**INTERACTION OF A COASTING BEAM AND A
BUNCHED BEAM WITH FREQUENCY SLIP***

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ABSTRACT

Collisions between a coasting beam and a bunched beam can be so arranged that a particle in the coasting beam collides with the bunched beam only intermittently. A particle thus interacts with a bunch for a given number of revolutions, say q , and then slips away into the interbunch space. The process, with each cycle taking say, n revolutions, then continues. The possible effects of this intermittent type force are considered. The linear stopbands introduced are dealt with in detail. Growth rates are obtained as a function of q , assuming a Gaussian fall-off in the force. The stabilizing influence of an azimuthally constant octupole field component is demonstrated. Isolated nonlinear resonances excited by the intermittent force are discussed. No quantitative treatment is given as far as concerns the stochastic nature of highly nonlinear systems. Also, the influence of feeding mechanisms, such as a time variation of the tune is neglected. If the number of revolutions between kicks, $\sim n$, is large, then there is the possibility of loss of correlation between kicks. The resulting diffusion is analyzed and diffusion rates are estimated. Conclusions are as follows: (1) Growth due to linear instabilities are effectively damped by making q large enough. The perturbation is turned on and off adiabatically and leaves no record of itself from cycle to cycle. The growth rate is linear in $\Delta\nu$, the beam-beam linear tune shift per interaction and per revolution. (2) The diffusion induced by random

* Work supported by the U.S. Atomic Energy Commission.

processes causing the kicks to be uncorrelated produces a beam growth rate which is quadratic in Δv . Although it is smaller than the gradient stopband growth (for low q), it is essentially independent of tune (and so occurs for all particles) and it has no drop-off with q . In fact, for large q , it increases like q^2 . And (3), an azimuthally constant octupole field component is an effective stabilizing influence against resonant growth.

1. Introduction

A significant consequence of a coasting beam colliding with a bunched beam is the fact that it can be so arranged that a particle in the coasting beam collides with the bunched beam only intermittently. So, a particular proton interacts with a bunch for a given number of revolutions and, if the frequencies are properly chosen, will slip away from the bunch into the interbunch space. One can consider the process to consist of cycles where the coasting particle first interacts with a bunch for a certain number of revolutions, then misses the bunch, then interacts again, and so on. This can occur also, to some extent, in the case of two bunched beams if they have sufficiently different bunch lengths and collide with a nonzero crossing angle.¹

The system we are considering is that of a time-varying force, where the changes in the force occur from revolution to revolution. The distinction between this type of force and the usual beam-beam force² is that the one we are considering has a periodicity which is not that of the physical structure, but, in general, much longer. A natural question which immediately arises concerns the influence of the intermittent force on the beam-beam interaction. Is the maximum allowable beam-beam tune shift affected? Is it increased or decreased? However, a simple answer is not possible without

a theoretical framework for the beam-beam interaction - a framework which, in its present state, is not adequate. It is obviously a complicated issue. On the one hand, the resonances become dense, as we will see below; but, on the other hand, their effective range of tune is very small, their strength low. Furthermore, the time scale is now cycles, i.e., n revolutions, rather than revolutions. Without a clear theory for stochastic and near-stochastic systems, it is not possible to decide on the relative influence of these factors - except to point out in a general way that "sufficiently" long cycle times must ultimately lead to longer lifetimes on the time scale of revolutions. This statement is not, however, a substitute for a detailed dependence of the long time behavior of such nonlinear, intermittent force systems on both the on, off times as well as the total cycle time. In the absence of a theory one must appeal to computer simulations and physical experiments.

