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THE YIELD STRESS AND PLASTIC STRAIN THEORY
FOR ANISOTROPIC MATERIALS

D.C. Bogue

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FOR ANISOTROPIC MATERIALS

D. C. Bogue

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NOTATION

Roman Letters:

A_j^i , a material tensor, see Eq. (2).

B_{ik}^{jl} , a material tensor, see Eq. (2).

B_{ik}^{*jl} , a material tensor in the strain theory, see Eq. (28).

c , a material constant.

f , the plastic potential function.

g_{ij} , the metric tensor.

K , a lumped parameter, see Appendix C.

k_{+ij} , the ratio of the contractile strain in the j -direction to the applied tensile strain in the j - (or test) direction.

k_{-ij} , the ratio of the expansive strain in the j -direction to the applied compressive strain in the i - (test) direction.

$p = -\frac{1}{3} \sigma_k^k$, scalar stress (pressure).

Y_{ci} , the magnitude of the compressive yield strength in the i -direction.

Y_{ti} , the tensile yield strength in the i -direction.

Greek Letters:

α_i , a material tensor, an abbreviation for A_j^i .

α_{ii} , β_{ij} , γ_{ij} , material tensors, abbreviations for certain groupings of B_{ik}^{jl} , see Eq. (6).

δ_j^i , δ_{ij} , the Kronecker delta, $\delta_{ij} = 0$ for $i \neq j$; $\delta_{ij} = 1$ for $i = j$.

δe_{ij} , an increment of the linear strain tensor.

e_j^i , e_{ij} , a nonlinear strain tensor.

σ_j^i , σ_{ij} , the total stress tensor.

τ_j^i , τ_{ij} , the deviator stress tensor.

τ_0 , the yield stress in von Mises' theory.

χ , the angle between the anisotropic axes and the rotated axes.

Abbreviations and Subscripts:

1 = rolling direction (RD).

2 = transverse direction (TD).

3 = normal direction (ND).

$$\det [\tau_n^m] \equiv \tau_{123}^1 \tau_{231}^2 + \tau_{231}^1 \tau_{123}^2 + \tau_{123}^2 \tau_{231}^3 - \tau_{321}^1 \tau_{123}^2 - \tau_{213}^1 \tau_{231}^3 - \tau_{321}^2 \tau_{231}^1.$$

Symbols with overbars denote components in an arbitrary coordinate system. Symbols without overbars denote general components or components in the anisotropic coordinate system, depending on the context.

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THE YIELD STRESS AND PLASTIC STRAIN THEORY FOR ANISOTROPIC MATERIALS

D. C. Bogue¹

ABSTRACT

A yield stress and plastic strain theory is developed for anisotropic materials that show the Bauschinger effect. The theory is a generalization of Hill's anisotropic theory. The quadratic (Hill) terms are common to and predominate in both the yield stress criterion and the plastic strain equations; there are additional constants in both cases that are not interrelated. The formulation is motivated by the behavior of Zircaloy-2 and provides a good correlation of experimental data from a highly textured material.

INTRODUCTION

The present work deals with developing a phenomenological yield stress criterion and a plastic strain theory for anisotropic materials, particularly for zirconium-base alloys. In a broad sense it is a generalization of the von Mises^{2,3} yield criterion for isotropic materials; more specifically it is a generalization of the Hill⁴ theory for anisotropic materials.

Introducing anisotropy into a yield theory is relatively straightforward; one allows the coefficients in the quadratic stress criterion to be variables (fourth-order tensors) instead of constants. While

¹Consultant from the Department of Chemical and Metallurgical Engineering, the University of Tennessee.

²R. von Mises, Nachr. Akad. Wiss Göttingen, Math. Physik. Kl. 1913, 582-592.

³C. Truesdell and R. A. Toupin, "The Classical Field Theories," pp. 226-793 in Handbuch der Physik, Vol. III/1, ed. by S. Flügge, Springer-Verlag, Berlin, 1960.

⁴R. Hill, Proc. Roy. Soc. (London) Ser. A 193 (1948); The Mathematical Theory of Plasticity, Oxford University Press, London, 1950.

Hill does not use tensor notation, this is the essence of his generalization. The additional difficulty of accommodating compressive and tensile yield strengths of unequal magnitudes (the Bauschinger effect) is less straightforward. One must in some way introduce an odd (not squared) term into the stress criterion or introduce the plastic strain in a sign-sensitive form. Since there is no mathematically unique way of doing this, one looks for a simple (and hopefully physically reasonable) form.

Edelman and Drucker,⁵ Drucker,⁶ and later Yoshimura⁷ have dealt with the problem of the Bauschinger effect in several, rather general ways by introducing the plastic strain explicitly. They used the plastic strain rather than the elastic since the stress state reflects the elastic portion of the deformation. This is in contrast to the present work, in which additional stress (but not strain) terms are introduced. There are advantages to both approaches. In terms of a yield stress criterion alone, it is more satisfying to deal directly in terms of the material parameters and the imposed stress state. One would not want to include a plastic strain that occurs after the yield starts; that is, the plastic strain is best thought of as a dependent variable. However, if one wishes to introduce the yield function into a strain theory, there is definitely some advantage to having the plastic strain appear explicitly. One now has the possibility of treating complex loading patterns in the plastic region.

Plastic strain theory capable of including work-hardening effects (but not the Bauschinger effect) has been considered by Lehmann,⁸ who cites related work due to Melan, Prager, and Shield and Ziegler. Roughly speaking, this work deals with setting down equations for the plastic potential function, which, in isotropic theory, is just the yield stress

⁵F. Edelman and D. C. Drucker, J. Franklin Inst. 251, 581-605 (1951).

⁶D. C. Drucker, U.S. Congress of Applied Mechanics, First, Proceedings, 1951, pp. 487-491, American Society of Mechanical Engineers, New York, 1952.

⁷Y. Yoshimura, Aeronautical Research Institute, Tokyo University, Report No. 349, November 1959.

⁸T. Lehmann, Rheol. Acta 3, 281-285 (1964).

function; in more general theory, it includes in addition some history-dependent terms. Finally a recent book by Thomas⁹ treats plastic theories in the very general framework of continuum mechanics.

In the present work a yield stress criterion is established in its own right as a means of correlating yields in variously oriented and loaded samples. The Bauschinger effect is included. Relationships between the yield criterion and plastic strain behavior are then exploited so far as possible, although it is recognized that there are independent factors in both cases.

THE YIELD STRESS CRITERION

Formulation of the Criterion

The following basic assumptions will be made.

1. The yield criterion depends on the present stress state only. It will be assumed that the stress (or strain) history does not affect the yield strength except insofar as the history determines the present stress state.
2. The yield criterion is a scalar quantity. That is, although the stresses are second-order tensors, some scalar combination of them determines when yield will occur.
3. Only the deviator stresses (over and above some scalar stress, the scalar "pressure") affect the yield. This scalar stress is not necessarily the hydrostatic pressure.
4. The material has three planes of anisotropy. A plane of anisotropy is a plane about which one does not observe changes of material properties during reflection. The three planes of anisotropy intersect in three axes, called the anisotropic axes. In the general statement it will not be assumed that the axes are orthogonal, but in all applications of the theory orthogonal axes will be assumed.

⁹T. Y. Thomas, Plastic Flow and Fracture of Solids, Academic Press, New York, 1961.

Within this framework one can write down a general "tensor polynomial" of the following form:¹⁰

$$A' \tau_i^i + A''^j \tau_j^i + B' \tau_j^i \tau_i^j + B''^j \tau_j^{\ell k} + B'''^j \tau_{ik}^j \tau_j^{\ell k} + c \det[\tau_n^m] + c' \tau_j^i (\tau_j^{\ell k} \tau_k^{\ell} + \tau_i^k \tau_{\ell}^i \tau_k^{\ell}) + \dots = 1. \quad (1)$$

The stress τ_j^i is the deviator stress, that over and above a scalar stress to be defined later. Yield will occur when the left-hand side exceeds unity. The coefficients A' , A''^j etc. are material parameters. Since they are tensor quantities (like stress), they can be determined in one coordinate system and transformed to another. It is not necessary to use the anisotropic coordinate system, although this system will produce the simplest results.

