UCID - 16660

This is an informal report intended primarily for internal or limited external distribution. The opinions and conclusions stated are those of the author and may or may not be those of the laboratory.

LAWRENCE LIVERMORE LABORATORY

University of California/Livermore, California

RQGPAK: A PACKAGE OF FORTRAN SUBPROGRAMS TO COMPUTE
MACROSCOPIC PROPERTIES OF RELATIVISTIC QUANTUM GASES

J. PETER VAJK

NOVEMBER 1974

MASTER

NOTICE

This report was prepared as an account of work sponsored by the United States Government. Neither the United States nor the United States Atomic Energy Commission, nor any of their employees, nor any of their contractors, subcontractors, or their employees, makes any warranty, express or implied, or assumes any legal liability or responsibility for the accuracy, completeness or usefulness of any information, apparatus, product or process disclosed, or represents that its use would not infringe privately owned rights.

Prepared for U.S. Atomic Energy Commission under contract no. W-7405-Eng-48

DISTRIBUTION OF THIS DOCUMENT IS UNLIMITED

KB

DISCLAIMER

This report was prepared as an account of work sponsored by an agency of the United States Government. Neither the United States Government nor any agency Thereof, nor any of their employees, makes any warranty, express or implied, or assumes any legal liability or responsibility for the accuracy, completeness, or usefulness of any information, apparatus, product, or process disclosed, or represents that its use would not infringe privately owned rights. Reference herein to any specific commercial product, process, or service by trade name, trademark, manufacturer, or otherwise does not necessarily constitute or imply its endorsement, recommendation, or favoring by the United States Government or any agency thereof. The views and opinions of authors expressed herein do not necessarily state or reflect those of the United States Government or any agency thereof.

DISCLAIMER

Portions of this document may be illegible in electronic image products. Images are produced from the best available original document.

RQGPAK: A PACKAGE OF FORTRAN SUBPROGRAMS TO COMPUTE
MACROSCOPIC PROPERTIES OF RELATIVISTIC QUANTUM GASES*

J. Peter Vajk

University of California, Lawrence Livermore Laboratory
Livermore, California 94550

November 1974

ABSTRACT

The theory of relativistic gases of particles obeying Bose-Einstein, Fermi-Dirac, or Maxwell-Boltzmann statistics is reviewed briefly. A package of FORTRAN subprograms which compute the macroscopic thermodynamic properties of such gases for arbitrary non-vanishing densities and temperatures (provided the independent particle model is applicable) is then described. These are especially useful for astrophysical problems in which particle-antiparticle pairs must be correctly treated.

*Work was performed under the auspices of the U.S. Atomic Energy Commission.

Introduction

In astrophysical calculations, it is often necessary to compute the macroscopic properties of boson or fermion gases in quasi-static equilibrium. Such quantities as the number density n , the mass-energy density ρ , the isotropic pressure P , and the specific entropy s may be expressed as integrals over the particle distribution function in which the particle rest mass m , the spin statistical weight w , the chemical potential μ , and the kinetic temperature T appear as parameters.

In certain limiting cases, the expressions for the macroscopic properties of such quantum gases can be reduced to closed form, suitable for very rapid numerical evaluation in digital computer calculations. For sufficiently low densities, for example, one may approximate the Bose-Einstein or the Fermi-Dirac statistics of the actual particles under consideration by Maxwell-Boltzmann statistics to obtain either the classical or the ultra-relativistic closed-form expressions.

In the general case, however, such approximations are invalid. In this report we shall summarize the theory of relativistic quantum gases in quasi-static thermodynamic and statistical equilibrium. We shall describe a package of FORTRAN subprograms which compute the macroscopic properties of such gases for arbitrary non-vanishing densities and temperatures. These routines are particularly useful in the "moderately relativistic" regime in which none of the usual approximations is appropriate, but in which conditions are sufficiently extreme that particle-antiparticle pairs must be correctly treated. The subprograms are available at LLL from the Elephant storage system under the directory .910575:RQGPAK or from the author.