Although the fundamental mechanism by which the nonlinear beam-beam force leads to slow amplitude growth is not known, certain effects introduced by the intermittent-type force can be estimated by standard techniques. We will be concerned, in particular, with two possible types of consequences. Consider, then, a system with a periodicity of n revolutions. Let q be the number of revolutions per period (cycle) in which the force operates. If n is large, the set of resonances in the basic machine becomes very dense, since the machine tune (ν_0) at which they occur goes as

$$\nu_0 \sim \frac{1}{n} .$$

For example, gradient resonances occur at³

$$\nu_0 = \frac{p}{n} , \quad \frac{p+1}{n} ; \quad p, \text{ integer} .$$

However, the stopband widths also go like $1/n$, and so the probability of being in a stopband is unchanged. If a particle is not in a stopband, there will be a

$\mathcal{R}$ -function modulation, but no continuing growth as occurs when within the stopband. We will here treat the unstable case and compute the growth rate due to these half-integer resonances, treated individually. A more realistic, although more difficult calculation would be to include v -fluctuating terms. Because of the high density of stopbands, particles would then be fed to the resonances resulting in a diffusion-like growth.

Aside from the resonance calculation we will also consider (1) the possible stabilizing influence of a 0th harmonic nonlinear component in the machine structure and, (2) in a qualitative way, the higher order resonances arising out of the intermittent nature of the beam-beam interaction.

A possible effect of large n is related to what is loosely called the decorrelation time, n_d . If $n > n_d$, then the beam conditions from one cycle to the next are uncorrelated. Thus, instead of resonance growth, we are concerned with a random-type process which leads to diffusion. We will estimate the diffusion rate, approximating the beam-beam force by its linear component and assuming no correlation between successive cycles.

In treating the two cases of resonant growth and diffusion, we essentially are considering the two extremes of, on the one hand, correlation of beam conditions over n revolutions (governed by an energy conserving Hamiltonian system); and, on the other hand, complete lack of correlation over n revolutions (arising from unspecified random processes, i.e., noise).

2. Resonant Growth with an Intermittent Force

Consider a particle in a coasting beam, designated by its betatron phase space variables x, x' . Differentiation is with respect to the azimuthal length, s . The intermittent nature of the force is defined by its periodicity of n revolutions and the distance in s between the bunches causing the force, denoted Δ .

If there is only one bunch, Δ is the circumference, C . The slip in s per revolution is simply Δ/n .

In a given cycle, assuming one interaction per circumference, the equation of motion for the coasting particle during the m^{th} revolution can be written⁴

$$x'' + K(s) x = - \frac{2N_0 r_0}{\gamma} \lambda_m(s) \frac{[1 - e^{-(s^2 \alpha^2)/2\sigma^2}]}{s^2 \alpha^2} x, \quad (2.1)$$

where s is restrained by $0 \leq s < C$, $s = 0$ being the center of the interaction region,

N_0 is the number of particles per bunch,

r_0 is the classical radius ($= e^2/4\pi m_0$) for particles in the bunched beam,

γ is the energy in units of $m_0 c^2$,

β ($= v/c$) is taken to be one,

σ is the rms transverse beam size,

α is the crossing angle,

K is the machine gradient forcing function ($= \text{gradient}/\text{momentum}$)

$\lambda_m(s)$ is the longitudinal density of the bunch on the m^{th} revolution of the cycle,

and the beams are assumed to have charge of opposite sign. We have taken only the gradient component of the beam-beam force and we have assumed a circular (transverse) beam in deriving the force. The assumed crossing angle is a formal mechanism for separating the beams. For a Gaussian longitudinal distribution, the particle density function over the cycle is

$$\lambda_m(s) = \frac{1}{\sqrt{2\pi}} \frac{1}{\sigma_\ell} \exp\left(-2\left[\frac{s}{\sigma_\ell} - \frac{m}{n} \frac{\Delta}{\sigma_\ell}\right]^2\right) \quad (2.2)$$

where σ_ℓ is the rms longitudinal size, and we have taken account of the apparent bunch contraction (since the velocity of the coasting particle

relative to the bunch $\approx 2c$) by using an rms of $\frac{1}{2}\sigma_f$ in the exponential factor.