The above series will be truncated after the first cubic term (that with the coefficient c) because this is the simplest form that predicts the desired inequality of compressive and tensile yield strengths. Fortunately, some collapsing of the remaining coefficients is possible. Rewriting Eq. (1), one has

$$(A' \delta_i^j + A''^j \tau_j^i) + (B' \delta_k^j \delta_i^{\ell} + B''^j \delta_k^{\ell} + B'''^j \tau_{ik}^j) \tau_j^{\ell k} + c \det[\tau_n^m] = 1. \quad (2)$$

Let

$$A_i^j \equiv A' \delta_i^j + A''^j \tau_j^i$$

and

$$B_{ik}^{j\ell} \equiv B' \delta_k^j \delta_i^{\ell} + B''^j \delta_k^{\ell} + B'''^j \tau_{ik}^j.$$

Thus finally one has the following yield criterion:

$$A_i^j \tau_j^i + B_{ik}^{j\ell} \tau_j^{\ell k} + c \det[\tau_n^m] = 1. \quad (3)$$

¹⁰For those not familiar with this type of tensor notation, it should be noted that the up and down positioning of the indices can be dispensed with in Cartesian coordinates. Thus, in Cartesian coordinates one has $\tau_2^1 = \tau_1^2 = \tau_{12} = \tau_{21}$. Also one should note the summation convention: a repeated index implies summation. Thus, for example, τ_k^k means $\sum_{k=1}^3 \tau_k^k$.

In this form there are $9 + 81 + 1 = 91$ coefficients. However, many of these can be eliminated by the fourth assumption, as will be done in subsequent sections.

To have a useful yield criterion one must define explicitly how the deviator stress τ_j^i is to be obtained from the applied total stress σ_j^i ; that is, one must define a scalar stress p . In linear theory one defines the scalar stress by

$$p = -\frac{1}{3} \sigma_k^k. \quad (4)$$

However, when one develops a nonlinear theory, one should in principle permit a nonlinear definition of p . For example

$$p = -\frac{1}{3} \sigma_k^k + a \sigma_j^k \sigma_k^j + \dots$$

The practical consequence of these nonlinear terms is that the yield criterion will depend on the hydrostatic pressure. Since this effect, if any, has yet to be measured, the conventional definition of p [Eq. (4)] will be used.

Specialization to Orthogonal Coordinate Systems

In principle one can carry out arguments directly on Eq. (3) having to do with the symmetry of the material. For example, during reflection about some plane 2-3, one ought to be able to interchange the positive 1-axis and the negative 1-axis without affecting Eq. (3). This will result in many of the coefficients being zero. To do this in fact, however, requires some physical interpretation of the tensor components (i.e., it requires expressing them as physical components). For non-orthogonal coordinate systems there is no unique way of doing this by mathematical arguments, although Truesdell¹¹ has presented a procedure based on physical arguments. However, that level of generality will not be used in the present report and attention will be limited to orthogonal (but not necessarily Cartesian) coordinates. Until noted

¹¹C. Truesdell, Z. Angew. Math. Mech. 33, 345-356 (1953); 34, 69-70 (1954).

later it will also be assumed that these coordinates coincide with the anisotropic axes.

It is hoped that orthogonal anisotropic axes will provide a reasonable description of such processed anisotropic materials as rolled or drawn Zircalloys. Despite the complexities of the anisotropy on the crystalline scale, one does have roughly orthogonal anisotropy on a macroscopic scale, oriented with respect to the rolling, normal, and transverse directions.

Because of the orthogonality assumption one may write

$$\tau_j^i = \frac{g_{jj}}{g_{ii}} \tau_i^j \quad (i, j \text{ not summed}). \quad (5)$$

That is, a component τ_j^i differs from a component τ_i^j only by some factor that involves the coordinate transformation. This means, then, that by invoking the fourth assumption, one can eliminate all terms from Eq. (3) in which τ_j^i or τ_i^j appear linearly. For example, one will permit terms like $\tau_2^1 \tau_1^2$ but not terms like $\tau_2^1 \tau_2^2$. Physically this means that a change from a positive to a negative shear stress if applied along the axes of anisotropy will not affect the yield. Equation (3) then reduces to

$$\begin{aligned} & \frac{\alpha_1}{A_1^1} \tau_1^1 + \frac{\alpha_2}{A_2^2} \tau_2^2 + \frac{\alpha_3}{A_3^3} \tau_3^3 + \frac{\alpha_{11}}{R_{11}^{11}} (\tau_1^1)^2 + \frac{\alpha_{22}}{B_{22}^{22}} (\tau_2^2)^2 + \frac{\alpha_{33}}{B_{33}^{33}} (\tau_3^3)^2 + \frac{\gamma_{12}}{(D_{12}^{12} + D_{21}^{21})} \tau_1^1 \tau_2^2 \\ & + \frac{\gamma_{13}}{(B_{13}^{13} + B_{31}^{31})} \tau_1^1 \tau_3^3 + \frac{\gamma_{23}}{(B_{23}^{23} + B_{32}^{32})} \tau_2^2 \tau_3^3 + \frac{\beta_{12}}{(2B_{21}^{12} + 2B_{12}^{21})} \tau_2^1 \tau_1^2 \\ & + \frac{\beta_{13}}{(2B_{31}^{13} + 2B_{13}^{31})} \tau_3^1 \tau_1^3 + \frac{\beta_{23}}{(2B_{32}^{23} + 2B_{23}^{32})} \tau_3^2 \tau_2^3 + c \det[\tau_n^m] = 1. \quad (6) \end{aligned}$$

Certain new symbols have been assigned to the coefficients as shown.

Note that these new symbols are symmetric in their indices. (Thus, for example, $\beta_{12} = \beta_{21}$.) Thus use of the fourth assumption has reduced the number of coefficients to 13.

Now, because of Eq. (4) not all of the deviator stresses are independent. That is, in principle one can solve for (say) τ_3^3 , given τ_1^1 and τ_2^2 . As a consequence one ought to be able to write Eq. (6) omitting

the terms in τ_3^3 without any loss of generality. This in effect says that there are actually four fewer independent constants. One is free then to assume relations among some of the constants to eliminate four of them. From dimensional and other considerations the following selection is convenient:

$$\alpha_{11} = -\frac{1}{2}(\gamma_{12} + \gamma_{13}) [\text{stress}^{-2}] , \quad (7)$$

$$\alpha_{22} = -\frac{1}{2}(\gamma_{12} + \gamma_{23}) [\text{stress}^{-2}] , \quad (8)$$

$$\alpha_{33} = -\frac{1}{2}(\gamma_{23} + \gamma_{13}) [\text{stress}^{-2}] , \quad (9)$$

$$\alpha_1 + \alpha_2 + \alpha_3 = 0 [\text{stress}^{-1}] . \quad (10)$$

The remaining constant "c" has units of $[\text{stress}^{-3}]$. An alternate statement of Eqs. (7), (8), and (9) is

$$\gamma_{12} = \alpha_{33} - \alpha_{11} - \alpha_{22} , \quad (11)$$

$$\gamma_{13} = \alpha_{22} - \alpha_{11} - \alpha_{33} , \quad (12)$$

$$\gamma_{23} = \alpha_{11} - \alpha_{22} - \alpha_{33} . \quad (13)$$

Finally then Eq. (6) reduces to

$$\begin{aligned} & \alpha_1 \tau_1^1 + \alpha_2 \tau_2^2 + \alpha_3 \tau_3^3 - \frac{1}{2} \gamma_{12} (\tau_1^1 - \tau_2^2)^2 - \frac{1}{2} \gamma_{13} (\tau_1^1 - \tau_3^3)^2 - \frac{1}{2} \gamma_{23} (\tau_2^2 - \tau_3^3)^2 \\ & + \beta_{12} \tau_2^1 \tau_1^2 + \beta_{13} \tau_3^1 \tau_1^3 + \beta_{23} \tau_3^2 \tau_2^3 + c \det [\tau_n^m] = 1 , \end{aligned}$$

where $\alpha_1 + \alpha_2 + \alpha_3 = 0$.

(14)

Equation (14) represents the most useful, general statement of the theory (in the anisotropic coordinate system). There are 9 material constants:

c

$\beta_{12}, \beta_{13}, \beta_{23}$

either $\alpha_{11}, \alpha_{22}, \alpha_{33}$, or $\gamma_{12}, \gamma_{13}, \gamma_{23}$

any two of $\alpha_1, \alpha_2, \alpha_3$.

Having shown that there are only 9 independent constants in the anisotropic coordinate system, one can state that there are only 9 independent constants in any coordinate system. The transformation equations to other coordinate systems are presented in Appendix A.

