\left(\frac{2}{v^2 r^2} - 1 \right) J_1(vr) - \frac{1}{vr} J_0(vr) \right] \quad (4.12b)$$

$$= \left(\frac{1}{r^2} - v^2 \right) H - \frac{1}{r} H'$$

$$H''' = C_1 v^3 \left[\left(\frac{3}{v^2 r^2} - 1 \right) \left(J_0(vr) - \frac{2}{vr} J_1(vr) \right) \right] \quad (4.12c)$$

$$= \left(\frac{3}{r^2} - v^2 \right) \left(H' - \frac{H}{r} \right)$$

$$M = \frac{B_{z0}}{B} = C_2 J_0(vr) \quad (4.13)$$

$$M' = -C_2 v J_1(vr) \quad (4.13a)$$

$$M'' = -C_2 v^2 \left[J_0(vr) - \frac{1}{vr} J_1(vr) \right] = - \left(v^2 M + \frac{M'}{r} \right), \quad (4.13b)$$

where C_1 and C_2 are arbitrary constants which can be used to scale the magnitudes of the equilibrium fields, and v is a constant which can be used to adjust the zeroes of the Bessel functions. Note that the BFM remains force free as long as $C_1 = C_2$.

Since the equilibrium field is specified by $\vec{B}_0(r) = \hat{\theta} B_{\theta 0}(r) +$

$\hat{z}B_{z0}(r)$, using $\nabla \times \vec{B} = 4\pi \vec{j}/c$, the current profiles for j_θ and j_z are given by

$$j_\theta = -\frac{c}{4\pi} \frac{\partial B_{z0}}{\partial r}, \quad (4.14)$$

$$j_z = \frac{c}{4\pi r} \frac{\partial}{\partial r} (rB_{\theta 0}). \quad (4.15)$$

Taking $C_1 = C_2 = v = 1$, and using (4.10) and (4.11), the special case of the FFBFM yields the current profiles

$$j_\theta(r) = \frac{c}{4\pi} J_1(r) = \frac{c}{4\pi B_0} B_{\theta 0}, \quad (4.16)$$

$$j_z(r) = \frac{c}{4\pi} J_0(r) = \frac{c}{4\pi B_0} B_{z0}, \quad (4.17)$$

i.e., the analytic forms of the θ and z components of the current density have the same shape as the θ and z components, respectively, of the equilibrium magnetic fields which were shown in Fig. 4.1, and the FFBFM has the feature that the ratio of the current density to the magnetic field is constant throughout the plasma.

4.2 Computational Results for the Incompressible Case

Using the force-free Bessel function model for the equilibrium magnetic fields, the regimes of stability/instability as predicted by analytic theory are compared with the computational results for the incompressible case. Growth rates are calculated for various (m, k_z) modes and the results are presented in the form of stability diagrams. Additional parametric studies are

also performed to determine the specific dependence of the growth rates on the position of outer conducting wall R_w , on the axial wave number k_z , and on the magnetic Reynolds number S .

A small perturbation of the form given in Eqs. (3.79) and (3.80) with $C_0 = 75$ is applied to the equilibrium configuration, and the growth rate p is computed from $p = (\partial\psi/\partial\tau)/\psi$, where ψ is the perturbed radial magnetic field. A problem is run until p becomes constant, i.e., until the growth becomes exponential. For a growing mode the perturbed variables increase with time, and for an exponential growth, p becomes a positive constant. For a stable mode the perturbed variables decay with time, and for an exponential decay, p becomes a negative constant. For some stable cases it was observed that the perturbed variables oscillate about the equilibrium values, never attaining an exponential growth or exponential decay. A mesh size of 101 points is used for all the problems to be discussed, and for a point at the center of the mesh, the time dependence of the perturbed radial magnetic field is shown in Fig. 4.2 for a typical growing mode, decaying mode, and oscillatory mode. The magnitudes of the three profiles have been arbitrarily scaled to fit the same plot. It can be seen in Fig. 4.2 that for the mesh point shown, the growing mode and the decaying mode settle down to an exponential growth and exponential decay, respectively. All other points on the mesh also settle down to an exponential growth or decay, but may do so at slightly earlier or later times than the point shown.

For an arbitrarily chosen set of parameters, the growth of a

typical perturbation profile ψ is shown in Fig. 4.3. Except for the first few time steps, the perturbation retains its shape as it grows. For the same problem, the behavior of the growth rate p with time is shown in Figure 4.4, and it can be seen that after some initial fluctuation, the growth rate becomes constant. Since the linearized hydromagnetic model excludes all nonlinear effects, the growth rate will not saturate and the perturbed variables will continue to grow (or decay) at an exponential rate for as long as the calculation is continued.

As described in Chapters II and III, the incompressible case assumes that perturbations in plasma resistivity result only from convection, i.e.,

$$\frac{\partial \eta_1}{\partial t} + (\bar{v}_1 \cdot \nabla) \eta_0 = 0. \quad (4.18)$$

For the results to be discussed in this Chapter, the equilibrium resistivity profile $\eta_0(r)$ is taken to be constant throughout the plasma, i.e., $\tilde{\eta} = 1$. Thus for the incompressible case, $\partial \eta_1 / \partial t = 0$, and if η_1 is initially zero, then η_1 is zero throughout the calculation. The equilibrium density profile $\rho_0(r)$ is also taken to be constant throughout the plasma, i.e., $\tilde{\rho} = 1$, and the characteristic length is taken to be $a = 1$. Since the time scale of the instabilities is short compared with the time during which the equilibrium values change significantly, the source terms are usually not required to maintain the equilibrium and are therefore neglected.

Figure 4.2. Radial magnetic field behavior for a growing mode, a decaying mode and an oscillatory mode.

Figure 4.3. Radial magnetic field profiles for the incompressible BFM with $m = 1$, $R_W = 5$, $k_z = .3$, $S = 100$.

Figure 4.4. Variation of the growth rate p with time for the incompressible BFM with $m = 1$, $R_W = 5$, $k_z = .3$, $S = 100$.

4.2.1 Stability Diagrams

In previous theoretical treatments of resistive tearing instabilities,^{5,6,7} the equilibrium plasma configuration is usually divided into two regions, a narrow resistive layer centered about the radius R where $\bar{k} \cdot B = 0$, or

$$F = mB_{\theta 0} + k_z R B_{z0} = 0, \quad (4.19)$$

in which the plasma can move without bending the field lines, and the rest of the plasma. Inside this layer finite resistivity equations are used, while in the rest of the plasma infinite conductivity or ideal MHD equations are used, and the solutions are joined at the boundaries. The change in the solutions across the resistive layer is formulated in terms of a quantity Δ' defined by

$$\Delta' = \left. \frac{\psi'}{\psi} \right|_{R^+} - \left. \frac{\psi'}{\psi} \right|_{R^-}, \quad (4.20)$$

where ψ is the perturbed radial magnetic field. If $\Delta' < 0$, the equilibrium plasma configuration is stable to the particular mode being examined; if $\Delta' > 0$, the mode is unstable, while the marginal case of $\Delta' = 0$ represents the boundary between stability/instability. In order to more clearly explain the relationship between ideal MHD instabilities and finite resistivity tearing modes for the special case of FFBFM, a brief summary of ideal MHD stability theory will be presented.

Comprehensive stability analyses using ideal MHD theory for cylindrical plasma configurations were first done by Newcomb⁸ and

Suydam.⁹ Their analyses are based on the hydromagnetic energy principle¹⁰ which yields an expression for the change in the potential energy of the system, δW , when it is subjected to a vector displacement ξ . If $\delta W > 0$ for every displacement ξ satisfying the boundary conditions, then the equilibrium configuration is stable; if there is any allowable ξ for which $\delta W < 0$, then the equilibrium configuration is unstable. Small amplitude oscillations about equilibrium are described in terms of the normal modes of the system, which can be written in the form

$$\xi = \xi_0(r) e^{i(m\theta + k_z z)},$$

where m can take any integral value and k_z any real value. For the equilibrium to be stable against all perturbations it must be stable for all modes (m, k_z) independently. It has been shown¹⁰ that it is stable for all modes if it is stable for the modes $m = 0$ as $k_z \rightarrow 0$ and for $m = 1$ for all values of k_z .

The resulting displacement equation in ξ has singular points at values of R where $F = 0$. Ideal MHD theory predicts that if the solution of the ξ equation has a zero before reaching a singular point, then the particular (m, k_z) mode is MHD unstable. Voslamber and Callebaut¹¹ have examined the MHD stability of the FFBFM and found that using the transformation

$$\psi = F\xi, \quad (4.21)$$

the equation for the perturbed radial magnetic field ψ has no singularity (ξ , however, is still singular wherever $F = 0$) and

has solutions in terms of Bessel functions, i.e.,¹¹

$$\psi = \frac{k_z r}{1 - k_z^2} \sqrt{1 - k_z^2} Z_{m-1} \left(r \sqrt{1 - k_z^2} \right) + m Z_m \left(r \sqrt{1 - k_z^2} \right) \quad (4.22)$$

where

$$Z_m(r) = a J_m(r) + b Y_m(r) \quad (4.23)$$

and where a and b are constants and Y_m is a Bessel function of the second kind. Applying regularity at the origin, Voslammer and Callebaut¹¹ combine Eqs. (4.19), (4.21), (4.22), and (4.23) to get

$$\xi = \frac{\frac{k_z r}{1 - k_z^2} \sqrt{1 - k_z^2} J_{m-1} \left(r \sqrt{1 - k_z^2} \right) + m J_m \left(r \sqrt{1 - k_z^2} \right)}{k_z r J_0(r) + m J_1(r)}, \quad (4.24)$$

and for instability ξ must have a zero before its first singularity on the r interval considered. Thus the criteria for MHD stability/instability for the special case of the FFBFM reduces to examining the relative positions of the zeroes of ψ and F .

If $\psi = 0$ at a smaller radius than $F = 0$, then the (m, k_z) mode is MHD unstable; if $\psi = 0$ at a larger radius than $F = 0$, then the (m, k_z) mode is MHD stable. Applying this criteria to the FFBFM, the regimes of both MHD instability and resistive instability are illustrated on the stability diagrams for the various m modes.

For the marginal case ($\Delta' = 0$), it has been shown¹¹ that the ξ equation (4.24) has no zeroes for $k_z^2 > 1$ except at $r = 0$. Therefore, ideal instability can occur only for $k_z^2 < 1$, and this is the range of k_z values which is discussed in the following

stability diagrams.

4.2.1.1 Stability Diagram for $m = 0$

For $m = 0$, the ξ equation (4.24) reduces to

$$\xi = \frac{\sqrt{1 - k_z^2} J_1 \left(r \sqrt{1 - k_z^2} \right)}{k_z - 1 J_0(r)}, \quad (4.25)$$

and since the first zero of $J_0(r)$ is always less than the first zero of $J_1 \left(r \sqrt{1 - k_z^2} \right)$, ξ has a singularity before it has a zero. Therefore, from ideal MHD stability theory, ψ has a zero at a larger r value than F has a zero, and thus the $m = 0$ mode for the FFBFM is MHD stable for all k_z and R_w .

For the finite resistivity case, Gibson and Whiteman¹² have studied the FFBFM and by matching the ideal solutions across the resistive layer have found that the equation for ψ yields for the marginal case ($\Delta' = 0$)

$$J_1 \left(r \sqrt{1 - k_z^2} \right) = 0, \quad (4.26)$$

i.e.

$$r = \frac{J_{1,1}}{\sqrt{1 - k_z^2}} = \frac{3.832}{\sqrt{1 - k_z^2}} \quad (4.27)$$

where $J_{1,1}$ is the first zero of J_1 . Also, for $m = 0$ the finite resistivity condition $F = 0$ reduces to $J_0(r) = 0$, which has its first zero at $J_{0,1} = 2.405$. Thus, the $F = 0$ curve is a straight line at $r = 2.405$, while the $\Delta' = 0$ curve described by (4.27) lies above the $F = 0$ line and has a minimum at $r = 3.832$, $k_z = 0$.

The $F = 0$ and $\Delta' = 0$ curves are shown in Fig. 4.5. Resistive instability requires $\Delta' > 0$, so tearing modes can occur for values of k_z and R_w which lie above the $\Delta' = 0$ curve, i.e., there are resistive instabilities where no ideal or MHD instabilities exist. Note that for $m = 0$ these tearing modes are axisymmetric, i.e., the problem is symmetric in k_z for all R_w since $\vec{k} \cdot \vec{B} = k_z B_{z0}$ has identical roots for $\pm k_z$, and the linearised equations contain k_z only in the form k_z^2 . For $m \neq 0$, however, the problem is not symmetric in k_z since $\vec{k} \cdot \vec{B} = 0$ is not symmetric in k_z .

Using the hydromagnetic model developed for the incompressible case, the (k_z, R_w) region shown in Fig. 4.5 was tested for resistive instabilities, and the numerical growth rates obtained are shown superimposed on Fig. 4.5 for the case $S = 100$. The decimal point in the p value marks the (k_z, R_w) point where that value of p occurs. It can be seen that the numerical growth rates occur only in the region above the $\Delta' = 0$ curve, in agreement with theoretical predictions, and, as predicted, there is a distinct boundary between resistive stability and resistive instability. Moreover, the resistive instabilities completely fill the region above the $\Delta' = 0$ curve.

As stated earlier, in order to keep the equilibrium magnetic fields realistic, only one reversal in B_z is considered. For the BFM this implies $R_w < 5.52$, which from Eq. (4.27) requires $k_z < 0.72$, and this limiting value is indicated by the vertical dotted lines in Fig. 4.5.

4.2.1.2 Stability Diagram for $m = 1$

For the $m = 1$ mode, the singularities in the ξ equation are given by

$$F = k_z r J_0(r) + J_1(r) = 0, \quad (4.28)$$

and the $\psi = 0$ or $\Delta' = 0$ curve is given by

$$J_1\left(r\sqrt{1 - k_z^2}\right) + \frac{k_z}{1 - k_z^2} r\sqrt{1 - k_z^2} J_0\left(r\sqrt{1 - k_z^2}\right) = 0. \quad (4.29)$$

From ideal MHD theory, if ψ has a zero at a smaller radius than F has a zero, then that particular (m, k_z) mode is MHD unstable. For the $m = 1$ case this occurs between the double intersection of the $\Delta' = 0$ and $F = 0$ curves at $(k_z = 0.272, R_w = 3.176)$ and $(k_z = -0.237, R_w = 4.744)$ as shown in Fig. 4.6. Between these two points the $\Delta' = 0$ curve lies below the $F = 0$ curve except at the point $(k_z = 0, R_w = 3.832)$ where the two curves are tangent to each other. Thus there is MHD stability for $R_w < 3.176$, and MHD instability for $R_w > 3.176$ and k_z in the range $-0.237 < k_z < 0.272$.

Resistive instability theory predicts that the tearing mode alone will occur whenever R_w is greater than the value of r at which $\Delta' = 0$ and outside the range of k_z values which are MHD unstable. Testing the (k_z, R_w) region for resistive instabilities for the $m = 1$ mode, the numerical growth rates obtained are shown in Fig. 4.7 for the case $S = 100$. As in the $m = 0$ case, the tearing modes occur only in the region above the $\Delta' = 0$ curve,

Figure 4.5. Stability diagram for the incompressible BFM with $m = 0$, $S = 100$.

Figure 4.6. Intersection region of the $\Delta' = 0$ and $F = 0$ curves for the BFM with $m = 1$.

and there is again a distinct boundary between resistive stability and resistive instability. The region of MHD instability is indicated by the vertical dotted lines at $k_z = -0.237$ and $k_z = 0.272$ in Fig. 4.7, but the numerical growth rates shown are for the resistive tearing modes. Note that resistive instabilities do not develop for $\alpha = [k_z^2 a^2 + (m^2 a^2 / R^2)]^{1/2} > 1$, which is consistent with the sheet pinch results for the tearing mode.⁵

In order to find out how much closer the walls must be moved to the plasma to eliminate resistive instabilities, the minimum in $\Delta' = 0$ curve must be found. A numerical solution of Eq. (4.29) yields a minimum in the $\Delta' = 0$ curve at ($k_z = 0.397$, $R_w = 3.112$). This result differs slightly from the result of Gibson and Whiteman¹² who found a minimum at ($k_z = 0.397$, $R_w = 3.104$). Thus resistive instabilities alone occur in the small region

$$3.112 < R_w < 3.176$$

$$0.272 < k_z < 0.511$$

and the wall position for complete stability from both MHD and resistive tearing modes is reduced from $R_w = 3.176$ to $R_w = 3.112$ for the $m = 1$ mode. Examining this small region more closely, it was found that the computer code could satisfactorily predict resistive tearing modes over such a small region, and for the case $S = 100$, the numerical growth rates obtained in this region are shown in Fig. 4.8.

The nature of the EFM is such that resistivity does not greatly affect the R_w value for overall stability of the $m = 1$ mode. In addition, the small separation between the ψ and F

Figure 4.7. Stability diagram for the incompressible BFM with $m = 1$, $S = 100$.

Figure 4.8. Stability diagram for the incompressible BFM with $m = 1$, $S = 100$ in the region of the minimum of the $\Delta' = 0$ curve.

curves as shown in Fig. 4.6 means that a small change in the equilibrium magnetic field could displace the $\psi = 0$ curve, making the equilibrium magnetic field configuration completely stable to MHD modes.

