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BEAM-BEAM RESONANCES EXCITED
BY
PROTON-PROTON COLLISIONS

M. Month
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ABSTRACT

Observations at the ISR on the beam-beam interaction of two coasting proton beams give clear evidence of non-linear resonance excitation. The two main features are: (1) the observation of odd-ordered resonances excited by small vertical orbit misalignments at the collision points; and (2) the strong dependence of the loss rate on the position of the physical aperture boundary. Although there is a hint of the 8th order resonance, the strongest effects are seen in the regions of a stack corresponding to the odd-ordered resonances, in particular the 5th and the 7th. To describe these observations, a theory of beam-beam resonances is proposed, one which takes proper account of their peculiar nature. The basic mechanism for beam growth and loss is the familiar lock-in process. Loss rates are estimated as a function of the magnitude of the misalignments and the distance of the physical aperture boundary. The theory of lock-in requires a time variation of tune. In a coasting beam, there is no obvious periodic modulation, such as is caused by synchrotron motion in a bunched beam. Scattering processes do however produce such a modulation. In particular, intrabeam scattering causes a momentum diffusion, which, via the machine chromaticity, induces a drifting tune. Particles lock-in, drift through the resonance and are transported to the aperture limit. In addition to tune drift, the diffusion process also provides a feeding mechanism whereby particles are fed into the resonance tune range and from there are channelled to the aperture boundary as their betatron amplitude grows. The results obtained here are in general agreement with ISR observations. Some extrapolations to future p-p machines are given.

I. INTRODUCTION

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Observations at the ISR on the beam-beam interaction of two coasting proton beams give clear evidence of the excitation of non-linear resonances.⁽¹⁻⁴⁾ The two main features are: (1) the observation of odd-ordered resonances excited by small orbit misalignments at the collision points; and (2) the strong dependence of the loss rate on the position of the physical aperture boundary. Although there is a hint of the 8th order resonance, the strongest effects are seen in the odd-ordered resonances, in particular the 5th and the 7th.

It has become apparent that resonant beam loss at the ISR is mediated by some feeding process. It has been pointed out that scattering processes can induce momentum diffusion and through the chromaticity tune drift.⁽⁵⁾ In this way particles are fed into the resonant tune range. Momentum diffusion arising from intrabeam scattering is the dominant of such scattering processes in the ISR.

However, high-order non-linear resonances are stable in a system with static tune.⁽⁶⁾ In a coasting proton beam, where there is no periodic tune modulation, such as is caused by the synchrotron motion in a bunched beam, one might expect that in the presence of sufficient non-linear detuning, the resonances would be quite harmless, producing only a slight betatron amplitude modulation. Bringing the particles into resonance is not enough. Thus, it has been suggested that in addition to feeding the resonance, the combination of tune diffusion and resonant streaming can produce amplitude growth sufficient to reach the physical aperture boundary.^(7,8)

One approach to this problem, used in References 7 and 8, is to include the effect of the resonance as streaming terms in a diffusion equation for the tune drift. The main drawback with this method is that it is difficult to solve the equation using the proper resonance characteristics. Drastic approximations must be made. For example, the betatron amplitude dependence of the resonant width is neglected in Ref. (7). In Ref. (8), an approximate linear dependence is assumed. To the extent that the real effect depends on the details of the resonance characteristics, specifically the amplitude dependence of both the resonance width and the non-linear detuning, this approach could give distorted results.

Our approach is in direct contrast. Rather than superimpose the resonance as a streaming term in the diffusion equation, we treat the diffusion as a mechanism for tune variation and we then determine the influence of this tune variation on the behavior of a beam near resonance. In this way the amplitude dependence of the resonant width and non-linear detuning can be properly included. As we will see, this amplitude dependence is a critical factor in the theory we propose to explain the ISR observations.

The key postulate of our analysis is that the mechanism by which the beam-beam resonances influence the betatron amplitude of a particle is lock-in. Thus, a particle passing through resonance either will be trapped in the passing island thereby changing its amplitude or it will remain essentially unaffected. This is not entirely correct for two reasons: In the first place, even if the islands pass by rapidly, there may be a residual effect. We presume "trapping" to be the dominant source of amplitude growth and we will simply neglect the influence of "fast crossing."⁽⁶⁾ Secondly, there is obviously no sharp boundary for the island speed which can give us a well-defined trap/no-trap criterion. The boundary is undoubtedly fuzzy. This being the case, results and conclusions sensitive to the precise boundary location must be interpreted with this limitation in mind. However, it is not our purpose here to provide accurate loss rates, but only order-of-magnitude estimates. Furthermore, it is our primary aim here to show qualitatively how the special features of beam-beam resonances manifest themselves in beam growth and beam loss. To achieve these ends, our assumption of a sharp lock-in boundary seems reasonable.

The particle loss mechanism we propose for coasting proton beams is the migration in betatron amplitude of particles trapped in islands whose motion is governed by the tune drifting in a random walk. In addition to the manifest properties of the drifting tune, a knowledge of the rate of tune variation for particles in their many different paths is also required. For it is this rate of tune change which determines whether or not a given particle will be trapped in an island drifting by or whether it will be bypassed by the island or whether it will be dropped by an island which has become leaky, this occurring if the tune change rate increases sufficiently. Let us consider the quantity $\delta\nu_{\text{rev}}$, which is the rms tune drift in one particle revolution. The question

is whether this quantity gives information about the rate of tune variation along the multitude of paths as the diffusion process proceeds. We claim that it does in the following sense: Let D_{rev} = the diffusion coefficient in units of (tune)² per revolution. In terms of the usual diffusion coefficient, D_p in units (momentum drift)²/second, we have

$$D_{rev} = \xi^2 D_p T_{rev} \quad (1.1)$$

where ξ is the absolute chromaticity,

$$\xi = p \frac{\partial \nu}{\partial p} \quad , \quad (1.2)$$

and T_{rev} is the revolution period of the machine; and thus, the rms tune drift rate over r revolution is simply⁽⁹⁾

$$(\delta \nu)_{r \text{ rev}} = \sqrt{\frac{2D_{rev}}{r}} \quad . \quad (1.3)$$

If $r = 1$, then we have,

$$\delta \nu_{rev} = \sqrt{2D_{rev}} = \sqrt{2\xi^2 D_p T_{rev}} \quad . \quad (1.4)$$

Now, we must emphasize that the number of momentum scatterings per revolution is a very large number, which implies two things: first, that the step size in the tune random walk is much smaller than $\delta \nu_{rev}$ and second, that the sum of the momentum changes in absolute value is much larger than the rms momentum drift (corresponding in tune to $\delta \nu_{rev}$). Thus, although the instantaneous momentum of any given particle changes many times during a revolution, they tend to cancel and we end up with a much smaller drift. In terms of tune, these momentum steps are equivalent to small quadrupoles tending to cancel each other. The tune is by definition a vector sum of these quadrupoles over a revolution and thus the tune shift is represented, not by the quadrupole fluctuations within the revolution period, but rather by the sum, or the tune drift. In other words the distribution in $\delta \nu$ after 1 revolution (many steps), is a good representation of the distribution of tune speeds. As representative of this distribution, we can choose the rms value, $\delta \nu_{rev}$, and use this as the rate of tune variation during any particle diffusion path. Actually, we can

expect that for any particle path, there is a 95% probability that the tune speed will not be larger than twice this value. Averaging over more than 1 revolution ($\tau > 1$) would result in a smaller mean tune speed.

In section 2, we review the trapping theory⁽¹⁰⁾ for beam-beam resonances,⁽¹¹⁾ emphasizing the distinctive amplitude dependence of both the non-linear detuning and the resonant widths. We use an approximate "head-on" model⁽¹¹⁾ to simulate the beam-beam interaction. We discuss the difference between odd-ordered and even-ordered beam-beam resonances. We show how lock-in and particle transport can occur as a consequence of the drifting (diffusing) tune in colliding, coasting proton beams. In section 3, we combine the beam-beam resonance theory with tune diffusion to construct a model for beam growth and loss. We relate the diffusion coefficient to the parameters for intrabeam scattering.⁽¹²⁾ Using a random walk model in the presence of an absorbing barrier,⁽⁹⁾ we estimate the loss rate to the physical aperture boundary due to an odd-order beam-beam resonance and discuss the closed orbit misalignment tolerance at the collision points. We then show qualitatively how beam halos can form. Applying our model to the ISR, we find our results in general agreement with ISR observations.⁽¹⁻⁴⁾ Finally, we make some extrapolations to future p-p colliding beam machines.⁽¹³⁾

II. TRAPPING THEORY FOR BEAM-BEAM RESONANCES

Invariant for Beam-Beam Resonances

The resonant invariant for a beam-beam resonance of order p can in general be written⁽¹¹⁾

$$C = \frac{\bar{v}}{M\xi} \alpha + U(\alpha) + \left[\int_E V_p(\alpha) + \int_O \bar{V}_p(\alpha) \right] \cos p\psi \quad (2.1)$$

$$\text{where } \bar{v} = v - m/p \quad (2.2)$$

$\bar{v}$ is the deviation of the tune, v , from the resonance,

p is the order of the resonance,

m is the azimuthal harmonic exciting the resonance,

M is the number of collisions in the ring,

ξ is the beam-beam strength of each collision (assumed identical),

α is the squared amplitude in units of rms squared amplitude, I_{rms} ,

$$\alpha = I/I_{\text{rms}} \quad , \quad (2.3)$$

with

$$I_{\text{rms}} = b^2/\beta^* \quad , \quad (2.4)$$

β^* is value of the β function⁽¹⁴⁾ at the collision point,

$$b = \sqrt{2} \sigma^* \quad , \quad (2.5)$$

σ^* is the rms beam size at the crossing point, and ψ is a phase canonically conjugate to the squared amplitude,⁽¹¹⁾ α .