Several special cases of Eq. (14) are important. For the special case of $\alpha_1, \alpha_2, \alpha_3$, and c all zero, one has Hill's six-constant theory. If furthermore

$$-3\gamma_{12} = -3\gamma_{13} = -3\gamma_{23} = 3\alpha_{11} = 3\alpha_{22} = 3\alpha_{33} = \beta_{12} = \beta_{13} = \beta_{23} = 1/(\tau_0)^2,$$

one has the von Mises' criterion for isotropic materials:

$$\frac{1}{6} \left(\tau_1^1 - \tau_2^2 \right)^2 + \frac{1}{6} \left(\tau_1^1 - \tau_3^3 \right)^2 + \frac{1}{6} \left(\tau_2^2 - \tau_3^3 \right)^2 + \tau_{21}^1 \tau_{12}^2 + \tau_{31}^1 \tau_{13}^3 + \tau_{23}^2 \tau_{32}^3 = (\tau_0)^2$$

or

$$\frac{1}{2} \tau_j^i \tau_i^j = (\tau_0)^2.$$

Treatment of Various Yield Experiments

Tensile-Compressive Tests Along Anisotropic Axes

Consider a tensile test along the 1-axis, letting Y_{t1} be the yield strength. For the various stresses one has

$$\sigma_{11} = Y_{t1}, \quad \sigma_{22} = 0, \quad \sigma_{33} = 0,$$

$$p = -Y_{t1}/3, \quad \tau_{11} = 2Y_{t1}/3, \quad \tau_{22} = -Y_{t1}/3, \quad \tau_{33} = -Y_{t1}/3.$$

Substituting in Eq. (14) and using Eqs. (7) and (10), one has finally

$$\alpha_1 Y_{t1} + \alpha_{11} (Y_{t1})^2 + \frac{2}{27} c (Y_{t1})^3 = 1. \quad (15)$$

Consider now a compression test along the 1-axis, letting Y_{c1} be the absolute value of the yield strength. Similar to above, one has

$$\sigma_{11} = -Y_{c1}, \quad \sigma_{22} = 0, \quad \sigma_{33} = 0,$$

$$p = Y_{c1}/3, \quad \tau_{11} = -2Y_{c1}/3, \quad \tau_{22} = Y_{c1}/3, \quad \tau_{33} = Y_{c1}/3.$$

Thus finally

$$-\alpha_1 Y_{c1} + \alpha_{11} (Y_{c1})^2 - \frac{2}{27} c (Y_{c1})^3 = 1. \quad (16)$$

Shear Tests Along Anisotropic Planes

Consider shear along an anisotropic plane (as in a torsion specimen), where Y_{s12} is the yield strength in 1-2 shear. The only total stress component is σ_{12} . Thus

$$\beta_{12} (Y_{s12})^2 = 1. \quad (17)$$

Similar relations exist for 2-3 and 3-1 shear.

General Tests not Along Anisotropic Axes

The value of a phenomenological formulation lies, of course, in its ability to predict yield in any specimen, independent of orientation. Also such orientations might be used in determining some of the constants. The transformation to an arbitrarily rotated coordinate system is shown in Appendix A.

Simplified Theory

It was noted in applying the general theory that the constants α_1 , α_2 , and α_3 made only a minor contribution (2 to 3%) to the yield criterion for the schedule 8 Zircaloy-2. It is therefore proposed that

a simplified theory may sometimes be used in which these constants are set equal to zero. Equation (14) then reduces to

$$\begin{aligned}
 & -\frac{1}{2} \gamma_{12} (\tau_1^1 - \tau_2^2)^2 - \frac{1}{2} \gamma_{13} (\tau_1^1 - \tau_3^3)^2 - \frac{1}{2} \gamma_{23} (\tau_2^2 - \tau_3^3)^2 \\
 & + \beta_{12} \tau_2^1 \tau_1^2 + \beta_{13} \tau_3^1 \tau_1^3 + \beta_{23} \tau_3^2 \tau_2^3 + c \det[\tau_n^m] = 1.
 \end{aligned} \tag{18}$$

where $\tau_j^i = +p\delta_j^i + \sigma_j^i$,

$$p = -\frac{1}{3} \sigma_k^k.$$

Table 1 summarizes the constants involved in the various forms of the theories.

Table 1. Summary of Constants Involved in Various Experiments

	General Theory	Simplified Theory ^a
Tensile-compressive tests along anisotropic axes (or shear tests 45° removed)	Any 2 of $\alpha_1, \alpha_2, \alpha_3$. $\gamma_{12}, \gamma_{13}, \gamma_{23}$ (or $\alpha_{11}, \alpha_{22}, \alpha_{33}$) and c (6)	$\gamma_{12}, \gamma_{13}, \gamma_{23}$, (or $\alpha_{11}, \alpha_{22}, \alpha_{33}$) and c (4)
Shear tests along anisotropic planes	$\beta_{12}, \beta_{13}, \beta_{23}$ (3)	$\beta_{12}, \beta_{13}, \beta_{23}$ (3)
General tests not along anisotropic directions	All of above (9)	All of above (7)

^aThe simplified theory reduces to Hill's theory if $c = 0$. It reduces to von Mises' theory if furthermore
 $-3\gamma_{12} = -3\gamma_{13} = -3\gamma_{23} = 3\alpha_{11} = 3\alpha_{22} = 3\alpha_{33} = \beta_{12} = \beta_{13} = \beta_{23} = 1/(\tau_0)^2$.

Some Qualitative Comments

The following comments may be helpful in gaining qualitative understanding of the theory:

1. If the simplified theory is to hold, the difference between the magnitudes of the compressive and tensile yield strengths must be proportional to the levels of these yield strengths. Specifically,

$$Y_{ci} - Y_{ti} = \frac{2}{27} c \left(Y_{ti} \right)^2 \left(Y_{ci} \right)^2 .$$

The relation, if it proves to be satisfactory, may have relevance in assigning a mechanistic meaning to c .

2. One cannot completely specify the yield behavior of an anisotropic material by presenting one yield surface in space (with axes lined up with the anisotropic axes). Such a surface does not involve the constants β_{12} , β_{13} , β_{23} . To bring in these constants one must have a second yield surface in space rotated in some manner with respect to the anisotropic axes. Stated in another way: data from shear yielding (along anisotropic axes) cannot be plotted on a yield surface whose axes are σ_{11} , σ_{22} , and σ_{33} (1, 2, and 3 being anisotropic axes).

3. While changing the sign of the shearing force (stress) has no effect on the yield when the test directions coincide with the anisotropic axes, this is not true when they do not coincide. Thus, for testing in a rotated sample, $+\tau_{12}$ at yield does not equal $-\tau_{12}$ at yield (see Appendix A).

THE PLASTIC STRAIN THEORY

Formulation of the Theory

Possible Formulations

The classical means of formulating a strain theory in the plastic region is to introduce a plastic potential function (f) such that

$$\delta \epsilon_{ij} = \text{const} \frac{\partial f}{\partial \tau_{ij}} , \quad (19)$$

where $\delta\epsilon_{ij}$ is an increment of the linear strain tensor and τ_{ij} is the stress tensor, the strain referring to the plastic strain only (excluding the elastic portion). The plastic potential function f is taken to be just the yield criterion in Hill's theory¹² or various modifications of it in the theories of Edelman and Drucker.¹³ The motivation for using the plastic potential is largely mathematical, although the results of course must be physically reasonable. One needs in some way to operate on the scalar function f so as to produce a second-order tensor. There is a mathematical (although not physical) analogy in the theory of elasticity, where the role of f is played by W , the elastic energy function.¹⁴ An important consequence of Eq. (19) is the automatic incorporation of the anisotropic yield parameters (which come directly from the yield stresses) in the strain theory, the general idea being that a direction that is "strong" in yield will be "resistant" to strain.

Now it is appealing to use Eq. (19) and the yield criterion of the present work [Eq. (14)] to produce a plastic strain theory. If one in fact does this, one obtains the following results for the strains in tensile-compressive testing along anisotropic axes:

$$\delta\epsilon_{11} = \text{const} \left[-\gamma_{12} \left(\tau_{11} - \tau_{22} \right) - \gamma_{13} \left(\tau_{11} - \tau_{33} \right) + c\tau_{22}\tau_{33} \right], \quad (20)$$

$$\delta\epsilon_{22} = \text{const} \left[-\gamma_{12} \left(\tau_{22} - \tau_{11} \right) - \gamma_{23} \left(\tau_{22} - \tau_{33} \right) + c\tau_{11}\tau_{33} \right], \quad (21)$$

$$\delta\epsilon_{33} = \text{const} \left[-\gamma_{13} \left(\tau_{33} - \tau_{11} \right) - \gamma_{23} \left(\tau_{33} - \tau_{22} \right) + c\tau_{11}\tau_{22} \right]. \quad (22)$$

The small terms involving α_1 , α_2 , and α_3 have been neglected. This theory does not assume incompressibility; that is, the ϵ_{ii} do not sum

¹²R. Hill, Proc. Roy. Soc. (London) Ser. A **193** (1948); The Mathematical Theory of Plasticity, Oxford University Press, London, 1950.