4.2.2 Parametric Studies

Using the FFBEM in the incompressible case, a variety of parametric studies were performed to investigate the dependence of the growth rates of the resistive instabilities upon the parameters

m = azimuthal wave number,

k_z = axial wave number,

R_w = position of outer conducting wall,

S = ratio of resistive diffusion time to hydromagnetic transit time.

p vs k_z

For both the $m = 0$ and $m = 1$ modes, the dependence of the growth rate p upon k_z is shown in Fig. 4.9 for various wall positions R_w for S fixed at 100. It is observed that for any R_w and k_z , the growth rates for the $m = 1$ mode are much larger than the corresponding growth rates for the $m = 0$ mode, in agreement with analytic theory. Since the $m = 0$ mode is symmetric in k_z , the $m = 0$ growth rates have been reflected through the $k_z = 0$ axis in Fig. 4.9.

There is also a strong dependence of p on k_z , and p has a definite maximum. For the $m = 1$ mode at large R_w , $p_{\max}$ occurs at approximately $k_z = 0.25$, while at smaller R_w , $p_{\max}$ shifts to

slightly higher k_z values, with $p_{\max}$ occurring at approximately $k_z = 0.3$ for $R_w = 3.5$. For the $m = 0$ mode, $p_{\max}$ occurs at approximately $k_z = 0.3$ for both $R_w = 5$ and $R_w = 5.5$. These $p_{\max}$ values are for $S = 100$; slightly different $p_{\max}$ values are obtained for other values of S .

For the $m = 1$ mode at large R_w (see $R_w = 5.5$, Fig. 4.9), the beginnings of a second peak can be detected at negative k_z values. This result is similar to the p vs k_z profiles obtained for the ideal MHD case which is discussed in Section 4.2.4.

p vs R_w

For various k_z values, the dependence of the growth rate p upon the position of the conducting wall R_w is shown in Fig. 4.10 for both the $m = 0$ and $m = 1$ modes for $S = 100$, and it can be seen that the instabilities vanish as the wall is moved closer to the plasma. As was evident in Fig. 4.9, the growth rates for the $m = 1$ mode are generally larger than those for the $m = 0$ mode for corresponding R_w values, and, in addition, the wall must be much closer to the plasma to eliminate the $m = 1$ mode. For the $m = 1$ mode, the larger k_z values (e.g., $k_z = 0.6, 0.7$) show a stronger dependence of p on R_w than the smaller k_z values. For MHD stability it is necessary that the longitudinal flux ϕ_z be positive, and the wall position where ϕ_z goes to zero is marked on Fig. 4.10. Typical profiles for B_{r1} , $B_{\theta1}$, v_{r1} , and $v_{\theta1}$ are shown in Figs. 4.11(a)-4.11(d) for $m = 1$, $R_w = 5.5$, $k_z = 0.4$, and $S = 100$.

S Dependence

The dependence of the growth rate p on the magnetic Reynolds

Figure 4.9. Variation of the growth rate p with k_z for the incompressible BFM with $m = 0, 1$ at various R_w for $S = 100$.

Figure 4.10. Variation of the growth rate p with R_w for the incompressible BFM with $m = 0, 1$ at various k_z for $S = 100$.

Figure 4.11. Perturbed profiles for the incompressible BFM with $m = 1$, $R_w = 5.5$, $k_z = .4$, $S = 100$.

number S has been examined over the range $S = 50$ to $S = 10^5$.

Typical results are shown in Fig. 4.12 for the $m = 0$ mode for two wall positions, $R_w = 5$ and $R_w = 5.5$, each for three values of k_z .

Similar results for the $m = 1$ mode are given in Fig. 4.13 for

$R_w = 3.5$ and in Fig. 4.14 for $R_w = 5$.

Using a linear least squares curve fit over the entire range of S examined, the dependence of p on S was cast in the form

$$p = cS^\zeta, \quad (4.30)$$

where c and ζ are constants. For ordinary diffusion, $\zeta \rightarrow 0$, while for the infinite conductivity case, $\zeta \rightarrow 1$ and $p \sim S$. Since resistive diffusion is faster than ordinary diffusion but slower than the infinite conductivity modes, ζ is in the range $0 < \zeta < 1$ for finite resistivity tearing modes, and for the case of a planar sheet pinch FKR⁵ have found $p \sim S^{2/5}$. For the data in Figs. 4.12-4.14, the least squares curve fits, using Eq. (4.30), yield the following values for c and ζ :

		$R_w = 5$ $\underline{m = 0}$			$R_w = 5.5$		
k_z		0.1	0.3	0.5	0.1	0.3	0.5
c		0.03	0.07	0.03	0.05	0.16	0.11
ζ		0.61	0.53	0.54	0.64	0.54	0.52

		$R_w = 3.5$ $\underline{m = 1}$			$R_w = 5$				
k_z		0.1	0.3	0.5	-0.2	0.2	0.4	0.6	0.7
c		0.002	0.26	0.32	0.08	0.20	0.48	0.37	0.21
ζ		1.05	0.68	0.50	0.8	0.75	0.55	0.48	0.48

Figure 4.12. Variation of the growth rate p with S for the incompressible BFM with $m = 0$, $R_W = 5$, 5.5 at various k_z values.

Figure 4.14. Variation of the growth rate p with S for the incompressible BFM with $m = 1$, $R_W = 5$ at various k_z values.

Figure 4.13. Variation of the growth rate p with S for the incompressible BFM with $m = 1$, $R_W = 3.5$ at various k_z values.

The above results indicate that true resistive tearing seems to occur only at the higher axial wave numbers, i.e., for $k_z \gtrsim 0.3$. For example, for $m = 1$ and $R_w = 3.5$, the $k_z = 0.1$ curve in Fig. 4.13 has $\xi \sim 1$ which resembles MHD instability, while the $k_z = 0.5$ curve has $\xi = 0.5$, which is representative of resistive instability. These observations can be compared with the stability diagram of Fig. 4.7 for the $m = 1$ mode which shows a region of MHD instability at $k_z = 0.1$, and a region of resistive instability alone at $k_z = 0.5$; in fact, resistive instability alone occurs in the region $k_z > 0.272$. Thus the dependence of p on S is related to the stability diagrams previously derived, and there are cases where the resistive tearing modes resemble the ideal MHD modes.

For the cylindrical FFBFM studied here, the value of ξ for the pure tearing modes is somewhat higher than that found for the planar sheet case⁵ ($\xi = 0.4$). This trend is true at least for the range $S = 50$ to $S = 10^5$; however, growth rates $\sim S^{2/5}$ have been reported¹³ for $S > 10^5$.

4.2.3 Higher Mode Numbers

For $m = 2$, the singularities in the ξ equation (4.24) for the FFBFM are given by

$$F = k_z r J_0(r) + 2J_1(r) = 0, \quad (4.31)$$

and the $\Delta' = 0$ curve is given by

$$\frac{k_z r}{1 - k_z^2} \sqrt{1 - k_z^2} J_1\left(r\sqrt{1 - k_z^2}\right) + 2J_2\left(r\sqrt{1 - k_z^2}\right) = 0, \quad (4.32)$$

where J_1 is evaluated using Eq. (4.9), and J_2 is

$$J_2(r) = \frac{1}{2!} \left(\frac{r}{2} \right)^2 \left[1 - \frac{(r/2)^2}{1!(3)} + \frac{(r/2)^4}{2!(3)(4)} - \frac{(r/2)^6}{3!(3)(4)(5)} + \dots \right]. \quad (4.33)$$

The $F = 0$ and $\Delta' = 0$ curves for $m = 2$ are shown on the stability diagram in Fig. 4.15. Since the $F = 0$ curve is below the $\Delta' = 0$ curve for all k_z and R_w , there are no regions of MHD instability. Furthermore, there is MHD stability for all $m \geq 2$ since for higher m values the $\Delta' = 0$ curves are displaced upward more and more and never intersect their corresponding $F = 0$ curves.

The FFBFM is, however, unstable with respect to tearing modes for every value of m . Since the $\Delta' = 0$ curves are displaced upward for higher m values, the resistively unstable region is also displaced upward since resistive instability occurs in the region where $\Delta' > 0$. Resistive growth rates for $m = 2$ have been computed and are shown in Fig. 4.15 for $S = 100$. Comparing the stability diagram of Fig. 4.15 for the $m = 2$ mode with the stability diagram of Fig. 4.7 for the $m = 1$ mode, it can be seen that the resistive instabilities appear over a smaller range of (k_z, R_w) as m increases. Since the equilibrium magnetic fields are not realistic for $R_w > 5.52$, m values greater than 2 are not meaningful for the FFBFM.

A comparison of the growth rates for the $m = 0, 1$, and 2 modes is shown in Fig. 4.16 for $R_w = 5.5$, $S = 100$; the S dependence of these three modes is shown in Fig. 4.17 for $R_w = 5.5$, $k_z = 0.3$. The ζ value for each of the three modes is labeled on

Figure 4.15. Stability diagram for the incompressible BFM with $m = 2$, $S = 100$.

Figure 4.16. Growth rate comparison for the $m = 0, 1, 2$ modes for the incompressible BFM with $R_w = 5.5$, $S = 100$.

Figure 4.17. Growth rate comparison for the $m = 0, 1, 2$ modes for the incompressible BFM with $R_w = 5.5$, $k_z = .3$.

each curve. As predicted analytically, the $m = 1$ modes have the fastest growth rates. Since the $\Delta' = 0$ curve for $m = 2$ lies above the $\Delta' = 0$ curve for $m = 1$, values of $R_w > 3.112$ can be used before $m = 2$ resistive instabilities are encountered, i.e., the higher mode numbers $m \geq 2$ produce a less restrictive stability criterion on R_w .

4.2.4 Infinite Conductivity Case

To further assess the extent of applicability of the numerical solution of the hydromagnetic equations, the FFBFM is used with the assumption of infinite conductivity to see if the resistive equations for the incompressible case reduce to the ideal MHD case. Taking $\eta = 0$, the $m = 0$ and $m = 1$ modes are tested. For the $m = 0$ mode, no instabilities are found for any k_z or R_w , in agreement with ideal MHD theory.¹¹ For the $m = 1$ mode, MHD growth rates are obtained and some typical values are shown on the stability diagram in Fig. 4.18. Except for some very small growth rates at $k_z = 0.3$, and a few anomalously large and unexplained growth rates for $k_z < -0.237$, the numerical growth rates obtained for the ideal MHD case fall within the analytically predicted MHD unstable region (between the vertical dotted lines in Fig. 4.18).

A more detailed examination of the growth rates in the MHD unstable region, and a comparison between the ideal MHD and resistive growth rates is shown in Fig. 4.19 for $S = 100$. For the MHD case, two peaks in the growth rates are evident, positive k_z values being more unstable than negative k_z values,^{13,14} and it is seen that for low values of S the MHD modes can grow much slower

Figure 4.18. Stability diagram for the BFM for the infinite conductivity case with $m = 1$, $S = 100$.

Figure 4.19. Comparison between the growth rates of a resistive tearing mode and an MHD mode for the incompressible BFM with $m = 1$, $R_w = 5.5$, $S = 100$.

than their resistive counterparts.¹³ Also, for the ideal MHD modes $p \sim S$, and this is verified in Fig. 4.20 for $m = 1$, $R_w = 5.5$, $k_z = 0.2$, which yields $p = 0.0112S$.

It has been found¹³ that ideal MHD modes and resistive tearing modes can often be distinguished by examining the profiles of the perturbed radial magnetic field B_{r1} , and such a comparison is given in Fig. 4.21(a). For a resistive mode, the B_{r1} profile does not change sign at the singular point $F = 0$, while the MHD profile does change sign. This technique of differentiation has its limitations, however, since it has been found¹³ that at low values of S the resistivity broadens the B_{r1} profiles of the ideal MHD modes so that the point where B_{r1} changes sign is no longer observable, and MHD modes at low S resemble resistive modes. This is illustrated in Fig. 4.21(b) for $S = 100$ and $S = 1000$ for $m = 1$, $k_z = 0.2$, $R_w = 5$, and the resemblance with Fig. 4.21(a) is evident.

4.3 Computational Results for the General Case¹⁵

For the general case which includes the effects of compressibility, viscosity, and thermal conductivity, the force-free Bessel function model is again used to test the applicability and limitation of the computer model for the examination of resistive instabilities in diffuse pinch geometries. Compared with the computer code for the incompressible case which requires approximately 0.06 sec per time step on a CDC 6600 computer, the computer code for the general case requires approximately 0.25-0.30 sec per time step, depending on the number of iterations on ρ_1 , and a typical problem for the general case requires approximately

Figure 4.20. Variation of the growth rate p with S for the BFM for the infinite conductivity case with $m = 1$, $R_w = 5.5$, $k_z = .2$.

Figure 4.21(a). Comparison between B_{r1} profiles for a resistive tearing mode and an MHD mode for $R_w = 5.5$, $k_z = .2$, $S = 100$.

Figure 4.21(b). Comparison between B_{r1} profiles for $S = 100$, $S = 1000$ for the incompressible BFM with $m = 1$, $R_w = 5$, $k_z = .2$.

5'-10' of CDC 6600 computer time. Since the computer code for the general case requires 4-5 times as much computer time as the computer code for the incompressible case in order to obtain a growth rate, exhaustive parametric studies are usually not possible, so for this and subsequent chapters, only sample results will be presented for the general case.

As described in Section 3.4, in addition to specifying values for m , a , and k_z as required for the incompressible case, the computer code for the general case also requires values for the number density n , temperature T , magnetic field B , and ratio of specific heats γ . Unless otherwise stated, the following values are used for the results presented in this Section: $a = 1$, $n = 10^{15}/\text{cm}^3$, $T = 16.7 \text{ eV}$, $B = 1950 \text{ G}$, $\gamma = 5/3$, $T_0 = \rho_0 = \eta_0 = \text{const}$, and all source terms are zero. These particular values for n , T , and B are chosen to approximate the parameters in a typical diffuse pinch with $\beta \approx 0.18$, as well as to correspond to a value of $S \approx 100$.

Using the FTBFM with the computer code for the general case, it was found that the instabilities occurred generally in the same region of (k_z, R_w) space as was found for the incompressible case, and the same general type of dependence of the growth rate on k_z and R_w was observed. Some typical results for the $m = 0$ and $m = 1$ modes are shown in Figs. 4.22 and 4.23 which are similar to Figs. 4.9 and 4.10, respectively, of the incompressible case. It was found, however, that for the BFM, the computer code for the general case would not produce reliable results for the $m = 1$ mode for $k_z \lesssim 0.1$. Also, in contrast to the incompressible results

Figure 4.22. Variation of the growth rate p with k_z for the BFM for the general case with $m = 0, 1$, $R_W = 5.5$, $n = 10^{15}/\text{cm}^3$, $T = 16.7$ eV, $B = 1950$ gauss.

Figure 4.23. Variation of the growth rate p with R_W for the BFM for the general case with $m = 0, 1$ for $n = 10^{15}/\text{cm}^3$, $T = 16.7$ eV, $B = 1950$ gauss.

shown in Fig. 4.10, the larger growth rates for the general case shown in Fig. 4.23 tend to peak at small positive k_z values and small R_w values which are near the (k_z, R_w) point where $F = \bar{k} \cdot \bar{B} = 0$. This effect, which was not observed for the incompressible case, may possibly be attributed to the fact that the boundary conditions as defined in Section 2.4 are valid for the general case only if $F \neq 0$ near the wall, which is violated for $k_z \lesssim 0.4$, $R_w \sim 3$. For example, for $k_z = 0.4$, F has a zero at $R_w = 3.03$ which is close to R_w values where large growth rates are occurring, while for $k_z = 0.6, 0.7$, F has zeroes at $R_w = 2.89, 2.84$, respectively, which are much smaller than the R_w values where the growth rates occur. This peaking effect is even more pronounced for $k_z < 0.4$ (not shown) for which the roots of $F = 0$ occur at progressively higher R_w values and are even closer to the region with large growth rates.

Referring back to Table 3.3, the dimensionless parameters β , Γ , and Λ are functions of (nT/B^2) , while $S \sim T^{3/2}(n/B^2)^{-1/2}$. Thus, for a given temperature T , the growth rate is unique for a given ratio of (n/B^2) , and the growth rates shown in Figs. 4.22 and 4.23 for $T = 16.7$ eV can also apply to any other plasma configuration with $T = 16.7$ eV and with $(n/B^2) \sim 2.6 \times 10^8$. This generalization is not strictly accurate since S is also a function of the Coulomb logarithm which is in turn a function of both n and T , but the slowly varying logarithmic function does not markedly affect this 'rule of thumb'. Varying n , T , or B individually represents a change in both the value of S and the plasma β which

in turn represents a new plasma configuration which yields new values for the growth rates. Since a particular value of S does not correspond to a unique plasma configuration, it is not possible to make a direct comparison between the results of the general case and the results of the incompressible case.

A test was made to determine if the equations for the general case could be used to examine the incompressible case by taking $\rho_0 = \text{constant}$, $\rho_1 = 0$, $\nabla \cdot \bar{v}_1 = 0$, and by letting γ become large. This approach was tried both with and without the viscosity and thermal conductivity terms, and it was found that reliable results could not be obtained for any of these cases.

It was also noted that when the dissipative terms are set to zero, the equations of motion and the energy equation are no longer second order in space for the dependent variables $\bar{v}$ and T , i.e., the diffusion type terms vanish. Thus, if large spatial gradients in $\bar{v}$ and/or T occur over a distance of the order of the mesh spacing, then accurate results cannot be expected using the present computer model, and for most cases tested using the BFM without one or both of the dissipation terms, meaningful results could not be obtained. It was possible, however, to obtain growth rates for the $m = 0$ mode case in which viscosity is neglected, and Fig. 4.24 compares these growth rates with those of the viscous case. As might be expected, the addition of viscosity slows down the growth rate and in addition the most unstable wave number shifts to a slightly lower value for the viscous case.