The Independent Particle Model

Let us begin our review by considering a gas of N identical particles in a container of volume V . If we assume that each particle moves about in the container independently of any other particle so that there are no collective motions of two or more particles ("quasi-particles"), then we may think of each particle as moving about in the container under the combined influence of the mean potentials induced by the $(N-1)$ other particles and of any externally imposed potentials, including those due to the container walls themselves. (Clearly, this assumption is invalid for sufficiently extreme conditions for any particular system.)

The various states of motion accessible to an individual particle in this independent particle model can then be described by various single-particle wave functions $\psi_1, \psi_2, \psi_\alpha, \dots$ which are indexed by any convenient complete set of quantum numbers which specify each allowable state of motion uniquely.

Under the above assumption, the state of the gas in the container is then described by a composite wave function Ψ which is a linear combination of products of N single-particle wave functions ψ_α , i.e.,

$$\Psi = \sum_{\alpha_1} \sum_{\alpha_2} \sum_{\alpha_3} \dots \sum_{\alpha_N} C_{\alpha_1 \alpha_2 \alpha_3 \dots \alpha_N} \psi_{\alpha_1} \psi_{\alpha_2} \psi_{\alpha_3} \dots \psi_{\alpha_N} \quad (1)$$

The coefficients in the above sum of products are not completely arbitrary, but are restricted by consideration of what happens to Ψ if we were to exchange, say, the particle in state α_i with the particle in state α_j . If

all the particles are identical and indistinguishable, such an exchange must leave unchanged the expectation values of all quantum numbers for the macroscopic state Ψ . Thus the magnitudes of $C \begin{smallmatrix} 1 & 2 & \dots & i & \dots & j & \dots & N \\ \alpha_1 & \alpha_2 & \dots & \alpha_i & \dots & \alpha_j & \dots & \alpha_N \end{smallmatrix}$ and $C \begin{smallmatrix} 1 & 2 & \dots & j & \dots & i & \dots & N \\ \alpha_1 & \alpha_2 & \dots & \alpha_j & \dots & \alpha_i & \dots & \alpha_N \end{smallmatrix}$ must be the same, but their complex phases may differ by any multiple of π without affecting any of the expectation values for the state Ψ . Thus we have two cases to consider for a gas of N identical and indistinguishable particles:

- (A) The matrix of coefficients $C \begin{smallmatrix} 1 & 2 & \dots & i & \dots & j & \dots & N \\ \alpha_1 & \alpha_2 & \dots & \alpha_i & \dots & \alpha_j & \dots & \alpha_N \end{smallmatrix}$ is symmetric under exchange of any two particles i and j . This leads to Bose-Einstein statistics and such particles are commonly called bosons.
- (B) The matrix of coefficients $C \begin{smallmatrix} 1 & 2 & \dots & i & \dots & j & \dots & N \\ \alpha_1 & \alpha_2 & \dots & \alpha_i & \dots & \alpha_j & \dots & \alpha_N \end{smallmatrix}$ is antisymmetric under exchange of any two particles i and j . This leads to Fermi-Dirac statistics and such particles are commonly called fermions.

(As an intellectual exercise, one may also consider the case of identical but distinguishable particles, in which case there is no such simple relation among the elements of the matrix of coefficients. This leads to Maxwell-Boltzmann statistics, and such particles are commonly referred to as classical particles. The discussion presented below is not properly legitimate for such particles; we shall only quote the results from classical statistical mechanics for these particles.)

For a given state vector Ψ for the gas, it is then meaningful to ask: what is the occupation number of the single-particle state ψ_α ? (Equivalently, what is the expectation value of the number of particles having the quantum numbers α ?) Consideration of the grand canonical ensemble, in which the container of gas exchanges both energy and particles with a reservoir described by a temperature kT and a chemical potential μ , leads to the result that

$$n_j = \frac{1}{\exp\left(\frac{\epsilon_j - \mu}{kT}\right) + \sigma} \quad (2)$$

where n_j is the occupation number for the single-particle state ψ_{α_j} , ϵ_j is the expectation value of the energy of the single-particle state ψ_{α_j} , kT is the temperature of the reservoir, μ is the chemical potential of the reservoir, and $\sigma = -1$ for bosons, $+1$ for fermions, and 0 for classical particles. All three cases are shown in Figure 1.