¹³F. Edelman and D. C. Drucker, J. Franklin Inst. **251**, 581-605 (1951).

¹⁴C. Truesdell and R. A. Toupin, "The Classical Field Theories," pp. 226-793 in Handbuch der Physik, Vol. III/1, ed. by S. Flügge, Springer-Verlag, Berlin, 1960, see p. 731.

to zero. To incorporate the incompressibility feature, one might reasonably add a scalar term S (to be determined) to Eq. (19) to give

$$\delta\epsilon_{ij} = \text{const} \frac{\partial f}{\partial \tau_{ij}} + S\delta_{ij}. \quad (23)$$

For tensile-compressive testing along the anisotropic 1-axis, the results of this kind of adaptation are as follows:¹⁵

$$k_{+12} \equiv \left. \frac{\delta\epsilon_{22}}{\delta\epsilon_{11}} \right|_{+1} = \frac{\gamma_{12} - \frac{1}{9} cY_{t1}}{2\alpha_{11} + \frac{2}{9} cY_{t1}}, \quad (24)$$

and

$$k_{-12} \equiv \left. \frac{\delta\epsilon_{22}}{\delta\epsilon_{11}} \right|_{-1} = \frac{-\gamma_{12} - \frac{1}{9} cY_{c1}}{-2\alpha_{11} + \frac{2}{9} cY_{c1}} \quad (25)$$

Drucker¹⁶ deals with the problem in a somewhat different manner, but still within the framework of the plastic potential concept. His conclusion is finally that the strain ratios are related to certain tangents on the yield surfaces. In particular

$$k_{+12} = - \left. \frac{d\sigma_{11}}{d\sigma_{22}} \right|_{P+}, \quad (26)$$

and

$$k_{-12} = - \left. \frac{d\sigma_{11}}{d\sigma_{22}} \right|_{P-}. \quad (27)$$

Point $P+$ (or $P-$) is the point where the curve on a yield-surface section intersects the $+x_1$ (or $-x_1$ axis); that is, the points corresponding to $\sigma_{22} = 0$. Again the terms involving α_1 , α_2 , and α_3 have been neglected.

¹⁵The symbol k_{+12} designates the ratio of the contractile strain in the 2-direction to an applied tensile strain in the 1- (or test) direction. Similarly k_{-12} designates the ratio of the expansive strain in the 2-direction to an applied compressive strain in the 1- (or test) direction. The k_{ij} are always negative.

¹⁶D. C. Drucker, U.S. Congress of Applied Mechanics, First, Proceedings, 1951, pp. 487-491, American Society of Mechanical Engineers, New York, 1952.

Applying Drucker's relations to Eq. (C-1) (Appendix C) gives identically the results of Eqs. (24) and (25). In other words, the modification of the plastic potential equation with incompressibility assumed is equivalent to Drucker's tangent method.

Equations (24) and (25) (and their counterparts in the other directions) are qualitatively correct, but no better than that. Anticipating the experimental results of the next section for schedule 8 Zircaloy-2, one has:

		<u>Calculated</u>	<u>Experimental</u>
k_{+12}	$= \frac{+(-408) - \frac{1}{9} (8214)(0.055)}{2(282) + \frac{2}{9} (8214)(0.055)}$	$= -0.69$	-0.885
k_{-12}	$= \frac{-(-408) - \frac{1}{9} (8214)(0.066)}{-2(282) + \frac{2}{9} (8214)(0.066)}$	$= -0.79$	-0.714

A better but still not satisfactory result is obtained by deleting the Bauschinger (or c) term from the theory:

$$k_{+12} = k_{-12} = -0.73 .$$

If one carries out similar calculations for testing in the 2- and 3- directions, the general conclusion is that the quadratic parameters (the γ_{ij} and the α_{ii}) predict the correct trends from one direction to another but that the Bauschinger (or c) term introduces a considerable error. These facts will motivate the formulation of the theory to be presented in the next section.

Presentation of the Theory

The following points will be considered in constructing the plastic strain theory.

1. The quadratic parameters (the γ_{ij} and α_{ii}) from the yield criterion will be included in the strain theory. However, there will also be new parameters having to do with the Bauschinger effect in strain (the nonequality of k_{+ij} and k_{-ij}), which are not related to the Bauschinger effect in yield.

2. Although the stress-strain relation through the plastic region is highly nonlinear (stress varies with strain to some small power), the ratios of the various strains (the k_{ij}) are quite constant. That is, the anisotropy manifests itself in the same way throughout the entire plastic region.

3. The strain behavior in yield is incompressible.

These points can be incorporated into a theory of the following form:

$$\epsilon_j^i = \lambda(\text{inv}) B_{j\ell}^{*ik} \tau_k^\ell, \quad (28)$$

where ϵ_j^i = a nonlinear logarithmic strain tensor, zero strain corresponding to the start of yield,

$\lambda(\text{inv})$ = a scalar strongly dependent on the stress or strain invariants or on the history but unspecified for the present purposes,

$B_{j\ell}^{*ik}$ = material parameters closely associated with (but not equal to) the $B_{j\ell}^{ik}$ in the yield theory, and
 τ_k^ℓ = the deviator stress, zero at the start of yield.

The deviator stress can be defined as follows:

$$\begin{array}{ccccccc} \sigma_k^\ell & = & -p\delta_k^\ell & + & \tau_k^\ell & + & \tau_{Ek}^\ell \\ \text{total} & & \text{scalar} & & \text{deviator} & & \text{deviator elastic} \\ \text{applied} & & \text{stress} & & \text{stress} & & \text{stress at start} \\ \text{stress} & & \text{("pressure")} & & \text{above yield} & & \text{of yield} \end{array} \quad (29)$$

$$\text{where } p = -\frac{1}{3} \sigma_k^k.$$

A general nonlinear definition of ϵ_j^i is given by Truesdell and Toupin¹⁷ but need not concern us here. In simple extension and compression it is the usual natural strain:

$$\epsilon_{ii} = \ln(L_i/L_{i_0}),$$

¹⁷C. Truesdell and R. A. Toupin, "The Classical Field Theories," pp. 226-793 in Handbuch der Physik, Vol. III/1, ed. by S. Flügge, Springer-Verlag, Berlin, 1960, see p. 271.

where L_i = present length of the specimen in the i -direction,

L_{i0} = original length of the specimen in the i -direction at the beginning of yield.

The B_{jl}^{*ik} will be assumed to be related to the B_{jl}^{ik} by the following relations, good in the anisotropic coordinate system (i.e., 1, 2, and 3 corresponding to axes of anisotropy):

$$B_{11}^{*11} = B_{11}^{11} - \xi_1^1 \sigma_1^1, \quad (30)$$

$$B_{22}^{*22} = B_{22}^{22} - \xi_2^2 \sigma_2^2, \quad (31)$$

$$B_{33}^{*33} = B_{33}^{33} - \xi_3^3 \sigma_3^3, \quad (32)$$

$$B_{12}^{*12} + B_{21}^{*21} = B_{12}^{12} + B_{21}^{21} + \xi_1^1 \sigma_1^1 + \xi_2^2 \sigma_2^2 - \xi_3^3 \sigma_3^3, \quad (33)$$

$$\begin{matrix} \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot \end{matrix}$$

$$B_{21}^{*12} + B_{12}^{*21} = B_{21}^{12} + B_{12}^{21}, \quad (34)$$

where ξ_i^i = material parameters to be determined. It is not immediately clear how these relations can be generalized to tensor notation. The form assures incompressibility.