For a given ratio of (n/B^2) , S varies with $T^{3/2}$, and the

variation of the growth rate p with S in the incompressible case corresponds to the effect of increasing T or β in the general case. For $n = 10^{15}/\text{cm}^3$ and $B = 1950$ gauss, β is in the range $0.1 \lesssim \beta \lesssim 1$ for $0.01 \lesssim T$ (keV) $\lesssim 0.1$, and the corresponding values for S are in the range $50 \lesssim S \lesssim 1200$. It was found that using the computer code for the general case with the above values for n and B , growth rates for values of $S \gtrsim 200$ required extremely small time steps of the order 10^{-4} - 10^{-5} sec, and corresponding computer times of 30-60 min. i.e., $p\Delta t \lesssim O(10^{-3})$. Thus, only limited results were possible, and some representative values are given in Fig. 4.25 for the $m = 0$ and $m = 1$ modes. A least squares fit using Eq. (4.30) is recorded on the curves.

For the $m = 1$ mode at $k_z = 0.4$, the profiles of the perturbed variables B_{r1} , $B_{\theta 1}$, v_{r1} , $v_{\theta 1}$, v_{z1} , T_1 , η_1 , ρ_1 , and p_1 are shown in Figs. 4.26(a)-(h). For the same set of parameters, comparison of Figs. 4.26(a)-(d) with Figs. 4.11(a)-(d) shows that the resulting profiles for B_{r1} , $B_{\theta 1}$, v_{r1} , and $v_{\theta 1}$ are consistent for both the general case and the incompressible case.

In summary, it was found that the computer models for both the general case and the incompressible case can be used to examine resistive tearing instabilities for reversed field pinch configurations, and that the computed growth rates for the special case of the force-free Bessel function model occur in the regions of (k_z, R_w) space predicted by analytic theory. The inclusion of finite resistivity in the analysis of the FFBFM yields the result that the conducting wall must be placed approximately 2% closer

Figure 4.24. Comparison between growth rates for viscous and non-viscous cases for the BFM for $m = 0$, $R_w = 5.5$, $n = 10^{15}/\text{cm}^3$, $T = 16.7$ eV, $B = 1950$ gauss.

Figure 4.25. Variation of the growth rate p with S , T , and β for the general case with $m = 0, 1$, $R_w = 5.5$, $k_z = .4$, $n = 10^{15}/\text{cm}^3$, $B = 1950$ gauss.

Figure 4.26. Perturbed profiles for the BFM for the general case with $m = 1$, $R_w = 5.5$, $k_z = .4$, $n = 10^{15}/\text{cm}^3$, $T = 16.7 \text{ eV}$, $B = 1950 \text{ gauss}$.

to the plasma in order to ensure stability against resistive tearing modes as well as ideal MHD modes. Since the BFM as used here is a force-free equilibrium, there are no pressure gradients as usually exist in real plasmas, so a more realistic reversed field pinch model which contains a pressure gradient will be discussed in the next chapter.

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V. PITCH AND PRESSURE MAGNETIC FIELD MODEL

5.1 Description of the Model

In general, diffuse pinches have $B_z \sim B_0$, and both shear and conducting walls are usually necessary for stability. In the previous chapter, equilibrium fields of this type were represented by a force-free Bessel function model (FFBFM), and it was found that such fields are unstable to resistive tearing modes in regions where no ideal MHD modes exist. Another force-free model, the force-free paramagnetic model (FFPM), has been examined in detail for MHD stability¹ and found to compare well with experiment.² Like the FBFM, the FFPM is also a solution to the static equilibrium equations for an ideal fluid, and it represents the static equilibrium which would be produced if the transverse component of conductivity were zero and the longitudinal component non-zero. It was found that a slightly modified form of the FFPM could stably support a small plasma pressure.³ Such force-free models are not completely realistic, however, since they do not contain pressure gradients as real plasmas usually do.

For high beta plasmas, D. C. Robinson⁴ used the hydromagnetic energy principle⁵ to examine the MHD stability of diffuse pinch configurations for infinitely conducting plasmas surrounded by perfectly conducting walls. He found that plasmas which have moderately high values of β ($\sim 30-40\%$) with a vacuum region outside the plasma can be stable to all ideal MHD modes provided that the following conditions are satisfied:

- (1) The axial field B_{z0} reverses outside the plasma;

(2) the axial flux $\Phi_z = 2\pi \int_0^{R_W} r B_{z0} dr$ does not reverse;

(3) the radial variation of the pitch $P = r B_{z0} / B_{\theta 0}$ does not have a minimum and decreases away from the magnetic axis ($r = 0$);

(4) the MHD criterion for local stability holds, i.e., the pressure gradient is not negative near the magnetic axis.

Using these criteria, Robinson⁴ constructed a reversed field configuration which has $\beta \sim 31\%$, a pitch that decreases away from the axis, a pressure gradient that vanishes at the axis, and a longitudinal flux which vanishes at $R_W \approx 2.65$. The pitch, pressure, and equilibrium magnetic field profiles for this model are shown in Fig. 5.1 as a function of the normalized radius, and this configuration will be referred to as the pitch and pressure model (PPM). Given the equilibrium magnetic fields, the pressure and pressure gradient are determined from

$$\nabla p = \frac{\bar{J} \times \bar{B}}{c} = \frac{1}{4\pi} [(\nabla \times \bar{B}) \times \bar{B}] \quad (5.1)$$

or

$$\nabla p = - \frac{1}{8\pi} \frac{\partial}{\partial r} [B_{\theta 0}^2 + B_{z0}^2] - \frac{B_{\theta 0}^2}{4\pi r}, \quad (5.2)$$

and

$$p = - \frac{1}{8\pi} [B_{\theta 0}^2 + B_{z0}^2] - \frac{1}{4\pi} \int_0^r dr \frac{B_{\theta 0}^2}{r}. \quad (5.3)$$

Computationally, the pressure gradient is evaluated using Eq.

(5.2), but instead of Eq. (5.3) it is convenient to evaluate the pressure using the finite difference approximation

$$\frac{p_{j+1} - p_j}{\Delta r} = \frac{1}{2} \left[\nabla p|_{j+1} + \nabla p|_j \right], \quad (5.4)$$

which has a truncation error $\epsilon = \frac{1}{12} p_j''' (\Delta r)^2$.

Ideal MHD theory for such reversed field configurations predicts⁴ that a necessary but not sufficient condition for stability is that the pitch $P(R_w) > -3P(0)$, or that $|B_{z0}(R_w)|/B_{z0}(0) \lesssim 0.3$. This implies that at the wall $B_{\theta 0}$ cannot be too small relative to B_{z0} or B_{z0} too strongly reversed, and these conditions are met by the pitch and pressure model of Fig. 5.1. Such fields have also been generated⁶ from a paramagnetic model by including an azimuthal electric field E_θ . Using a numerical procedure developed by Copley and Whiteman,⁷ it has been shown⁴ that the pitch and pressure model is MHD stable for $R_w \lesssim 2.45$, which is close to the point where the axial flux reverses ($R_w \approx 2.65$). For comparison, a similar analysis applied to the FFBFM predicts stability for $R_w < 3.35$ as compared with the actual⁸ stability criterion of $R_w < 3.176$.

The experimental existence for such reversed field configurations which satisfy MHD stability has been pointed out by Robinson and King⁶ who attribute the improved stability in ZETA to the reversed axial B field in the outer region, and stability was found for $\beta \sim 10\%$ in ZETA where the configuration set itself up after instability and wall contact. Bodin, et al.⁹ have experimentally set up reversed field pinches in ZETA which were stable for $\beta \sim 30-40\%$ and which became unstable when a negative pressure

gradient near the axis gave rise to a non-localized mode as theoretically predicted. Other reversed field configurations with even higher β , $\sim 80\%$, have been set up¹⁰ without wall contact or instability and which had lifetimes which corresponded to the classical resistive decay time of the fields. In many diffuse pinch experiments it is believed that the residual instabilities may not be hydromagnetic in origin and may possibly be of the resistive type.^{6,11,12} In this chapter, the pitch and pressure model as detailed in Fig. 5.1 is tested for resistive tearing modes using the computational models described in Chapters II and III for the case where the plasma fills the entire region to the wall.

5.2 Computational Results for the Incompressible Case

Using the PFM model of Fig. 5.1 for the equilibrium magnetic fields, the configuration is examined for resistive tearing modes using the computer model for the incompressible case, and the results are compared with those for the ideal MHD case. As in the case of the FFBFM, the region of instability in (k_z, R_w) space is related to the $F = \bar{k} \cdot \bar{B} = 0$ curves. For the PFM, the $F = 0$ curves for various m values are shown in Fig. 5.2. Except for $m = 0$ for which the $F = 0$ curve is a straight line at $R_w = 1.445$, the $F = 0$ curves are singular in a region which includes the origin $k_z = 0$ and the interval $\pm k_z$, the exact value of k_z , i.e., the width of the region, depending upon the value of m . For $m = 1$, the singular region is $-0.5 \leq k_z \leq 0.5$, while for $m = 2$ the region widens to $-1 \leq k_z \leq 1$, and for higher m values the region widens by

Figure 5.1. B_θ , B_z , pressure and pitch profiles for the Pitch and Pressure model (PPM).

Figure 5.2. $F = \mathbf{k} \cdot \mathbf{B} = 0$ curves for the PPM.

$k_z \rightarrow k_z \pm 0.5$ as $m \rightarrow m + 1$ ($m \neq 0$). Since the computer models developed here require a finite root for $F = 0$, these singular regions cannot be tested for resistive instabilities using these models, and such singular regions will be identified on the stability diagrams presented in this chapter.

For the incompressible case, the equilibrium resistivity and density are chosen to be $\eta_0(r) = \text{constant}$, $\rho_0(r) = \text{constant}$, and the characteristic length is taken to be $'a' = 1$. A small perturbation of the form given in Eqs. (3.79) and (3.80) with $C_0 = 75$ is applied to the equilibrium configuration, and a mesh size of 101 points is used throughout.

5.2.1 Parametric Studies

For the PPM, a variety of parametric studies are performed to determine the specific dependence of the growth rates on the azimuthal wave number m , the axial wave number k_z , the position of the outer conducting wall R_w , the magnetic Reynolds number S , and the value of β .

p vs k_z

For the $m = 0$ mode, the variation of the growth rate p with k_z is shown in Fig. 5.3 for $R_w = 2$ and $R_w = 3$ with S fixed at 100. Since the $m = 0$ mode is symmetric with respect to $\pm k_z$, the growth rates in Fig. 5.3 have been reflected through the $k_z = 0$ axis, and since the equilibrium fields do not change much for $R_w \gtrsim 3$, larger R_w values are not considered. Similar results for the $m = 1$ mode are shown in Fig. 5.4 where the growth rates for the $m = 0$, $R_w = 3$ case are included for comparison (see dotted curves). For $m = 1$,

Figure 5.3. Variation of the growth rate p with k_z for the incompressible PPM with $m = 0$, $S = 100$.

Figure 5.4. Variation of the growth rate p with k_z for the incompressible PPM with $m = 0, 1$ at various R_w for $S = 100$.

the region in which F is singular is shown as a hatched area in Fig. 5.4, and comparison of Fig. 5.4 with the $F = 0$ curves in Fig. 5.2 shows that for $R_w = 2$, the resistive growth rates for the $m = 1$ mode have 2 peaks, one on either side of the upper $F = 0$ curve.

For the pitch and pressure model, MHD theory predicts⁴ that there is ideal stability for $R_w \lesssim 2.45$, so the results shown for $R_w = 2$ represent a regime of resistive instability alone. In addition, ideal theory also predicts that the growth rates for the $m = 1$ mode are larger than the corresponding growth rates for the $m = 0$ mode, and that for $m = 1$ positive wave numbers are more unstable than negative wave numbers, the most unstable wave number being $\sim \frac{1}{3} P(0) \sim 0.67$. It can be seen in Fig. 5.4 that these conditions are also satisfied by the resistive tearing modes. For the $m = 0$ mode, the maximum growth rate $p_{\max}$ occurs at approximately $k_z = \pm 1$, and this maximum is relatively independent of R_w . Slightly larger $p_{\max}$ values are obtained for larger values of S , and this effect will be shown in more detail later in this section. In contrast to the FFBFM for which resistive instabilities do not develop for $\alpha > 1$, for the PFM the unstable regions for the $m = 1$ mode correspond to $0.5 \lesssim \alpha \lesssim 3.0$.

p vs R_w

For various k_z values and with S fixed at 100, the variation of the growth rate p with the position of the conducting wall R_w is shown in Fig. 5.5 for the $m = 0$ mode and in Fig. 5.6 for the $m = 1$ mode. In Fig. 5.6, the growth rates for the $m = 0$ mode

Figure 5.5. Variation of the growth rate p with R_w for the incompressible PPM with $m = 0$, $S = 100$. The maximum wall position for complete MHD stability is denoted by R_1 .

Figure 5.6. Variation of the growth rate p with R_w for the incompressible PPM with $m = 0, 1$ and $S = 100$. The maximum wall position for complete MHD stability is denoted by R_1 .

for the most unstable axial wave number ($k_z = 1$) are included for comparison. Again, it can be seen that the $m = 1$ mode has the larger growth rate, and the wall must be moved much closer to the plasma to eliminate this mode.

For MHD stability the longitudinal flux ϕ_z must be positive, and the wall position where ϕ_z vanishes ($R_w \approx 2.65$) is marked on Figs. 5.5 and 5.6 along with the maximum stable wall position ($R_w \approx 2.45$) as determined numerically by Robinson.⁴ For either value it can be seen that resistive tearing modes occur at much smaller R_w values, with the negative k_z values of the $m = 1$ mode being the last to stabilize. Based on the results shown in Figs. 5.5 and 5.6, complete stability for positive k_z values requires $R_w \approx 1.90$ or approximately 22% smaller than the MHD stability criterion, while complete stability for negative k_z values requires $R_w \approx 1.15$ or approximately 50% closer than the MHD prediction. Thus resistive instabilities occur at R_w values which are predicted to be stable using ideal MHD theory, and resistive instabilities also occur in the MHD unstable region $R_w \gtrsim 2.45$. Compared with the FFBFM for which resistivity does not greatly affect the R_w value for overall stability, resistivity has a greater impact on the stable wall position for the PFM. Typical profiles for the perturbed variables B_{r1} , $B_{\theta1}$, v_{r1} , and $v_{\theta1}$ are shown in Figs. 5.7 (a)-(d) for $m = 1$, $R_w \approx 2$, $k_z = -1.2$, and $S = 100$.

S Dependence

The dependence of the growth rate p on the magnetic Reynolds number S is examined for the approximate range $S = 50$ to $S = 10^5$. Typical results are shown in Fig. 5.8 for the $m = 0$ mode for two wall positions: $R_w = 2$ which corresponds to a region of resistive

Figure 5.7. Perturbed profiles for the incompressible PPM with $m = 1$, $R_w = 2$, $k_z = -1.2$, $S = 100$.

instability alone, and $R_W = 3$ which corresponds to a region with both MHD and resistive instabilities, each for two values of k_z . Similar results for the $m = 1$ mode are given in Fig. 5.9 for $R_W = 2$ and in Fig. 5.10 for $R_W = 3$.

Using the form (4.30), $p = cS^\xi$, a linear least squares curve fit to the data in Figs. 5.8-5.10 yields the following values for c and ξ :

<u>$m = 0$</u>				
$R_W = 2$			$R_W = 3$	
k_z	0.4	1.0	0.4	1.0
c	0.026	0.026	0.074	0.12
ξ	0.88	0.93	0.80	0.80

<u>$m = 1$</u>						
$R_W = 2$			$R_W = 3$			
k_z	0.8	-1.2	-2.4	-1.2	0.8	1.0
c	0.11	0.13	0.07	0.15	0.24	0.15
ξ	0.97	0.82	0.87	0.83	0.97	0.94

The above results for ξ are much larger than that found for the planar sheet case¹³ (i.e., $p \sim S^{2/5}$) and indicate that for the range of S examined, ξ is close to 1, which resembles ideal MHD modes. This seems to be true even for $R_W = 2$ which represents a region of resistive instability alone. As in the case of the FFBM, however, true resistive tearing seems to occur for $S \gtrsim 10^5$, e.g. for the $m = 1$ mode with $k_z = -2.4$ and $R_W = 3$, ξ has the value 0.38 for the range $S = 3 \times 10^4 - 3 \times 10^5$.

Figure 5.8. Variation of the growth rate p with S for the incompressible PPM with $m = 0$.

Figure 5.9. Variation of the growth rate p with S for the incompressible PPM with $m = 1$, $R_W = 2$.

Figure 5.10. Variation of the growth rate p with S for the incompressible PPM with $m = 1$, $R_w = 3$.

Figure 5.11. Variation of the growth rate p with k_z for the incompressible PPM with $m = 0$, $R_w = 3$ for various S values.

As pointed out earlier, the maximum growth rate tends to occur at slightly larger axial wave numbers as the value of S increases. This effect is demonstrated in Fig. 5.11 for the $m = 0$ mode with $R_w = 3$. As is also evident from Fig. 5.11, the instabilities cover a broader range of axial wave numbers as S increases.