Figure 1. Occupation number as a function of energy. (a) Bosons. (b) Fermions. Allowable particle states are indicated by cross-hatching; antiparticle states are indicated by horizontal shading. (c) Classical particles.

Note that for bosons, μ must not be positive if the occupation number n_j is to be positive-definite.

For fermions, there is no such restriction on μ , and vacant single-particle states with $\epsilon < 0$ are interpreted as antiparticles ("holes in the Fermi sea") with energy $|\epsilon|$, the occupation number being confined to the range $0 \leq n_j \leq 1$. This limitation on n_j is a direct consequence of the antisymmetry of the matrix of coefficients $C_{\alpha_1 \alpha_2 \alpha_3 \dots \alpha_N}^{1 \ 2 \ 3 \ \dots \ N}$ and is often referred to as the Pauli exclusion principle: no more than one fermion can occupy a single-particle state ψ_α described by a complete set α of quantum numbers. The energy for which $n_j = 1/2$ is just μ , the chemical potential, which is also called the Fermi energy in discussions of fermion gases. If the fermions have non-vanishing rest mass m , there is a forbidden band: states with energies in the band $-mc^2 < \epsilon < +mc^2$ cannot be occupied. For very low temperature ($kT \rightarrow 0$ or $kT \ll mc^2$), the occupation number as a function of ϵ becomes a step function $\theta(\mu - \epsilon)$: all states with energy less than μ are occupied by one particle; all states with energy greater than μ are vacant. The fermion gas is then "completely degenerate", with all fermions packed into the lowest energy states allowed by the Pauli exclusion principle. In a gas of antiparticle fermions with chemical potential μ , the occupation number for a single-particle state with energy $|\epsilon|$ is the same as the occupation number for a single-particle state with energy $|\epsilon|$ in a gas of particle fermions with chemical potential $-\mu$.

For classical particles, the occupation number reduces to the familiar Boltzmann factor, and the chemical potential μ is simply a constant of normalization.

Macroscopic Properties of the Gas

Armed with the occupation number for a single-particle state ψ_α , we may now compute the macroscopic properties of the gas, provided we have sufficient information about the system to compute the density of single-particle states in the phase space of the system. By considering free particles moving in a large box with perfectly reflecting walls in the limit of an infinitely large box, one deduces the well-known result that the density of states for particles in a constant potential is

$$\rho(\epsilon) d^3p = \frac{4\pi w p^2 dp}{h^3} \quad (3)$$

per unit volume of the box where p is the momentum of the particle, w is the statistical weight of the particle, and h is Planck's constant. (This result is valid for both the Newtonian and the relativistic cases -- but in the latter case, p is the relativistic momentum.)

The macroscopic number density of the gas is then given by

$$n = \frac{N}{V} = \int_{p=0}^{\infty} \frac{4\pi w p^2 dp}{h^3} \frac{1}{\left[\exp\left(\frac{\epsilon(p) - \mu}{kT}\right) + \sigma \right]} \quad (4)$$

Similarly, the total internal energy density of the gas is

$$\rho = \frac{U}{V} = \int_{p=0}^{\infty} \frac{4\pi w p^2 dp}{h^3} \frac{\epsilon(p)}{\left[\exp\left(\frac{\epsilon(p) - \mu}{kT}\right) + \sigma \right]}, \quad (5)$$

while the isotropic pressure of the gas is

$$p = \int_{p=0}^{\infty} \frac{4\pi p^2 dp}{h^3} \frac{\frac{p^2 c^2}{3\epsilon(p)}}{\left[\exp\left(\frac{\epsilon(p) - \mu}{kT}\right) + \sigma \right]} \quad (6)$$

and the specific entropy is

$$s = \frac{S}{V} = \int_{p=0}^{\infty} \frac{4\pi p^2 dp}{h^3} \frac{k}{\left[\exp\left(\frac{\epsilon(p) - \mu}{kT}\right) + \sigma \right]} \log \left[\exp\left(\frac{\epsilon(p) - \mu}{kT}\right) + \sigma \right] \quad (7a)$$

in order to evaluate these integrals, we must specify the functional relation $\epsilon(p)$ between the momentum and the energy of a particle. In the classical (Newtonian) case, this is