Now writing out Eq. (28) and retaining only nonzero values of B_{jl}^{*ik} , as before, one has

$$\epsilon_1^1 = \lambda(\text{inv}) \left[\overbrace{2B_{11}^{*11}}^{\alpha_{11}^*} \tau_1^1 + \left(\overbrace{B_{12}^{*12} + B_{21}^{*21}}^{\gamma_{12}^*} \right) \tau_2^2 + \left(\overbrace{B_{13}^{*13} + B_{31}^{*31}}^{\gamma_{13}^*} \right) \tau_3^3 \right], \quad (35)$$

$$\epsilon_2^2 = \lambda(\text{inv}) \left[\left(\overbrace{B_{12}^{*12} + B_{21}^{*21}}^{\gamma_{12}^{*2}} \right) \tau_1^1 + \overbrace{2B_{22}^{*22}}^{2\alpha_{22}^*} \tau_2^2 + \left(\overbrace{B_{23}^{*23} + B_{32}^{*32}}^{\gamma_{23}^{*2}} \right) \tau_3^3 \right], \quad (36)$$

$$\epsilon_3^3 = \lambda(\text{inv}) \left[\left(\overbrace{B_{13}^{*13} + B_{31}^{*31}}^{\gamma_{13}^{*3}} \right) \tau_1^1 + \left(\overbrace{B_{23}^{*23} + B_{32}^{*32}}^{\gamma_{23}^{*3}} \right) \tau_2^2 + \overbrace{2B_{33}^{*33}}^{2\alpha_{33}^*} \tau_3^3 \right], \quad (37)$$

$$\epsilon_2^1 = \lambda(\text{inv}) \left[\overbrace{2B_{21}^{12} + 2B_{12}^{21}}^{\beta_{12}} \right] \tau_2^1. \quad (38)$$

Similar relations hold for ϵ_3^1 and ϵ_3^2 . Using Eqs. (30) through (34) and abbreviating ξ_i^1 by ξ_i one has in the anisotropic coordinate system:

$$\alpha_{11}^* = \alpha_{11} - \xi_1 \sigma_1^1, \quad (39)$$

$$\alpha_{22}^* = \alpha_{22} - \xi_2 \sigma_2^2, \quad (40)$$

$$\alpha_{33}^* = \alpha_{33} - \xi_3 \sigma_3^3, \quad (41)$$

$$\gamma_{12}^* = \gamma_{12} - \xi_3 \sigma_3^3 + \xi_1 \sigma_1^1 + \xi_2 \sigma_2^2, \quad (42)$$

$$\gamma_{13}^* = \gamma_{13} - \xi_2 \sigma_2^2 + \xi_1 \sigma_1^1 + \xi_3 \sigma_3^3, \quad (43)$$

$$\gamma_{23}^* = \gamma_{23} - \xi_1 \sigma_1^1 + \xi_2 \sigma_2^2 + \xi_3 \sigma_3^3. \quad (44)$$

In most experiments the anisotropic coordinate system is Cartesian, in which case one may write $\epsilon_1^1 = \epsilon_{11}$ and $\tau_1^1 = \tau_{11}$, etc. Rearranging somewhat, Eqs. (35) through (38) become:

$$\epsilon_{11} = \lambda(\text{inv}) \left[-\gamma_{12}^* (\tau_{11} - \tau_{22}) - \gamma_{13}^* (\tau_{11} - \tau_{33}) \right], \quad (45)$$

$$\epsilon_{22} = \lambda(\text{inv}) \left[-\gamma_{12}^* (\tau_{22} - \tau_{11}) - \gamma_{23}^* (\tau_{22} - \tau_{33}) \right], \quad (46)$$

$$\epsilon_{33} = \lambda(\text{inv}) \left[-\gamma_{13}^* (\tau_{33} - \tau_{11}) - \gamma_{23}^* (\tau_{33} - \tau_{22}) \right]. \quad (47)$$

Equations (42) through (47) constitute the working equations in a Cartesian anisotropic coordinate system. Incompressibility is assured since Eqs. (35), (36), and (37) [or Eqs. (45), (46), and (47)] sum to zero. That is,

$$\epsilon_1^1 + \epsilon_2^2 + \epsilon_3^3 = 0$$

for any deformation.

Tensile-Compressive Testing

Consider an experiment in which a tensile stress T_1 (over and above the yield) is applied to the specimen in the 1-direction. The total

stress on the specimen is $\sigma_{11} = \lambda_{t1} + T_1 \approx Y_{t1}$. The stress situation can be summarized by the following relations:

$$\sigma_{11} \approx Y_{t1}, \quad \sigma_{22} = 0, \quad \sigma_{33} = 0,$$

$$\tau_{11} = 2T_1/3, \quad \tau_{22} = -T_1/3, \quad \tau_{33} = -T_1/3.$$

Thus

$$\epsilon_{11} = \lambda(\text{inv})[2\alpha_{11} - 2\xi_1 Y_{t1}]T_1,$$

$$\epsilon_{22} = \lambda(\text{inv})[\gamma_{12} + \xi_1 Y_{t1}]T_1,$$

$$\epsilon_{33} = \lambda(\text{inv})[\gamma_{13} + \xi_1 Y_{t1}]T_1.$$

Formation of a ratio as before yields

$$k_{+12} = \frac{\epsilon_{22}}{\epsilon_{11}} \Big|_{+1} = \frac{\gamma_{12} + \xi_1 Y_{t1}}{2\alpha_{11} - 2\xi_1 Y_{t1}}. \quad (48)$$

By similar steps one can obtain the following:

$$k_{-12} = \frac{-\gamma_{12} + \xi_1 Y_{c1}}{-2\alpha_{11} - 2\xi_1 Y_{c1}}, \quad (49)$$

$$k_{+21} = \frac{+\gamma_{12} + \xi_2 Y_{t2}}{2\alpha_{22} - 2\xi_2 Y_{t2}}, \quad (50)$$

$$k_{-21} = \frac{-\gamma_{12} + \xi_2 Y_{c2}}{-2\alpha_{22} - 2\xi_2 Y_{c2}}, \quad (51)$$

$$k_{+31} = \frac{+\gamma_{13} + \xi_3 Y_{t3}}{2\alpha_{33} - 2\xi_3 Y_{t3}}, \quad (52)$$

$$k_{-31} = \frac{-\gamma_{13} + \xi_3 Y_{c3}}{-2\alpha_{33} - 2\xi_3 Y_{c3}}. \quad (53)$$

These six k_{ij} are those commonly used. Relationships for the remaining ratios may be obtained in a similar manner.

The following interrelationship is sometimes useful and holds generally:

$$k_{+ij} + k_{+ik} = -1 \quad \text{for } k \neq j. \quad (54)$$

The following interrelationships rigorously hold only for the case of no Bauschinger effect ($\xi_1 = \xi_2 = \xi_3 = 0$), although they may hold approximately in other cases because of the canceling effect of the small terms involving the ξ_i :

$$\frac{k_{-12} k_{-23} k_{-31}}{k_{-13} k_{-21} k_{-32}} = \frac{k_{+12} k_{+23} k_{+31}}{k_{+13} k_{+21} k_{+32}} = 1. \quad (55)$$

Although the index notation is being used here, it should be noted that the k_{ij} are not tensors.

APPLICATION TO ZIRCALOY-2

General Remarks

Rittenhouse¹⁸ has reported yield stress and strain ratio data for several samples of Zircaloy-2 of differing textures. The yield stress data in Table 2 were reported for schedule 8, the most anisotropic of those studied. A revised estimate of Y_{t3} (shown in Table 2) was used. The strain ratio data for this schedule are as follows:

$$\begin{array}{lll} k_{+12} = -0.885 & k_{-12} = -0.714 & k_{+31} = -0.498 \\ k_{-31} = -0.488 & k_{+21} = -0.844 & k_{-21} = -0.701 \end{array}$$

These data will now be processed through the theoretical equations to illustrate their use.

¹⁸P. L. Rittenhouse, Characterization of the Three-Dimensional Anisotropy of Yielding and Flow in Zircaloy-2 from a Single Uniaxial Test, ORNL-TM-1553 (August 1966).

Table 2. Yield Stress Data for Schedule 8 Zircaloy-2

Direction	Yield Stress, Mpsi ^a	
	Y_{ti}	Y_{ci}
1 (RD)	0.055	0.066
2 (TD)	0.058	0.067
3 (ND)	0.076 ^b	0.118

^aSince the unit millions of pounds per square inch is convenient to use here and in subsequent discussion, it will be abbreviated as Mpsi.

^bRevised estimate.

Determination of Constants and Presentation of Yield Surfaces

Using the equations of Appendix B, one can calculate the values of the material constants shown in Table 1, starting with values for the yield stresses in tension and yield. For schedule 8 Zircaloy-2, the results are:

$$\begin{aligned}
 \alpha_{11} &= 282 \text{ Mpsi}^{-2} & \gamma_{12} &= -408 \text{ Mpsi}^{-2} & \alpha_1 &= 0.82 \text{ Mpsi}^{-1} \\
 \alpha_{22} &= 263 \text{ Mpsi}^{-2} & \gamma_{23} &= -118 \text{ Mpsi}^{-2} & \alpha_2 &= -0.05 \text{ Mpsi}^{-1} \\
 \alpha_{33} &= 137 \text{ Mpsi}^{-2} & \gamma_{13} &= -156 \text{ Mpsi}^{-2} & \alpha_3 &= -0.77 \text{ Mpsi}^{-1} \\
 c &= 8214 \text{ Mpsi}^{-3}
 \end{aligned}$$

Having available these constants, one can now predict the shapes of the cross sections in the various planes of the yield surface. The equations are shown in Appendix C and the calculated results are shown in Figs. 1, 2, and 3. The dotted portions of the lines result from a difficulty in convergence of the digital computer program and not from a discontinuity in the equations. The curves must of necessity pass through the yield stresses in uniaxial testing; thus the real test of such curves must await comparison with yield data obtained by biaxial or other complex loadings.