Higher Mode Numbers

For higher mode numbers $m \geq 2$, the upper and lower branches of the $F = 0$ curves of Fig. 5.2 become more and more separated as m increases, and this has the effect of displacing the unstable regions to larger (k_z, R_w) values. That is, resistive instabilities will occur over a smaller range of (k_z, R_w) values as m increases, and for a given value of R_w the growth rates of the higher mode numbers $m \geq 2$ become smaller as shown in Fig. 5.12, where the $m = 0, 1$, and 2 modes are compared for $R_w = 3$ and $S = 100$. The singular regions for both $m = 1$ and $m = 2$ have been indicated on Fig. 5.12. Since the equilibrium fields change very little for $R_w \gtrsim 3$, and since the growth rates of the higher modes are extremely small and generally vanish anyway within the range of k_z for which the $m = 0$ and $m = 1$ modes occur, values of $m > 2$ have not been considered. The results indicate that larger R_w values can be used before $m = 2$ resistive instabilities are encountered, i.e., higher mode numbers $m \geq 2$ produce a less restrictive stability criterion on R_w .

β Dependence

All of the previous results discussed in this section have been obtained for the PFM with a value of $\beta = 31\%$, and the equi-

Figure 5.12. Growth rate comparison for the $m = 0$, 1, and 2 modes for the incompressible PPM with $R_w = 3$, $S = 100$.

Figure 5.13. Equilibrium magnetic field profiles for the PPM for various values of β .

librium configuration for this β value was shown in Fig. 5.1. However, similar results can be obtained for other β values by adjusting the equilibrium magnetic field configurations as shown in Figs. 5.13 (a) and (b). These field configurations are adjusted so that the pressure gradient vanishes at $r = 0$ and $r = R_w$. As the equilibrium magnetic field changes, the corresponding equilibrium current profile also changes, and Fig. 5.14 shows the axial current profiles which correspond to the three β values of Fig. 5.13. Since the longitudinal flux Φ_z vanishes at $R_w \approx 2.65$, $R_w = 2$ represents a region of resistive instability alone, and for $R_w = 2$ the variation of the growth rate p with β is given in Fig. 5.15 for the $m = 0, 1$, and 2 modes at the k_z values which represent the most unstable cases.

5.2.2 Infinite Conductivity Case

Making the assumption of infinite conductivity ($\eta = 0$) for the PFM, the $m = 0$ and $m = 1$ modes are tested using the computer model for the incompressible case. For the $m = 0$ mode, no instabilities are found for any k_z or R_w , in agreement with ideal MHD theory. For the $m = 1$ mode, MHD growth rates are obtained and a comparison between the ideal MHD and resistive growth rates is shown in Fig. 5.16 for $R_w = 3$, $S = 100$. The resistive growth rates are found to occur over a much wider range of k_z values than the MHD growth rates. For both cases, however, positive k_z values are found to be more unstable than negative k_z values, as theoretically predicted. At positive k_z values the resistive growth

Figure 5.14. Current density profiles for the PPM for various values of β .

Figure 5.15. Variation of the growth rate p with β for the incompressible PPM with $R_w = 2$, $S = 100$.

Figure 5.16. Comparison between resistive and infinite conductivity growth rates for the incompressible PPM with $m = 1$, $R_W = 3$, $S = 100$.

Figure 5.17. Variation of the growth rate ρ with S for the incompressible PPM for the infinite conductivity case with $m = 1$, $R_W = 2$, $k_z = -0.8$.

rates are comparable to the MHD growth rates, while for the smaller negative k_z values, the ideal MHD modes grow faster than their resistive counterparts. For the ideal MHD modes, $p \sim S$, and this is verified in Fig. 5.17 for $m = 1$, $k_z = -0.08$, and $R_w = 2$, which yields $p = 0.25S$.

As discussed in the previous chapter, ideal MHD modes and resistive tearing modes can often be distinguished by examining the profiles of the perturbed radial magnetic field B_{r1} , and it was shown in Fig. 4.21 that the B_{r1} profile for a resistive mode does not change sign at the singular point where $F = 0$, while the MHD profile does change sign. This effect is also observed for the PFM for which a sign change at $F = 0$ occurs for all infinite conductivity cases tested.

5.2.3 Alternate Boundary Conditions

The boundary conditions used for the incompressible case have been discussed in detail in Chapters II and III. At $r = 0$ the dependent variables are set to zero if $m \neq 1$, while for $m = 1$ the derivatives of the dependent variables are set to zero. At $r = R_w$, the dependent variables are set to zero if $m = 0$, while for $m \neq 0$, $B_{r1} = v_{r1} = 0$ and

$$B_{\theta 1}(R_w, t) = \frac{imR_w}{m^2 + k_z^2 R_w^2} \frac{\partial B_{r1}}{\partial r} \bigg|_{r=R_w} \quad (2.58)$$

$$v_{\theta 1}(R_w, t) = \frac{imR_w}{m^2 + k_z^2 R_w^2} \frac{\partial v_{r1}}{\partial r} \bigg|_{r=R_w} \quad (2.59)$$

A detailed study was performed to examine the effect that the proper

choice of boundary conditions may have on the results for the incompressible case. For this study, an alternate set of boundary conditions was chosen so that for all values of m the dependent variables vanish at $r = R_w$, and the derivatives of the dependent variables vanish at $r = 0$. The original and alternate sets of boundary conditions are listed in Table 5.1.

For the $m = 0$ mode, little difference was found between the two sets of boundary conditions since the perturbed profiles tend to adjust themselves so that their derivatives vanish at $r = 0$ anyway, and at $r = R_w$ the two sets of boundary conditions agree. For the $m = 1$ mode the two sets of boundary conditions are identical at $r = 0$ but not at $r = R_w$, and a definite difference in the results was observed for the two sets. This difference is shown in Fig. 5.18 for the $m = 1$ mode at $S = 100$, where the solid curves are the growth rates for the original set of boundary conditions and the dotted curves are the results for the alternate set. For both positive and negative k_z values, the alternate set of boundary conditions tends to enlarge the (k_z, R_w) region of the resistive instabilities and thus demand a more restrictive criterion on R_w for complete stability. This same effect is displayed in Fig. 5.19 which shows how the growth rates are affected as a function of k_z . The peaks in the growth rates which occur at $R_w = 1.5$ and $R_w = 2$ with the original boundary conditions are no longer evident using the alternate boundary conditions, and the growth rates steadily increase as the k_z values approach the region where $F = 0$ is singular, with the greatest difference between the two

Table 5.1. Original and alternate boundary conditions for the incompressible case.

Original boundary conditions	Alternate boundary conditions
$r = 0$	
$B'_{rl} = B'_{\theta l} = v'_{rl} = v'_{\theta l} = 0$	$m = 1$
$B'_{rl} = B'_{\theta l} = v'_{rl} = v'_{\theta l} = 0$	$m \neq 1$
$B_{rl} = B_{\theta l} = v_{rl} = v_{\theta l} = 0$	$m \neq 1$
$B_{rl} = B_{\theta l} = v_{rl} = v_{\theta l} = 0$	$m \neq 1$
$r = R_w$	
$B_{rl} = B_{\theta l} = v_{rl} = v_{\theta l} = 0$	$m = 0$
$B_{rl} = B_{\theta l} = v_{rl} = v_{\theta l} = 0$	$m = 0$
$B_{rl} = v_{rl} = 0$ $B_{\theta l} = \frac{imR_w}{m^2 + k_z^2 R_w^2} \frac{\partial B_{rl}}{\partial r} \Big _{R_w}$ $v_{\theta l} = \frac{imR_w}{m^2 + k_z^2 R_w^2} \frac{\partial v_{rl}}{\partial r} \Big _{R_w}$	$m \neq 0$
$B_{rl} = B_{\theta l} = v_{rl} = v_{\theta l} = 0$	$m \neq 0$

Figure 5.18. Comparison of growth rates for the two sets of boundary conditions in Table 5.1 for the incompressible PPM with $m = 1$, $S = 100$ at various k_z values. The maximum wall position for complete MHD stability is denoted by R_2 .

Figure 5.19. Comparison of growth rates for the two sets of boundary conditions in Table 5.1 for the incompressible PPM with $m = 1$, $S = 100$ at various R_w values.

sets of boundary conditions occurring near the $F = 0$ curves. This same effect, displayed in Figs. 5.18 and 5.19, was also observed with the FFBFM. Thus, it is seen that the alternate set of boundary conditions allows resistive instabilities to occur in regions which are found to be stable with the original set of boundary conditions. Since the results are quite sensitive to the proper choice of boundary conditions, the boundary conditions must be carefully chosen for the examination of resistive instabilities in diffuse pinches. The original set of boundary conditions are used for all computational results presented in this report.

5.3 Computational Results for the General Case

Using the pitch and pressure model for the equilibrium magnetic field, the computer model for the general case which includes compressibility, viscosity, and thermal conductivity is used to examine the configuration for resistive tearing modes. Unless otherwise stated, the following values are used for the results presented in this section: $\eta_0(r) = \text{constant}$, $\rho_0(r) = \text{constant}$, $a = 1$, $n = 1.7 \times 10^{15}/\text{cm}^3$, $T = 18.5 \text{ eV}$, $B = 2000 \text{ G}$, $\gamma = 5/3$, and all source terms are zero. These particular values for n , T , and B represent the parameters in a typical diffuse pinch with $\beta = 31\%$, and also correspond to a value of $S = 100$.

As found with the FFBFM, using the PPM with the computer code for the general case produces instabilities which occur generally in the same region of (k_z, R_w) space as was found for the incompressible case, and the same general type of dependence of the growth rate on k_z and R_w is observed. Some typical results for the $m = 0$

and $m = 1$ modes are shown in Figs. 5.20 and 5.21, which are similar to Figs. 5.4 and 5.6, respectively, of the incompressible case. In contrast to the FFBFM results, which were shown in Fig. 4.23, the PPM growth rates for the general case shown in Fig. 5.21 do not tend to peak at small positive k_z values and small R_w values. Again assuming the Coulomb logarithm to be nearly constant, the growth rates given in Figs. 5.20 and 5.21 for $T = 18.5$ eV can also apply to any other plasma configuration with $T = 18.5$ eV and with $(n/B^2) \sim 4.25 \times 10^8$.

The same limitations in the computer model for the general case that were detailed in conjunction with the FFBFM in Chapter IV are also found to apply to the PPM. This is not unexpected since both models are quite similar and both represent reversed field configurations. As in the case of the FFBFM, growth rates are obtained for the $m = 0$ mode for the non-viscous case, and Fig. 5.22 shows the comparison between the viscous and non-viscous cases for the PPM. As theoretically predicted,¹⁴ viscosity tends to inhibit the growth rate, and the most unstable wave number shifts to a slightly lower value for the viscous case.

For a given ratio of (n/B^2) , S varies with $T^{3/2}$, and the variation of the growth rate p with S in the incompressible case corresponds to the effect of increasing T or β in the general case. For $n = 1.7 \times 10^{15}/\text{cm}^3$ and $B = 2000$ G, β is in the range $0.1 \lesssim \beta \lesssim 1$ for T in the range $6 \lesssim T$ (eV) $\lesssim 60$, and the corresponding values for S are in the range $20 \lesssim S \lesssim 500$. As with the FFBFM, it is found for the PPM that with the above values for n and B ,

Figure 5.20. Variation of the growth rate p with k_z for the PPM for the general case with $n = 1.7 \times 10^{15}/\text{cm}^3$, $T = 18.5$ eV, $B = 2000$ gauss.

Figure 5.21. Variation of the growth rate p with R_w for the PPM for the general case with $n = 1.7 \times 10^{15}/\text{cm}^3$, $T = 18.5$ eV, $B = 2000$ gauss. The maximum wall position for complete MHD stability is denoted by R_1 .

growth rates for values of $S \gtrsim 200$ require time steps of the order 10^{-4} sec, and $p\Delta t \lesssim (10^{-3})$. For a fixed equilibrium configuration as given in Fig. 5.1, the dependence of the growth rate on β , T , and S is given in Fig. 5.23 for the $m = 0$ and $m = 1$ modes with $R_w = 3$, $k_z = 0.7$. A least squares fit using Eq. (4.30) is recorded on the curves.

For the $m = 1$ mode at $k_z = -1.2$, the profiles of the perturbed variables B_{r1} , $B_{\theta 1}$, v_{r1} , $v_{\theta 1}$, v_{z1} , T_1 , η_1 , ρ_1 , and p_1 are shown in Figs. 5.24 (a)-(h). For the same set of parameters, comparison of Figs 5.24 (a)-(d) with Figs. 5.7 (a)-(d) shows that the profiles for B_{r1} , $B_{\theta 1}$, v_{r1} , and $v_{\theta 1}$ for both the general case and the incompressible case are consistent.

In summary it is found that the computer models for both the general case and the incompressible case can be used to examine resistive tearing instabilities for the reversed field pinch configuration represented by the pitch and pressure model. The inclusion of finite resistivity in the analysis of the PPM indicates that for positive k_z values the conducting wall must be placed approximately 22% closer to the plasma in order to ensure stability against resistive tearing modes as well as ideal MHD modes, and up to 50% closer to ensure stability for negative k_z values. In contrast to the FFBPM for which the resistive instabilities did not develop for $\alpha > 1$, for the PPM the unstable regions for the $m = 1$ mode correspond to $0.5 \lesssim \alpha \lesssim 3.0$.

Figure 5.22. Comparison between growth rates for viscous and non-viscous cases for the PPM for $m = 0$, $R_W = 3$, $n = 1.7 \times 10^{15}/\text{cm}^3$, $T = 18.5 \text{ eV}$, $B = 2000 \text{ gauss}$.

Figure 5.23. Variation of the growth rate p with S , T , and β for the PPM for the general case with $R_W = 3$, $k_z = .7$, $n = 1.7 \times 10^{15}/\text{cm}^3$, $B = 2000 \text{ gauss}$.

Figure 5.24. Perturbed profiles for the PPM for the general case with $m = 1$, $R_w = 2$, $k_z = -1.2$, $n = 1.7 \times 10^{15}/\text{cm}^3$, $T = 18.5 \text{ eV}$, $B = 2000 \text{ gauss}$.

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VI. TOKAMAK MAGNETIC FIELD MODELS

The previous two chapters have dealt with equilibrium magnetic field configurations in which the axial field B_z reversed direction. In this chapter, magnetic field configurations which represent toroidal pinch discharges in which the axial field does not reverse direction are examined for resistive tearing modes. In a toroidal configuration, the magnetic fields are obtained from two currents, a poloidal current creating a toroidal (along z) magnetic field, and a toroidal current creating a poloidal (along θ) magnetic field. The net magnetic field created has the feature that the field lines are displaced as they traverse the torus. The angular displacement of a flux line is the rotational transform ι (iota). In a tokamak configuration, plasma currents instead of external conductors are used to convert a purely toroidal field into a poloidal-toroidal field with the desired value for ι . A low temperature, low density toroidal plasma is usually pulsed by a strong toroidal electric field which drives a toroidal current that produces the poloidal magnetic field and heats the plasma to high temperatures.

Experimental observations of magnetic field perturbations in tokamak discharges have indicated that kink-like magnetohydrodynamic instabilities may exist.¹⁻³ These magnetohydrodynamic modes have been examined by various authors for current-carrying plasmas in tokamak systems. The infinite conductivity kink modes have been discussed by Shafranov,⁴ Rutherford and Furth,⁵ and Sinclair, et al.,⁶ and the finite resistivity tearing mode has

been described by Furth, et al.^{7,8} It has also been shown by Shafranov⁴ that these kink-like instabilities can be reasonably described using cylindrical geometry and neglecting toroidal effects.

In this chapter, the linearized hydromagnetic computer models which have been developed to study resistive instabilities in cylindrical geometry are applied to study specific magnetic field profiles which are applicable to tokamak systems. These magnetic field profiles have been taken from the work of Furth, Rutherford, and Selberg,⁸ who examined the effects of the shape of the profiles on the ranges of instability for the finite-resistivity tearing mode. In the present work, a different computational technique than that used in Ref. 8 is applied to the equilibrium magnetic field profiles, and the results⁹ are compared with those of Ref. 8.

6.1 Description of the Models

The equilibrium magnetic field is given by $\vec{B} = \hat{\theta} B_{\theta}(r) + \hat{z} B_z(r)$. Following Ref. 8, three different B_{θ} profiles are examined. These radial profiles for B_{θ} correspond to increasingly flattened current distributions, and the equilibrium fields are expressed analytically as follows:

$$\text{Peaked Model: } B_{\theta} = \frac{r}{1 + r^2} \quad (6.1)$$

$$\text{Rounded Model: } B_{\theta} = \frac{r}{(1 + r^4)^{1/2}} \quad (6.2)$$

$$\text{Flattened Model: } B_{\theta} = \frac{r}{(1+r^8)^{1/4}} \quad (6.3)$$

From Maxwell's equation, $\nabla \times \vec{B} = (4\pi/c)\vec{J}$,

$$\frac{4\pi}{c} J_z = \frac{dB_{\theta}}{dr} + \frac{B_{\theta}}{r}, \quad (6.4)$$

thus the current distributions for the 3 models are given by

$$\text{Peaked Model: } \frac{2\pi}{c} J_z = \frac{1}{(1+r^2)^2} \quad (6.5)$$

$$\text{Rounded Model: } \frac{2\pi}{c} J_z = \frac{1}{(1+r^4)^{3/2}} \quad (6.6)$$

$$\text{Flattened Model: } \frac{2\pi}{c} J_z = \frac{1}{(1+r^8)^{5/4}} \quad (6.7)$$

The B_{θ} magnetic field profiles for the 3 models are shown in Fig. 6.1, and the corresponding current distributions are shown in Fig. 6.2. These tokamak magnetic field profiles correspond to a plasma with a beta of a few percent and a pitch which is finite on the axis and monotonically increases with r , as shown in the insert in Fig. 6.1

If it is assumed that the magnetic fields represent a force free configuration, then the B_z fields can be determined from $\nabla p = (\vec{J} \times \vec{B})/c = 0$, or since $\vec{B}$ is a function of r only, B_z can be expressed as

Figure 6.1. B_θ profiles for the 3 tokamak models. A typical pitch profile is shown in the insert.