$$\epsilon(p) = p^2/2m \quad ; \quad (8a)$$

in the relativistic case, it is

$$\epsilon(p) = [(pc)^2 + (mc^2)^2]^{1/2} \quad (8b)$$

Before proceeding to evaluate the integrals (4) through (7), we note that an alternative, easier to evaluate, expression for the entropy per unit volume can be obtained by considering the thermodynamic potential defined in thermodynamics as

$$\Omega = -PV = U - TS - \mu N \quad (9)$$

in terms of which the number density and the specific entropy are given by

$$n = -\frac{1}{V} \frac{\partial \Omega}{\partial \mu} \quad (10a)$$

$$s = \frac{S}{V} = -\frac{1}{V} \frac{\partial \Omega}{\partial T} \quad (10b)$$

(In the statistical mechanics treatment of the grand canonical ensemble, the thermodynamic potential is

$$\Omega = -kT \cdot \log \Xi \quad (11a)$$

where Ξ is the grand canonical partition function. For $\sigma \neq 0$, we obtain

$$\Omega = -\sigma \sum_i kT \cdot \log \left[1 + \sigma \exp \left(\frac{\epsilon_i - \mu}{kT} \right) \right] \quad (11b)$$

where the summation **extends** over all single-particle states ψ_{α_i} ..)

Using the above expressions, we find for the entropy per unit volume

$$\frac{S}{V} = k \int_{p=0}^{\infty} \frac{4\pi p^2 dp}{h^3} \cdot \frac{4\epsilon^2(p) + 3\mu\epsilon(p) - mc^2}{3\epsilon(p)} \cdot \frac{1}{\exp\left(\frac{\epsilon(p) - \mu}{kT}\right) + \sigma} \quad (7b)$$

an expression somewhat easier to work with than (7a).

Upon inserting the relativistic energy formula (8b) into the integrals (4) through (7), and upon defining dimensionless quantities a , b , and x as

$$a \equiv \frac{mc^2}{kT}, \quad b \equiv \frac{\mu}{kT}, \quad x \equiv \frac{\epsilon(p)}{kT} \quad (12)$$

respectively, the macroscopic properties of the gas may be reduced to the convenient forms

$$n = \frac{4\pi w}{(hc)^3} (kT)^3 \int_a^{\infty} \frac{x(x^2 - a^2)^{1/2} dx}{\exp(x - b) + \sigma}, \quad (13)$$

$$\rho = \frac{4\pi w}{(hc)^3} (kT)^4 \int_a^{\infty} \frac{x^2(x^2 - a^2)^{1/2} dx}{\exp(x - b) + \sigma}, \quad (14)$$

$$p = \frac{4\pi w}{(hc)^3} \frac{(kT)^4}{3} \int_a^\infty \frac{(x^2 - a^2)^{3/2} dx}{\exp(x - b) + \sigma} \quad (15)$$

$$s = k \cdot \frac{4\pi w}{(hc)^3} \frac{(kT)^3}{3} \int_a^\infty \frac{(4x^2 - 3nx - a^2)(x^2 - a^2)^{1/2} dx}{\exp(x - b) + \sigma} \quad (16)$$

The Chemical Potential

While such concepts as temperature, number density, pressure, and energy density are fairly clear intuitively, the meaning of the chemical potential is not so clear. If we consider a gas of classical particles, i.e., the case $\sigma = 0$, it is clear that $\exp(\mu/kT)$ can be factored out of each of the integrals (13) through (16), so that $\exp(\mu/kT)$ is just a normalization constant for the particle distribution function: pressure, energy density, and entropy density scale proportionately to the number density for any given temperature.

For fermions or for bosons, we cannot factor the integrals in this manner, but if we rewrite (9) in the form

$$\mu = \frac{U}{N} + P \frac{V}{N} - T \frac{S}{N} \quad (17)$$

we may interpret the chemical potential as the net energy which must be expended to add to the system one more particle in equilibrium with the original system. The first term is just the average energy of the particles already in the system, the second term is the compression work that must be done to make room for one more particle, and the last term is the heat which must be extracted from the system in order to perform the compression isothermally.