To get an estimate of the effect of the beam-beam perturbation, we replace the s -variation of the force by a Dirac δ -function centered, at $s = 0$. With an interaction length of $4\sigma_f$, Eq. (2.1) becomes

$$x'' + K(s) x = - \frac{4N_o r_o}{\sqrt{2\pi} \gamma \sigma^2} \exp\left(-2\left[\frac{\pi \Delta}{n\sigma_f}\right]^2\right) \delta(s) x \quad (2.3)$$

This equation can be expressed in terms of the beam-beam "tune shift" when the coasting particle passes through the bunch center,

$$\Delta\nu = \frac{N_o r_o \mu^*}{\pi \sqrt{2\pi} \gamma \sigma^2} \quad (2.4)$$

where μ^* is the value of the β -function at the center of the interaction region ($s = 0$). We thus obtain

$$x'' + K(s) x = - \frac{4\pi}{\beta^*} \Delta\nu \exp\left(-2\left[\frac{\pi \Delta}{n\sigma_f}\right]^2\right) \delta(s) x \quad (2.5)$$

We will treat the intermittent force as a perturbation. Consider the unperturbed system as one composed of n times the basic machine, for which the tune is denoted ν_o . If ν_u is the unperturbed ν -value, then the condition that the perturbation can lead to a linear stopband (and thus beam growth) is simply

$$\nu_u = n\nu_o = p \text{ or } p + \frac{1}{2}, \quad p \text{ integer} \quad (2.6)$$

For large n , the number of such resonances in terms of the machine tune, ν_o , becomes very dense. We consider the worst possible situation and compute the maximum growth rate, which occurs at the stopband center. Denoting the e-folding growth rate by μ^* and using standard stopband theory,³ we find,

restricting ourselves to the case $v_u \approx p$,

$$\cosh \mu^* = 1 + \frac{|J_{2p}|^2}{8}, \quad (2.7)$$

where

$$J_{2p} = \sum_{m=0}^n \int_0^C ds \beta(s) k_m(s) e^{-12p\varphi}, \quad (2.8)$$

and

$$\varphi = \frac{1}{v_u} \int_0^s \frac{dp}{\beta(p)} + \frac{2\pi m}{n}. \quad (2.9)$$

The quantity $k_m(s)$ is a perturbation of the gradient function $K(s)$, and $\beta(s)$ is the usual β -function. For $k_m(s)$ given in (2.5), we have

$$J_{2p} = 4\pi \Delta v \sum_{m=-n/2}^{n/2} \exp\left(-2\left[\frac{m\Delta}{n\sigma_L}\right]^2\right) e^{-14\pi p m/n}. \quad (2.10)$$

This can be simplified a little. Introduce q , the number of consecutive revolutions in one cycle that a coasting particle collides with the bunch (of length $4\sigma_L$). Thus, q is given by

$$\frac{q}{n} = \frac{4\sigma_L}{\Delta}. \quad (2.11)$$

For large n , we can replace p by nv_0 and extend the sum to infinity.

Substituting into (2.10), we obtain

$$J_{2p} = J = 4\pi \Delta v \sum_{m=-\infty}^{\infty} e^{-32m^2/q^2} e^{-14\pi v_0 m}, \quad (2.12)$$

which is independent of p .

Equation (2.7) can be solved formally:

$$\mu^* = \cosh^{-1} \left[1 + \frac{|J|^2}{8} \right]. \quad (2.13)$$

For small $|J|$,

$$\mu^* \approx \frac{|J|}{2}. \quad (2.14)$$

The e-folding lifetime in units of revolution number is

$$n_{\tau} = \frac{n}{\mu^*} \approx \frac{2n}{|J|} \quad (\text{for small } |J|) \quad (2.15)$$

We can evaluate the sum in J approximately, in the limit of large and small q. For q = 1, only the m = 0 term is significant, and we have

$$J(q = 1) = 4\pi\Delta v \quad (2.16)$$

In the case

$$\frac{1}{q} \ll 1, \quad J \approx 4\pi\Delta v q \int_{-\infty}^{\infty} dt e^{-32t^2} e^{-14\pi v_0 q t} \quad (2.17)$$

or

$$J \approx \frac{\pi^{3/2}}{\sqrt{2}} \Delta v q e^{-\pi^2 v_0^2 q^2 / 8} \quad (2.18)$$