The quantities Γ_E and Γ_O are the effective resonant excitation terms for odd and even ordered resonances, respectively:

$$\Gamma_E = \frac{1}{M} \left| \sum_{\ell=0}^{M-1} \left(\frac{\delta \xi}{\xi} \right)_\ell e^{im\theta_\ell} \right| \quad (2.6)$$

$$\Gamma_O = \frac{1}{M/2} \left| \sum_{\ell=0}^{M-1} \sigma_\ell e^{im\theta_\ell} \right| \quad , \quad (2.7)$$

where $\left(\frac{\delta \xi}{\xi} \right)_\ell$ is the deviation of the strength of the ℓ^{th} intersection from the mean strength,

θ_ℓ is the azimuth of the ℓ^{th} intersection from some arbitrary reference, and σ_ℓ is the deviation of the closed orbit of the weak beam with respect to the strong beam center in units of rms beam size at the collision point,

$$\sigma_\ell = \frac{\sigma_{EO}}{\sigma^*} \quad . \quad (2.8)$$

We have assumed that the beams cross at an angle α_c and the resonance is in the non-crossing plane. This situation corresponds approximately to head-on collisions with the effective size in the crossing dimension much greater than the beam size in the non-crossing one. Thus, the strength parameter, ξ , is given by

$$\xi = \left(\frac{I}{ec} \right) \frac{2r_p \beta^*}{\pi \gamma \alpha_c \sigma^*} \quad , \quad (2.9)$$

where I is the current in the coasting beam,

γ is the energy in proton mass units,

and r_p is the classical proton radius. Also, the beam-beam resonance functions are, (11)

$$U(\alpha) = 2 \left\{ e^{-\alpha/2} \left[(1 + \alpha) I_0(\alpha/2) + \alpha I_1(\alpha/2) \right] - 1 \right\} , \quad (2.10)$$

$$V_p(\alpha) = -4 (-1)^{\frac{p}{2}} \frac{e^{-\alpha/2}}{p^2 - 1} \left[(1 + \alpha) I_{p/2}(\alpha/2) + \alpha I'_{p/2}(\alpha/2) \right] , \quad (2.11)$$

and

$$\bar{V}_p(\alpha) = (-1)^{\frac{p-1}{2}} \sqrt{2} \int_0^{\alpha/2} dx \frac{e^{-x}}{\sqrt{x}} \left\{ \left(1 - \frac{1}{p}\right) I_{\frac{p+1}{2}}(x) - \left(1 + \frac{1}{p}\right) I_{\frac{p-1}{2}}(x) \right\} , \quad (2.12)$$

where we have taken $U, V_p, \bar{V}_p \rightarrow 0$ as $\alpha \rightarrow 0$.

Note that $V_p(\alpha) \neq 0$ only if p is even,

and $\bar{V}_p(\alpha) \neq 0$ only if p is odd.

In other words, closed orbit deviations excite odd-ordered resonances, while deviations in the strength of the interactions excite even-ordered resonances.

Note also that $U(\alpha) \approx \alpha$ as $\alpha \rightarrow 0$. This term corresponds to the linear tune shift from the beam-beam interaction. The Bessel functions and their derivatives are defined by

$$I_n(x) = \frac{1}{\pi} \int_0^\pi e^{x \cos \theta} \cos n \theta d\theta , \quad (2.13)$$

and

$$I'_n(x) = \frac{1}{2} \left(I_{n+1}(x) + I_{n-1}(x) \right), \quad n \geq 1 . \quad (2.14)$$

Criteria for Trapping

The mechanism we propose for how beam-beam resonances can change particle amplitudes is the lock-in process.^(6,11) There are 3 particle parameters which determine the effect of the resonance:

$\bar{\nu}$, the tune with respect to the resonant tune,

$\delta \nu_{\text{rev}}$, the rate of tune variation,

and α , the particle amplitude (i.e., betatron amplitude squared).

The value of $\bar{\nu}$ determines the amplitude of the lock-in islands, α_F . Particles are influenced by the resonance only when their amplitudes are in the vicinity of the islands, i.e.,

$$\alpha \approx \alpha_F \quad . \quad (2.15)$$

As has been previously pointed out [See for example Ref.(11)], the detuning function $U(\alpha)$ is much larger than the resonance functions $V_p(\alpha)$ and $\bar{V}_p(\alpha)$. In this case the islands extend only over a small amplitude range and their position in amplitude, α_F , is determined almost entirely by the detuning function. In fact, the island position is obtained from the solution of the equation,

$$\frac{\partial}{\partial \alpha} U + U'(\alpha) = 0 \quad , \quad (2.16)$$

where $U'(\alpha)$ is the derivative of U and is given by

$$U'(\alpha) = e^{-\alpha/2} \left[I_0(\alpha/2) + I_1(\alpha/2) \right] \quad . \quad (2.17)$$

Note that we can without ambiguity use the symbol α to correspond to both island position and particle amplitude, as long as we remember that only particles at the island position are to be considered. That is, (2.15) is implicitly assumed. However, even if at some instant a particle has amplitude α and tune $\bar{\nu}$ approximately satisfying (2.16), lock-in or trapping need not necessarily follow. In other words, the condition (2.16) is not necessarily maintained at later times. In order for trapping to result, the resonant amplitude must be changing at least as fast as the passing islands. This leads to a trapping criterion,⁽¹¹⁾ which can be stated as an upper limit on the rate of tune variation: for even-ordered resonances, we have

$$\frac{\delta v_{\text{rev}}}{M^2 \xi^2 \Gamma_E} \leq R_p(\alpha) \quad , \quad (2.18)$$

while for odd ordered resonances,

$$\frac{\delta v_{\text{rev}}}{M^2 \xi^2 \Gamma_0} \leq \bar{R}_p(\alpha) \quad , \quad (2.19)$$

where we have written

$$R_p(\alpha) = 2\pi p |v_p(\alpha) U''(\alpha)| \quad , \quad (2.20)$$

$$\text{and } \bar{R}_p(\alpha) = 2\pi p |\bar{v}_p(\alpha) U''(\alpha)| \quad . \quad (2.21)$$

Note that the tune rate criterion is amplitude dependent. For example, given δv_{rev} at different times, the boundary implied in (2.18) or (2.19) might at a particular value of α be crossed, thus changing a "capture" condition into a "drop" condition or vice versa. The quantity U'' can be obtained from U' by differentiation, giving

$$U''(\alpha) = -\frac{1}{\alpha} e^{-\alpha/2} I_1(\alpha/2) \quad . \quad (2.22)$$

The resonance functions are universal in the sense that they are strength independent and functions of amplitude only. We plot in Fig. 1 the function

$$D = -U' \quad , \quad (2.23)$$

and the functions R_p and $\bar{R}_p$ in Figs. 2 and 3 respectively for various resonances. Instead of the squared amplitude, α , we use for the independent variable σ , the displacement in units of rms displacement. It is related to α by

$$\sigma = \sqrt{2\alpha} \quad . \quad (2.24)$$

The function $D(\sigma)$ determines the range over which trapped particles can be transported. If $\tilde{\nu}_i$ and $\tilde{\nu}_f$ are initial and final tunes respectively, and if σ_i corresponds to the amplitude of a particle trapped at $\tilde{\nu}_i$, then the amplitude to which this particle can be transported, σ_f , can be found from

$$D(\sigma_f) - D(\sigma_i) = \frac{\tilde{\nu}_i - \tilde{\nu}_f}{M \xi} \quad . \quad (2.25)$$

This gives the limits of the transport for a given tune shift. For a particle to actually move with the islands, the rate criterion must be satisfied. Namely, for an even-ordered resonance,

$$\delta\nu_{\text{rev}} \leq M^2 \xi^2 \Gamma_E R_p(\sigma) \quad (2.26)$$

must be satisfied for all σ along the path from σ_i to σ_f ; while for an odd-ordered resonance

$$\delta\nu_{\text{rev}} \leq M^2 \xi^2 \Gamma_O \bar{R}_p(\sigma) \quad (2.27)$$

must be satisfied.

Trapping Probability

If we take account of the distribution of particles in the amplitude, α , we can estimate the fraction of particles that lock-in to the resonance as the islands pass through the beam. Consider an odd-ordered resonance. At any instant particles in a certain restricted amplitude range can lock-in. This range is simply the amplitude extent of the islands; or put another way, it is the maximum amplitude modulation for trapped particles. At a given amplitude α , we have for the range,⁽¹¹⁾

$$\alpha_{\pm} = \alpha \pm 2\sqrt{\Gamma_O \left| \frac{\tilde{\nu}_p}{\tilde{\nu}_\sigma} \right|} \quad . \quad (2.28)$$

If we assume a Gaussian distribution in transverse particle displacement, then the fraction of total particles contained within this amplitude range is given by

$$P_T = e^{-\alpha_-^2} - e^{-\alpha_+^2} \quad . \quad (2.29)$$

If the particles at this value of α satisfy the trapping criterion (2.27), then this number represents the fraction of particles that can be instantaneously trapped. Actually it is a small overestimate, since the islands do not occupy the entire ring between α_- and α_+ . Actually, we are not so much interested in the instantaneous trapping, but rather in the cumulative effect after the islands pass through some amplitude range, perhaps a few rms amplitudes. This is clearly a complex process with particles entering and leaving the islands (at the unstable fixed point which are always non-adiabatic regions and therefore potential leaky regions) as the islands move through the beam. However, the dynamical system is Hamiltonian and must satisfy Liouville's well known theorem, stating that the local phase space density does not change. Thus we might expect that over a small area of phase space, as for example the island ring, the average density, as distinct from the local density, will not vary too much. It follows that we might estimate the number of trapped particles by some typical value of the "instantaneous trapped fraction." That is, an estimate of the trapping probability could be some typical value of P_T in the region of α where trapping can occur. We plot in Fig. 4 P_T as a function of σ for various α' -ordered beam-beam resonances of various strengths. Notice the slow maximum. As the islands pass through this region of σ the ring is increasing in size but the phase area which can be added to the islands has a lower density. It is not likely that the fraction P_T will increase significantly from this maximum value. It could of course be somewhat higher if just the right exchange takes place before the maximum. We disregard this possibility and take as an approximate trapping probability the maximum value of the instantaneous trapping fraction in the region of σ where lock-in can occur, i.e., where (2.27) is satisfied. Such an approach to a quantitative estimate of trapping has been successfully applied to the case of conventional resonances in Ref. (10). Thus, we take for the trapping probability,

$$P_T = P_T(\alpha) \Big|_{\text{max. value of (2.29) in trapping region.}} \quad (2.30)$$

Uncorrelated Errors

If there are variations in the M collision strengths and if we assume them to be uncorrelated, we can relate Γ to the rms value of the error. Consider, for example, the closed orbit errors at the intersections, σ_ℓ ,

to be uncorrelated. Then, (2.7) for Γ_o becomes

$$\Gamma_o = \frac{1}{M/2} \sqrt{\sum_{\ell=0}^{M-1} \sigma_{\ell}^2} .$$

If we define the rms deviation $\langle \sigma \rangle_{rms}$ by

$$\langle \sigma \rangle_{rms}^2 = \frac{1}{M} \sum_{\ell=0}^{M-1} \sigma_{\ell}^2 , \quad (2.31)$$

then

$$\Gamma_o = \frac{1}{\sqrt{2M}} \langle \sigma \rangle_{rms} . \quad (2.32)$$

We measure the strength of odd-ordered resonances, which are excited by these closed orbit deviations, by the rms value. The trapping probability, P_T , which is a function of the resonant strength, shown in Fig. 4, is plotted against the amplitude for various values of $\langle \sigma \rangle_{rms}$.