In practical calculations, it is usually more convenient to specify the state of a gas by giving its temperature and number density, for example, than to specify its temperature and chemical potential. Since each of the integrals (13) to (16) is monotonic in μ for fixed T , one can solve for μ given the temperature and any one of the number density, the energy density, the pressure, or the entropy density by successive trial and error methods. The remaining three macroscopic quantities can then be computed directly from the integrals, using the solution for the chemical potential together with the already specified temperature.

Numerical Technique

The integrals (13) through (16) cannot be reduced to elementary functions in closed form. The numerical integration technique which has been used in a set of FORTRAN subroutines is based on Gaussian integration. Note first that each of the integrands vanishes at $x = a$ and at $x = \infty$, reaching a single maximum at $x = x_m > 0$, where x_m depends on the parameters a and b according to some transcendental relation. Without elaborate analysis, however, it can be seen that x_m is comparable in magnitude to some one of the three quantities $|a|$, $|b|$, and $|a-b|$. Accordingly, as the simplest algorithm, we choose a number d equal to half the smallest non-vanishing one of the above three quantities, and begin successive eight-point Gaussian integrations from the lower limit of the integrals toward increasing x , using the value d as the interval of integration the first time, and then increasing the interval by 5% for each successive Gaussian integration. (The choice of d and the pattern of successive interval sizes is an arbitrary choice, but it is sufficiently conservative to guarantee good accuracy.)

After each integration step, the contribution of the latest interval to the total integral is examined: when the contribution becomes smaller than the limit of machine accuracy, the integration is terminated at a large (but finite) value of x . The subroutines which compute the integrands check whether x is sufficiently large to cause an underflow for the integrand; whenever this would occur, the integrand is set to zero, so that the next integration step would produce zero contribution to the total integration, even if the preceding step had resulted in a contribution which was still significant within the limits of the machine's word length.

For very elaborate calculations in which it is necessary to hold machine running time down to the absolute minimum possible, it is suggested that some experiments be done on optimizing the choice of integration intervals. As an indication of the running time needed with the present coding, a number of test calculations for elementary particles ranging in mass from the electron to the neutron at number densities in the range 10^{20} to 10^{35} cm^{-3} at temperatures from 10^9 to 10^{14} °K have always terminated the integrations in less than 15 integration intervals, often using as few as 8.

Because of the major algebraic simplifications in the integrals (13) through (16) which result for the special cases $\sigma = 0$ (classical particles); $\sigma = -1$ and $m = 0$ (photons, which are massless bosons); $\sigma = +1$, $m = 0$ (neutrinos, which are massless fermions); or $\sigma = +1$, $kT = 0$ (completely degenerate fermion gas), separate subroutines have been written for these cases. Some of these cases can be integrated analytically in closed form, so they do not require the triple subroutine structure of the general case, which

has a subroutine which controls the integration intervals and tallies the total integral, an eight-point Gaussian integration subroutine which is common to the entire package of routines, and a subroutine which evaluates the integrand for a given x .

Often in an astrophysical problem one needs derivatives with respect to chemical potential or with respect to temperature of some macroscopic thermodynamic variable. To avoid severe numerical noise problems, the derivatives of the integrals (13) through (16) have been computed analytically, and subroutines written to numerically integrate the resulting expressions.

Because of loss-of-significance due to subtraction of large, nearly equal numbers for cases at low temperatures or small values of the chemical potential, routines to compute the total energy, total pressure, total entropy, total number, and net number of particle and antiparticle gases in statistical equilibrium have also been programmed for the fermion cases. (The net electron charge, for example, is the total number of electrons minus the total number of positrons.)

One subroutine set is also included for computing the net number of particles for a mixture of two different fermion species having different masses but the same chemical potential. (This was needed in a particular problem in which the net proton-antiproton charge was to be balanced by the net muon-antimuon and electron-positron charge.) These routines are designed to prevent loss-of-significance at the single-precision level on the CDC 6600 and CDC 7600 computers. (Both of these machines use 60-bit words, corresponding to 15 or 16 decimal place accuracy.) For greater or lesser precision, the necessary changes in coding should be evident from the FORTRAN listings.

A table of all the subroutines and their calling arguments is given in Appendix I.