Comparing (2.18) with (2.16), it is clear that we obtain a dramatic reduction of the growth rate by having the coasting particle slowly slip away from the bunch (i.e., $q \gg 1$). For $q \approx 10$, (taking $v_0 \approx 20$), the value of J in (2.18) is already an exceedingly small number. The meaning of this result is physically clear. It corresponds to turning the perturbation on and off adiabatically; thus the perturbation leaves no record of itself from cycle to cycle. The drop-off with q is shown in Fig. 1, where μ^* [from (2.13)] is plotted as a function of q. J is taken from (2.12). The sum is performed numerically. The curves are plotted for $\Delta v = 0.01$, and $v_0 = 20.25$, and 20.17.

3. Diffusion with an Intermittent Force

We have considered in the previous section the effect of an intermittent perturbation, acting only every n revolutions. It was assumed that from cycle to cycle, between perturbations, the motion in betatron phase space could be described by a Hamiltonian formalism. We simply used linear forces. The effects of nonlinear terms will be discussed in the next section.

However, if n (the number of revolutions per cycle) is large, one might anticipate the presence in any real system of stochastic-type processes which would tend to randomize the motion.⁵ If such processes exist, then since the perturbing force is a function of the particle coordinate, the force itself becomes randomized. The basic assumption here is that the decorrelation time, n_d , (that is, the number of revolutions it takes for the stochastic forces, the noise, to decorrelate the particle coordinates from their initial values) is smaller than the cycle time, n . This is a strong assumption and not at all obvious, even for n values as large as 10^3 - 10^4 .

Nevertheless, setting aside the doubts about the validity of assuming $n_d < n$, we take the following model. Let the perturbing force be a random variable, the randomization deriving from the decorrelation of the particle coordinate. Using the force as given in (2.5), the equation with decorrelated perturbations is

$$x'' + K(s)x = -\frac{4\pi}{\beta^*} \Delta v e^{-32m^2/q^2} \delta(s) x_r, \quad (3.1)$$

where q is the number of consecutive turns that the coasting particle encounters the bunch defined to be of length 4 standard deviations. Over q turns, the motion is assumed to be correlated, and x_r is just the particle coordinate, x . However over n revolutions, that is, between the groups of q turns, the correlation is lost.

Introducing the coordinate

$$u = x \sqrt{\frac{\beta^*}{\beta}}, \quad (3.2)$$

where β^* is the value of β at the interaction point, $s = 0$, and

$$\varphi_m = \frac{1}{v_0} \int_0^s \frac{d\rho}{\beta(\rho)} + 2\pi m, \quad (3.3)$$

Eq. (3.1) becomes

$$u'' + v_0^2 u = -4\pi v_0^2 \Delta v \beta^* e^{-32m^2/q^2} \delta(s) u_{m,r} . \quad (3.4)$$

The variable $u_{m,r}$ is random, except correlated over the q revolutions during which the force is consecutively acting on the particle designated by u . We can find the value of u at $s = 0$ after ℓ revolutions. Denoting this quantity u_ℓ , we have

$$u_\ell = u_0 - 4\pi \Delta v \sum_m e^{-32m^2/q^2} \sin v_0(\varphi_\ell - \varphi_m) u_{m,r} , \quad (3.5)$$

where u_0 is the initial value of u at $s = 0$; φ_m and φ_ℓ are also to be evaluated at $s = 0$. The solution is only formal, in the sense that $u_{m,r}$ has not been specified. If we look for a solution to first order in the perturbing Δv , we can replace $u_{m,r}$ by the unperturbed solution,