III. BEAM-BEAM EFFECT IN COASTING PROTON BEAMS

The Diffusion of Tune from Intrabeam Scattering

The multiple scattering of protons in a beam due to Coulomb forces, that is, intrabeam scattering, causes a diffusion in longitudinal momentum.¹² The diffusion coefficient in the lab frame of reference is derived in Ref. 12 and can be written

$$D_p = \frac{d}{dt} \left(\frac{\delta p_{rms}}{p} \right)^2 = \frac{\sqrt{\pi} c r_p^2 \rho}{\beta^3 \gamma^3 \varphi_{rms}} f(x_o) , \quad (3.1)$$

where δp_{rms} is the rms momentum spread,
 p is the longitudinal momentum,
 ρ is the average particle density,
 β is the velocity in units of c ,
 γ is the energy in proton mass units,
 r is the classical proton radius ($= 1.54 \times 10^{-18}$ m),
 φ_{rms} is the rms dispersion in the betatron angle.

The quantities δp_{rms} , p and ρ are to be evaluated in the lab frame of reference. The definitions of ρ , φ_{rms} and $f(x_0)$ are:

$$\rho = N/V \quad , \quad (3.2)$$

where N is the number of particles in the volume V ,

$$V = Cdh \quad , \quad (3.3)$$

$C = 2\pi R$ is the circumference,

d is the total beam width,

$$d = \sqrt{4\epsilon_x \beta_x + \left(x_p \frac{\delta p}{p}\right)^2} \quad , \quad (3.4)$$

h is the total beam height,

$$h = \sqrt{4\epsilon_y \beta_y} \quad (3.5)$$

with β_x , β_y the average horizontal and vertical β -function values,

$$\beta_x = R/v_x \quad , \quad \beta_y = R/v_y \quad , \quad (3.6)$$

ϵ_x , ϵ_y the 95% horizontal and vertical emittances (Area = $\pi\epsilon$), v_x , v_y the horizontal and vertical tunes;

$$\varphi_{rms} = \frac{1}{2} \sqrt{\frac{\epsilon_x}{\beta_x} + \frac{\epsilon_y}{\beta_y}} \quad ; \quad (3.7)$$

and

$$f(x_0) = \int_{x_0}^{\infty} \frac{1}{x} e^{-x} \ln \frac{x}{x_0} dx \quad , \quad (3.8)$$

where

$$x_0 = \frac{2r}{\varphi_{rms}} \left(\frac{\rho}{\beta_y} \right)^{1/3} \quad . \quad (3.9)$$

The number of particles N is related to the current I by

$$I = \frac{e N c \beta}{C} \quad . \quad (3.10)$$

The function f can be estimated from a graph given in Ref. 12, which we sketch in Fig. 5.

By way of the chromaticity, ξ_{ch} , defined as tune change per unit fractional momentum change, momentum is related to betatron tune. The distributions can be connected through the relation

$$\delta v_{rms} = \xi_{ch} \frac{\delta p_{rms}}{p} \quad . \quad (3.11)$$

Thus, the diffusion coefficient for tune

$$D_v = \xi_{ch}^2 D_p \quad . \quad (3.12)$$

As already stated in section 1, the rms tune spread in 1 revolution is given by

$$\delta v_{rev} = \sqrt{2D_v T_{rev}} = \xi_{ch} \sqrt{2D_p T_{rev}} \quad . \quad (3.13)$$

Further, we have argued that this quantity, δv_{rev} , can be interpreted as the mean rate of tune variation. This means that at any given instant, we expect about 95% of the particles to have a tune spread less than twice this value. Of course, a small amount will have a faster speed; on the other hand, many will have a speed significantly slower than the mean.

Recall that we have presumed that the number of collisions per revolution, N_{rev} , is large. This is true for the cases of interest here. An expression for N_{rev} can be obtained with the help of Ref. 12:

$$N_{rev} = \sqrt{\pi} C \left(\rho / \beta \gamma \right)^{1/3} \phi_{rms} \quad . \quad (3.14)$$

Random Walk with an Absorbing Barrier

We conceive of particles being fed into the resonance by a random walk in tune via intrabeam scattering, transferred to the aperture (in betatron amplitude) by the resonance and thereby lost. The simple random walk in tune must be modified since not all particles crossing the resonance will be trapped and transported to larger amplitudes. The two restrictions are those already mentioned, namely the rate of tune variation must be such as to satisfy the trapping criterion and secondly, the amount of particles trapped must be consistent with the preservation of local particle density in betatron phase space. In fact, we have defined P_T , see Eq. (2.30), with exactly these two constraints in mind. If we assume that the two processes, trapping and random walk are independent, we can estimate the loss rate by simply multiplying the loss rate due to diffusion by P_T .

Thus, the loss rate for a set of particles at tune ν in a range $d\nu$ at time T is given by⁽⁹⁾

$$\frac{d^2 N}{dT d\nu} = - \frac{\rho(\nu) P_T(\nu_A - \nu)}{2T(\pi D_\nu T)^{1/2}} e^{-(\nu_A - \nu)^2 / 4D_\nu T} \quad , \quad (3.15)$$

where ν_A is the tune corresponding to the absorbing barrier for a particle trapped,

D_ν is the tune diffusion coefficient, related to the intrabeam scattering parameters through (3.12) and (3.1),

N is the total number of particles in the beam, and $\rho(\nu)$ is the tune density.

For a uniform density beam in tune,

$$\rho = N/\Delta \quad , \quad (3.16)$$

with Δ the total beam tune spread.

Notice that (3.15) for the loss rate depends only on the distance of ν from the tune aperture ν_A . Now, to obtain the total loss rate, it is not sufficient simply to integrate over ν , since in order to be trapped, particles must pass through the resonant tune, ν_R . Let us assume therefore that the resonant tune lies within the beam, meaning

$$\nu_- < \nu_R < \nu_+ \quad , \quad (3.17)$$

where ν_+ and ν_- are the upper and lower edges of the beam in tune, with

$$\Delta = \nu_+ - \nu_- \quad . \quad (3.18)$$

Since the loss rate depends only on the distance to the tune aperture, then we can accomodate the case where ν lies between ν_R and ν_A at $T = 0$ by reflecting the distribution around ν_R , thus forcing the particles to travel the increased distance to ν_R and then to ν_A . Thus, with $\nu_A > \nu_R$, we can integrate (3.15) over ν with the replacements

$$\nu_A - \nu \rightarrow \nu_A - \nu \text{ if } \nu_- \leq \nu \leq \nu_R \quad ,$$

$$\text{and } \nu_A - \nu \rightarrow \nu_A - 2\nu_R + \nu \text{ if } \nu_R \leq \nu \leq \nu_+ \quad .$$

In the case that the distance from the resonance to the aperture,

$$\delta = \nu_A - \nu_R \quad , \quad (3.19)$$

is much smaller than the distance from resonance to either beam edge ν_+ or ν_- , the loss rate, $\frac{\dot{N}}{N}$, is approximately given by

$$\frac{\dot{N}}{N} = -2 \frac{P_T}{\Delta} \sqrt{\frac{D}{\pi T}} e^{-\delta^2/4D T} \quad . \quad (3.20)$$

This function has a slow variation with T and a broad maximum at

$$T = \delta^2/2D \quad . \quad (3.21)$$

An estimate of the loss rate, independent of T , can therefore be taken to be the value at the maximum

$$\frac{\dot{N}}{N} = -0.97 \frac{P_T D}{\Delta \delta} \sim - \frac{P_T D}{\Delta \delta} \quad . \quad (3.22)$$

Orbit Misalignment Tolerance at Collision Points

For small islands and odd-ordered resonances, we can approximate P_T , using (2.28), (2.29) and (2.32), by

$$P_T = 4\sqrt{\langle \sigma \rangle_{\text{rms}}} \sqrt{\frac{\bar{v}_p}{2M|U''|}} e^{-\alpha_0} \quad (3.23)$$

where the resonance functions are to be evaluated at α_0 , the maximum of P_T in the trapping amplitude region. Thus, we expect P_T to be approximately proportional to the square root of the rms orbit error - i.e.

$$P_T = C_p \sqrt{\langle \sigma \rangle_{\text{rms}}} \quad (3.24)$$

We can estimate C_p for the different order resonances from Fig. 4.

To obtain an upper limit for $\langle \sigma \rangle_{\text{rms}}$, we set an upper limit for the loss rate, $\dot{n}_{\text{max}}$. Thus, from (3.22), we have for an odd resonance of order p ,

$$\langle \sigma \rangle_{\text{rms}} < \frac{\dot{n}_{\text{max}} \Delta \delta}{D_v C_p} \quad (3.25)$$

Beam Halos

For a freshly scraped beam, we can see qualitatively how halos can form. The theoretical basis is the presence of a trapping boundary. Since particles can have v rates which are slow, they can be trapped at very small amplitudes. Note however that the mean rate is given by δv_{rev} . Thus some particles will have small rates at small amplitudes, be trapped and transported to larger amplitudes, then cross the trapping boundary and be dropped from the islands, thereby forming a halo. It can be seen from Fig. 3 that the higher the order of the resonance, the slower the halo will develop. This is because in order to follow a path of the type we have indicated, the rate would have to be relatively slower at small amplitudes the higher the resonance order. Thus, particles would take longer to reach the halo region and we could expect that the mean time for halo formation should be roughly proportional to this time.

Application to the ISR

With the ISR parameters, $\gamma = 26$, $R = 150$ m, $I = 15$ A ($N = 2.94 \times 10^{14}$), $\nu = 8.6$, $e_x = 1.5 \times 10^{-6}$ rad-m, $e_y = 0.75 \times 10^{-6}$ rad-m, $\Delta p/p = 0.025$, and $\xi_{ch} = 1.6$, we obtain from (3.12) and Fig. 5, $D = 5.54 \times 10^{-11}$ /sec. Using (3.13), we thus have $\delta_{\nu_{rev}} = 1.866 \times 10^{-8}$. Note that the number of collisions per revolution can be computed from (3.14), showing that N_{rev} is indeed a large number. The ISR collision parameters, $\beta_v^* = 14$ m, $\sigma^* = 1.62$ mm, and $\alpha_c = 258.1$ mrad lead to a beam-beam strength [from (2.9)], $\xi = 3.94 \times 10^{-4}$ per interaction. Since there are 8 collision points at the ISR, $M=8$, and the trapping region, from (2.27) for odd-ordered resonances, can be written

$$\bar{R}_p(\sigma) \geq \frac{7.51 \times 10^{-3}}{\langle \sigma \rangle_{rms}}, \quad (3.26)$$

where we have used (2.32) relating Γ_0 and $\langle \sigma \rangle_{rms}$. (Recall that even-ordered resonances are excited by asymmetries in the interaction strength ξ at the M collision points. Since, from the symmetry of the ring design, this strength asymmetry should be small, then, we expect the even-ordered resonances to be only weakly excited. Observations at the ISR indicate only weak even-ordered resonances; and this simply reflects that the strength asymmetry is indeed small.)