Figure 6.2. Current density profiles for the tokamak models.

$$B_z^2 = -B_\theta^2 - 2 \int_0^r \frac{B_\theta^2}{r} dr. \quad (6.8)$$

For the 3 tokamak models of Fig. 6.1, B_z for the force-free case becomes

$$\text{Peaked Model: } B_z = \sqrt{B_0^2 - 1 + \frac{1}{(1+r^2)^2}} \quad (6.9)$$

$$\text{Rounded Model: } B_z = \sqrt{B_0^2 - \frac{r^2}{1+r^4} - \tan^{-1} r^2} \quad (6.10)$$

$$\text{Flattened Model: } B_z = \sqrt{B_0^2 - \frac{r^2}{(1+r^8)^{1/2}} - 2 \int \frac{r dr}{(1+r^8)^{1/2}}} \quad (6.11)$$

where B_0 is a normalization constant such that $B_z = B_0$ at $r = 0$.

The integral in the flattened model is evaluated numerically.

For all 3 tokamak models and for values of $B_0 > 1$, B_z is a very slowly varying function of r as demonstrated in Fig. 6.3 for the peaked model of Eq. (6.9). Taking B_z to be constant removes the force-free feature of the equilibrium field configuration and introduces a small pressure gradient, and using Eqs. (5.2) and (5.4) the resulting pressure profiles for the 3 models are given in Fig. 6.4. It will be shown in the next section that a constant B_z results in only a small change in the growth rates as compared with the force-free case, so for all results discussed in this chapter, B_z is taken to be constant.

As pointed out in Chapter III, the computer codes require the derivatives of the equilibrium functions; in particular, the

Figure 6.3. B_z profiles for the peaked model for the force free case.

Figure 6.4. Pressure profiles for the tokamak models for $B_z = \text{constant}$.

following functions are required:

$$B_{\theta}, B'_{\theta}, B''_{\theta}, B'''_{\theta}, B_z, B'_z, B''_z.$$

For the 3 tokamak models, the equilibrium fields and their derivatives are:

Peaked Model

$$B_{\theta} = \frac{r}{1+r^2}$$

$$B_z = \sqrt{B_0^2 - 1 + \frac{1}{(1+r^2)^2}}$$

$$B'_{\theta} = \frac{1-r^2}{(1+r^2)^2}$$

$$B'_z = \frac{-2r}{B_z(1+r^2)^3}$$

$$B''_{\theta} = \frac{2r(r^2-3)}{(1+r^2)^3}$$

$$B''_z = \frac{2rB'_z(1+r^2) + 2B_z(5r^2-1)}{B_z^2(1+r^2)^4}$$

$$B'''_{\theta} = \frac{6(6r^2-r^4-1)}{(1+r^2)^4}$$

Rounded Model

$$B_{\theta} = \frac{r}{(1+r^4)^{1/2}}$$

$$B_z = \sqrt{B_0^2 - \frac{r^2}{1+r^4} - \tan^{-1}r^2}$$

$$B'_{\theta} = \frac{1-r^4}{(1+r^4)^{3/2}}$$

$$B'_z = \frac{-2r}{B_z(1+r^4)^2}$$

$$B''_{\theta} = \frac{2r^3(r^4-5)}{(1+r^4)^{5/2}}$$

$$B''_z = \frac{2rB'_z(1+r^4) + 2B_z(7r^4-1)}{B_z^2(1+r^4)^3}$$

$$B'''_{\theta} = \frac{6r^2(14r^4-r^8-5)}{(1+r^4)^{7/2}}$$

Flattened Model

$$\begin{aligned}
 B_{\theta} &= \frac{r}{(1+r^8)^{1/4}} & B_z &= \sqrt{B_0^2 - \frac{r^2}{(1+r^8)^{1/2}} - 2 \int \frac{r dr}{(1+r^8)^{1/2}}} \\
 B'_{\theta} &= \frac{1-r^8}{(1+r^8)^{5/4}} & B'_z &= \frac{-2r}{B_z(1+r^8)^{3/2}} \\
 B''_{\theta} &= \frac{2r^7(r^8-9)}{(1+r^8)^{9/4}} & B''_z &= \frac{2B_z(11r^8-1) + 2rB'_z(1+r^8)}{B_z^2(1+r^8)^{5/2}} = \\
 B'''_{\theta} &= \frac{6r^6(38r^8-r^{16}-21)}{(1+r^8)^{13/4}} & &= \frac{2(11r^8-1)}{B_z(1+r^8)^{5/2}} - \frac{B_z'^2}{B_z}
 \end{aligned}$$

As discussed in Chapter IV, it has been shown⁷ that for the treatment of resistive instabilities, the ideal hydromagnetic equations provide a satisfactory description of the plasma apart from narrow layers about the singular radii r_s at which

$$F = \bar{k} \cdot \bar{B} = \frac{m}{r_s} B_{\theta} + k_z B_z = 0. \quad (6.12)$$

Inside this layer, finite resistivity equations are used, while in the rest of the plasma infinite conductivity or ideal MHD equations are used, and the solutions are joined at the boundaries. In Ref. 8, the 3 tokamak models are examined using this formulation, while in this chapter the 3 models are tested for the case in which the resistive equations hold throughout the plasma. The equilibrium configurations are tested for resistive instabilities in the parameter space about which $F = 0$, and the growth rates are obtained for a variety of mode numbers (m, k_z) and wall positions R_w .

6.2 Computational Results for the Incompressible Case

Using the tokamak models of Fig. 6.1 for the equilibrium magnetic fields, the configurations are examined for resistive tearing modes using the computer model for the incompressible case. In many analyses of toroidal systems, stability diagrams are formulated in terms of the 'safety factor'

$$q = \left| \frac{r B_z}{R B_\theta} \right| = \frac{2\pi}{\iota}, \quad (6.13)$$

where ι is the rotational transform and R is the major radius of the torus. At the singularity r_s , $q(r_s) = m$, since q is related to F by

$$F = \left(\frac{B_z}{r_s} \right) \left(1 - \frac{m \iota}{2\pi} \right). \quad (6.14)$$

For the results to be discussed here, it is convenient to evaluate q at the characteristic length 'a', i.e.,⁸

$$q = \frac{m a B_z(a) B_\theta(r_s)}{r_s B_z(r_s) B_\theta(a)}. \quad (6.15)$$

For given fields B_θ and B_z , the singularity r_s is a function of the wave number k_z , therefore for a given m value, q is a function of k_z , and the growth rates for the 3 tokamak models will be presented in terms of both q and k_z .

For the incompressible case, the equilibrium resistivity and density are chosen to be $\eta_0(r) = \text{constant}$, $\rho_0(r) = \text{constant}$,

and the characteristic length is taken to be ' a ' = 1. A small perturbation of the form given in Eqs. (3.79) and (3.80) with $C_0 = 75$ is applied to the equilibrium configuration, and a mesh size of 101 points is used throughout. Since B_z does not change sign in the tokamak models, only negative k_z values are valid so that F can have a null. Except where specified, the normalization constant for B_z is arbitrarily chosen to be $B_0 = 7$. As will be demonstrated later, this value for B_0 allows a wide range of axial wave numbers (i.e., q) to be examined.

6.2.1 Parametric Studies

Using the 3 tokamak models for the equilibrium magnetic fields in the incompressible case, a variety of parametric studies are performed to determine the specific dependence of the growth rates on the azimuthal wave number m , the axial wave number k_z (or q), the position of the outer conducting wall R_w , the magnetic Reynolds number S , and the magnitude of the axial field B_z .

In pinch geometries without periodicity, the most unstable cases occur for the $m = 1$ mode, so it is usually not necessary to consider higher mode numbers.¹⁰ However, in a toroidal device or in a cylindrical pinch of finite length which has a pitch length comparable with the length of the system, the $m = 1$ mode may be absent due to the periodicity of the system. In these cases, the stability of the system depends on the higher mode numbers. For example, in a toroid of major radius R , the axial wave number k_z has discrete values n/R , $n = 1, 2, \dots$, and $q = P/R$, where P is the pitch. The singularity condition becomes⁴ $nq - m = 0$, so as

q becomes greater than one, at some point the $m = 1$ mode is no longer possible. As shown in Ref. 8, the $m = 1$ kink modes are unstable even with infinite conductivity, so except for a few cases in which the $m = 1$ mode is included for comparison, only modes with $m > 1$ are examined here for resistive instability for the 3 tokamak models.

As in the case of the reversed field models, the region of instability in (k_z, R_w) space is related to the $F = \bar{K} \cdot \bar{B} = 0$ curves. For all 3 models, F cannot have a null for the $m = 0$ mode since the axial field does not reverse direction. For other values of m , the $F = 0$ curves are displaced to regions of progressively larger R_w and larger (negative) $|k_z|$ values as m increases. For the peaked tokamak model, $F = 0$ curves for various m values are shown in Fig. 6.5. Similar results are also obtained for the rounded and flattened models, but for these latter models the $F = 0$ curves reach $R_w = 0$ at slightly different k_z values. In addition, the curves in Fig. 6.5 correspond to the case with $B_z = \text{constant}$, and again the $F = 0$ curves reach $R_w = 0$ at slightly different k_z values for the force-free case, i.e., with B_z as given in Eqs. (6.9)-(6.11).

Pea Model

For the peaked tokamak model, the numerical values obtained for the growth rate p for the $m = 2$ mode with $S = 100$ are shown in Fig. 6.6. The decimal point in the p value marks the (k_z, R_w) point where that value of p occurs. It is seen that the instabilities begin at the large (negative) $|k_z|$ end of the $F = 0$ curve at

Figure 6.5. $F = \bar{k} \cdot \bar{B} = 0$ curves for the peaked tokamak model.

Figure 6.6. Stability diagram for the incompressible peaked model with $m = 2$, $S = 100$.

$k_z \sim -0.29$ and go through a maximum at some smaller $|k_z|$ value, the exact k_z value of the maximum depending on the wall position R_w . For $m = 2$, these growth rate profiles as a function of k_z are presented in Fig. 6.7 for 3 wall positions, and the dependence of the growth rate on R_w is shown in Fig. 6.8 for various k_z values. For the $m = 2$ mode at large R_w values, $p_{\max}$ occurs at $k_z \approx -0.23$, while at smaller R_w values $p_{\max}$ shifts to slightly larger $|k_z|$. Again it is evident that the instabilities vanish as the wall is moved closer to the plasma. Since the equilibrium fields are similar and do not change much for $R_w \gtrsim 3$, $R_w = 6$ is equivalent to having the wall at infinity.

Similar results for the $m = 3$ mode are shown in Fig. 6.9. As observed for the $m = 2$ mode, the $m = 3$ instabilities begin at the large (negative) $|k_z|$ end of the $F = 0$ curve which is now at $k_z \sim -0.43$, i.e., the unstable region is displaced to more negative k_z values as the value of m increases. For the $m = 3$ mode, $p_{\max}$ occurs at $k_z \approx -0.36$ for $R_w = 6$ and $p_{\max}$ shifts to $k_z \approx -0.39$ for $R_w = 2$.

Results for even higher mode numbers $m > 3$ show that the instabilities occur at increasingly larger $|k_z|$ values as the $F = 0$ curves shift to larger $|k_z|$ values. A comparison of the growth rates for the peaked model for modes $m = 2$ through $m = 5$ is given in Fig. 6.10 for $R_w = 4$ (lower curves) and $R_w = 6$ (upper curves). Figure 6.11 shows similar results for $R_w = 2$ where the $m = 1$ mode has been included for comparison. In agreement with theoretical prediction, the growth rates are largest for the $m = 1$ mode for a

Figure 6.7. Variation of the growth rate p with k_z for the incompressible peaked model with $m = 2$, $S = 100$.

Figure 6.8. Variation of the growth rate p with R_w for the incompressible peaked model with $m = 2$, $S = 100$.

Figure 6.9. Variation of the growth rate p with k_z for the incompressible peaked model with $m = 3$, $S = 100$.

Figure 6.10. Variation of the growth rate p with k_z for the incompressible peaked model with $m = 2, 3, 4, 5$ at $R_w = 4$ and $R_w = 6$ for $S = 100$.

Figure 6.11. Variation of the growth rate p with k_z for the incompressible peaked model with $m = 1, 2, 3, 4, 5$ at $R_w = 2$ for $S = 100$.

given value of R_w . The next largest mode is $m = 2$, while for $m \geq 3$ the maximum growth rate for each m value decreases only slightly as m increases.^{6,10} This effect can be seen in Figs. 6.10 and 6.11 for $m \leq 5$, and has been verified for m values up through $m = 10$. In addition, Fig. 6.10 also shows that there is considerable overlap of the azimuthal modes, i.e., more than one mode can occur for a given value of k_z . These effects also are in agreement with the results of Ref. 8. For the peaked model, typical profiles for the perturbed variables B_{r1} , $B_{\theta1}$, v_{r1} , and $v_{\theta1}$ are shown in Figs. 6.12 (a)-(d) for $m = 2$, $k_z = -0.16$, $R_w = 2$, $S = 100$.

Rounded and Flattened Models

For the rounded and flattened models, it is found that both the growth rate profiles and the profiles of the perturbed variables are similar to those of the peaked model, and some representative growth rates for the $m = 2$ mode are shown in Figs. 6.13 and 6.14 for the rounded and flattened models, respectively.

S Dependence

The dependence of the growth rate p on the magnetic Reynolds number S has been examined over the range $S \approx 50$ to $S = 10^5$. Typical results are shown in Fig. 6.15 (a) for the peaked model with $m = 2$ and $R_w = 6$ for various values of k_z . Using the form $p = cS^\xi$, a linear least squares fit to the data in Fig. 6.15 (a) yields values for ξ that are greater than unity, i.e., the S dependence of the growth rate is characteristic of MHD type modes. Furthermore, if the perturbed profiles for B_{r1} are examined, it is found

Figure 6.12. Perturbed profiles for the incompressible peaked model with $m = 2$, $R_w = 2$, $k_z = -0.16$, $S = 100$.

Figure 6.13. Variation of the growth rate p with k_z for the incompressible rounded model with $m = 2$, $S = 100$.

Figure 6.14. Variation of the growth rate p with k_z for the incompressible flattened model with $m = 2$, $S = 100$.

that many of these profiles change sign at or near the r value where $F = 0$, i.e., many of the perturbed profiles resemble MHD modes rather than resistive tearing modes. Using the B_{r1} profiles as a guide, the data presented in Figs. 6.6-6.15 were re-examined in an attempt to identify the instabilities as either resistive type modes or MHD modes. For convenience, B_{r1} profiles which do not change sign between $r = 0$ and $r = R_w$ will be referred to as resistive type profiles, while B_{r1} profiles which do change sign will be referred to as non-resistive type profiles, and for the case in which the profiles change sign at the r value where $F = 0$, the profiles will be referred to as MHD modes.

It is found that for the $m = 2$ mode of the peaked model, resistive tearing profiles occur generally at the smaller $|k_z|$ values, while at the larger $|k_z|$ values the profiles are characteristic of MHD modes. Between these two regions, i.e., for the middle values of $k_z \sim -0.2$, there is a transition region in which the B_{r1} profiles change sign but at an r value that is not near the r value where $F = 0$. For the $m = 3$ mode, the peaked model has non-resistive type profiles which extend throughout most of the unstable region except for $R_w \lesssim 3$, $|k_z| \lesssim 0.34$. For higher mode numbers $m > 3$, the non-resistive type profiles completely fill the unstable region of the peaked model. For the rounded and flattened models, the non-resistive type modes do not cover as much of the unstable region as for the peaked model, and for the higher mode numbers $m \geq 3$, the non-resistive modes do not completely fill the unstable region.

For the S dependence shown in Fig. 6.15 (a) for the peaked model, it is found that for $k_z = -0.24$ the B_{r1} profiles are all of the MHD type, while for $k_z = -0.20$ the profiles are characteristic of the transition region, i.e., they change sign but not near the r value where $F = 0$. For $k_z = -0.16$, the B_{r1} profiles resemble resistive modes except for $S \gtrsim 2000$ where the profiles are of the MHD type, and these MHD type modes at large S could account for ζ being close to unity. Since the flattened model yields mostly resistive type profiles throughout the unstable region, it might be expected that for small ($|k_z|$, R_w) values the S dependence of the flattened model would have ζ values less than unity, and this is verified in Fig. 6.15 (b) where $\zeta = 0.9$ for $k_z = -0.20$ and $\zeta = 0.6$ for $k_z = -0.16$, and all perturbed profiles are of the resistive type.

For the reversed field models it was found that the maximum growth rate occurs at slightly larger $|k_z|$ values as the value of S increases. This same effect is also observed for the tokamak models and is displayed in Fig. 6.16 for the flattened model with $m = 2$, $R_w = 2$. In contrast with the reversed field models, however, the instabilities do not cover a broader range of axial wave numbers as S increases.

B_z Dependence

The normalization constant B_0 is used to adjust the magnitude of the axial field B_z . For a given value of B_0 , the $F = 0$ curves shift to higher $|k_z|$ values as the value of m increases, and this effect was shown in Fig. 6.5 for $B_0 = 7$. For most of the results

Figure 6.15(a). Variation of the growth rate p with S for the incompressible peaked model with $m = 2$, $R_W = 6$.