$$u_{m,r} = a_r \cos(v_0 \varphi_m + \delta_r) . \quad (3.6)$$

We can assume that a_r and δ_r are uncorrelated random variables. Including the randomization in this way allows us to properly correlate the perturbing kicks (through the phase variable φ) since a_r and δ_r are constants (to 0th order in Δv). The mean square of u_ℓ can be found, using

$$\langle a_{r_1} a_{r_2} \cos \delta_{r_1} \cos \delta_{r_2} \rangle = \frac{1}{2} a_{r_1} a_{r_2} \langle a^2 \rangle ,$$

$$\langle a_{r_1} \cos \delta_{r_2} \rangle = 0 ,$$

and other similar relations.

We find for the mean square of u after n revolutions (1 cycle),

$$\langle u_n^2 \rangle = u_o^2 + 8\pi^2 \Delta v^2 \langle a^2 \rangle \times \\ \times \sum_{m, m'} \exp(-32(m^2 + m'^2)/q^2) \sin v_o(\varphi_n - \varphi_m) \sin v_o(\varphi_n - \varphi_{m'}) \cos v_o(\varphi_m - \varphi_{m'}) . \quad (3.7)$$

If we average over the phase φ_n , we obtain the mean square amplitude after n revolutions, $\langle a_n^2 \rangle$, in terms of the mean square of initial amplitudes, $\langle a_o^2 \rangle$.

So, averaging over particle phases, there results

$$\langle a_n^2 \rangle = \langle a_o^2 \rangle + 8\pi^2 \Delta v^2 \langle a_o^2 \rangle \sum_{m, m'} \exp(-32(m^2 + m'^2)/q^2) \cos^2 v_o(\varphi_m - \varphi_{m'}) . \quad (3.8)$$

Thus, the diffusion growth time (per cycle, n revolutions) for the rms amplitude is

$$\mu_D^* = (2\pi \Delta v)^2 \sum_{m, m'} \exp(-32(m^2 + m'^2)/q^2) \cos^2 v_o(\varphi_m - \varphi_{m'}) . \quad (3.9)$$

We can estimate μ_D^* for the two cases, $q = 1$, and $q \gg 1$. If $q = 1$, only the term $m = m' = 0$ is significant, and we have⁵

$$\mu_D^* (q = 1) = (2\pi \Delta v)^2 . \quad (3.10)$$

For $q \gg 1$, we can replace the sums by integrals. Setting $1/q = dt$ and $m/q = t$, and noting that

$$\varphi_m = 2\pi m ,$$

we obtain

$$\mu_D^* (q \gg 1) = (2\pi \Delta v)^2 q^2 \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} dt dt' \times \\ \times e^{-32(t^2 + t'^2)} \cos^2 2\pi v_o q(t - t') . \quad (3.11)$$

For large q , this can be approximated by

$$\mu_D^* (q \gg 1) \approx \frac{\pi^3}{16} (\Delta v)^2 q^2 . \quad (3.12)$$

Thus, with increasing q the diffusion will ultimately get worse. This result is opposite to the one obtained for the resonance growth where there was a

dramatic decrease in the growth rate with increasing q . Physically, this means that there is a mean square residual effect after the period of q perturbing forces, even though there is a slow on and off process (i.e., $q \gg 1$). μ_D^* is plotted as a function of q , taken from (3.9), in Fig. 2, for $\Delta v = 0.01$ and $v_0 = 20.25$ and 20.17 .

4. Effects of Nonlinearities

There are two broad classes of nonlinearities which we consider. First, there are the nonlinearities arising from the intermittent beam-beam force. We will discuss these in a rather rough, general way; and secondly, there are machine nonlinearities. We neglect random-type machine errors, which lead to all harmonics of the azimuthal field distribution, and treat only systematic errors which give a predominantly 0th harmonic contribution. Specifically, we look at the influence of a uniform azimuthal octupole component of the magnetic field on the intermittent beam-beam interaction.