Let us consider (3.26) in relation to Fig. 3. For $\langle \sigma \rangle_{rms} = 1$, from Fig. 3(b), we can see that the trapping region starts at $\sigma = 0.85, 1.75, 2.50$ and 3.70 for 5th, 7th, 9th and 11th order resonances respectively. From Fig. 4, we can obtain P_T : $P_T = 0.20, 0.056, 0.014$ and 9×10^{-4} for $p = 5, 7, 9$ and 11 respectively. Note that in the case of 9th and 11th order resonances, the trapping probabilities do not occur at the peak of the trapping function. In order to evaluate the loss rate, we must determine the tune aperture distance from the resonance, δ . Using (2.25) with Fig. 1 and taking the aperture at $\sigma \approx 7$, we obtain $\delta = M\xi(.65, .50, .35 \text{ and } .2) = (2.1, 1.6, 1.1, \text{ and } 0.6) \times 10^{-3}$ for $p = 5, 7, 9$ and 11 respectively. Loss rates can now be obtained from (3.22): $\dot{N}/N = - (7.9, 2.9, 1.1, \text{ and } 0.1) \times 10^{-6}/\text{min}$ for 5th, 7th, 9th and 11th order resonances, where the total beam tune spread is $\Delta = 0.04$.

It must be remembered that this loss rate is induced by particles in a selective betatron amplitude range depending on which resonance is excited.

For example, the loss rate due to the 5th order resonance involves particles with $\sigma \gtrsim 0.85$, i.e. near the beam center, while, for the 11th order resonance, only particles with $\sigma \gtrsim 3.7$ are instantaneously affected. Thus, the higher order resonances are not as harmful because they depend on a superimposed feeding mechanism to refill the amplitude region depleted by the resonance. The loss rate over a long time will therefore be determined primarily by the feeding process, a part of which could be the halo formation mechanism previously mentioned.

As the excitation strength, i.e. $\langle \sigma \rangle_{\text{rms}}$, is reduced, the loss rate diminishes. Actually it is a complicated function of the strength. The trapping probability, as previously shown and verified in Fig. 4, decreases with $\langle \sigma \rangle_{\text{rms}}^{\frac{1}{2}}$ and so the loss rate tends to decrease similarly. However, the trapping region threshold increases as $\langle \sigma \rangle_{\text{rms}}$, meaning that δ increases, causing an increase in the loss rate. This tendency to increase the loss rate is only superficial since the betatron amplitude threshold has increased and fewer particles are vulnerable. Thus, as $\langle \sigma \rangle_{\text{rms}}$ diminishes, the dependence on some halo formation mechanism becomes stronger.

An interesting aspect of beam-beam resonances is the existence of a threshold strength. This arises because of the maximum in the function $\bar{R}_p(\sigma)$ as seen in Fig. 3(a). If $\langle \sigma \rangle_{\text{rms}}$ is sufficiently small, the trapping region will thus shrink to zero, leading therefore to a tolerance on $\langle \sigma \rangle_{\text{rms}}$. From (3.26) and Fig. 3(a) we can see that this tolerance is given by

$$\langle \sigma \rangle_{\text{rms}} < \frac{7.51 \times 10^{-3}}{[R_p]_{\text{max}}} \quad . \quad (3.27)$$

Since $[R_p]_{\text{max}} = [0.22, 0.09, 0.04, 0.02 \text{ and } 0.01]$ for $p = 5, 7, 9, 11$ and 13 respectively, we obtain

$$\langle \sigma \rangle_{\text{rms}} < \begin{cases} .03, & p = 5 \\ .08, & p = 7 \\ .19, & p = 9 \\ .38, & p = 11 \\ .75, & p = 13 \end{cases} \quad . \quad (3.28)$$

Thus, for the 5th order resonance, with $\sigma^* = 1.62$ mm, the tolerance on the vertical rms orbit alignment for the crossing beams is $(\sigma_{EQ})_{rms} \lesssim 0.05$ mm. For the 7th order resonance, $(\sigma_{EQ})_{rms} \lesssim 0.13$ mm.

Application to the ISABELLE Design

Typical parameters for a future p-p colliding beam device can be found in Ref. (13). Taking the ISABELLE design parameters, $\gamma = 214$ (200 GeV), $R = 428.1$ m, $I = 10$ A ($N = 5.6 \times 10^{14}$ protons), $v = 20.4$, $e_x = e_y = 5.85 \times 10^{-8}$ rad.m, $\frac{\Delta p}{p} = 3 \times 10^{-3}$, and $\xi_{ch} = 6$ (i.e. $\Delta = \xi_{ch} \Delta p/p = .018$), we obtain from (3.12) and Fig. 5, $D_v = 12.25 \times 10^{-11} \text{ sec}^{-1}$, and from (3.13), $\delta v_{rev} = 4.69 \times 10^{-8}$. The ISA collision parameters $\beta^* = 2.2$ m, $\sigma^* = 0.18$ mm, and $\alpha_c = 2.34$ mrad lead to a beam-beam strength [from (2.9)], $\xi = 5 \times 10^{-3}$ per interaction. For 4 collision points ($M = 4$), we obtain a trapping region for odd-ordered resonances [using (2.27) and (2.32)],

$$\bar{R}_p(\sigma) \geq \frac{3.32 \times 10^{-4}}{\langle \sigma \rangle_{rms}} \quad (3.29)$$

We see from Fig. 3(b) that if $\langle \sigma \rangle_{rms} \approx 1$, the trapping region for the 5th order resonance begins at $\sigma \approx 0.4$ and for $p = 7$, the start is $\sigma = 0.95$. Even for the 13th order resonance, the trapping region starts at a σ value as low as $\sigma = 2.75$. The significance of this is that achieving the condition on $\langle \sigma \rangle_{rms}$ that would reduce the trapping region to zero, that is,

$$\langle \sigma \rangle_{rms} \leq \frac{3.32 \times 10^{-4}}{[\bar{R}_p]_{max}} \quad (3.30)$$

implies an extremely low tolerance on $\langle \sigma \rangle_{rms}$. For $p = 5$, it means, $(\sigma_{EQ})_{rms} \leq 2.7 \times 10^{-4}$ mm! For the 13th order resonance, $(\sigma_{EQ})_{rms} \leq 0.006$ mm, still very small. However, let us estimate the loss rates for $\langle \sigma \rangle_{rms} = 1$, which corresponds to an rms orbit error, $\langle \sigma_{EQ} \rangle_{rms} = 0.18$ mm. From Fig. 4, we deduce $P_T = (0.20, 0.056, 0.017, 0.0052 \text{ and } 0.0016)$ for 5th, 7th, 9th, 11th and 13th order resonances respectively. Using (2.25) with Fig. 1 and Fig. 3(b) and

taking the aperture at $\sigma = 10$ (the results are insensitive to the aperture value of σ when it is this large), we have $\delta = M\pi(.75, .7, .6, .5 \text{ and } .4)$ or $\delta = (.015, .014, .012, .010, \text{ and } .008)$ for $p = 5, 7, 9, 11 \text{ and } 13$. Thus, with $\Delta = 0.018$, we obtain $\dot{N}/N = -(5.4, 1.6, 0.58, 0.21 \text{ and } 0.08) \times 10^{-6}/\text{min.}$ for 5th, 7th, 9th, 11th and 13th order resonances, respectively.

Conclusions

We have presented a theory describing the effects of beam-beam resonances induced by the collisions of coasting proton beams. Our results, concerning the nature of the resonances, the magnitude of the orbit misalignments required to excite them, and the magnitude of the loss rates all agree reasonably well with observations at the ISR. Computations using the ISABELLE design parameters indicate that very fine orbit control at the collision points will be required, i.e. $\langle \sigma_{EQ} \rangle_{\text{rms}} < 0.1 \text{ mm}$ at least, and that careful working line control to avoid high order resonances (perhaps as high as the 11th order) will be needed to keep the loss rate below $10^{-6}/\text{min.}$

REFERENCES

1. ISR Performance Report, Run 372, Oct. 12 (1973).
2. ISR Performance Report, Run 376, Oct. 30 (1973).
3. ISR Performance Report, Run 400, Dec. 6 (1973).
4. ISR Performance Report, Run 403, Dec. 11 (1973).
5. E. Keil, III All-Union National Conf. on Part. Accel., October, 1972;
E. Keil, ISR Running-In Reports, 8.3.1971 and 17.5. 1971.
6. A. Schoch, CERN Report, CERN 57-21 (1957).
7. H.G. Hereward, CERN Report, CERN/ISR-DI/72-26 (1972).
8. J. LeDuff, "Group Anneaux de Collisions," Linear Accelerator Laboratory,
Orsay, RT/6-72 (1972).
9. S. Chandrasekhar, Rev. of Mod. Phys., 15, 1, p.1 (1943).
10. A.W. Chao and M. Month, Nucl. Instrum. Methods 121, 129 (1974).
11. M. Month, BNL Report, CRISP 75-1 (1975).
12. H. Bruck, Accélérateurs Circulaires De Particules, Presses Univer-
sitaires De France, Paris (1966).
13. See, for example, "A Proposal for Construction of a Proton-Proton
Storage Accelerator Facility, ISABELLE," BNL Report, BNL 18891 (1974).
14. E.D. Courant and H.S. Snyder, Ann. Phys. 3, 1 (1958).

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FIGURE CAPTIONS

- Fig. 1: Detuning function, $D(\sigma)$. σ is the displacement from the strong beam center in units of the strong beam rms size.
- Fig. 2: Resonance Functions $R_p(\sigma)$ for even-ordered resonances.
- Fig. 3: Resonance Functions $\bar{R}_p(\sigma)$ for odd-ordered resonances. (a) Behavior to $\sigma = 10$. (b) Small σ behavior.
- Fig. 4: Trapping Probability Function $P_T(\sigma)$.
- Fig. 5: Form Factor in Momentum Diffusion Coefficient for Intrabeam Scattering.