Fig. 1. Yield Surface Cross Section in the 1-2 Plane (Schedule 8 Zircaloy-2).

Fig. 2. Yield Surface Cross Section in the 2-3 Plane (Schedule 8 Zircaloy-2).

Fig. 3. Yield Surface Cross Section in the 3-1 Plane (Schedule 8 Zircaloy-2).

If, instead of the complete theory [Eq. (14)], one used the simplified theory [Eq. (18)], the following constants should be used:

$$\begin{aligned}
 \alpha_{11} &= 282 \text{ Mpsi}^{-2} & \gamma_{12} &= -408 \text{ Mpsi}^{-2} \\
 \alpha_{22} &= 263 \text{ Mpsi}^{-2} & \gamma_{23} &= -118 \text{ Mpsi}^{-2} & c &= 8000 \text{ Mpsi}^{-3} \\
 \alpha_{33} &= 137 \text{ Mpsi}^{-2} & \gamma_{13} &= -156 \text{ Mpsi}^{-2} & \alpha_1 &= \alpha_2 = \alpha_3 = 0
 \end{aligned}$$

A "compromise" value of c was chosen to best accommodate all of the directions. The extent to which this simplification influences the results is shown in Table 3.

Table 3. Comparison of Yields Stresses from Complete and Simplified Theories

Theory	Yield Stress, Mpsi					
	Y_{t1}	Y_{c1}	Y_{t2}	Y_{c2}	Y_{t3}	Y_{c3}
Complete (and experimental results)	0.055	0.066	0.058	0.067	0.076	0.118
Simplified	0.056	0.065	0.059	0.067	0.074	0.125

Interrelation of Yield Stresses and Strain Ratios

The problem of calculating the yield stresses from the strain ratios, or vice versa, centers on the trial-and-error solution of the equations relating the quadratic parameters, which are common to the two theories, and assuming or empirically correlating the Bauschinger parameters, which are not common. Also, in working with the equations, it is immediately clear that the strain ratios are a much more sensitive measure of the anisotropy than are the yield stresses; that is, small errors in the stresses are magnified tremendously in calculating strain ratios.

These considerations led to the following procedure: take as given all of the measured strain ratios (the k_{ij}) and one yield stress (Y_{t1}), and assume a reasonable value of c [$c = 10,000 \text{ Mpsi}^{-3}$]. (Preliminary work with other schedules of Zircaloy-2 indicates that c may not vary too widely from schedule to schedule.) By a trial-and-error procedure (shown in Appendix D), one can then calculate the remaining material constants and the remaining yield stresses. The results of these calculations are given in Table 4. Preliminary work with the other schedules suggests that the agreement may not be this satisfactory in all cases.

Were one to take two of the yield stresses as given (say Y_{t1} and Y_{t2}), more accuracy would be possible in calculating the remaining yield stresses. The algebra is somewhat formidable, but one should be able to set up an iterative computer solution similar to that shown in Appendix D.

Table 4. Calculation of Material Constants from Strain Ratios, One Yield Stress, and an Assumed c

Direction	Values Calculated for $c = 10,000 \text{ Mpsi}^{-3}$				Experimental (Mpsi)	
	ξ_i (Mpsi^{-3})	α_i (Mpsi^{-2})	Y_{ti} (Mpsi)	Y_{ci} (Mpsi)	Y_{ti}	Y_{ci}
1 (RD)	1419	290	0.055	0.064	0.055	0.066
2 (TD)	1592	294	0.055	0.064	0.058	0.067
3 (ND)	0	131	0.074	0.124	0.076	0.118

CONCLUDING COMMENTS ON THE APPLICATION OF THE THEORIES

The yield criterion can accommodate exactly the yield stresses in the three anisotropic directions, including the Bauschinger effect. The six independent material constants can be calculated and used to predict yield surface cross sections as shown in Figs. 1, 2, and 3.

The strain theory and yield stresses are interrelated through the quadratic parameters (the α_{ii} or the γ_{ij}). Additional Bauschinger parameters appear in each of the two cases and are not interrelated, although experience may suggest some empirical relationships. Assuming a reasonable value for c and taking as given all of the experimental strain ratios and one experimental yield stress, one can predict the remaining yield stresses. For schedule 8 Zircaloy-2 the theory agreed with the experimental yield stresses to within about 5%.

The practical difficulties of carrying out the trial-and-error algebra between the two theories may limit their usefulness in the prediction of yield stresses from strain ratios, or vice versa. Hopefully, however, the correlation of data in this framework will help identify the similarities and the differences of the anisotropy in the two experiments.

APPENDICES

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APPENDIX A

Transformation to an Arbitrary Coordinate System

The material tensors A_j^i and B_{ik}^{jl} are second- and fourth-order tensors, respectively. They can be transformed from the anisotropic coordinate system x_i to an arbitrary coordinate system $\bar{x}_i$ by the usual transformation laws:

$$\bar{A}_q^p = A_j^i \frac{\partial \bar{x}^p}{\partial x^i} \frac{\partial x^j}{\partial \bar{x}_q}$$

$$\bar{B}_{rs}^{pq} = B_{ik}^{jl} \frac{\partial \bar{x}^p}{\partial x^j} \frac{\partial \bar{x}^q}{\partial x^l} \frac{\partial x^i}{\partial \bar{x}^r} \frac{\partial x^k}{\partial \bar{x}^s}$$

As an example of this transformation, consider a coordinate system which is rotated by angle χ about the 3-axis in the 1-2 plane:

The transformation partial derivatives are:

$$\frac{\partial \bar{x}_1}{\partial x_1} = \cos \chi, \quad \frac{\partial \bar{x}_2}{\partial x_1} = -\sin \chi, \quad \frac{\partial \bar{x}_1}{\partial x_2} = \sin \chi, \quad \frac{\partial \bar{x}_2}{\partial x_2} = \cos \chi.$$

Because one is dealing with Cartesian coordinates, it is not necessary to distinguish between covariant and contravariant (i.e. down and up) positioning of indices. Thus the transformation equation for the material tensor B_{ik}^{jl} is:

$$\bar{B}_{pqrs} = B_{jlik} \frac{\partial \bar{x}_p}{\partial x_j} \frac{\partial \bar{x}_q}{\partial x_l} \frac{\partial \bar{x}_r}{\partial x_i} \frac{\partial \bar{x}_s}{\partial x_k}$$

If, for example, one wished to find $\bar{\gamma}_{12}$ in the rotated coordinate system one would write

$$\bar{\gamma}_{12} \equiv \bar{B}_{1212} + \bar{B}_{2121} = B_{jlik} \frac{\partial \bar{x}_1}{\partial x_j} \frac{\partial \bar{x}_2}{\partial x_l} \frac{\partial \bar{x}_1}{\partial x_i} \frac{\partial \bar{x}_2}{\partial x_k} + B_{jlik} \frac{\partial \bar{x}_2}{\partial x_j} \frac{\partial \bar{x}_1}{\partial x_l} \frac{\partial \bar{x}_2}{\partial x_i} \frac{\partial \bar{x}_1}{\partial x_k}.$$

Retaining only the B_{jlik} that have values [see Eq. (6)] one has

$$\begin{aligned} \bar{\gamma}_{12} = & B_{1111}(\cos^2 X \sin^2 X) + B_{1122}(-\sin^2 X \cos^2 X) + B_{2211}(-\sin^2 X \cos^2 X) \\ & + B_{2222}(\sin^2 X \cos^2 X) + B_{1212}(\cos^4 X) + B_{1221}(-\sin^2 X \cos^2 X) \\ & + B_{2112}(-\sin^2 X \cos^2 X) + B_{2121}(\sin^4 X) \\ & + B_{1111}(\cos^2 X \sin^2 X) + B_{1122}(-\sin^2 X \cos^2 X) + B_{2211}(-\sin^2 X \cos^2 X) \\ & + B_{2222}(\sin^2 X \cos^2 X) + B_{1212}(\sin^4 X) + B_{1221}(-\sin^2 X \cos^2 X) \\ & + B_{2112}(-\sin^2 X \cos^2 X) + B_{2121}(\cos^4 X). \end{aligned}$$