Figure 6.15(b). Variation of the growth rate p with S for the incompressible flattened model with $m = 2$, $R_W = 2$.

Figure 6.16. Variation of the growth rate p with k_z for the incompressible flattened model with $m = 2$, $R_W = 2$ for various s values.

discussed in this chapter, a value of $B_0 = 7$ is chosen so that the instabilities can be examined over a wide range of k_z wave numbers since the instabilities occur at the large $|k_z|$ end of the $F = 0$ curve. For a given value of m and for $B_z = \text{constant}$, the $F = 0$ curves shift to smaller $|k_z|$ values as the value of B_0 increases, and this effect is shown in Fig. 6.17 for the peaked model for the $m = 2$ mode. That is, increasing the value of B_0 narrows the unstable region for all values of m . Similar $F = 0$ curves are obtained for the rounded and flattened models, except that for these models the $F = 0$ curves approach $R_w = 0$ at slightly different k_z values. Similar curves are also obtained for the force-free case except that F does not have a root for $B_0 = 1$ for $m = 1$ or $m = 2$.

The effect on the growth rates of changing the normalization constant B_0 is shown in Fig. 6.18 for the flattened model with $R_w = 2$, $k_z = -0.16$, $S = 100$. As $|B_z|$ is increased, the growth rates at a given m value pass through a maximum and eventually disappear, giving way to instabilities at higher m values. As was observed with the S dependence for the flattened model for these same (k_z, R_w) values, all of the B_{r1} profiles obtained for the results shown in Fig. 6.18 are of the resistive type.

Force-Free Case

For the force-free case, B_z is not constant and is evaluated analytically for the 5 tokamak models using Eqs. (6.9)-(6.11). For the peaked model with $m = 2$, $S = 100$, the difference between the growth rates for the force-free case and for the case with constant B_z is shown in Fig. 6.19, where the solid curves represent

Figure 6.17. $F = \bar{k} \cdot \bar{B} = 0$ curves for the peaked model with $B_z = \text{constant}$, $m = 2$ for various B_0 values.

Figure 6.18. Variation of the growth rate p with $|B_z|$ for the incompressible peaked model with $k_z = -.16$, $R_w = 2$, $S = 100$.

Figure 6.19. Comparison between growth rates for the force free case (dotted curves) and the constant B_z case (solid curves) for the peaked model with $m = 2$, $S = 100$.

the constant B_z case and the dotted curves represent the force-free case. It can be seen that the introduction of a small pressure gradient increases the growth rates and slightly increases the size of the unstable region, in agreement with earlier analyses.⁶ Similar results are also obtained for the rounded and flattened models.

6.2.2 Comparison of Models

A comparison of the growth rates for the 3 tokamak models is given in Fig. 6.20 for the $m = 1$ mode with $R_w = 2$ and $S = 100$. It can be seen that the growth rates for the flattened model are higher than the corresponding growth rates of the rounded model, which are in turn higher than those of the peaked model, and for all 3 models the instabilities occur over the same range of k_z values. Similar results for the $m = 2$ and $m = 3$ modes are given in Fig. 6.21 where this same qualitative difference occurs, except that for the $m = 3$ mode the growth rates for the peaked and rounded models are now comparable. Results for higher mode numbers $m > 3$ are similar to those for $m = 3$, except that the magnitudes of the growth rates become smaller as m increases. Similar results also hold for other wall positions R_w .

For all 3 models for a given R_w value, the $m = 1$ mode has the largest growth rate as expected. The $m = 2$ mode is the next most dominant mode for the rounded and flattened models, while for the peaked model the maximum growth rates are comparable for $m = 2$ through 5. For the flattened model, the maximum growth rates are comparable for $m = 5$ through $m = 10$. For $m = 2$ through $m = 5$,

Figure 6.20. Comparison of growth rates for the 3 tokamak models for $m = 1$, $R_W = 2$, $S = 100$.

Figure 6.21. Comparison of growth rates for the 3 tokamak models for $m = 2$ and $m = 3$ with $R_W = 2$, $S = 100$.

the maximum growth rates of the peaked model occur at increasingly larger $|k_z|$ values as R_w decreases. In the rounded model, this same effect occurs for $m = 3$ through $m = 5$, but for the $m = 2$ mode, the maximum growth rates occur at slightly smaller $|k_z|$ values as R_w decreases. In the flattened model for $m = 4$ and $m = 5$, the maximum p values shift to larger $|k_z|$ values as R_w decreases, as in the other 2 models, but for $m = 2$ and $m = 3$ modes the maximum p 's shift to smaller $|k_z|$ values as R_w decreases.

It has been shown⁸ that the $m = 1$ kink mode is unstable even for the infinite conductivity case, but examination of the B_{r1} profiles for the results given in Fig. 6.20 shows that the B_{r1} profiles are of the resistive type for $|k_z| < 0.06, 0.08, 0.10$ for the peaked, rounded, and flattened models, respectively. At larger $|k_z|$ values, the B_{r1} profiles are of the non-resistive and/or MHD type.

In all 3 models for all m values tested, the growth rates decrease as the outer conducting wall is moved closer to the plasma. The variation of the growth rate p with R_w is given in Fig. 6.22 for the $m = 1$ mode with $k_z = -0.04$, and in Fig. 6.23 for the $m = 2$ mode with $k_z = -0.20$. For the $m = 1$ mode the growth rates vanish at approximately the same R_w value in all 3 models, and for the growth rates presented in Fig. 6.22, resistive type B_{r1} profiles are obtained only for $R_w \lesssim 2$. For the results shown in Fig. 6.23 for the $m = 2$ mode, the growth rates again vanish at approximately the same R_w value in all 3 models, but now all of

the B_{r1} profiles are of the resistive type.

In the preceding comparison of the 3 tokamak models, only results for the case with $S = 100$ have been presented. For the $m = 2$ mode at $k_z = -0.16$ and $R_w = 2$, the S dependence for all 3 models is shown in Fig. 6.24. The ζ value for each of the 3 models is listed on the curves. For the range $S \lesssim 10^4$, the growth rates are again largest for the flattened model and smallest for the peaked model, but for $S \gtrsim 10^4$ the growth rates for the flattened model become smaller than those of the peaked model. For the peaked model, $\zeta \sim 1$ which is characteristic of ideal MHD modes, but the B_{r1} profiles are characteristic of resistive modes. However, recall that the peaked model has a smaller region of instability which is characterized by resistive type profiles as compared with the other 2 models, and it may be possible that the resistivity has broadened the MHD type profiles for this case. For the rounded and flattened models, the B_{r1} profiles are of the resistive type for the entire S range shown, and the ζ values are now more representative of resistive tearing modes. For $S \gtrsim 10^4$, the ζ value becomes $\zeta = 0.57$ for the rounded model and $\zeta = 0.42$ for the flattened model, and these ζ values are close to that of the planar sheet case ($\zeta = 0.4$), i.e., true resistive tearing seems to occur for $S \gtrsim 10^4$ for the rounded and flattened models.

6.3 General Case

For the general case which includes the effects of compressibility, viscosity, and thermal conductivity, the 3 tokamak equi-

Figure 6.22 Variation of the growth rate p with R_w for the three tokamak models with $m = 1$, $k_z = -.04$, $S = 100$.

Figure 6.23 Variation of the growth rate p with R_w for the 3 tokamak models with $m = 2$, $k_z = .20$, $S = 100$.

Figure 6.24. Variation of the growth rate p with S for the 3 tokamak models with $m = 2$, $R_w = 2$, $k_z = -.16$.

librium configurations were used to test the computer model for the examination of resistive instabilities in toroidal pinch geometries. For typical tokamak systems, the equilibrium configurations are characterized by $n \sim 10^{13}$ - $10^{14}/\text{cm}^3$, $T \sim 100$ eV-1 keV, $B \sim 10$ -30 kG, and betas of a few percent. For these parameter ranges, the minimum magnetic Reynolds number is $S \sim 20\,000$. As was pointed out in connection with the reversed field models, problems with $S \gtrsim 200$ require extremely small time steps in order to obtain either a growth rate or the profiles for the perturbed variables. In particular, it was found that large values of S required Δt such that $S\Delta t \sim 10^{-2}$. For the tokamak models with a minimum value of $S \sim 20\,000$, a time increment of $\Delta t \sim 0(10^{-6})$ is required, and this would result in prohibitive amounts of computer time. Therefore, it has not been possible to obtain any results for the general case for the tokamak type profiles.

6.4 Summary

It has been found that the 3 tokamak models as presented in Fig. 6.1 are unstable to resistive tearing modes for large values of m . The resistive instabilities appear over comparable ranges of (k_z, R_w) for all values of m up through $m = 10$, and the maximum growth rate for each value of m decreases as m increases. Examination of the B_{r1} profiles for all 3 models shows that resistive type profiles occur for $R_w \sim 2$ and at the small $|k_z|$ end of the unstable region for each m value. In addition, the unstable regions of the flattened model show the largest portion of resistive type B_{r1} profiles, while the peaked model shows the smallest fraction. In

contrast to the usual stability windows which appear⁴ for the infinite conductivity case, the present calculations show considerable overlap of the azimuthal modes, and the stability windows appear only as reduced growth rates.

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VII. SUMMARY AND CONCLUSIONS

Numerical studies of resistive instabilities in diffuse pinch geometries have been performed. Two numerical models in cylindrical geometry have been developed: a general model which includes the effects of compressibility, finite resistivity, viscosity, and thermal conductivity, and a simpler incompressible model which neglects the dissipative effects but includes finite resistivity. The same numerical technique has been used to solve the equations for both the general and incompressible models, and both computational models have been applied to diffuse pinch geometries to study the resistive tearing mode. The computational models have been applied to three different types of equilibrium magnetic field configurations: a force-free Bessel function model, a high-beta reversed field configuration and three toroidal or tokamak-type pinch configurations in which the axial field does not reverse direction. The equilibrium configurations have been tested for resistive tearing instabilities in the parameter space about which $\bar{\mathbf{k}} \cdot \bar{\mathbf{B}} = 0$, growth rates have been calculated for various (m, k_z) modes and the results have been summarized in the form of stability diagrams. In addition, a wide variety of parametric studies have been performed to determine the specific dependence of the growth rates on the position of the outer conducting wall R_w , the axial wave number k_z , and the magnetic Reynolds number S .

For the incompressible case it is found that the inclusion of finite resistivity in the hydromagnetic analysis of the force-free Bessel function model (FFBFM) causes the FFBFM to be unstable

to resistive tearing modes for all values of m , as compared with MHD instability only for $m = 1$. That is, resistive tearing modes occur where no ideal MHD modes exist, and for the $m = 1$ mode, the outer conducting wall must be placed approximately 2% closer to the plasma in order to ensure stability against resistive tearing modes as well as ideal MHD modes. Thus for the FFBFM, resistivity does not greatly affect the R_w value for overall stability of the $m = 1$ mode. A small change in the equilibrium magnetic field could make the plasma completely stable to MHD modes, but the configuration would still be unstable to resistive modes.

For the high-beta reversed field pitch and pressure model (PPM) it is again found that resistive tearing modes occur at R_w values predicted stable on ideal MHD theory. Complete stability for positive k_z values requires R_w approximately 22% smaller than the MHD stability criterion, while complete stability for negative k_z values requires R_w approximately 50% smaller. Compared with the FFBFM for which resistivity does not greatly affect the R_w value for overall stability, resistivity has a large impact on the stable wall position for the PPM.

For the Bessel function model and the pitch and pressure model, both of which represent reversed field configurations, the equilibrium magnetic fields were tested for ideal MHD instability by taking $\eta = 0$ in the incompressible model, and the results are found to agree well with ideal MHD theory. For both of these reversed field models for $m = 1$, positive axial wave numbers are found to be more unstable than negative axial wave numbers to both

MHD and resistive tearing modes, and the unstable regions are displaced to larger ($|k_z|$, R_w) values as m increases. In addition, the resistive instabilities occur over a smaller range of ($|k_z|$, R_w) values as m increases, and for a given value of R_w , the growth rates of the higher mode numbers ($m \geq 2$) become smaller as m increases. Thus, larger R_w values can be used before $m \geq 2$ resistive instabilities are encountered, i.e., higher mode numbers produce a less restrictive stability criterion on R_w .

As compared with the derived boundary conditions based on incompressible MHD theory, a set of boundary conditions was tested which, for all values of m , the dependent variables vanish at $r = R_w$ and the derivatives of the dependent variables vanish at $r = 0$. Using these alternate boundary conditions, the ($|k_z|$, R_w) region of the resistive instabilities becomes enlarged, and resistive instabilities now occur in regions previously found to be stable with the original set of boundary conditions. Thus a more restrictive criterion for complete stability is placed on R_w with the alternate set of boundary conditions.

For the 3 tokamak models it is found that all 3 models are unstable to resistive tearing modes for large values of m . For given values of m and R_w , the growth rates for the flattened model are higher than the corresponding growth rates of the rounded model which are in turn higher than those of the peaked model, and the instabilities occur at more negative k_z values as m increases. The resistive instabilities appear over comparable ranges of (k_z , R_w) for all values of m up through $m = 10$, and the

maximum growth rate for each value of m decreases as m increases. The introduction of a small pressure gradient increases the growth rates and slightly increases the size of the unstable regions. Examination of the perturbed magnetic field profiles for all 3 models shows that resistive type profiles occur for $R_w \sim 2$ at the small $|k_z|$ end of the unstable region for each m value, while at the larger $|k_z|$ values the profiles are characteristic of MHD type modes. In addition, the unstable regions of the flattened model show the largest portion of resistive type profiles, while the peaked model shows the smallest. In contrast to the usual stability windows which appear for the infinite conductivity case, the present calculations show considerable overlap of the azimuthal modes, i.e., more than one m mode can occur for a given value of k_z , and the stability windows now appear only as reduced growth rates.

For all of the configurations tested, the growth rates for the $m = 1$ mode are the largest, and the instabilities vanish as the outer conducting wall is moved closer to the plasma. The maximum growth rates occur at slightly larger $|k_z|$ values as the value of S increases, and for the reversed field models the instabilities cover a broader range of axial wave numbers as S increases. For all of the models, true resistive tearing modes seem to occur only for $S \geq 10^4 - 10^5$.

For the general case which includes the effects of compressibility, finite resistivity, viscosity, and thermal con-

ductivity, plasma parameters characteristic of actual pinch experiments are selected, and it is found that the instabilities occur generally in the same region of (k_z, R_w) space as found for the incompressible case, and the same general type of dependence of the growth rate on k_z , R_w , and S is observed. For the same set of parameters, the perturbed profiles produced by the computer model for the general case are consistent with the perturbed profiles of the incompressible case.

Since the computer code for the general case requires 4 to 5 times as much computer time as that of the incompressible case, extensive parameter studies have not been possible using the general model. In particular, plasma parameters for which $S \gtrsim 200$ require extensive amounts of computer time, so only representative results have been presented.

By neglecting the viscous terms in the general set of equations, it is found that viscosity tends to inhibit the growth rate, and the most unstable wave number shifts to a slightly lower value for the viscous case. In trying to separate the other effects of compressibility and thermal conductivity, reliable results could not be obtained. It was noted that when the dissipative terms vanish the hydromagnetic equations are no longer diffusion-type equations, and large spatial gradients in the dependent variables occur. These effects could possibly be overcome by increasing the number of mesh points and/or decreasing the size of the time step, but the corresponding increase in computer time and computer storage would be prohibitive.

It is not feasible to attempt to examine the stability of all possible equilibrium configurations proposed or in existence in controlled fusion research. It has been shown, however, that the computer models developed here for both the general case and the incompressible case can be used to study the resistive instability of a wide class of equilibrium configurations. Any equilibrium field, temperature, density or resistivity configuration that can be expressed in analytic or tabular form can be examined for resistive tearing instabilities, and for cases in which a gradient in the resistivity exists, the rippling mode can also be studied.

Future Work

In actual pinch experiments, the equilibrium profiles do not remain static as assumed in the present analysis, but can diffuse as the experiment progresses. Diffusion of an initially stable configuration can cause an increase in the plasma beta and a decrease in the shear which can lead to an unstable configuration. Improved results in the study of resistive instabilities may possibly be achieved by coupling the present numerical models with a transport model which would allow diffusion of the equilibrium quantities.

Another area of future work is that of non-linear effects. Most stability theories assume that departure of the system from equilibrium is infinitesimal and the resulting linearized equations are tested for growth or decay. However, an unstable disturbance will not remain infinitesimal, and it is important to

know what happens when the disturbance becomes large. The detailed study of non-linear effects is now receiving increased attention, and future successes in this area should greatly enhance the study of resistive instabilities.