Fig. 1

Fig. 2

$$\max \bar{R}_5^* (3) = 0.22$$

Fig. 3(a)

Fig. 3(b)

(a) 5th Order Resonance ($p = 5$)

Fig. 4(a)

(b) 7th Order Resonance ($p = 7$)

CRISP 75-2

Fig. 4(b)

(c) 9th Order Resonance ($\theta \sim \pi$)

$$\epsilon_{\text{res}} = \begin{bmatrix} 0.5 \text{ (I)} \\ 1.0 \text{ (II)} \\ 1.5 \text{ (III)} \\ 2.0 \text{ (IV)} \end{bmatrix}$$

Fig. 4(c)

(d) 11th Order Resonance ($p = 11$)

$$\sigma^2_{rms} = \begin{cases} 0.5 \text{ (I)} \\ 1.0 \text{ (II)} \\ 1.5 \text{ (III)} \\ 2.0 \text{ (IV)} \end{cases}$$

Fig. 4(d)

(e) 13th Order Resonance ($p = 13$)

$$\sigma_{rms} = \begin{bmatrix} 0.5 \text{ (I)} \\ 1.0 \text{ (II)} \\ 1.5 \text{ (III)} \\ 2.0 \text{ (IV)} \end{bmatrix}$$

Fig. 4(e)

Fig. 5

I. INTRODUCTION

Observations at the ISR on the beam-beam interaction of two coasting proton beams give clear evidence of the excitation of non-linear resonances.⁽¹⁻⁴⁾ The two main features are: (1) the observation of odd-ordered resonances excited by small orbit misalignments at the collision points; and (2) the strong dependence of the loss rate on the position of the physical aperture boundary. Although there is a hint of the 8th order resonance, the strongest effects are seen in the odd-ordered resonances, in particular the 5th and the 7th.

It has become apparent that resonant beam loss at the ISR is mediated by some feeding process. It has been pointed out that scattering processes can induce momentum diffusion and through the chromaticity tune drift.⁽⁵⁾ In this way particles are fed into the resonant tune range. Momentum diffusion arising from intrabeam scattering is the dominant of such scattering processes in the ISR.

However, high-order non-linear resonances are stable in a system with static tune.⁽⁶⁾ In a coasting proton beam, where there is no periodic tune modulation, such as is caused by the synchrotron motion in a bunched beam, one might expect that in the presence of sufficient non-linear detuning, the resonances would be quite harmless, producing only a slight betatron amplitude modulation. Bringing the particles into resonance is not enough. Thus, it has been suggested that in addition to feeding the resonance, the combination of tune diffusion and resonant streaming can produce amplitude growth sufficient to reach the physical aperture boundary.^(7,8)

One approach to this problem, used in References 7 and 8, is to include the effect of the resonance as streaming terms in a diffusion equation for the tune drift. The main drawback with this method is that it is difficult to solve the equation using the proper resonance characteristics. Drastic approximations must be made. For example, the betatron amplitude dependence of the resonant width is neglected in Ref. (7). In Ref. (8), an approximate linear dependence is assumed. To the extent that the real effect depends on the details of the resonance characteristics, specifically the amplitude dependence of both the resonance width and the non-linear detuning, this approach could give distorted results.

Our approach is in direct contrast. Rather than superimpose the resonance as a streaming term in the diffusion equation, we treat the diffusion as a mechanism for tune variation and we then determine the influence of this tune variation on the behavior of a beam near resonance. In this way the amplitude dependence of the resonant width and non-linear detuning can be properly included. As we will see, this amplitude dependence is a critical factor in the theory we propose to explain the ISR observations.

The key postulate of our analysis is that the mechanism by which the beam-beam resonances influence the betatron amplitude of a particle is lock-in. Thus, a particle passing through resonance either will be trapped in the passing island thereby changing its amplitude or it will remain essentially unaffected. This is not entirely correct for two reasons: In the first place, even if the islands pass by rapidly, there may be a residual effect. We presume "trapping" to be the dominant source of amplitude growth and we will simply neglect the influence of "fast crossing."⁽⁶⁾ Secondly, there is obviously no sharp boundary for the island speed which can give us a well-defined trap/no-trap criterion. The boundary is undoubtedly fuzzy. This being the case, results and conclusions sensitive to the precise boundary location must be interpreted with this limitation in mind. However, it is not our purpose here to provide accurate loss rates, but only order-of-magnitude estimates. Furthermore, it is our primary aim here to show qualitatively how the special features of beam-beam resonances manifest themselves in beam growth and beam loss. To achieve these ends, our assumption of a sharp lock-in boundary seems reasonable.

The particle loss mechanism we propose for coasting proton beams is the migration in betatron amplitude of particles trapped in islands whose motion is governed by the tune drifting in a random walk. In addition to the manifest properties of the drifting tune, a knowledge of the rate of tune variation for particles in their many different paths is also required. For it is this rate of tune change which determines whether or not a given particle will be trapped in an island drifting by or whether it will be bypassed by the island or whether it will be dropped by an island which has become leaky, this occurring if the tune change rate increases sufficiently. Let us consider the quantity $\delta\nu_{\text{rev}}$, which is the rms tune drift in one particle revolution. The question

is whether this quantity gives information about the rate of tune variation along the multitude of paths as the diffusion process proceeds. We claim that it does in the following sense: Let D_{rev} = the diffusion coefficient in units of (tune)² per revolution. In terms of the usual diffusion coefficient, D_p in units (momentum drift)²/second, we have

$$D_{\text{rev}} = \xi^2 D_p T_{\text{rev}} \quad (1.1)$$

where ξ is the absolute chromaticity,

$$\xi = p \frac{\partial n}{\partial p} \quad , \quad (1.2)$$

and T_{rev} is the revolution period of the machine; and thus, the rms tune drift rate over r revolution is simply⁽⁹⁾

$$(\delta v)_r \text{ rev} = \sqrt{\frac{2D_{\text{rev}}}{r}} \quad . \quad (1.3)$$

If $r = 1$, then we have,

$$\delta v_{\text{rev}} = \sqrt{2D_{\text{rev}}} = \sqrt{2\xi^2 D_p T_{\text{rev}}} \quad . \quad (1.4)$$

Now, we must emphasize that the number of momentum scatterings per revolution is a very large number, which implies two things: first, that the step size in the tune random walk is much smaller than δv_{rev} and second, that the sum of the momentum changes in absolute value is much larger than the rms momentum drift (corresponding in tune to δv_{rev}). Thus, although the instantaneous momentum of any given particle changes many times during a revolution, they tend to cancel and we end up with a much smaller drift. In terms of tune, these momentum steps are equivalent to small quadrupoles tending to cancel each other. The tune is by definition a vector sum of these quadrupoles over a revolution and thus the tune shift is represented, not by the quadrupole fluctuations within the revolution period, but rather by the sum, or the tune drift. In other words the distribution in δv after 1 revolution (many steps), is a good representation of the distribution of tune speeds. As representative of this distribution, we can choose the rms value, δv_{rev} , and use this as the rate of tune variation during any particle diffusion path. Actually, we can

expect that for any particle path, there is a 95% probability that the tune speed will not be larger than twice this value. Averaging over more than 1 revolution ($r > 1$) would result in a smaller mean tune speed.

In section 2, we review the trapping theory⁽¹⁰⁾ for beam-beam resonances,⁽¹¹⁾ emphasizing the distinctive amplitude dependence of both the non-linear detuning and the resonant widths. We use an approximate "head-on" model⁽¹¹⁾ to simulate the beam-beam interaction. We discuss the difference between odd-ordered and even-ordered beam-beam resonances. We show how lock-in and particle transport can occur as a consequence of the drifting (diffusing) tune in colliding, coasting proton beams. In section 3, we combine the beam-beam resonance theory with tune diffusion to construct a model for beam growth and loss. We relate the diffusion coefficient to the parameters for intrabeam scattering.⁽¹²⁾ Using a random walk model in the presence of an absorbing barrier,⁽⁹⁾ we estimate the loss rate to the physical aperture boundary due to an odd-order beam-beam resonance and discuss the closed orbit misalignment tolerance at the collision points. We then show qualitatively how beam halos can form. Applying our model to the ISR, we find our results in general agreement with ISR observations.⁽¹⁻⁴⁾ Finally, we make some extrapolations to future p-p colliding beam machines.⁽¹³⁾

II. TRAPPING THEORY FOR BEAM-BEAM RESONANCES

Invariant for Beam-Beam Resonances

The resonant invariant for a beam-beam resonance of order p can in general be written⁽¹¹⁾

$$C = \frac{\bar{v}}{M\xi} \alpha + U(\alpha) + \left[\int_E V_p(\alpha) + \int_0^{\bar{v}} \hat{V}_p(\alpha) \right] \cos p\psi \quad (2.1)$$

$$\text{where } \bar{v} = v - m/p \quad (2.2)$$

$\bar{v}$ is the deviation of the tune, v , from the resonance,

p is the order of the resonance,

m is the azimuthal harmonic exciting the resonance,

M is the number of collisions in the ring,

ξ is the beam-beam strength of each collision (assumed identical),

α is the squared amplitude in units of rms squared amplitude, I_{rms} ,

$$\alpha = I/I_{rms} \quad , \quad (2.3)$$

with

$$I_{rms} = b^2/\beta^* \quad , \quad (2.4)$$

β^* is value of the β function⁽¹⁴⁾ at the collision point,

$$b = \sqrt{2} \sigma^* \quad , \quad (2.5)$$

σ^* is the rms beam size at the crossing point, and Ψ is a phase canonically conjugate to the squared amplitude,⁽¹¹⁾ α .

The quantities Γ_E and Γ_O are the effective resonant excitation terms for odd and even ordered resonances, respectively:

$$\Gamma_E = \frac{1}{M} \left| \sum_{\ell=0}^{M-1} \left(\frac{\delta \xi}{\xi} \right)_\ell e^{im\theta_\ell} \right| \quad (2.6)$$

$$\Gamma_O = \frac{1}{M/2} \left| \sum_{\ell=0}^{M-1} \sigma_\ell e^{im\theta_\ell} \right| \quad , \quad (2.7)$$

where $\left(\frac{\delta \xi}{\xi} \right)_\ell$ is the deviation of the strength of the ℓ^{th} intersection from the mean strength,

θ_ℓ is the azimuth of the ℓ^{th} intersection from some arbitrary reference, and σ_ℓ is the deviation of the closed orbit of the weak beam with respect to the strong beam center in units of rms beam size at the collision point,

$$\sigma_\ell = \frac{\sigma_{EQ}}{\sigma^*} \quad . \quad (2.8)$$

We have assumed that the beams cross at an angle α_c and the resonance is in the non-crossing plane. This situation corresponds approximately to head-on collisions with the effective size in the crossing dimension much greater than the beam size in the non-crossing one. Thus, the strength parameter, ξ , is given by

$$\xi = \left(\frac{I}{ec} \right) \frac{2\pi \beta^*}{\pi \gamma \alpha_c \sigma^*} \quad , \quad (2.9)$$

where I is the current in the coasting beam,

γ is the energy in proton mass units,

and r_p is the classical proton radius. Also, the beam-beam resonance functions are, (11)

$$U(\alpha) = 2 \left\{ e^{-\alpha/2} \left[(1 + \alpha) I_0(\alpha/2) + \alpha I_1(\alpha/2) \right] - 1 \right\} , \quad (2.10)$$

$$V_p(\alpha) = -4 (-1)^{\frac{p}{2}} \frac{e^{-\alpha/2}}{p^2 - 1} \left[(1 + \alpha) I_{p/2}(\alpha/2) + \alpha I'_{p/2}(\alpha/2) \right] , \quad (2.11)$$

and

$$\bar{V}_p(\alpha) = (-1)^{\frac{p-1}{2}} \sqrt{2} \int_0^{\alpha/2} dx \frac{e^{-x}}{\sqrt{x}} \left\{ \left(1 - \frac{1}{p}\right) I_{\frac{p+1}{2}}(x) - \left(1 + \frac{1}{p}\right) I_{\frac{p-1}{2}}(x) \right\} , \quad (2.12)$$

where we have taken $U, V_p, \bar{V}_p \rightarrow 0$ as $\alpha \rightarrow 0$.