Noting that $B_{1122} = B_{1221} = B_{2112}$ etc., one has

$$\begin{aligned} \bar{\gamma}_{12} = & 2\alpha_{11} \cos^2 X \sin^2 X + 2\alpha_{22} \cos^2 X \sin^2 X + \gamma_{12} \cos^4 X + \gamma_{12} \sin^4 X \\ & - 2\beta_{12} \cos^2 X \sin^2 X. \end{aligned}$$

or

$$\bar{\gamma}_{12} = \gamma_{12}(\cos^2 X - \sin^2 X)^2 - (\gamma_{13} + \gamma_{23} + 2\beta_{12})\cos^2 X \sin^2 X.$$

Continuing in this manner one has finally the yield criterion in the rotated coordinate system:

$$\begin{aligned} & \bar{\alpha}_1 \bar{\tau}_{11} + \bar{\alpha}_2 \bar{\tau}_{22} + \bar{\alpha}_3 \bar{\tau}_{33} - \frac{1}{2} \bar{\gamma}_{12} (\bar{\tau}_{11} - \bar{\tau}_{22})^2 - \frac{1}{2} \bar{\gamma}_{13} (\bar{\tau}_{11} - \bar{\tau}_{33})^2 \\ & - \frac{1}{2} \bar{\gamma}_{23} (\bar{\tau}_{22} - \bar{\tau}_{33})^2 + \bar{\beta}_{12} \bar{\tau}_{12}^2 + \bar{\beta}_{13} (\bar{\tau}_{13})^2 + \bar{\beta}_{23} (\bar{\tau}_{23})^2 + c \det \bar{\tau}_{mn} \\ & + (\bar{a}_{12} + {}^1\bar{\Theta}_{12} \bar{\tau}_{11} + {}^2\bar{\Theta}_{12} \bar{\tau}_{22} + {}^3\bar{\Theta}_{12} \bar{\tau}_{33}) \bar{\tau}_{12} + {}^3\bar{\Theta}_{12} \bar{\tau}_{23} \bar{\tau}_{13} = 1, \end{aligned}$$

where

$$\begin{aligned}
 \bar{\alpha}_1 &= \alpha_1 \cos^2 \chi + \alpha_2 \sin^2 \chi , \\
 \bar{\alpha}_2 &= \alpha_1 \sin^2 \chi + \alpha_2 \cos^2 \chi , \\
 \bar{\alpha}_3 &= \alpha_3 , \\
 \bar{\gamma}_{12} &= \gamma_{12} (\cos^2 \chi - \sin^2 \chi)^2 - (\gamma_{13} + \gamma_{23} + 2\beta_{12}) \cos^2 \chi \sin^2 \chi \\
 \bar{\gamma}_{13} &= \gamma_{13} \cos^2 \chi + \gamma_{23} \sin^2 \chi , \\
 \bar{\gamma}_{23} &= \gamma_{13} \sin^2 \chi + \gamma_{23} \cos^2 \chi , \\
 \bar{\beta}_{12} &= -2(4\gamma_{12} + \gamma_{13} + \gamma_{23}) \cos^2 \chi \sin^2 \chi + \beta_{12} (\cos^2 \chi - \sin^2 \chi)^2 , \\
 \bar{\beta}_{13} &= \beta_{13} \cos^2 \chi + \beta_{23} \sin^2 \chi , \\
 \bar{\beta}_{23} &= \beta_{13} \sin^2 \chi + \beta_{23} \cos^2 \chi , \\
 \bar{a}_{12} &= (2 \cos \chi \sin \chi)(\alpha_2 - \alpha_1) , \\
 {}^1\bar{\Theta}_{12} &= (2 \cos \chi \sin \chi)[(2\gamma_{12} + \beta_{12})(\cos^2 \chi - \sin^2 \chi) \\
 &\quad + \gamma_{13} \cos^2 \chi - \gamma_{23} \cos^2 \chi] , \\
 {}^2\bar{\Theta}_{12} &= (2 \cos \chi \sin \chi)[(2\gamma_{12} + \beta_{12})(\sin^2 \chi - \cos^2 \chi) \\
 &\quad + \gamma_{13} \sin^2 \chi - \gamma_{23} \cos^2 \chi] , \\
 {}^3\bar{\Theta}_{12} &= (2 \cos \chi \sin \chi)(\gamma_{23} - \gamma_{13}) , \\
 {}^3\bar{\Phi}_{12} &= (2 \cos \chi \sin \chi)(\beta_{23} - \beta_{13}) .
 \end{aligned}$$

Also,

$$\begin{aligned}
 \bar{\alpha}_{11} &= -\frac{1}{2} [\bar{\gamma}_{12} + \bar{\gamma}_{13}] , \\
 \bar{\alpha}_{22} &= -\frac{1}{2} [\bar{\gamma}_{12} + \bar{\gamma}_{23}] , \\
 \bar{\alpha}_{33} &= -\frac{1}{2} [\bar{\gamma}_{13} + \bar{\gamma}_{23}] , \\
 \bar{\alpha}_1 + \bar{\alpha}_2 + \bar{\alpha}_3 &= 0 .
 \end{aligned}$$

These equations relate the material tensors $\bar{\alpha}_i$, $\bar{\gamma}_{ij}$, $\bar{\beta}_{ij}$, etc. in the new system to the corresponding entities in the anisotropic system. Thus, for example, if one performed a tensile yield test along a specimen whose test axis $\bar{x}_1$ is 30° from the anisotropic reference axis x_1 , the tensile yield strength $\bar{Y}_{t1}$ along the test axis would be given by

$$\bar{\alpha}_1 \bar{Y}_{t1} + \bar{\alpha}_{11} \left(\bar{Y}_{t1} \right)^2 + \frac{2}{27} c \left(\bar{Y}_{t1} \right)^3 = 1 ,$$

where

$$\begin{aligned}
 \bar{\alpha}_1 &= \alpha_1 \cos^2 30^\circ + \alpha_2 \sin^2 30^\circ , \\
 \bar{\alpha}_{11} &= -\frac{1}{2} [\bar{\gamma}_{12} + \bar{\gamma}_{13}] = -\frac{1}{2} [\gamma_{12} (\cos^2 30^\circ - \sin^2 30^\circ)^2 \\
 &\quad + \gamma_{13} \cos^4 30^\circ + \gamma_{23} \sin^4 30^\circ - 2\beta_{12} \cos^2 30^\circ \sin^2 30^\circ] , \\
 c &= \text{unchanged.}
 \end{aligned}$$

However, there is also an important difference between Eq. (A-1) and its anisotropic counterpart, Eq. (14), having to do with the appearance of the terms involving $\bar{a}_{12}$, $k_{ij}^{\bar{\Theta}}$, $k_{ij}^{\bar{\Phi}}$. These parameters have the effect of producing a difference in yield strength in shear when the sign of the shear stress is changed. This is not a big effect (in fact, a zero effect if $\alpha_i = 0$) except in the presence of a tensile or compressive stress. Consider, for example, $\bar{\tau}_{12}$ superimposed on the tension experiment $\bar{\tau}_{11} = \frac{2}{3} \bar{\sigma}_{11}$, $\bar{\tau}_{22} = \bar{\tau}_{33} = -\frac{1}{3} \bar{\sigma}_{11}$. At yield, one has

$$\begin{aligned} \bar{\alpha}_1 \bar{\sigma}_{11} + \bar{\alpha}_{11} \bar{\sigma}_{11}^2 + \bar{\beta}_{12} \bar{\tau}_{12}^2 + 2c(\bar{\sigma}_{11})^3/27 \\ + [\bar{a}_{12} + (\frac{2}{3} \bar{\Theta}_{12} - \frac{1}{3} \bar{\Theta}_{12} - \frac{1}{3} \bar{\Theta}_{12}) \bar{\sigma}_{11}] \bar{\tau}_{12} = 1. \end{aligned}$$

By similar transformations one can write down yield criteria for rotations in the 2-3 and 3-1 planes. More complicated transformations are possible; they would, to say the least, be cumbersome.