Concerning the beam-beam nonlinearities, the effect of intermittency is as follows: Let us recall that n is the number of revolutions between groups of beam-beam "kicks", and v_0 is the tune of the physical machine. Then an l^{th} harmonic of the beam-beam field will excite a resonance of order m at a tune, v_0 , given by³

$$v_0 = l/mn \quad , \quad (4.1)$$

where the beam-beam force (if the collisions are ideal) excites only even order resonances²; i.e.

$$m = 2, 4, 6 \dots \quad . \quad (4.2)$$

Since n could be very large, the resonances (in terms of the machine tune, v_0) are very dense. However the increase in density engendered by the intermittency of n revolutions is offset by the decrease in resonant widths by the factor $1/n$.

If the resonances are treated separately, that is, if we consider only the influence of isolated resonances, then the number n is of importance only insofar as the resonant widths are concerned. In effect, the isolated resonance approach assumes the density is not a significant quantity. It appears then that the effect of isolated resonances, for short times, should be greatly diminished. We will discuss this in more detail for the case of a gradient resonance ($m = 2$) below.

It is important to emphasize that in treating resonances as isolated, our conclusions can only be made for "short" times. The "long" time behavior of nonlinear systems, which manifest stochastic-like properties, which induce diffusion-like behavior, surely cannot be treated with an isolated resonance approach. In fact, to adequately treat "long" time systems, we are led to consider the interaction of many resonances² and this subject is outside the scope of this paper. However, there is a clear question which may be asked: What is the difference in the long-time behavior between, on the one hand, a system with a given set of resonances, and, on the other hand, a system derived from this one, but one where the density is increased by n and the resonant widths decreased by $1/n$? Of course this question is nothing but a rephrasing of the question of whether the highly nonlinear beam-beam force is more or less harmful if it is intermittent (but still periodic!).

We have shown in a previous section that a linear intermittent perturbation induces a density of linear stopbands proportional to n , the periodicity of the intermittent force. The widths of these stopbands are, however, proportional to $1/n$. In fact in terms of the machine tune, ν_0 , the stopband width is

$$\delta\nu_0 = \frac{|J|}{2\pi n} \quad , \quad (4.3)$$

where J is given by (2.12).

We can now ask what the effect of an azimuthally constant octupole field has on this linear intermittent-force system. We consider two cases: (1) ν_0 is outside the stopbands, and (2) ν_0 is at the center of a stopband. In the first case, the linear system, including the intermittent quadrupole perturbation, is stable. Thus, there exists a perturbed β -function and corresponding tune. It can be shown that such a system is stable for all time for all practical octupole field strengths.⁷ It must be emphasized that the only assumption required to reach this strong conclusion is the smooth approximation, which is simply the assumption that fluctuations in the β -function cannot induce long-time instability in a 0th harmonic octupole field. This may or may not be true; however, it should be remembered that the β -function distortion introduced by the intermittent linear kick is much smaller than the β -function fluctuations in any real machine. In fact, the maximum β -function change is given by

$$\left(\frac{\Delta\beta}{\beta} \right)_{\max} = \frac{|J|}{\pi}, \quad (4.4)$$

which for a single kick per cycle (the worst case) is

$$\left(\frac{\Delta\beta}{\beta} \right)_{\max} = 4|\Delta\nu|. \quad (4.5)$$

For $|\Delta\nu| = 0.01$, this amounts to a 4% fluctuation. It is interesting to compare this with the emittance fluctuation introduced by the 0th harmonic octupole. If we denote the emittance change by $\Delta\epsilon$, and the initial emittance by ϵ_0 , then⁶

$$\frac{\Delta\epsilon}{\epsilon_0} = \frac{1}{2} \frac{\epsilon_0 \langle \beta \rangle^3 b_3}{\rho}, \quad (4.6)$$

where $\langle \beta \rangle$ is the average value of β (smooth approximation),

ρ is the bending radius for the machine,

$b_3 = \frac{1}{6B_0} \frac{\partial^3 B}{\partial x^3}$ is the strength of the octupole field,

B_0 is the bending field,

and x is a transverse direction.

