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ON THE BEHAVIOR OF SOLUTIONS OF ORDINARY FINITE DIFFERENCE EQUATIONS

When finite difference equations are substituted for differential equations, it does not follow a priori that the behavior of the solution of the difference equation will approximate the behavior of the solution of the differential equation. This memo is not distributed because of the intrinsic value of the mathematics it contains but because it shows that even for a very simple ordinary differential equation, the corresponding difference equation solution can exhibit an anomalous character.

Consider the equation $y'' - \lambda^2 y = 0$. The character of the solution depends upon the value of λ^2 . Indeed, the linearly independent solutions are

$$e^{\lambda t} \text{ and } e^{-\lambda t}, \text{ when } \lambda^2 > 0;$$

$$t \text{ and } 1, \text{ when } \lambda^2 = 0;$$

$$e^{i|\lambda|t} \text{ and } e^{-i|\lambda|t}, \text{ when } \lambda^2 < 0.$$

Using the standard difference scheme, the corresponding finite difference equation is

$$y_{n+1} - 2y_n + y_{n-1} - \lambda^2 h^2 y_n = 0, \quad (h = \Delta t),$$

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which may be written in the form

$$y_{n+1} - 2\mu y_n + y_{n-1} = 0 .$$

Unlike the differential equation, the solutions of this equation fall into four categories.

By assuming solutions of the form

$$y_n = e^{\beta t_n} = e^{\beta n h}$$

where $h = \Delta t$, one can obtain an expression for βh as a function of μ .

Clearly,

$$e^{\beta h(n-1)} \left[e^{2\beta h} - 2\mu e^{\beta h} + 1 \right] = 0$$

hence,

$$e^{\beta h} = \mu \pm \sqrt{\mu^2 - 1} ,$$

or

$$\beta h = \log \left[\mu \pm \sqrt{\mu^2 - 1} \right] .$$

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As with the differential equation, the separate cases which arise depend upon the properties of the logarithm function.

Case I. $\mu > 1$. In this case, $\mu \pm \sqrt{\mu^2 - 1}$ is positive; hence, β_h is real. Moreover, the value of β obtained by using the + sign leads to an exponentially increasing solution, while the other sign leads to a solution which vanishes exponentially as n increases.

Case II. $\mu = +1$. One can verify directly that

$$y_n = An + B$$

is a solution of the difference equation.

Case III. $-1 \leq \mu < +1$. For all μ in this interval, it can be shown that

$$\left| \mu \pm \sqrt{\mu^2 - 1} \right|^2 = 1.$$

Indeed, with either sign,

$$\left| \mu \pm \sqrt{\mu^2 - 1} \right|^2 = \left| \mu \pm i\sqrt{1 - \mu^2} \right|^2 = (\mu + i\sqrt{1 - \mu^2})(\mu - i\sqrt{1 - \mu^2}) = 1.$$

Thus, β_h is a pure imaginary. In fact, there exists a θ such that

$$\beta_+ h = i\theta \quad \text{and} \quad \beta_- h = -i\theta,$$

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and hence the solution is a pure oscillation.

Case IV. $\mu < -1$. Now βh is complex, since $\mu + \sqrt{\mu^2 - 1}$ is real but negative. Thus,

$$\begin{aligned}\beta_{\pm} h &= \log \left| \mu + \sqrt{\mu^2 - 1} \right| + i\pi \\ &= \alpha_{\pm} + i\pi.\end{aligned}$$

Note that $\alpha_{+} > 0$ and $\alpha_{-} < 0$. The solutions, then, are oscillations with exponentially increasing or decreasing amplitudes. Note particularly that the frequency of oscillation of $y_n = e^{\beta h n}$ is independent of μ . In fact, the signs alternate at successive grid points.

For the difference equation obtained from the original differential equation,

$$\mu = 1 + \frac{h^2 \lambda^2}{2}, \quad (h = \Delta t).$$

Thus, it follows that for

(a) $\lambda^2 > 0$, the behavior of the difference solution has the same character as the differential solutions, independent of the size of Δt , while for

(b) $\lambda^2 < 0$, where the differential solution is a pure oscillation, the difference solution will only have this character provided Δt is sufficiently small. In fact, the necessary and sufficient condition is

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$$-1 \leq 1 + \frac{(\Delta t)^2 \lambda^2}{2} < 1$$

or, (since λ^2 is assumed to be negative)

$$2 \geq \frac{(\Delta t)^2 |\lambda|^2}{2} > 0$$

Hence,

$$(\Delta t)^2 \leq \frac{4}{|\lambda|^2} \quad \text{or} \quad \frac{1}{\Delta t} \geq \frac{|\lambda|}{2\pi} \pi$$

This is not surprising. It says that in order to approximate an oscillatory solution, the number of grid points per unit interval must exceed π times the number of oscillations in that interval.

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