APPENDIX B

Equations and Computer Scheme for Determining
Material Constants from Yield Stress Data

The most straightforward procedure for determining the material constants is to do compressive-tensile testing along anisotropic directions. One has available then six experimental values (Y_{t1} , Y_{t2} , Y_{t3} , Y_{c1} , Y_{c2} , Y_{c3}) and six constants to be determined from them (any two of α_1 , α_2 , and α_3 ; either γ_{12} , γ_{13} , and γ_{23} or α_{11} , α_{22} , and α_{33} ; and c). The algebra consists of solving the following equations simultaneously:¹⁹

$$\begin{aligned}\alpha_i Y_{ti} + \alpha_{ii} (Y_{ti})^2 + \frac{2}{27} c (Y_{ti})^3 &= 1 & i = 1, 2, 3, \\ -\alpha_i Y_{ci} + \alpha_{ii} (Y_{ci})^2 - \frac{2}{27} c (Y_{ci})^3 &= 1 & i = 1, 2, 3.\end{aligned}$$

The results are:

$$c = \frac{\frac{27}{2} \left(\frac{Y_{c1} - Y_{t1}}{Y_{t1} Y_{c1}} + \frac{Y_{c2} - Y_{t2}}{Y_{t2} Y_{c2}} + \frac{Y_{c3} - Y_{t3}}{Y_{t3} Y_{c3}} \right)}{Y_{t1} Y_{c1} + Y_{t2} Y_{c2} + Y_{t3} Y_{c3}} \quad (B-1)$$

$$\alpha_i = \frac{Y_{ci} - Y_{ti} - \frac{2}{27} c (Y_{ti})^2 (Y_{ci})^2}{Y_{ti} Y_{ci}} \quad i = 1, 2, 3 \quad (B-2)$$

$$\alpha_{ii} = \frac{1 - \frac{2}{27} c (Y_{ti})^3 - \alpha_i Y_{ti}}{(Y_{ti})^2} \quad i = 1, 2, 3 \quad (B-3)$$

Also

$$\gamma_{12} = \alpha_{33} - \alpha_{11} - \alpha_{22} \quad (B-4)$$

$$\gamma_{13} = \alpha_{22} - \alpha_{11} - \alpha_{33} \quad (B-5)$$

$$\gamma_{23} = \alpha_{11} - \alpha_{22} - \alpha_{33} \quad (B-6)$$

¹⁹Index (but not tensor) notation is being used here.

The constants β_{ij} cannot be obtained except from shear experiments [see Eq. (17)] or tensile-compressive testing on a rotated specimen (see Appendix A).

If the simplified theory is used (all $\alpha_i = 0$), one has the following simpler result for c :

$$\frac{2}{27} c = \frac{Y_{c1} - Y_{t1}}{(Y_{c1})^2 (Y_{t1})^2} = \frac{Y_{c2} - Y_{t2}}{(Y_{c2})^2 (Y_{t2})^2} = \frac{Y_{c3} - Y_{t3}}{(Y_{c3})^2 (Y_{t3})^2}.$$

One may have to compromise the three directions somewhat to obtain an average value for c . Equations (B-3) through (B-6) are the same as before.

The computer scheme uses the full theory and consists of taking the six given values of Y_{ti} and Y_{ci} and inserting them into Eqs. (B-1) through (B-6).

APPENDIX C

Equations and Computer Scheme for Yield Surface Cross Sections

Having determined the constants, one can now produce the yield surface cross sections that are shown in Figs. 1, 2, and 3. Consider, for example, biaxial loading where both σ_{11} and σ_{22} (along anisotropic directions) can take on values at the same time. The yield equation is

$$\alpha_1 \tau_{11} + \alpha_2 \tau_{22} + \alpha_3 \tau_{33} - \frac{1}{2} \gamma_{12} (\sigma_{11} - \sigma_{22})^2 - \frac{1}{2} \gamma_{13} (\sigma_{11})^2 - \frac{1}{2} \gamma_{23} (\sigma_{22})^2 + c \tau_{11} \tau_{22} \tau_{33} = 1, \quad (C-1)$$

where

$$\begin{aligned} \tau_{11} &= + 2\sigma_{11}/3 - \sigma_{22}/3, \\ \tau_{22} &= - \sigma_{11}/3 + 2\sigma_{22}/3, \\ \tau_{33} &= - (\sigma_{11} + \sigma_{22})/3. \end{aligned}$$

Since Eq. (C-1) is cubic, its direct solution is quite complex. However, an iterative computer scheme can be set up using the following "near solution" [a solution that assumes that the quadratic terms in Eq. (C-1) predominate]:

$$\sigma_{22} = \frac{-\gamma_{12}\sigma_{11} \pm \sqrt{(\gamma_{12})^2(\sigma_{11})^2 - 4\alpha_{22}K}}{2\alpha_{22}}, \quad (C-2)$$

where

$$\begin{aligned} K &= \alpha_{11}(\sigma_{11})^2 + c\tau_{11}\tau_{22}\tau_{33} + \alpha_1\tau_{11} + \alpha_2\tau_{22} + \alpha_3\tau_{33} - 1, \\ \tau_{11} &= + 2\sigma_{11}/3 - \sigma_{22}/3, \\ \tau_{22} &= - \sigma_{11}/3 + 2\sigma_{22}/3, \\ \tau_{33} &= - (\sigma_{11} + \sigma_{22})/3. \end{aligned}$$

The computer takes values of σ_{11} in equally spaced units, assumes a rough value for σ_{22} , and then iterates continually on Eq. (C-2) until

a solution is obtained for each value of σ_{11} . Similar equations may be used in the 2-3 and 3-1 planes. This procedure worked satisfactorily except in the extremities of the curves shown as the dotted lines in Figs. 1, 2, and 3.

APPENDIX D

Equations and Computer Scheme for Interrelating
Yield Stresses and Strain Ratios

The procedure for interrelating the yield stresses and the strain ratios is somewhat involved because the yield stresses appear explicitly in the equations for the strain ratios. Thus a trial-and-error (or iterative) scheme is necessary. Two special cases will be treated below.

1. Given six k_{ij} , one Y_{ti}

In addition, one assumes a priori a value for c and sets all $\alpha_i = 0$ (simplified theory). The program then calculates the remaining material constants and the other five yield stresses.

The computer program first makes rough estimates for the remaining yield stresses; it then iterates continuously on the following set of equations (in order) until a solution is obtained. These equations represent an algebraic rearrangement of Eqs. (11), (15), and (48) through (53). The yield stress Y_{t1} is given.

$$\alpha_{11} = [1 - \frac{2}{27} c(Y_{t1})^3] / (Y_{t1})^2, \quad (D-1)$$

$$\xi_1 = \frac{2\alpha_{11}(k_{+12} - k_{-12})}{Y_{t1}(1 + 2k_{+12}) + Y_{c1}(1 + 2k_{-12})}, \quad (D-2)$$

$$\gamma_{12} = 2k_{+12}\alpha_{11} - \xi_1 Y_{t1}(1 + 2k_{+12}), \quad (D-3)$$

$$\xi_2 = \frac{\gamma_{12}(k_{+21} - k_{-21})}{Y_{t2}k_{-21}(2k_{+21} + 1) + Y_{c2}k_{+21}(2k_{-21} + 1)}, \quad (D-4)$$

$$\alpha_{22} = \frac{\gamma_{12} + \xi_2 Y_{t2}(1 + 2k_{+23})}{2k_{+21}}, \quad (D-5)$$

$$\alpha_{33} = \gamma_{12} + \alpha_{11} + \alpha_{22}. \quad (D-6)$$

$$\xi_3 = \frac{2\alpha_{33}(k_{+31} - k_{-31})}{Y_{t3}(1 + 2k_{+31}) + Y_{c3}(1 + 2k_{-31})}, \quad (D-7)$$

$$Y_{ci} = (\alpha_{ii} - \frac{2}{27} c Y_{ci})^{-\frac{1}{2}} \quad i = 1, 2, 3 \quad (D-8)$$

$$Y_{ti} = (\alpha_{ii} + \frac{2}{27} c Y_{ti})^{-\frac{1}{2}} \quad i = 2, 3. \quad (D-9)$$

A difficulty occurs if a direction i is nearly isotropic ($k_{-ij} = k_{+ij} = -0.5$). The equations for ξ_i blow up, both numerator and denominator going to zero. Physically, however, the ξ_i must go to zero for this case. It was thus arbitrarily decided that if both $|k_{-ij}|$ and $|k_{+ij}|$ lay between 0.48 and 0.52, ξ_i was set equal to zero.

The final results are values for Y_{t2} , Y_{t3} , Y_{c1} , Y_{c2} , Y_{c3} , all γ_{ij} and α_{ii} , and all ξ_i . This procedure was used to calculate the values shown in the discussion of the strain ratio results.

2. Given six k_{ij} , one set of Y_{ti} and Y_{ci}

The procedure is exactly the same as before except that now one calculates a value for c by the equation

$$\frac{2}{27} c = (Y_{c1} - Y_{t1}) / (Y_{c1})^2 (Y_{t1})^2.$$

This procedure has not been programmed.

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