Using some typical storage ring values, $\epsilon_0 = 10^{-6}$ rad-m, $\langle \beta \rangle = 20$ m, $\rho = 400$ m,

$b_3 = 1 \text{ m}^{-3}$, we have

$$\frac{\Delta \epsilon}{\epsilon_0} = 10^{-5} \quad , \quad (4.7)$$

which is much smaller than the β -function fluctuation, given in (4.5).

We next consider the case where ν_0 is at the center of the stopband.

In this case, we must keep the intermittent kick as an explicit perturbation, since, by assumption, no stable betatron system would exist if we included it in the linear system of the machine. Therefore, with the two perturbations acting simultaneously, we use an expansion approach. It can be shown that there exists an invariant of the motion to first order in the strengths of both the octupole component and the kick. If we transform from the physical coordinates $[x, x' = dx/ds]$ to normalized coordinates $[\eta, \dot{\eta} = d\eta/d\varphi]$:

$$\eta = \frac{x}{\sqrt{\beta}} \quad , \quad (4.8)$$

$$\varphi = \frac{1}{\nu} \int_0^s \frac{ds'}{\beta(s')} \quad ; \quad (4.9)$$

and thence to phase-amplitude variables:

$$\eta = a \sin \alpha \quad , \quad (4.10a)$$

$$\dot{\eta} = a \nu \cos \alpha \quad ; \quad (4.10b)$$

we find the invariant (with the unperturbed tune at the center of the linear stopband),

$$u = \frac{\Delta \nu}{n} a^2 \cos 2\alpha + \frac{3}{16} \langle \beta \rangle^2 b_3 a^4 \quad . \quad (4.11)$$

The motion represented by this invariant is bounded. However, it should be remembered that it is only valid for "short" times. Keeping this in mind, we have, noting that a^2 is just the emittance, ϵ , that the emittance fluctuation is given by

$$\frac{\Delta\epsilon}{\epsilon_0} = \frac{16\Delta\nu}{3n \langle\beta\rangle^2 b_3\epsilon_0} \quad (4.12)$$

With the previously used values for $\langle\beta\rangle$, b_3 , and ϵ_0 , and setting $\Delta\nu = 0.01$, we have

$$\frac{\Delta\epsilon}{\epsilon_0} = \frac{400}{3n} \quad (4.13)$$

This means for $n \approx 100$, the stabilizing octupole will allow an emittance growth (due to the gradient stopband) of about a factor of 2. If $n \approx 1000$, then only a 10% growth is allowed. Thus, even a single kick is effectively stabilized for this value of n .

We can therefore make the following conclusions concerning the influence of a uniform octupole component on a system with a linear intermittent kick:

- (1) Outside the stopband, the system is "long time" stable; and
- (2) Inside the stopband, the octupole is "short time" stabilizing for the case of a single kick. For more kicks per cycle, we have seen that the stopband widths drop off dramatically, and so the octupole is essentially acting on a stable system.

Acknowledgment

We have discussed the subject of this paper with a great many people during and after the 1973 PEP Summer Study. Some of them agreed with our point of view, others disagreed. We thank all of them.

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7. M. Month, BNL Accelerator Dept. Report CRISP 73-16 (1973).

Figure Captions

Fig. 1. Growth rate per cycle (n revolutions) at the center of a linear resonance stopband. The tune shift due to the point perturbation is $\Delta\nu$. The variable q is the number of kicks per cycle, distributed as a Gaussian. Curves are drawn for two values of tune, ν_0 (20.25 and 20.17).

Fig. 2. Growth rate per cycle (n revolutions) due to diffusion arising from uncorrelated linear kicks. The tune shift due to the point perturbation is $\Delta\nu$. The variable q is the number of kicks per cycle, distributed as a Gaussian. Curves are drawn for two values of tune, ν_0 (20.25 and 20.17).

Diffusion Growth Rate: $\Delta v = 0.01$