Note that $V_p(\alpha) \neq 0$ only if p is even,

and $\bar{V}_p(\alpha) \neq 0$ only if p is odd.

In other words, closed orbit deviations excite odd-ordered resonances, while deviations in the strength of the interactions excite even-ordered resonances.

Note also that $U(\alpha) \approx \alpha$ as $\alpha \rightarrow 0$. This term corresponds to the linear tune shift from the beam-beam interaction. The Bessel functions and their derivatives are defined by

$$I_n(x) = \frac{1}{\pi} \int_0^\pi e^{x \cos \theta} \cos n \theta d\theta , \quad (2.13)$$

and

$$I'_n(x) = \frac{1}{2} (I_{n+1}(x) + I_{n-1}(x)), \quad n \geq 1 . \quad (2.14)$$

Criteria for Trapping

The mechanism we propose for how beam-beam resonances can change particle amplitudes is the lock-in process.^(6,11) There are 3 particle parameters which determine the effect of the resonance:

$\bar{\nu}$, the tune with respect to the resonant tune,

$\delta \nu_{\text{rev}}$, the rate of tune variation,

and α , the particle amplitude (i.e., betatron amplitude squared).

The value of $\bar{\nu}$ determines the amplitude of the lock-in islands, α_F . Particles are influenced by the resonance only when their amplitudes are in the vicinity of the islands, i.e.,

$$\alpha \approx \alpha_F \quad . \quad (2.15)$$

As has been previously pointed out [See for example Ref.(11)], the detuning function $U(\alpha)$ is much larger than the resonance functions $V_p(\alpha)$ and $\bar{V}_p(\alpha)$. In this case the islands extend only over a small amplitude range and their position in amplitude, α_F , is determined almost entirely by the detuning function. In fact, the island position is obtained from the solution of the equation,

$$\frac{\bar{\nu}}{M\xi} + U'(\alpha) = 0 \quad , \quad (2.16)$$

where $U'(\alpha)$ is the derivative of U and is given by

$$U'(\alpha) = e^{-\alpha/2} \left[I_0(\alpha/2) + I_1(\alpha/2) \right] \quad . \quad (2.17)$$

Note that we can without ambiguity use the symbol α to correspond to both island position and particle amplitude, as long as we remember that only particles at the island position are to be considered. That is, (2.15) is implicitly assumed. However, even if at some instant a particle has amplitude α and tune $\bar{\nu}$ approximately satisfying (2.16), lock-in or trapping need not necessarily follow. In other words, the condition (2.16) is not necessarily maintained at later times. In order for trapping to result, the resonant amplitude must be changing at least as fast as the passing islands. This leads to a trapping criterion,⁽¹¹⁾ which can be stated as an upper limit on the rate of tune variation: for even-ordered resonances, we have

$$\frac{\delta v_{\text{rev}}}{M^2 \xi^2 \Gamma_E} \leq R_p(\alpha) \quad , \quad (2.18)$$

while for odd ordered resonances,

$$\frac{\delta v_{\text{rev}}}{M^2 \xi^2 \Gamma_0} \leq \bar{R}_p(\alpha) \quad , \quad (2.19)$$

where we have written

$$R_p(\alpha) = 2\pi p |v_p(\alpha) U''(\alpha)| \quad , \quad (2.20)$$

$$\text{and } \bar{R}_p(\alpha) = 2\pi p |\bar{v}_p(\alpha) U''(\alpha)| \quad . \quad (2.21)$$

Note that the tune rate criterion is amplitude dependent. For example, given δv_{rev} at different times, the boundary implied in (2.18) or (2.19) might at a particular value of α be crossed, thus changing a "capture" condition into a "drop" condition or vice versa. The quantity U'' can be obtained from U' by differentiation, giving

$$U''(\alpha) = -\frac{1}{\alpha} e^{-\alpha/2} I_1(\alpha/2) \quad . \quad (2.22)$$

The resonance functions are universal in the sense that they are strength independent and functions of amplitude only. We plot in Fig. 1 the function

$$D = -U' \quad , \quad (2.23)$$

and the functions R_p and $\bar{R}_p$ in Figs. 2 and 3 respectively for various resonances. Instead of the squared amplitude, α , we use for the independent variable σ , the displacement in units of rms displacement. It is related to α by

$$\sigma = \sqrt{2\alpha} \quad . \quad (2.24)$$

The function $D(\sigma)$ determines the range over which trapped particles can be transported. If $\bar{\nu}_i$ and $\bar{\nu}_f$ are initial and final tunes respectively, and if σ_i corresponds to the amplitude of a particle trapped at $\bar{\nu}_i$, then the amplitude to which this particle can be transported, σ_f , can be found from

$$D(\sigma_f) - D(\sigma_i) = \frac{\bar{\nu}_i - \bar{\nu}_f}{M \xi} \quad . \quad (2.25)$$

This gives the limits of the transport for a given tune shift. For a particle to actually move with the islands, the rate criterion must be satisfied. Namely, for an even-ordered resonance,

$$\delta\nu_{\text{rev}} \leq M^2 \xi^2 \Gamma_E R_p(\sigma) \quad (2.26)$$

must be satisfied for all σ along the path from σ_i to σ_f ; while for an odd-ordered resonance

$$\delta\nu_{\text{rev}} \leq M^2 \xi^2 \Gamma_O \bar{R}_p(\sigma) \quad (2.27)$$

must be satisfied.

Trapping Probability

If we take account of the distribution of particles in the amplitude, α , we can estimate the fraction of particles that lock-in to the resonance as the islands pass through the beam. Consider an odd-ordered resonance. At any instant particles in a certain restricted amplitude range can lock-in. This range is simply the amplitude extent of the islands; or put another way, it is the maximum amplitude modulation for trapped particles. At a given amplitude α , we have for the range, ⁽¹¹⁾

$$\alpha_{\pm} = \alpha \pm 2\sqrt{\Gamma_O \left| \frac{\bar{\nu}_p}{\bar{\nu}''} \right|} \quad . \quad (2.28)$$

If we assume a Gaussian distribution in transverse particle displacement, then the fraction of total particles contained within this amplitude range is given by

$$P_T = e^{-\alpha^2} - e^{-\alpha_{-}^2} \quad . \quad (2.29)$$

If the particles at this value of α satisfy the trapping criterion (2.27), then this number represents the fraction of particles that can be instantaneously trapped. Actually it is a small overestimate, since the islands do not occupy the entire ring between α_- and α_+ . Actually, we are not so much interested in the instantaneous trapping, but rather in the cumulative effect after the islands pass through some amplitude range, perhaps a few rms amplitudes. This is clearly a complex process with particles entering and leaving the islands (at the unstable fixed point which are always non-adiabatic regions and therefore potential leaky regions) as the islands move through the beam. However, the dynamical system is Hamiltonian and must satisfy Liouville's well known theorem, stating that the local phase space density does not change. Thus we might expect that over a small area of phase space, as for example the island ring, the average density, as distinct from the local density, will not vary too much. It follows that we might estimate the number of trapped particles by some typical value of the "instantaneous trapped fraction." That is, an estimate of the trapping probability could be some typical value of P_T in the region of α where trapping can occur. We plot in Fig. 4 P_T as a function of σ for various odd-ordered beam-beam resonances of various strengths. Notice the slow maximum. As the islands pass through this region of σ the ring is increasing in size but the phase area which can be added to the islands has a lower density. It is not likely that the fraction P_T will increase significantly from this maximum value. It could of course be somewhat higher if just the right exchange takes place before the maximum. We disregard this possibility and take as an approximate trapping probability the maximum value of the instantaneous trapping fraction in the region of σ where lock-in can occur, i.e., where (2.27) is satisfied. Such an approach to a quantitative estimate of trapping has been successfully applied to the case of conventional resonances in Ref. (10). Thus, we take for the trapping probability,

$$P_T = P_T(\alpha) \mid \text{max. value of (2.29) in trapping region.} \quad (2.30)$$

Uncorrelated Errors

If there are variations in the M collision strengths and if we assume them to be uncorrelated, we can relate Γ to the rms value of the error. Consider, for example, the closed orbit errors at the intersections, ϵ_L ,

to be uncorrelated. Then, (2.7) for Γ_0 becomes

$$\Gamma_0 = \frac{1}{M/2} \sqrt{\sum_{l=0}^{M-1} \sigma_l^2} .$$

If we define the rms deviation $\langle \sigma \rangle_{rms}$ by

$$\langle \sigma \rangle_{rms}^2 = \frac{1}{M} \sum_{l=0}^{M-1} \sigma_l^2 , \quad (2.31)$$

then

$$\Gamma_0 = \frac{1}{\sqrt{2M}} \langle \sigma \rangle_{rms} . \quad (2.32)$$

We measure the strength of odd-ordered resonances, which are excited by these closed orbit deviations, by the rms value. The trapping probability, P_T , which is a function of the resonant strength, shown in Fig. 4, is plotted against the amplitude for various values of $\langle \sigma \rangle_{rms}$.