Acknowledgement

I wish to acknowledge the assistance of James F. Barbieri, who has helped debug these routines and who wrote several of the second-derivative subroutines in the package.

Appendix I. Function Subprograms and Calling Arguments

The tables in this appendix list the available function subprograms and their calling arguments. The principal routines have names consisting of three parts: (a) a two-character prefix designating the types of particles considered; (b) a single-letter root designating the macroscopic quantity being evaluated; and (c) a suffix which, if present, designates a certain derivative of the basic function. All results are in c.g.s. units.

Prefixes:

B0	bosons with chemical potential μ , temperature kT , rest mass mc^2 , and statistical weight w
CL	classical particles with chemical potential μ , temperature kT , rest mass mc^2 , and statistical weight w
F1	the positive energy states for a single species of fermion with chemical potential μ , temperature kT , rest mass mc^2 , and statistical weight w .
PH	photons - i.e., bosons with chemical potential zero, rest mass zero, temperature kT , and statistical weight 2
F0	neutrinos - i.e., fermions with chemical potential zero, rest mass zero, temperature kT , and statistical weight 1
FD	the positive energy states for a single species of fermion with chemical potential μ , zero temperature (i.e., completely degenerate), rest mass mc^2 , and

statistical weight w . (If $\mu < 0$, the function returns the value $-FD(|\mu|)$, i.e., the number of negative energy states with a minus sign.)

For fermions, since there are both positive and negative energy states, a number of additional functions are useful, and have the following prefixes and definitions:

$$G1(\mu) \equiv F1(\mu) + F1(-\mu)$$

$$H1(\mu) \equiv F1(\mu) - F1(-\mu)$$

$$H2(\mu, m_1, m_2) \equiv H1(\mu, m_1) + H1(\mu, m_2)$$

Roots:

- N number density (cm^{-3})
- U relativistic mass-energy density (erg cm^{-3})
- S entropy density ($\text{erg (K}^\circ)^{-1} \text{cm}^{-3}$)
- P pressure (dyne cm^{-2})

Suffixes

- C first derivative with respect to chemical potential
- CC second derivative with respect to chemical potential
- CT mixed second derivative with respect to chemical potential and temperature kT
- T first derivative with respect to temperature kT
- TT second derivative with respect to temperature kT

Besides the eight-point Gaussian integration routine GAUSS, several additional auxiliary mathematical routines are listed in Table V. These routines compute certain combinations of hyperbolic functions which appear

repeatedly in the evaluation of the macroscopic properties of these gases, and have been designed to minimize loss-of-significance errors due to the finite word-length of any computer.

TABLE I. NUMBER DENSITY

Function names						Calling arguments
BON	BONC	BONCC	BONCT	BONT	BONTT	(μ, kT, mc^2, w)
CLN	CLNC	CLNCC	CLNCT	CLNT	CLNTT	
FLN	FLNC	FLNCC	FLNCT	FLNT	FLNTT	
GLN	FLNC	GLNCC	GLNCT	GLNT	GLNTT	
H1N	H1NC	H1NCC	H1NCT	H1NT	H1NTT	
H2N	H2NC	H2NCC	H2NCT	H2NT	H2NTT	
PHN				PHNT	PHNTT	$(\mu, kT, m_1c^2, m_2c^2, w)$
F0N				F0NT	F0NTT	(kT)
FDN	FDNC	FDNCC	FDNCT	FDNT	FDNTT	(kT)
						(μ, mc^2, w)

TABLE II. ENERGY DENSITY

Function names						Calling arguments
BOU	BOUC	BOUCC	BOUCT	BOUT	BOUTT	(μ, kT, mc^2, w)
CLU	CLUC	CLUCC	CLUCT	CLUT	CLUTT	
FLU	FLUC	FLUCC	FLUCT	FLUT	FLUTT	
GLU	GLUC	GLUCC	GLUCT	GLUT	GLUTT	
PHU				PHUT	PHUTT	
F0U				F0UT	F0UTT	
FDU	FDUC	FDUCC	FDUCT	FDUT	FDUTT	(kT)
						(kT)
						(μ, mc^2, w)