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APPENDICES

A. Matrix Elements for the General Case

For the general case with compressibility, viscosity, and thermal conductivity, the six dimensionless component Eqs. (3.22)-(3.27) for the dependent variables B_{r1} , $B_{\theta1}$, T_1 , v_{r1} , $v_{\theta1}$, v_{z1} can be put into the form of a single vector difference equation

$$-A_j X_{j+1}^{n+1} + B_j X_j^{n+1} - C_j X_{j-1}^{n+1} = d_j^n, \quad (A.1)$$

where A, B, C are 6 x 6 matrices, d is a 6-component vector, and

$$X = \begin{bmatrix} B_{r1} \\ B_{\theta1} \\ T_1 \\ v_{r1} \\ v_{\theta1} \\ v_{z1} \end{bmatrix} \rightarrow \begin{bmatrix} \psi \\ \phi \\ T \\ W \\ U \\ V \end{bmatrix}. \quad (A.2)$$

Arranging the six-component equations in the order

- (1) r component of the magnetic field equation--(3.22),
- (2) θ component of the magnetic field equation--(3.23),
- (3) energy equation--(3.24),
- (4) r component of the equation of motion--(3.25),
- (5) θ component of the equation of motion--(3.26),
- (6) z component of the equation of motion--(3.27),

the expanded form of the vector difference equation (A.1) becomes

$$\begin{aligned}
 & \begin{bmatrix} a_{11} & 0 & 0 & 0 & 0 & 0 \\ 0 & a_{22} & 0 & a_{24} & 0 & 0 \\ a_{31} & a_{32} & a_{33} & a_{34} & 0 & 0 \\ a_{41} & a_{42} & a_{43} & a_{44} & a_{45} & a_{46} \\ a_{51} & 0 & 0 & a_{54} & a_{55} & 0 \\ a_{61} & 0 & 0 & a_{64} & 0 & a_{66} \end{bmatrix}_j \begin{bmatrix} \psi \\ \phi \\ T \\ W \\ U \\ V \end{bmatrix}_{j+1}^{n+1} \\
 & + \begin{bmatrix} b_{11} & b_{12} & 0 & b_{14} & 0 & 0 \\ b_{21} & b_{22} & 0 & b_{24} & b_{25} & b_{26} \\ b_{31} & b_{32} & b_{33} & b_{34} & b_{35} & b_{36} \\ b_{41} & b_{42} & b_{43} & b_{44} & b_{45} & 0 \\ b_{51} & b_{52} & b_{53} & b_{54} & b_{55} & b_{56} \\ b_{61} & b_{62} & b_{63} & b_{64} & b_{65} & b_{66} \end{bmatrix}_j \begin{bmatrix} \psi \\ \phi \\ T \\ W \\ U \\ V \end{bmatrix}_j^{n+1} \\
 & - \begin{bmatrix} c_{11} & 0 & 0 & 0 & 0 & 0 \\ 0 & c_{22} & 0 & c_{24} & 0 & 0 \\ c_{31} & c_{32} & c_{33} & c_{34} & 0 & 0 \\ c_{41} & c_{42} & c_{43} & c_{44} & c_{45} & c_{46} \\ c_{51} & 0 & 0 & c_{54} & c_{55} & 0 \\ c_{61} & 0 & 0 & c_{64} & 0 & c_{66} \end{bmatrix}_j \begin{bmatrix} \psi \\ \phi \\ T \\ W \\ U \\ V \end{bmatrix}_{j-1}^{n+1} = \begin{bmatrix} d_1 \\ d_2 \\ d_3 \\ d_4 \\ d_5 \\ d_6 \end{bmatrix}_j^n \quad (A.3)
 \end{aligned}$$

where the elements of the matrices A, B, C and the vector d are as follows:

$$a_{11} = l \tilde{\eta}_j \left[\frac{1}{(\Delta\mu)^2} + \frac{1}{2\mu_j \Delta\mu} \right]$$

$$a_{22} = a_{11} + \frac{l}{2\Delta\mu} \tilde{\eta}'_j$$

$$a_{24} = I \frac{l H_j}{2\alpha \Delta\mu}$$

$$a_{31} = -2l(\gamma - 1) \tilde{\eta}_j M'_j \left[\frac{1}{(\Delta\mu)^2} + \frac{1}{2\mu_j \Delta\mu} \right] = -2(\gamma - 1) M'_j (a_{11})$$

$$a_{32} = \frac{l(\gamma - 1)}{\Delta\mu} \tilde{\eta}_j (\kappa_z N_j - \frac{m}{\mu_j} M'_j)$$

$$a_{33} = \frac{l \beta \kappa_z \Lambda}{2} \left[\frac{1}{(\Delta\mu)^2} + \frac{1}{2\mu_j \Delta\mu} \right]$$

$$a_{34} = I \frac{l(\gamma - 1) \beta \kappa_z \tilde{\rho}_j \tilde{T}_j}{4\alpha \Delta\mu}$$

$$a_{41} = -l \left\{ M_j \left[\frac{1}{(\Delta\mu)^2} + \frac{1}{2\mu_j \Delta\mu} \right] + \frac{M'_j}{2\Delta\mu} \right\}$$

$$a_{42} = \frac{l}{2\Delta\mu} (\kappa_z H_j - \frac{m}{\mu_j} M_j)$$

$$a_{43} = \frac{l \beta \kappa_z \tilde{\rho}_j}{4\Delta\mu}$$

$$a_{44} = \left(1 + \frac{I}{3} \right) \frac{l \kappa_z \tilde{\rho}_j \Gamma}{\alpha s^2} \left[\frac{1}{(\Delta\mu)^2} + \frac{1}{2\mu_j \Delta\mu} \right] = \left(1 + \frac{I}{3} \right) (a_{55})$$

$$a_{45} = I \frac{l m \kappa_z \tilde{\rho}_j \Gamma}{6\alpha s^2 \mu_j \Delta\mu}$$

$$a_{46} = I \frac{\ell \kappa_z^2 \rho_j \Gamma}{608^2 \Delta \mu}$$

$$a_{51} = \frac{\ell m H_j}{2 \mu_j \Delta \mu}$$

$$a_{54} = -a_{45}$$

$$a_{55} = \frac{a_{44}}{\left(1 + \frac{1}{3}\right)}$$

$$a_{61} = -\frac{\ell m H_j}{2 \mu_j \Delta \mu}$$

$$a_{64} = -a_{46}$$

$$a_{66} = a_{55}$$

$$b_{11} = \frac{1}{\Delta \tau} + \ell \tilde{\eta}_j \left[\frac{2}{(\Delta \mu)^2} - \frac{m^2 + 1}{\mu_j^2} + \kappa_z^2 \right]$$

$$b_{12} = \ell \frac{2m}{\mu_j^2} \tilde{\eta}_j$$

$$b_{14} = \ell F_j$$

$$b_{21} = \ell \frac{m}{\mu_j} \left(\frac{2}{\mu_j} \tilde{\eta}_j - \tilde{\eta}'_j \right) = b_{12} - \ell \frac{m}{\mu_j} \tilde{\eta}'_j$$

$$b_{22} = b_{11} - \frac{\ell}{\mu_j} \tilde{\eta}'_j$$

$$b_{24} = \frac{\ell}{\alpha} \left[\left(2 \frac{H_j}{\mu_j} - N_j \right) - I \frac{H_j}{\mu_j} \right]$$

$$b_{25} = \frac{\ell}{\alpha} \left[(\kappa_z M_j + \frac{m}{\mu_j} H_j) - I \frac{m}{\mu_j} H_j \right]$$

$$b_{26} = -I \frac{\ell}{\alpha} \kappa_z H_j$$

$$b_{31} = -2\ell(\gamma - 1) \tilde{\eta}_j \left\{ M_j' \left[\frac{2}{(\Delta\mu)^2} + \frac{m^2 + 1}{\mu_j^2} + \kappa_z^2 \right] + \frac{m}{\mu_j} (\kappa_z N_j - \frac{m}{\mu_j} M_j') \right\}$$

$$b_{32} = -2\ell(\gamma - 1) \frac{\tilde{\eta}_j}{\mu_j} (\kappa_z N_j + \frac{m}{\mu_j} M_j')$$

$$b_{33} = \frac{\beta \kappa_z}{2} \left\{ \frac{\tilde{\rho}_j}{\Delta\tau} + \ell \lambda \left[\frac{2}{(\Delta\mu)^2} + \frac{m^2}{\mu_j^2} + \kappa_z^2 \right] \right\}$$

$$b_{34} = -\frac{\ell \beta \kappa_z \tilde{\rho}_j}{2\alpha} \left[\tilde{T}_j' + I(\gamma - 1) \frac{\tilde{T}_j}{\mu_j} \right]$$

$$b_{35} = -I \frac{\ell(\gamma - 1) m \beta \kappa_z \tilde{\rho}_j \tilde{T}_j}{2\alpha \mu_j}$$

$$b_{36} = -I \frac{\ell(\gamma - 1) \beta \kappa_z \tilde{\rho}_j \tilde{T}_j}{2\alpha} = -2\kappa_z \Delta\mu (a_{34})$$

$$b_{41} = \ell \left\{ \frac{M_j'}{\mu_j} - \alpha \kappa_z F_j - M_j \left[\frac{2}{(\Delta\mu)^2} + \frac{1}{\mu_j^2} \right] \right\}$$

$$b_{42} = \ell \left[\frac{m}{\mu_j} \left(M_j' - \frac{M_j}{\mu_j} \right) - \kappa_z \left(N_j + \frac{H_j}{\mu_j} \right) \right]$$

$$b_{43} = -\frac{\ell \beta \kappa_z \tilde{\rho}_j'}{2}$$

$$b_{44} = \frac{\kappa_z \tilde{\rho}_j}{\alpha s^2} \left\{ \frac{1}{\Delta \tau} + \ell \Gamma \left[\left(\frac{2}{(\Delta \mu)^2} + \frac{m^2 + 1}{\mu_j^2} + \kappa_z^2 \right) + \frac{1}{3} \left(\frac{2}{(\Delta \mu)^2} + \frac{1}{\mu_j^2} \right) \right] \right\}$$

$$b_{45} = \left(2 + \frac{1}{3} \right) \frac{\ell m \kappa_z \tilde{\rho}_j \Gamma}{\alpha s^2 \mu_j^2}$$

$$b_{51} = -\ell \left(\frac{m}{\mu_j^2} M_j + \kappa_z N_j \right)$$

$$b_{52} = -\ell M_j \left(\frac{m^2}{\mu_j^2} + \kappa_z^2 \right)$$

$$b_{53} = \frac{\ell \beta m \kappa_z \tilde{\rho}_j}{2 \mu_j}$$

$$b_{54} = b_{45}$$

$$b_{55} = \frac{\kappa_z \tilde{\rho}_j}{\alpha s^2} \left\{ \frac{1}{\Delta \tau} + \ell \Gamma \left[\left(\frac{2}{(\Delta \mu)^2} + \frac{m^2 + 1}{\mu_j^2} + \kappa_z^2 \right) + \frac{1}{3} \frac{m^2}{\mu_j^2} \right] \right\}$$

$$b_{56} = I \frac{\ell m \kappa_z \tilde{\rho}_j \Gamma}{3 \alpha s^2 \mu_j} = 2 \frac{m}{\mu_j} \Delta \mu (a_{46})$$

$$b_{61} = \ell \left(\frac{m}{\mu_j^2} H_j - \kappa_z M_j' \right)$$

$$b_{62} = \ell H_j \left(\frac{m^2}{\mu_j^2} + \kappa_z^2 \right)$$

$$b_{63} = \frac{\ell \beta \kappa_z \tilde{\rho}_j}{2}$$

$$b_{64} = I \frac{\ell \kappa_z^2 \tilde{\rho}_j \Gamma}{3 \alpha s^2 \mu_j} = 2 \frac{\Delta \mu}{\mu_j} (a_{46})$$

$$b_{65} = b_{56}$$

$$b_{66} = \frac{\kappa_z^2 \tilde{\rho}_j}{\alpha s^2} \left\{ \frac{1}{\Delta \tau} + \ell \Gamma \left[\frac{2}{(\Delta \mu)^2} + \frac{m^2}{\mu_j^2} + \left(1 + \frac{I}{3} \right) \kappa_z^2 \right] \right\}$$

$$c_{11} = \ell \tilde{\eta}_j \left[\frac{1}{(\Delta \mu)^2} - \frac{1}{2 \mu_j \Delta \mu} \right]$$

$$c_{22} = c_{11} - \frac{\ell}{2 \Delta \mu} \tilde{\eta}_j'$$

$$c_{24} = -a_{24}$$

$$c_{31} = -2 \ell (\gamma - 1) M_j' \tilde{\eta}_j \left[\frac{1}{(\Delta \mu)^2} - \frac{1}{2 \mu_j \Delta \mu} \right] = -2 (\gamma - 1) M_j' (c_{11})$$

$$c_{32} = -a_{32}$$

$$c_{33} = \frac{\ell \beta \kappa_z^2 \Lambda}{2} \left[\frac{1}{(\Delta \mu)^2} - \frac{1}{2 \mu_j \Delta \mu} \right]$$

$$c_{34} = -a_{34}$$

$$c_{41} = \ell \left\{ \frac{M_j'}{2 \Delta \mu} - M_j \left[\frac{1}{(\Delta \mu)^2} - \frac{1}{2 \mu_j \Delta \mu} \right] \right\}$$

$$c_{42} = -a_{42}$$

$$c_{43} = -a_{43}$$

$$c_{44} = \left(1 + \frac{I}{3} \right) \frac{\ell \kappa_z^2 \tilde{\rho}_j \Gamma}{\alpha s^2} \left[\frac{1}{(\Delta \mu)^2} - \frac{1}{2 \mu_j \Delta \mu} \right]$$

$$c_{45} = -a_{45}$$

$$c_{46} = -a_{46}$$

$$c_{51} = -a_{51}$$

$$c_{54} = a_{45}$$

$$c_{55} = \frac{c_{44}}{\left(1 + \frac{1}{j}\right)}$$

$$c_{61} = -a_{61}$$

$$c_{64} = -a_{64} = a_{46}$$

$$c_{66} = c_{55}$$

$$(d_1)_j^n = \frac{(1 - \ell)}{\ell} \left\{ (a_{11})_j \psi_{j+1}^n - \left[(b_{11})_j - \frac{1}{(1 - \ell)\Delta\tau} \right] \psi_j^n + (c_{11})_j \psi_{j-1}^n - (b_{12})_j \phi_j^n - (b_{14})_j w_j^n \right\} + (\alpha'_j + \frac{2m}{\mu_j} H_j) \left[\delta_j^{n+1} + (1 - \ell) \delta_j^n \right]$$

$$(d_2)_j^n = \frac{(1 - \ell)}{\ell} \left\{ -(b_{21})_j \psi_j^n + (a_{22})_j \phi_{j+1}^n - \left[(b_{22})_j - \frac{1}{(1 - \ell)\Delta\tau} \right] \phi_j^n + (c_{22})_j \phi_{j-1}^n + (a_{24})_j w_{j+1}^n - (b_{24})_j w_j^n + (c_{24})_j w_{j-1}^n - (b_{25})_j u_j^n - (b_{26})_j v_j^n \right\} - N'_j \left[\delta_j^{n+1} + (1 - \ell) \delta_j^n \right] - \frac{N_j}{2\Delta\tau} \left[\ell(\delta_{j+1}^{n+1} - \delta_{j-1}^{n+1}) + (1 - \ell)(\delta_{j+1}^n - \delta_{j-1}^n) \right]$$

$$\begin{aligned}
 (d_3)_j^n &= \frac{(1-\ell)}{\ell} \left\{ (a_{31})_j \psi_{j+1}^n - (b_{31})_j \psi_j^n + (c_{31})_j \psi_{j-1}^n \right. \\
 &\quad + (a_{32})_j \phi_{j+1}^n - (b_{32})_j \phi_j^n + (c_{32})_j \phi_{j-1}^n \\
 &\quad + (a_{33})_j \tau_{j+1}^n - \left[(b_{33})_j - \frac{\beta \kappa_z \tilde{\rho}_j}{2(1-\ell)\Delta\tau} \right] \tau_j^n + (c_{33})_j \tau_{j-1}^n \\
 &\quad + (a_{34})_j w_{j+1}^n - (b_{34})_j w_j^n + (c_{34})_j w_{j-1}^n - (b_{35})_j u_j^n - (b_{36})_j v_j^n \Big\} \\
 &\quad - \kappa_z (\gamma - 1) (M_j^2 + N_j^2) \left[\ell \delta_j^{n+1} + (1-\ell) \delta_j^n \right] + \frac{1}{2} \beta \kappa_z (\gamma - 1) \mathcal{L}_E \\
 (d_4)_j^n &= \frac{(1-\ell)}{\ell} \left\{ (a_{41})_j \psi_{j+1}^n - (b_{41})_j \psi_j^n + (c_{41})_j \psi_{j-1}^n \right. \\
 &\quad + (a_{42})_j \phi_{j+1}^n - (b_{42})_j \phi_j^n + (c_{42})_j \phi_{j-1}^n \\
 &\quad + (a_{43})_j \tau_{j+1}^n - (b_{43})_j \tau_j^n + (c_{43})_j \tau_{j-1}^n \\
 &\quad + (a_{44})_j w_{j+1}^n - \left[(b_{44})_j - \frac{\kappa_z \tilde{\rho}_j}{\alpha S^2 (1-\ell)\Delta\tau} \right] w_j^n + (c_{44})_j w_{j-1}^n \\
 &\quad + (a_{45})_j u_{j+1}^n - (b_{45})_j u_j^n + (c_{45})_j u_{j-1}^n \\
 &\quad + (a_{46})_j v_{j+1}^n + (c_{46})_j v_{j-1}^n \Big\} \\
 &\quad + \frac{\beta \kappa_z \tilde{m}_j}{2} \left[\ell \mathcal{R}_j^{n+1} + (1-\ell) \mathcal{R}_j^n \right] \\
 &\quad + \frac{\beta \kappa_z \tilde{m}_j}{4\Delta\mu} \left[\ell (\mathcal{R}_{j+1}^{n+1} - \mathcal{R}_{j-1}^{n+1}) + (1-\ell) (\mathcal{R}_{j+1}^n - \mathcal{R}_{j-1}^n) \right] \\
 &\quad + \kappa_z (\mathcal{L}_P)_r
 \end{aligned}$$

$$\begin{aligned}
 (d_5)_j^n &= \frac{(1-l)}{l} \left\{ (a_{51})_j \psi_{j+1}^n - (b_{51})_j \psi_j^n + (c_{51})_j \psi_{j-1}^n \right. \\
 &\quad - (b_{52})_j \theta_j^n - (b_{53})_j \tau_j^n - (b_{56})_j v_j^n \\
 &\quad + (a_{54})_j w_{j+1}^n - (b_{54})_j w_j^n + (c_{54})_j w_{j-1}^n \\
 &\quad \left. + (a_{55})_j u_{j+1}^n - \left[(b_{55})_j - \frac{\kappa_z \tilde{\rho}_j}{\alpha s^2 (1-l) \Delta \tau} \right] u_j^n + (c_{55})_j u_{j-1}^n \right\} \\
 &\quad - \frac{m \beta \kappa_z \tilde{\eta}_j}{2 \mu_j} \left[l \mathcal{R}_j^{n+1} + (1-l) \mathcal{R}_j^n \right] + \kappa_z (\mathcal{L}_p)_\theta \\
 (d_6)_j^n &= \frac{(1-l)}{l} \left\{ (a_{61})_j \psi_{j+1}^n - (b_{61})_j \psi_j^n + (c_{61})_j \psi_{j-1}^n \right. \\
 &\quad - (b_{62})_j \theta_j^n - (b_{63})_j \tau_j^n - (b_{65})_j u_j^n \\
 &\quad + (a_{64})_j w_{j+1}^n - (b_{64})_j w_j^n + (c_{64})_j w_{j-1}^n \\
 &\quad \left. + (a_{66})_j v_{j+1}^n - \left[(b_{66})_j - \frac{\kappa_z \tilde{\rho}_j}{\alpha s^2 (1-l) \Delta \tau} \right] v_j^n + (c_{66})_j v_{j-1}^n \right\} \\
 &\quad - \frac{\beta \kappa_z \tilde{\eta}_j}{2} \left[l \mathcal{R}_j^{n+1} + (1-l) \mathcal{R}_j^n \right] + \kappa_z (\mathcal{L}_p)_z
 \end{aligned}$$