III. BEAM-BEAM EFFECT IN COASTING PROTON BEAMS

The Diffusion of Tune from Intrabeam Scattering

The multiple scattering of protons in a beam due to Coulomb forces, that is, intrabeam scattering, causes a diffusion in longitudinal momentum.¹² The diffusion coefficient in the lab frame of reference is derived in Ref. 12 and can be written

$$D_p = \frac{d}{dt} \left(\frac{\delta p_{rms}}{p} \right)^2 = \frac{\sqrt{\pi} c r^2 \rho}{\beta^3 \gamma^3 \varphi_{rms}} f(x_0) , \quad (3.1)$$

where δp_{rms} is the rms momentum spread,

p is the longitudinal momentum,

ρ is the average particle density,

β is the velocity in units of c ,

γ is the energy in proton mass units,

r is the classical proton radius ($= 1.54 \times 10^{-18}$ m),

φ_{rms} is the rms dispersion in the betatron angle.

The quantities δp_{rms} , p and ρ are to be evaluated in the lab frame of reference. The definitions of ρ , φ_{rms} and $f(x_0)$ are:

$$\rho = N/V \quad , \quad (3.2)$$

where N is the number of particles in the volume V ,

$$V = Cdh \quad , \quad (3.3)$$

$C = 2\pi R$ is the circumference,

d is the total beam width,

$$d = \sqrt{4\epsilon_x \beta_x + \left(x_p \frac{\delta p}{p}\right)^2} \quad , \quad (3.4)$$

h is the total beam height,

$$h = \sqrt{4\epsilon_y \beta_y} \quad (3.5)$$

with β_x , β_y the average horizontal and vertical β -function values,

$$\beta_x = R/v_x \quad , \quad \beta_y = R/v_y \quad , \quad (3.6)$$

ϵ_x , ϵ_y the 95% horizontal and vertical emittances (Area = $\pi\epsilon$), v_x , v_y the horizontal and vertical tunes;

$$\varphi_{rms} = \frac{1}{2} \sqrt{\frac{\epsilon_x}{\beta_x} + \frac{\epsilon_y}{\beta_y}} \quad ; \quad (3.7)$$

and

$$f(x_0) = \int_{x_0}^{\infty} \frac{1}{x} e^{-x} \ln \frac{x}{x_0} dx \quad , \quad (3.8)$$

where

$$x_0 = \frac{2\pi}{\varphi_{rms}} \left(\frac{\rho}{\beta\gamma}\right)^{1/3} \quad . \quad (3.9)$$

The number of particles N is related to the current I by

$$I = \frac{e N c \beta}{C} \quad . \quad (3.10)$$

The function f can be estimated from a graph given in Ref. 12, which we sketch in Fig. 5.

By way of the chromaticity, ξ_{ch} , defined as tune change per unit fractional momentum change, momentum is related to betatron tune. The distributions can be connected through the relation

$$\delta v_{rms} = \xi_{ch} \frac{\delta p_{rms}}{p} \quad . \quad (3.11)$$

Thus, the diffusion coefficient for tune

$$D_v = \xi_{ch}^2 D_p \quad . \quad (3.12)$$

As already stated in section 1, the rms tune spread in 1 revolution is given by

$$\delta v_{rev} = \sqrt{2D_v T_{rev}} = \xi_{ch} \sqrt{2D_p T_{rev}} \quad . \quad (3.13)$$

Further, we have argued that this quantity, δv_{rev} , can be interpreted as the mean rate of tune variation. This means that at any given instant, we expect about 95% of the particles to have a tune speed less than twice this value. Of course, a small amount will have a faster speed; on the other hand, many will have a speed significantly slower than the mean.

Recall that we have presumed that the number of collisions per revolution, N_{rev} , is large. This is true for the cases of interest here. An expression for N_{rev} can be obtained with the help of Ref. 12:

$$N_{rev} = \sqrt{\pi} C \left(\rho / \beta \gamma \right)^{1/3} \phi_{rms} \quad . \quad (3.14)$$

Random Walk with an Absorbing Barrier

We conceive of particles being fed into the resonance by a random walk in tune via intrabeam scattering, transferred to the aperture (in betatron amplitude) by the resonance and thereby lost. The simple random walk in tune must be modified since not all particles crossing the resonance will be trapped and transported to larger amplitudes. The two restrictions are those already mentioned, namely the rate of tune variation must be such as to satisfy the trapping criterion and secondly, the amount of particles trapped must be consistent with the preservation of local particle density in betatron phase space. In fact, we have defined P_T , see Eq. (2.30), with exactly these two constraints in mind. If we assume that the two processes, trapping and random walk are independent, we can estimate the loss rate by simply multiplying the loss rate due to diffusion by P_T .

Thus, the loss rate for a set of particles at tune ν in a range $d\nu$ at time T is given by⁽⁹⁾

$$\frac{d^2N}{dTd\nu} = - \frac{\rho(\nu)P_T(\nu_A - \nu)}{2T(\pi D_\nu T)^{1/2}} e^{-(\nu_A - \nu)^2/4D_\nu T} \quad , \quad (3.15)$$

where ν_A is the tune corresponding to the absorbing barrier for a particle trapped,

D_ν is the tune diffusion coefficient, related to the intrabeam scattering parameters through (3.12) and (3.1),

N is the total number of particles in the beam, and $\rho(\nu)$ is the tune density.

For a uniform density beam in tune,

$$\rho = N/\Delta \quad , \quad (3.16)$$

with Δ the total beam tune spread.

Notice that (3.15) for the loss rate depends only on the distance of ν from the tune aperture ν_A . Now, to obtain the total loss rate, it is not sufficient simply to integrate over ν , since in order to be trapped, particles must pass through the resonant tune, ν_R . Let us assume therefore that the resonant tune lies within the beam, meaning

$$\nu_- < \nu_R < \nu_+ \quad , \quad (3.17)$$

where v_+ and v_- are the upper and lower edges of the beam in tune, with

$$\Delta = v_+ - v_- \quad . \quad (3.18)$$

Since the loss rate depends only on the distance to the tune aperture, then we can accommodate the case where v lies between v_R and v_A at $T = 0$ by reflecting the distribution around v_R , thus forcing the particles to travel the increased distance to v_R and then to v_A . Thus, with $v_A > v_R$, we can integrate (3.15) over v with the replacements

$$v_A - v \rightarrow v_A - v \text{ if } v_- \leq v \leq v_R \quad ,$$

$$\text{and } v_A - v \rightarrow v_A - 2v_R + v \text{ if } v_R \leq v \leq v_+ \quad .$$

In the case that the distance from the resonance to the aperture,

$$\delta = v_A - v_R \quad , \quad (3.19)$$

is much smaller than the distance from resonance to either beam edge v_+ or v_- , the loss rate, $\frac{\dot{N}}{N}$, is approximately given by

$$\frac{\dot{N}}{N} = - 2 \frac{P_T}{\Delta} \frac{\sqrt{\frac{D}{v}}}{\sqrt{\pi T}} e^{-\delta^2/4D T} \quad . \quad (3.20)$$

This function has a slow variation with T and a broad maximum at

$$T = \delta^2/2D \quad . \quad (3.21)$$

An estimate of the loss rate, independent of T , can therefore be taken to be the value at the maximum

$$\frac{\dot{N}}{N} \approx - 0.97 \frac{P_T D}{\Delta \delta} \approx - \frac{P_T D}{\Delta \delta} \quad . \quad (3.22)$$

Orbit Misalignment Tolerance at Collision Points

For small islands and odd-ordered resonances, we can approximate P_T , using (2.28), (2.29) and (2.32), by

$$P_T = 4\sqrt{\langle \sigma \rangle_{rms}} \sqrt{\frac{|\bar{v}_p|}{2M|U''|}} e^{-\alpha_0} \quad (3.23)$$

where the resonance functions are to be evaluated at α_0 , the maximum of P_T in the trapping amplitude region. Thus, we expect P_T to be approximately proportional to the square root of the rms orbit error - i.e.

$$P_T = C_p \sqrt{\langle \sigma \rangle_{rms}} \quad (3.24)$$

We can estimate C_p for the different order resonances from Fig. 4.

To obtain an upper limit for $\langle \sigma \rangle_{rms}$, we set an upper limit for the loss rate, $\dot{n}_{max}$. Thus, from (3.22), we have for an odd resonance of order p ,

$$\langle \sigma \rangle_{rms} < \frac{\dot{n}_{max} \Delta \delta}{D_v C_p} \quad (3.25)$$

Beam Halos

For a freshly scraped beam, we can see qualitatively how halos can form. The theoretical basis is the presence of a trapping boundary. Since particles can have ν rates which are slow, they can be trapped at very small amplitudes. Note however that the mean rate is given by $\delta \nu_{rev}$. Thus some particles will have small rates at small amplitudes, be trapped and transported to larger amplitudes, then cross the trapping boundary and be dropped from the islands, thereby forming a halo. It can be seen from Fig. 3 that the higher the order of the resonance, the slower the halo will develop. This is because in order to follow a path of the type we have indicated, the rate would have to be relatively slower at small amplitudes the higher the resonance order. Thus, particles would take longer to reach the halo region and we could expect that the mean time for halo formation should be roughly proportional to this time.

Application to the ISR

With the ISR parameters, $\gamma = 26$, $R = 150$ m, $I = 15$ A ($N = 2.94 \times 10^{14}$), $v = 8.6$, $\epsilon_x = 1.5 \times 10^{-6}$ rad-m, $\epsilon_y = 0.75 \times 10^{-6}$ rad-m, $\Delta p/p = 0.025$, and $\xi_{ch} = 1.6$, we obtain from (3.12) and Fig. 5, $D_v = 5.54 \times 10^{-11}$ /sec. Using (3.13), we thus have $\delta_{v,rev} = 1.866 \times 10^{-8}$. Note that the number of collisions per revolution can be computed from (3.14), showing that N_{rev} is indeed a large number. The ISR collision parameters, $\beta_v^* = 14$ m, $\sigma^* = 1.62$ mm, and $\alpha_c = 258.1$ mrad lead to a beam-beam strength [from (2.9)], $\xi = 3.94 \times 10^{-4}$ per interaction. Since there are 8 collision points at the ISR, $M=8$, and the trapping region, from (2.27) for odd-ordered resonances, can be written

$$\bar{R}_p(\sigma) \geq \frac{7.51 \times 10^{-3}}{\langle \sigma \rangle_{rms}}, \quad (3.26)$$

where we have used (2.32) relating Γ_0 and $\langle \sigma \rangle_{rms}$. (Recall that even-ordered resonances are excited by asymmetries in the interaction strength ξ at the M collision points. Since, from the symmetry of the ring design, this strength asymmetry should be small, then, we expect the even-ordered resonances to be only weakly excited. Observations at the ISR indicate only weak even-ordered resonances; and this simply reflects that the strength asymmetry is indeed small.)