TABLE III. PRESSURE

Function names						Calling arguments
BOP	BOPC	BOPCC	BOPCT	BOPT	BOPTT	(μ, kT, mc^2, w)
CLP	CLPC	CLPCC	CLPCT	CLPT	CLPTT	
FLP	FLPC	FLPCC	FLPCT	FLPT	FLPTT	
GLP	GLPC	GLPCC	GLPCT	GLPT	GLPTT	
PHP				PHPT	PHPTT	(kT)
FOP				FOPT	FOPTT	(kT)
FDP	FDPC	FDPC	FDPCT	FDPT	FDPTT	(μ, mc^2, w)

TABLE IV. ENTROPY DENSITY

Function names						Calling arguments
BOS	BOSC	BOSCC	BOSCT	BOST	BOSTT	(μ, kT, mc^2, w)
CLS	CLSC	CLSCC	CLSCT	CLST	CLSTT	
FLS	FLSC	FLSCC	FLSCT	FLST	FLSTT	
GLS	GLSC	GLSCC	GLSCT	GLST	GLSTT	
PHS				PHST	PHSTT	(kT)
FOS				FOST	FOSTT	(kT)
FDS	FDSC	FDSCC	FDSCT	FDST	FDSTT	(μ, mc^2, w)

TABLE V. AUXILIARY MATHEMATICAL SUBPROGRAMS

FUNCTION GAUSS (FFF, A, B)

Performs eight-point Gaussian integration of any programmed FUNCTION FFF (X) over the integration interval $A \leq X \leq B$.

FUNCTION TWOSNH (X)

Evaluates $(e^X - e^{-X})$ by either an exponential calculation or a series calculation. The series calculation is performed in a separately callable entry point, SR2SNH (X), for $|X| < 0.3466$. For larger values of X, the exponential calculation is valid past the limits of the ORDERLIB exponential function, which stops about 18 orders of magnitude below a machine overflow condition.

FUNCTION SOVCPC (Z, B)

Evaluates the function $\sinh(B)/[\cosh(Z) + \cosh(B)]$ for all values of Z and B from machine underflow to machine overflow, whether positive, negative, or zero.

FUNCTION COVCPC (Z, B)

Evaluates the function $\cosh(B)/[\cosh(Z) + \cosh(B)]$ for all values of Z and B from machine underflow to machine overflow, whether positive, negative, or zero.

FUNCTION CC1CPC2 (Z, B)

Evaluates the function $[\cosh(Z) \cosh(B) + 1]/[\cosh(Z) + \cosh(B)]^2$ for all values of Z and B from machine underflow to machine overflow, whether positive, negative, or zero.

Distribution

J. D. Anderson
J. F. Barbieri
G. A. Burginyon
J. S. Chang
G. Chapline
R. G. Couch
G. H. Dahlbacka
H. E. deWitt
P. G. Eltgroth
J. A. Fleck
R. J. Gelinas
H. C. Graboske
R. J. Grader
W. Grasberger
J. W. Green
R. C. Haight
P. T. Herman
R. W. Hill
J. T. Knox
J. M. LeBlanc
W. A. Lokke
W. E. Loewe
C. Max
M. S. Maxon
M. M. May
T. M. Palmieri
F. J. Rogers
F. D. Seward
J. W. Shearer
C. A. Sherman
J. C. Stevens
C. B. Tarter
E. Teller
R. F. Tooper
A. Toor
J. P. Vajk
R. E. vonHoldt
R. L. Wagner
T. Wainwright
T. A. Weaver
J. C. Weisheit

H. I. West
C. K. Westbrook
R. H. White
M. L. Wilkins
J. R. Wilson
L. L. Wood
D. A. Young
G. B. Zimmerman

T-Division (3)
TID (5)
CTR Library (5)
NMS (5)

NOTICE

"This report was prepared as an account of work sponsored by the United States Government. Neither the United States nor the United States Atomic Energy Commission, nor any of their employees, nor any of their contractors, subcontractors, or their employees, makes any warranty, express or implied, or assumes any legal liability or responsibility for the accuracy, completeness or usefulness of any information, apparatus, product or process disclosed, or represents that its use would not infringe privately-owned rights."