B. Matrix Elements for the Incompressible Case

For the incompressible model without dissipation, the four dimensionless component Equations (3.30)-(3.33) for the dependent variables B_{r1} , $B_{\theta 1}$, v_{r1} , $v_{\theta 1}$ can be put into the form of a single vector difference equation

$$-A_j X_{j+1}^{n+1} + B_j X_j^{n+1} - C_j X_{j-1}^{n+1} = d_j^n \quad (B.1)$$

where A, B, C are 4×4 matrices, d is a 4-component vector, and

$$X = \begin{bmatrix} B_{r1} \\ B_{\theta 1} \\ v_{r1} \\ v_{\theta 1} \end{bmatrix} \rightarrow \begin{bmatrix} \psi \\ \phi \\ W \\ U \end{bmatrix} . \quad (B.2)$$

Arranging the four-component equations in the order

- (1) r component of the magnetic field equation--(3.22),
- (2) r component of the equation of motion--(3.23),
- (3) θ component of the magnetic field equation--(3.24),
- (4) θ component of the equation of motion--(3.25),

the expanded form of the vector difference equation (B.1) becomes

$$\begin{aligned}
 & - \begin{bmatrix} a_{11} & 0 & 0 & 0 \\ a_{21} & 0 & a_{23} & 0 \\ 0 & a_{32} & 0 & 0 \\ a_{41} & a_{42} & 0 & a_{44} \end{bmatrix}_j \begin{bmatrix} \psi \\ \phi \\ W \\ U \end{bmatrix}_{j+1}^{n+1} \\
 & + \begin{bmatrix} b_{11} & b_{12} & b_{13} & 0 \\ b_{21} & b_{22} & b_{23} & b_{24} \\ b_{31} & b_{32} & b_{33} & b_{34} \\ b_{41} & b_{42} & b_{43} & b_{44} \end{bmatrix}_j \begin{bmatrix} \psi \\ \phi \\ W \\ U \end{bmatrix}_j^{n+1} \\
 & - \begin{bmatrix} c_{11} & 0 & 0 & 0 \\ c_{21} & 0 & c_{23} & 0 \\ 0 & c_{32} & 0 & 0 \\ c_{41} & c_{42} & 0 & c_{44} \end{bmatrix}_j \begin{bmatrix} \psi \\ \phi \\ W \\ U \end{bmatrix}_{j-1}^{n+1} = \begin{bmatrix} d_1 \\ d_2 \\ d_3 \\ d_4 \end{bmatrix}_j^n \quad (B.3)
 \end{aligned}$$

where the elements of matrices A, B, C and the vector d are as follows:

$$a_{11} = \hat{\eta}_j \left[\frac{1}{(\Delta\mu)^2} + \frac{1}{2\mu_j \Delta\mu} \right]$$

$$a_{21} = \ell F_j \left[\frac{1}{(\Delta\mu)^2} + \frac{1}{2\mu_j \Delta\mu} \right] - \frac{\ell}{\alpha} \frac{m}{\mu_j^2 \Delta\mu} H_j$$

$$a_{23} = -\frac{\tilde{\rho}_j}{\alpha^2 S^2 \Delta\tau} \left[\frac{1}{(\Delta\mu)^2} + \frac{1}{2\mu_j \Delta\mu} \right]$$

$$a_{32} = a_{11} + \frac{\ell}{2\Delta\mu} \tilde{\eta}'_j$$

$$a_{41} = \frac{\ell}{\alpha} \left\{ N_j \left[\frac{1}{(\Delta\mu)^2} + \frac{1}{2\mu_j \Delta\mu} \right] + \frac{N'_j}{\Delta\mu} \right\}$$

$$a_{42} = \ell F_j \left[\frac{1}{(\Delta\mu)^2} + \frac{1}{2\mu_j \Delta\mu} \right] + \frac{\ell}{\Delta\mu} (F'_j + \frac{m}{\alpha\mu_j^2} H_j)$$

$$a_{44} = a_{23}$$

$$b_{11} = \frac{1}{\Delta\tau} + \ell \tilde{\eta}_j \left[\frac{2}{(\Delta\mu)^2} + \frac{m^2 + 1}{\mu_j^2} + \kappa_z^2 \right]$$

$$b_{12} = \ell \frac{2m}{\mu_j^2} \tilde{\eta}_j$$

$$b_{13} = \ell F_j$$

$$b_{21} = \ell F_j \left[\frac{2}{(\Delta\mu)^2} + \frac{m^2 + 1}{\mu_j^2} + \kappa_z^2 \right] + \ell (F''_j - \frac{F'_j}{\mu_j} + \frac{4m}{\alpha\mu_j^2} N_j - \frac{6m}{\alpha\mu_j^3} H_j)$$

$$b_{22} = \ell \frac{2m}{\mu_j^2} F_j + \ell \frac{2H_j}{\alpha\mu_j} \left(\frac{m^2}{\mu_j^2} + \kappa_z^2 \right)$$

$$b_{23} = -\frac{\tilde{\rho}_j}{\alpha^2 s^2 \Delta \tau} \left[\frac{2}{(\Delta \mu)^2} + \frac{m^2 + 1}{\mu_j^2} + \kappa_z^2 \right]$$

$$b_{24} = -\frac{2m\tilde{\rho}_j}{\alpha^2 s^2 \mu_j^2 \Delta \tau}$$

$$b_{31} = \ell \frac{m}{\mu_j} \left(\frac{2}{\mu_j} \tilde{\eta}_j - \tilde{\eta}'_j \right) = b_{12} - \ell \frac{m}{\mu_j} \tilde{\eta}'_j$$

$$b_{32} = b_{11} - \frac{\ell}{\mu_j} \tilde{\eta}'_j$$

$$b_{33} = \frac{\ell}{\alpha} \left(\frac{2H_j}{\mu_j} - N_j \right)$$

$$b_{34} = b_{13}$$

$$b_{41} = \frac{\ell}{\alpha} N_j \left[\frac{2}{(\Delta \mu)^2} + \frac{m^2 + 1}{\mu_j^2} + \kappa_z^2 \right] + \ell \frac{2m}{\mu_j^2} F_j - \ell \left(\frac{2m}{\mu_j} F'_j + \frac{2m^2}{\alpha \mu_j^3} H_j + \frac{N'_j}{\alpha \mu_j} + \frac{N''_j}{\alpha} \right)$$

$$b_{42} = \ell F_j \left[\frac{2}{(\Delta \mu)^2} + \frac{m^2 + 1}{\mu_j^2} + \kappa_z^2 \right] - \ell \left(\frac{2m}{\alpha \mu_j^2} N_j - \frac{6m}{\alpha \mu_j^3} H_j + F''_j + \frac{F'_j}{\mu_j} \right)$$

$$b_{43} = b_{24}$$

$$b_{44} = b_{23}$$

$$c_{11} = \ell \tilde{\eta}_j \left[\frac{1}{(\Delta \mu)^2} - \frac{1}{2\mu_j \Delta \mu} \right]$$

$$c_{21} = \ell F_j \left[\frac{1}{(\Delta\mu)^2} - \frac{1}{2\mu_j \Delta\mu} \right] + \frac{\ell}{\alpha} \frac{m}{\mu_j^2 \Delta\mu} H_j$$

$$c_{23} = - \frac{\tilde{\rho}_j}{\alpha S^2 \Delta\tau} \left[\frac{1}{(\Delta\mu)^2} - \frac{1}{2\mu_j \Delta\mu} \right]$$

$$c_{32} = c_{11} - \frac{\ell}{2\Delta\mu} \tilde{\eta}_j$$

$$c_{41} = \frac{\ell}{\alpha} \left\{ N_j \left[\frac{1}{(\Delta\mu)^2} - \frac{1}{2\mu_j \Delta\mu} \right] - \frac{N'_j}{\Delta\mu} \right\}$$

$$c_{42} = \ell F_j \left[\frac{1}{(\Delta\mu)^2} - \frac{1}{2\mu_j \Delta\mu} \right] - \frac{\ell}{\Delta\mu} (F'_j + \frac{m}{\alpha\mu_j^2} H_j)$$

$$c_{44} = c_{23}$$

$$\begin{aligned} (d_1)_j^n = \frac{(1-\ell)}{\ell} \left\{ (a_{11})_j \psi_{j+1}^n - \left[(b_{11})_j - \frac{1}{(1-\ell)\Delta\tau} \right] \psi_j^n + (c_{11})_j \psi_{j-1}^n \right. \\ \left. - (b_{12})_j \phi_j^n - (b_{13})_j w_j^n \right\} \\ + (\alpha F'_j + \frac{2m}{\mu_j^2} H_j) \left[\ell \delta_j^{n+1} + (1-\ell) \delta_j^n \right] \end{aligned}$$

$$\begin{aligned} (d_2)_j^n = \frac{(1-\ell)}{\ell} \left\{ (a_{21})_j \psi_{j+1}^n - (b_{21})_j \psi_j^n + (c_{21})_j \psi_{j-1}^n - (b_{22})_j \phi_j^n \right. \\ \left. - (a_{23})_j w_{j+1}^n + (b_{23})_j w_j^n - (c_{23})_j w_{j-1}^n + (b_{24})_j u_j^n \right\} \end{aligned}$$

$$\begin{aligned} (d_3)_j^n = \frac{(1-\ell)}{\ell} \left\{ (a_{32})_j \phi_{j+1}^n - \left[(b_{32})_j - \frac{1}{(1-\ell)\Delta\tau} \right] \phi_j^n \right. \\ \left. + (c_{32})_j \phi_{j-1}^n - (b_{31})_j \psi_j^n - (b_{33})_j w_j^n - (b_{34})_j u_j^n \right\} \end{aligned}$$

$$\begin{aligned}
 & -N'_j \left[\delta_j^{n+1} + (1 - \ell) \delta_j^n \right] \\
 & - \frac{N_j}{2\Delta u} \left[\ell (\delta_{j+1}^{n+1} - \delta_{j-1}^{n+1}) + (1 - \ell) (\delta_{j+1}^n - \delta_{j-1}^n) \right] \\
 (d_4)_j^n = & \frac{(1 - \ell)}{\ell} \left\{ (a_{41})_j \psi_{j+1}^n - (b_{41})_j \psi_j^n + (c_{41})_j \psi_{j-1}^n \right. \\
 & \left. + (a_{42})_j \phi_{j+1}^n - (b_{42})_j \phi_j^n + (c_{42})_j \phi_{j-1}^n \right\} \\
 & + (b_{43})_j w_j^n - (a_{44})_j u_{j+1}^n + (b_{44})_j u_j^n - (c_{44})_j u_{j-1}^n
 \end{aligned}$$

C. Glossary of Symbols

A	matrix (4×4 or 6×6) in tri-diagonal difference scheme
a	characteristic length or thickness of the current layer
$\vec{B}$, B	magnetic induction
B	matrix (4×4 or 6×6) in tri-diagonal difference scheme
B_0	normalization constant for B_z field in tokamak models
C	matrix (4×4 or 6×6) in tri-diagonal difference scheme
C_0	adjustable constant for initial perturbation profiles
C_1 , C_2	arbitrary constants to scale magnitudes of equilibrium magnetic fields
c	speed of light
c	curve fit parameter
c_s	speed of sound
D	diffusion coefficient
d	column vector (4-element or 6-element) in tri-diagonal difference scheme
$\vec{E}$, E	electric field
E	matrix (4×4 or 6×6) in tri-diagonal difference scheme
e	electronic charge in statcoulombs
e	internal energy/mass
F	dimensionless notation for $(\vec{k} \cdot \vec{E}) = 0$
F	column vector (4-element or 6-element) in tri-diagonal difference scheme
$\vec{F}$, $\vec{f}$	force density
f	number of degrees of freedom
H	dimensionless θ component of equilibrium magnetic field

$\bar{I}$	unit tensor
i	$\sqrt{-1}$
$\bar{J}, J, \mathcal{J}$	current density
J	total number of mesh points
J_β	Bessel function of the first kind of order β
j	subscript denoting spatial mesh point in finite difference equations
K, K_T	coefficient of thermal conductivity
$\bar{k}$	wave number vector
k	total wave number
k_B	Boltzmann constant
k_z	axial wave number
L	characteristic length
L	dimensionless operator
ℓ	weighting factor for implicit difference scheme
M	mass of fluid
M	dimensionless z component of equilibrium magnetic field
m	azimuthal wave number
m_e	electronic mass
m_i	ionic mass
N	dimensionless notation used with equilibrium magnetic field models
n	superscript denoting time level in finite difference equations
n_e	electron number density
n_i	ion number density

P	pitch of the equilibrium magnetic field lines
P	dimensionless perturbed pressure
p	dimensionless growth rate
p	thermodynamic scalar pressure
$\bar{p}$	pressure tensor
$\tilde{p}$	dimensionless equilibrium pressure
q	tokamak 'safety factor'
R	universal gas constant
R	major radius of a torus
R, r_s	radius at which $(\bar{k} \cdot \bar{B}) = 0$
R_w	position of outer conducting wall
$\mathcal{R}$	dimensionless perturbed mass density
r	radial coordinate
r_0	adjustable constant to define zero-point of initial perturbation
S	magnetic Reynolds number
S_E	energy source term
S_M	mass source term
$\bar{S}_p, S_p$	momentum source term
$\mathcal{S}_E$	dimensionless energy source term
$\mathcal{S}_M$	dimensionless mass source term
$\mathcal{S}_p$	dimensionless momentum source term
T	temperature in energy/mass
$\tilde{T}$	dimensionless equilibrium temperature
t	time
U	dimensionless θ component of perturbed velocity

V	dimensionless z component of perturbed velocity
$\bar{v}, v$	fluid velocity
W	dimensionless r component of perturbed velocity
W	potential energy
X	column vector (4-element or 6-element) denoting dependent variables in tri-diagonal difference scheme
Y	Bessel function of the second kind
Z	valence
z	axial coordinate
α	dimensionless total wave number
β	plasma beta
Γ	dimensionless viscosity
Γ	gamma function
γ	ratio of specific heats
Δ'	change in hydromagnetic solutions across resistive layer
δ	dimensionless perturbed resistivity
δ_{ik}	Kronecker delta
ϵ	truncation error
ζ	second coefficient of viscosity
ξ	curve fit parameter
η	resistivity
$\tilde{\eta}$	dimensionless equilibrium resistivity
θ	temperature in degrees Kelvin
θ	azimuthal coordinate
ι	rotational transform
K_z	dimensionless axial wave number

Λ	dimensionless thermal conductivity
λ_c	mean free path between particle deflections
λ_c	Coulomb logarithm collision parameter
λ_D	Debye length
μ	dimensionless radial coordinate
μ_F	coefficient of dynamic viscosity
ν	coefficient of kinematic viscosity
ν	Bessel function parameter
ξ	vector displacement of equilibrium plasma configuration
ρ	mass density
$\tilde{\rho}$	dimensionless equilibrium mass density
ρ_e	volume charge density
τ	dimensionless time
τ_c	mean time between particle deflections
τ_H	hydromagnetic transit time
τ_R	resistive diffusion time
τ_{ik}	viscous stress tensor
T	dimensionless perturbed temperature
Φ	hydrodynamic dissipation function
Φ_z	axial magnetic flux
ϕ	dimensionless θ component of perturbed magnetic field
ψ	dimensionless r component of perturbed magnetic field
Ω_e	electron cyclotron frequency
Ω_i	ion cyclotron frequency
ω	characteristic frequency
ω_p	electron plasma frequency

Δ increment
 ∇ del or gradient operator
 d/dt convective derivative

Equilibrium quantities are denoted by the subscript 0.

Perturbed quantities are denoted by the subscript 1.

Primes denote differentiation with respect to the radial coordinate.