Let us consider (3.26) in relation to Fig. 3. For $\langle \sigma \rangle_{rms} = 1$, from Fig. 3(b), we can see that the trapping region starts at $\sigma = 0.85, 1.75, 2.50$ and 3.70 for 5th, 7th, 9th and 11th order resonances respectively. From Fig. 4, we can obtain $P_T : P_T = 0.20, 0.056, 0.014$ and 9×10^{-4} for $p = 5, 7, 9$ and 11 respectively. Note that in the case of 9th and 11th order resonances, the trapping probabilities do not occur at the peak of the trapping function. In order to evaluate the loss rate, we must determine the tune aperture distance from the resonance, δ . Using (2.25) with Fig. 1 and taking the aperture at $\sigma \approx 7$, we obtain $\delta = M\xi(.65, .50, .35 \text{ and } .2) = (2.1, 1.6, 1.1, \text{ and } 0.6) \times 10^{-3}$ for $p = 5, 7, 9$ and 11 respectively. Loss rates can now be obtained from (3.22) : $\dot{N}/N = - (7.9, 2.9, 1.1, \text{ and } 0.1) \times 10^{-6}/\text{min}$ for 5th, 7th, 9th and 11th order resonances, where the total beam tune spread is $\Delta = 0.04$.

It must be remembered that this loss rate is induced by particles in a selective betatron amplitude range depending on which resonance is excited.

For example, the loss rate due to the 5th order resonance involves particles with $\sigma \gtrsim 0.85$, i.e. near the beam center, while, for the 11th order resonance, only particles with $\sigma \gtrsim 3.7$ are instantaneously affected. Thus, the higher order resonances are not as harmful because they depend on a superimposed feeding mechanism to refill the amplitude region depleted by the resonance. The loss rate over a long time will therefore be determined primarily by the feeding process, a part of which could be the halo formation mechanism previously mentioned.

As the excitation strength, i.e. $\langle \sigma \rangle_{\text{rms}}$, is reduced, the loss rate diminishes. Actually it is a complicated function of the strength. The trapping probability, as previously shown and verified in Fig. 4, decreases with $\langle \sigma \rangle_{\text{rms}}^{\frac{1}{2}}$ and so the loss rate tends to decrease similarly. However, the trapping region threshold increases as $\langle \sigma \rangle_{\text{rms}}$, meaning that δ increases, causing an increase in the loss rate. This tendency to increase the loss rate is only superficial since the betatron amplitude threshold has increased and fewer particles are vulnerable. Thus, as $\langle \sigma \rangle_{\text{rms}}$ diminishes, the dependence on some halo formation mechanism becomes stronger.

An interesting aspect of beam-beam resonances is the existence of a threshold strength. This arises because of the maximum in the function $\tilde{R}_p(\sigma)$ as seen in Fig. 3(a). If $\langle \sigma \rangle_{\text{rms}}$ is sufficiently small, the trapping region will thus shrink to zero, leading therefore to a tolerance on $\langle \sigma \rangle_{\text{rms}}$. From (3.26) and Fig. 3(a) we can see that this tolerance is given by

$$\langle \sigma \rangle_{\text{rms}} < \frac{7.51 \times 10^{-3}}{[R_p]_{\text{max}}} \quad . \quad (3.27)$$

Since $[R_p]_{\text{max}} = [0.22, 0.09, 0.04, 0.02 \text{ and } 0.01]$ for $p = 5, 7, 9, 11$ and 13 respectively, we obtain

$$\langle \sigma \rangle_{\text{rms}} < \begin{cases} .03, & p = 5 \\ .08, & p = 7 \\ .19, & p = 9 \\ .38, & p = 11 \\ .75, & p = 13 \end{cases} \quad . \quad (3.28)$$

Thus, for the 5th order resonance, with $\sigma^* = 1.62$ mm, the tolerance on the vertical rms orbit alignment for the crossing beams is $(\sigma_{EQ})_{rms} \lesssim 0.05$ mm. For the 7th order resonance, $(\sigma_{EQ})_{rms} \lesssim 0.13$ mm.

Application to the ISABELLE Design

Typical parameters for a future p-p colliding beam device can be found in Ref. (13). Taking the ISABELLE design parameters, $\gamma = 214$ (200 GeV), $R = 428.1$ m, $I = 10A$ ($N = 5.6 \times 10^{14}$ protons), $v = 20.4$, $e_x = e_y = 5.85 \times 10^{-8}$ rad.m, $\frac{\Delta p}{p} = 3 \times 10^{-3}$, and $\xi_{ch} = 6$ (i.e. $\Delta = \xi_{ch} \Delta p/p = .018$), we obtain from (3.12) and Fig. 5, $D = 12.25 \times 10^{-11} \text{ sec}^{-1}$, and from (3.13), $\delta v_{rev} = 4.69 \times 10^{-8}$. The ISA collision parameters $\beta^* = 2.2$ m, $\sigma^* = 0.18$ mm, and $\alpha_c = 2.34$ mrad lead to a beam-beam strength [from (2.9)], $\xi = 5 \times 10^{-3}$ per interaction. For 4 collision points ($M = 4$), we obtain a trapping region for odd-ordered resonances [using (2.27) and (2.32)],

$$\bar{R}_p(\sigma) \geq \frac{3.32 \times 10^{-4}}{\langle \sigma \rangle_{rms}} \quad . \quad (3.29)$$

We see from Fig. 3(b) that if $\langle \sigma \rangle_{rms} \approx 1$, the trapping region for the 5th order resonance begins at $\sigma \approx 0.4$ and for $p = 7$, the start is $\sigma = 0.95$. Even for the 13th order resonance, the trapping region starts at a σ value as low as $\sigma = 2.75$. The significance of this is that achieving the condition on $\langle \sigma \rangle_{rms}$ that would reduce the trapping region to zero, that is,

$$\langle \sigma \rangle_{rms} \leq \frac{3.32 \times 10^{-4}}{[\bar{R}_p]_{max}} \quad , \quad (3.30)$$

implies an extremely low tolerance on $\langle \sigma \rangle_{rms}$. For $p = 5$, it means, $(\sigma_{EQ})_{rms} \leq 2.7 \times 10^{-4}$ mm! For the 13th order resonance, $(\sigma_{EQ})_{rms} \leq 0.006$ mm, still very small. However, let us estimate the loss rates for $\langle \sigma \rangle_{rms} = 1$, which corresponds to an rms orbit error, $\langle \sigma_{EQ} \rangle_{rms} = 0.18$ mm. From Fig. 4, we deduce $P_T = (0.20, 0.056, 0.017, 0.0052 \text{ and } 0.0016)$ for 5th, 7th, 9th, 11th and 13th order resonances respectively. Using (2.25) with Fig. 1 and Fig. 3(b) and

taking the aperture at $\sigma = 10$ (the results are insensitive to the aperture value of σ when it is this large), we have $\delta = M\pi(.75, .7, .6, .5 \text{ and } .4)$ or $\delta = (.015, .014, .012, .010, \text{ and } .008)$ for $p = 5, 7, 9, 11 \text{ and } 13$. Thus, with $\Delta = 0.018$, we obtain $\dot{N}/N = -(5.4, 1.6, 0.58, 0.21 \text{ and } 0.08) \times 10^{-6}/\text{min.}$ for 5th, 7th, 9th, 11th and 13th order resonances, respectively.

Conclusions

We have presented a theory describing the effects of beam-beam resonances induced by the collisions of coasting proton beams. Our results, concerning the nature of the resonances, the magnitude of the orbit misalignments required to excite them, and the magnitude of the loss rates all agree reasonably well with observations at the ISR. Computations using the ISABELLE design parameters indicate that very fine orbit control at the collision points will be required, i.e. $\langle \sigma_{EQ} \rangle_{rms} < 0.1 \text{ mm}$ at least, and that careful working line control to avoid high order resonances (perhaps as high as the 11th order) will be needed to keep the loss rate below $10^{-6}/\text{min.}$

REFERENCES

1. ISR Performance Report, Run 372, Oct. 12 (1973).
2. ISR Performance Report, Run 376, Oct. 30 (1973).
3. ISR Performance Report, Run 400, Dec. 6 (1973).
4. ISR Performance Report, Run 403, Dec. 11 (1973).
5. E. Keil, III All-Union National Conf. on Part. Accel., October, 1972;
E. Keil, ISR Running-In Reports, 8.3.1971 and 17.5. 1971.
6. A. Schoch, CERN Report, CERN 57-21 (1957).
7. H.G. Hereward, CERN Report, CERN/ISR-DI/72-26 (1972).
8. J. LeDuff, "Group Anneaux de Collisions," Linear Accelerator Laboratory,
Orsay, RT/6-72 (1972).
9. S. Chandrasekhar, Rev. of Mod. Phys., 15, 1, p.1 (1943).
10. A.W. Chao and M. Month, Nucl. Instrum. Methods 121, 129 (1974).
11. M. Month, BNL Report, CRISP 75-1 (1975).
12. H. Bruck, Accélérateurs Circulaires De Particules, Presses Univer-
sitaires De France, Paris (1966).
13. See, for example, "A Proposal for Construction of a Proton-Proton
Storage Accelerator Facility, ISABELLE," BNL Report, BNL 18891 (1974).
14. E.D. Courant and H.S. Snyder, Ann. Phys. 3, 1 (1958).

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FIGURE CAPTIONS

- Fig. 1: Detuning function, $D(\sigma)$. σ is the displacement from the strong beam center in units of the strong beam rms size.
- Fig. 2: Resonance Functions $R_p(\sigma)$ for even-ordered resonances.
- Fig. 3: Resonance Functions $\bar{R}_p(\sigma)$ for odd-ordered resonances. (a) Behavior to $\sigma = 10$. (b) Small σ behavior.
- Fig. 4: Trapping Probability Function $P_T(\sigma)$.
- Fig. 5: Form Factor in Momentum Diffusion Coefficient for Intrabeam Scattering.

Fig. 1

Fig. 2

$$\max \bar{R}_5^* (3) = 0.22$$

Fig. 3(a)

Fig. 3(b)

(a) 5th Order Resonance ($p = 5$)

Fig. 4(a)

Fig. 4(b)

(c) 9th Order Resonance ($p = 9$)

$$\langle \sigma \rangle_{rms} = \begin{bmatrix} 0.5 \text{ (I)} \\ 1.0 \text{ (II)} \\ 1.5 \text{ (III)} \\ 2.0 \text{ (IV)} \end{bmatrix}$$

Fig. 4(c)

Fig. 4(d)

(e) 13th Order Resonance ($p = 13$)

$$\zeta_p^* = \begin{bmatrix} 0.5 \text{ (I)} \\ 1.0 \text{ (II)} \\ 1.5 \text{ (III)} \\ 2.0 \text{ (IV)} \end{bmatrix}$$

Fig. 4(e)

Fig. 5